DESCRIPTION OF MY PAST RESEARCH

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1. Introduction

In this text, I speak about my past research. I intentionally left apart the results of [RW17a] and [RW17b] because I need them in my research project. Since these two papers contains by far my most consequent results, I advise the reader to have a look at section A.1 of my research project. Until 2016, I have been focusing mostly on the $\mathfrak{sl}_3$-homology and more specifically I have been studying the web-algebras attached to this theory.

2. Context: the $\mathfrak{sl}_3$-homology

The $\mathbb{C}(q)$-vector space $U_q(\mathfrak{sl}_3)$ is a deformation of the universal algebra of the Lie algebra $\mathfrak{sl}_3$. It is naturally endowed with a structure of Hopf algebra and its category of finite dimensional modules is monoidal rigid and braided. This gives rise to a polynomial invariant of oriented links (the $\mathfrak{sl}_3$-*invariant*) by interpreting a link diagram using the following dictionnary:

Up-going strand	$\leftrightsquigarrow$	Identity of V^+ the standard representation of $U_q(\mathfrak{sl}_3)$.
Down-going strand	$\leftrightsquigarrow$	Identity of V^+, the dual of V^-.
Cups and caps	$\leftrightsquigarrow$	Duality structures.
Crossings	$\leftrightsquigarrow$	Braidings.
Horizontal concatenation	$\leftrightsquigarrow$	Tensor products.
Vertical concatenation	$\leftrightsquigarrow$	Composition of morphisms.

If D is a link diagram, we denote by $\langle D\rangle$ the endomorphism of $\mathbb{C}(q)$ obtained using the dictionary. We identify $\langle D\rangle$ with an element of $\mathbb{C}(q)$. It turns out that the value of $\langle \bullet \rangle$ on the unknot is equal to $[3]_q := \frac{q^3-q^{-3}}{q-q^{-1}} = q^{-2}+1+q^2$ and that the following skein relation holds:

$$q^{-2}\left\langle \times \right\rangle - q^{2}\left\langle \times \right\rangle = (q-q^{-1})\left\langle)\ (\right\rangle .$$

One can use this relation to prove that $\langle \cdot \rangle$ is a Laurent polynomial in q: c'est l'invariant $\mathfrak{sl}_3$.

In order to compute the $\mathfrak{sl}_3$-invariant, the previous relation is not convenient. This is why we prefer the following alternative relations:

$$\left\langle \times \right\rangle = q^{3}\left\langle)\ (\right\rangle - q^{2}\left\langle \asymp \right\rangle, \qquad \left\langle \times \right\rangle = q^{-3}\left\langle)\ (\right\rangle - q^{-2}\left\langle \asymp \right\rangle .$$

The trivalent vertices represent the morphisms $V^+\otimes V^+ \to V^+ \wedge_q V^+ \simeq V^-$ and $V^-\otimes V^- \to V^- \wedge_q V^- \simeq V^+$. Thanks to these relations, it is enough to know how to evaluate bipartite, plane, trivalent graphs (we call these objects *webs*), such an evaluation is provided by Kuperberg [Kup96].

In 2003, Khovanov [Kho04] categorified the $\mathfrak{sl}_3$-invariant. This means that he associated with every link diagram D a complex $[\![D]\!]$ of graded $\mathbb{Z}$-modules. The homotopy type of $[\![D]\!]$ depends only on the link represented by D and its graded Euler characteristic is equal to the $\mathfrak{sl}_3$-invariant of the underlying link. The homology of the complex $[\![D]\!]$ is called the $\mathfrak{sl}_3$-*homology*. In order to construct $[\![\cdot]\!]$, Khovanov defines a (web, *foam*[1])-TQFT which categorifies Kuperberg's evaluation; he interprets crossings as homological cones using the alternative relations.

Categorification provides finer invariants: the $\mathfrak{sl}_3$-homology detects more links that the $\mathfrak{sl}_3$-invariant does. Moreover, it is functorial, this means in particular that is concerned with cobordisms between links. The functoriality reveals some topological information about links which was until then mysterious (see [Ras10], [Lew14]).

3. Extension to tangles

An (oriented) *tangle* is an (oriented) link in $\mathbb{R}^2 \times [0, 1]$ with boundary. In order to use a categorical vocabulary, it is convenient to divide the boundary of a tangle into its part in $\mathbb{R}^2 \times \{0\}$ and its part in $\mathbb{R}^2 \times \{1\}$: we speak of $(\varepsilon_0, \varepsilon_1)$-tangles where ε_0 and ε_1 are two sequences of signs. The classical $\mathfrak{sl}_3$-invariant extends to a tangle invariant. It is not anymore a polynomial invariant. If T is an $(\varepsilon_0, \varepsilon_1)$-tangle, $\langle T\rangle$ is an element of $\mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(V^{\otimes\varepsilon_0}, V^{\varepsilon_1})$, where $V^{\otimes\varepsilon_i}$ are tensor products of V^+ and V^-.

