Schubert Calculus according to Schubert

Felice Ronga

February 16, 2006

Abstract

We try to understand and justify Schubert calculus the way Schubert did it.

1 Introduction

In his famous book [7] "Kalkül der abzählende Geometrie", published in 1879, Dr. Hermann C. H. Schubert has developed a method for solving problems of enumerative geometry, called Schubert Calculus today, and has applied it to a great number of cases. This book is self-contained : given some aptitude to the mathematical reasoning, a little geometric intuition and a good knowledge of the german language, one can enjoy the many enumerative problems that are presented and solved.

Hilbert's 15th problems asks to give a rigourous foundation to Schubert's method. This has been largely accomplished using intersection theory (see [4],[5], [2]), and most of Schubert's calculations have been confirmed.

Our purpose is to understand and justify the very method that Schubert has used. We will also step through his calculations in some simple cases, in order to illustrate Schubert's way of proceeding.

Here is roughly in what Schubert's method consists. First of all, we distinguish basic elements in the complex projective space : points, planes, lines. We shall represent by symbols, say x, y, conditions (in german : *Bedingungen*) that some geometric objects have to satisfy; the product $x \cdot y$ of two conditions represents the condition that x and y are satisfied, the sum $x + y$ represents the condition that x or y is satisfied. The conditions on the basic elements that can be expressed using other basic elements (for example : the lines in space that must go through a given point) satisfy a number of formulas that can be determined rather easily by geometric reasoning.

In order to solve an enumerative problem, one tries to express it in terms of conditions on the basic elements, by resorting if necessary to moderatly degenerate situations, which are geometrically simpler to handle, but might require to take in account the multiplicities of the solutions found. For example : to find the number of tangents tha can be drawn from a point P in a plane to a conic in the same plane. If we take the conic to be degenerate into two distinct lines intersecting in a point Q, then the line through P and Q is the only solution, but it must be counted twice (see figure 1)[1]. If we degenerate the conic into a double line, then all the lines through P can be considered as tangent, and nothing can be concluded.

[1] In the figures, the elements that are used to express conditions are black, the others are gray

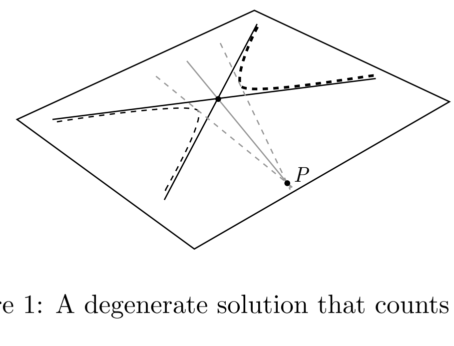

Figure 1: A degenerate solution that counts twice

Schubert justifies this procedure by the Principle of the conservation of the number (*Prinzip des Erhaltung der Anzahl*, [7, § 4, page 12]), which says roughly that that the number of solutions of an enumerative problem remains unchanged, when its parameters are varied, provided that this number remains finite. In turn, Schubert justifies this principle through its algebraic analog : the number of solutions of a polynomial equation (in one variable) doesn't change if the coefficients of the polynomial are varied, provided that multiplicities are taken into account, and that the equation doesn't become an identity (i.e. that the polynomial is not identically zero), in which case there are infinitely many solutions.

The strength of Schubert's approach resides in the fact that his symbolic notation, however ambiguous, contains in germ the notion of cohomology ring of a space (or of the Chow ring if one prefers). A condition x represents in fact a family of conditions, that can be interpreted as a cohomology class in a space of configurations. It goes without saying that when we write a product, say $x \cdot y$, the corresponding sets of objects satisfying the conditions must be in general position (or at least, their intersection must have the right dimension). For example, denote by p the condition, addressed to the points of $\mathbb{P}^3$, that they must lie in a plane; p_g denotes the condition that the points must lie on a line. Then the formula $p \cdot p = p_g$ holds. In other words, the plane expressing the condition p is generic, otherwise we would just have $p \cdot p = p$. The ambiguity of the symbolic notation is what makes it worthy. It must be noted that using some good principles and some rather simple geometric reasoning, Schubert has obtained plenty of remarkable results, whose justification, in accordance with today's standards of rigour, has required many great efforts.

Among others, Schubert has established what he has called *Coinzidenzformeln*, mainly formula 1), page 44 of [7], which is a prototype of the residual intersection formula as it can be found in [2, theorem 9.2]. He has used this formula to