It is natural to look for an extension of Khovanov's categorification of the $\mathfrak{sl}_3$-invariant to tangles. I succeeded to define this extension in [Rob12, Rob13a] (this construction has been independently defined and studied in [MPT14]). The idea is to restrict to some *admissible*[2] sequences of signs and to associate with every admissible sequence of signs ε an algebra K^ε. These algebras are called *Khovanov–Kuperberg algebras* or *web-algebras*. Once these algebras are defined one can define the categorification of the $\mathfrak{sl}_3$-invariant for tangles. We associate a complex of K^{ε_0}-modules-K^{ε_1} $[\![D]\!]$ with any diagram D of an $(\varepsilon_0, \varepsilon_1)$-tangle[3]. The homotopy type of $[\![D]\!]$ depends only of the tangle represented by D. Furthermore, if a diagram D of an $(\varepsilon_0, \varepsilon_2)$-tangle is obtained by stacking a diagram D_1 of an $(\varepsilon_0, \varepsilon_1)$-tangle and a diagram D_2 of an $(\varepsilon_1, \varepsilon_2)$-tangle along ε_1, we have $[\![D]\!] \simeq [\![D_1]\!] \otimes_{K^{\varepsilon_1}} [\![D_2]\!]$.

4. In search of indecomposable projective K^ε-modules

The Khovanov–Kuperberg algebras are the key element of the previous construction. An analogy with the $\mathfrak{sl}_2$-case[4] suggests that the indecomposable projective K^ε-modules are of high interest. Indeed, they should decategorify on the dual canonical base of $\mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(V^{\otimes\varepsilon}, \mathbb{C}(q))$. Moreover, the construction of the algebras K^ε's gives for free some projective K^ε-modules $M(w)$ associated with any ε-web w.

Some ε-webs are irreducible with respect to the combinatorial counterpart of the theory: these are the *non-elliptic* ε-webs[5]. However, Khovanov and Kuperberg observed [KK99] that the K^ε-module associated with some non-elliptic ε-webs are decomposable. The counter-example given in [KK99] is the smallest non-elliptic web containing a *nested face*. In [Rob13a], I proved that this is not a coincidence:

Theorem 1. *If a non-elliptic ε-web w does not contain any nested face, the projective K^ε-module $M(w)$ is indecomposable.*

This sufficient condition is nice since it is very easy to check, however it is far from being a necessary condition (especially when the length of ε grows). In

[1]Foams are the natural cobordisms between webs: surfaces with 1-dimensional singularities.

[2]A sequence of signs is *admissible* if its sum is divisible by 3.

[3]If D is a link diagram, we have $\varepsilon_0 = \varepsilon_1 = \emptyset$, $K^\emptyset = \mathbb{Z}$ and the complex $[\![D]\!]$ coincide with the one from section 2.

[4]The $\mathfrak{sl}_2$-invariant is the *Jones Polynomial* and its categorification is the *Khovanov homology.* The tangle version is defined and studied in [Kho02].

[5]An ε-web is *non-ellitptic* if it does not contain any circle, digon, or square.

[Rob15b], I proved that some combinatorial apsects of ε-webs have an algebraical counterparts. If w is an ε-web, the graded dimension[6] of $\mathrm{HOM}_{K^\varepsilon}(M(w), M(w))$ is equal to $q^{l(\varepsilon)}\left\langle w\overline{w}\right\rangle$ (where $\overline{w}$ is the mirror image of w, with reversed orientation). In particular, if the coefficient of q^0 in $q^{l(\varepsilon)}\left\langle w\overline{w}\right\rangle$ is 1, one can deduce that $M(w)$ is indecomposable. In [Rob15b], I proved that the converse is true:

Theorem 2. *A web-module $M(w)$ is indecomposable if and only if we can deduce it from the graded dimension of* $\mathrm{HOM}_{K^\varepsilon}(M(w), M(w))$.

The content of the paper is actually deeper. When $M(w)$ is decomposable, I provide an algorithm to start its decomposition. The complete decomposition seems to be a very difficult problem. However, we can compute the Grothendieck group of K^ε-proj, I did this in [Rob15c] (see [MPT14] as well).

Theorem 3. *The Grothendieck group of K^ε-proj is equal to*

$$\bigoplus_{w\in NE(\varepsilon)} \mathbb{Z}[M(w)],$$

where $NE(\varepsilon)$ is the set of non-elliptic ε-webs and $[M(w)]$ is the class of $M(w)$ in $K^0(K^\varepsilon\text{-proj})$.

The proof of this theorem relies on the combinatorics of toric foams and on a tool of algebraic K-theory called the Hattori–Stallings trace.

5. Colored homology

There is a colored version of the $\mathfrak{sl}_3$-invariant. One can color the components of an oriented framed link by any finite dimensional representation of $U_q(\mathfrak{sl}_3)$. This is the analogue of the colored Jones polynomial in the $\mathfrak{sl}_3$-case. In [Kho05], Khovanov categorified the colored Jones polynomial. His strategy is to use a *tensor resolution* of finite dimensional simple $\mathfrak{sl}_2$-modules and to interpret it in terms of cables and cobordisms. In [Rob16], I managed to define a tensor resolution of every finite dimensional simple $\mathfrak{sl}_3$-modules and to interpret it in terms of cables and cobordisms. Altogether, it provides a categorification of the colored $\mathfrak{sl}_3$-invariant.

6. Webs and colors

In [Rob13b], I proved that the evaluation of $\mathfrak{sl}_3$-webs can be seen as an enumeration of colorings[7] taking in account a certain degree. In addition I proved that all colorings of $\mathfrak{sl}_3$-webs are Kempe-equivalent. This means that one can transform any coloring into any other coloring by a finite sequence of semi-local moves. Furthermore, I show that these semi-local moves are well-behaved with respect to the degree.

In [Rob15a], I generalized the first result of [Rob13b] for $\mathfrak{sl}_N$-webs. The combinatorics is much more involved. This is can be seen as a combinatorial rewriting of [CKM14] and as a 1-dimensional analogue of [RW17a]. The proof gives some nice identities about quantum integers and quantum binomials.

References

[CKM14] Sabin Cautis, Joel Kamnitzer, and Scott Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014.

[Jae92] François Jaeger. A new invariant of plane bipartite cubic graphs. *Discrete Math.*, 101(1-3):149–164, 1992. Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II.

[6] I write HOM rather than Hom to emphasize that I do not assume the morphism to be degree-preserving.

[7] The same result without the degree has been obtained by Jaeger [Jae92].

[Kho02] Mikhail Khovanov. A functor-valued invariant of tangles. *Algebr. Geom. Topol.*, 2:665–741 (electronic), 2002.

[Kho04] Mikhail Khovanov. sl(3) link homology. *Algebr. Geom. Topol.*, 4:1045–1081, 2004.

[Kho05] Mikhail Khovanov. Categorifications of the colored Jones polynomial. *J. Knot Theory Ramifications*, 14(1):111–130, 2005.

[KK99] Mikhail Khovanov and Greg Kuperberg. Correction to: "Web bases for sl(3) are not dual canonical". *Pacific J. Math.*, 188(2):399, 1999.

[Kup96] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.

[Lew14] Lukas Lewark. Rasmussen's spectral sequences and the $\mathfrak{sl}_N$-concordance invariants. *Adv. Math.*, 260:59–83, 2014.

[MPT14] M. Mackaay, W. Pan, and D. Tubbenhauer. The $\mathfrak{sl}_3$-web algebra. *Math. Z.*, 277(1-2):401–479, 2014.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[Rob12] Louis-Hadrien Robert. Algebras for the $\mathfrak{sl}_3$-homology. In *Proceedings of the 12th Forum des jeunes mathématicien-ne-s*, Paris, 2012.

[Rob13a] Louis-Hadrien Robert. A large family of indecomposable projective modules for the Khovanov–Kuperberg algebra of sl_3–webs. *Journal of Knot Theory and Its Ramifications*, 22(11):1350062, 2013.

[Rob13b] Louis-Hadrien Robert. On edge-colorings of bicubic planar graphs. *ArXiv e-prints 1312.0361*, December 2013.

[Rob15a] Louis-Hadrien Robert. A new way to evaluate MOY graphs. *ArXiv e-prints*, December 2015.

[Rob15b] Louis-Hadrien Robert. A characterization of indecomposable web modules over Khovanov-Kuperberg algebras. *Algebr. Geom. Topol.*, 15(3):1303–1362, 2015.

[Rob15c] Louis-Hadrien Robert. Grothendieck groups of the Khovanov–Kuperberg algebras. *Journal of Knot Theory and Its Ramifications*, 0(0):1550070, 2015.

[Rob16] Louis-Hadrien Robert. Categorification of the colored $\mathfrak{sl}_3$-invariant. *J. Knot Theory Ramifications*, 25(7):1650038, 43, 2016.

[RW17a] Louis-Hadrien Robert and Emmanuel Wagner. A closed formula for the evaluation of $\mathfrak{sl}_n$-foams, 2017.

[RW17b] Louis-Hadrien Robert and Emmanuel Wagner. Symmetric Khovanov–Rozansky link homologies, 2017.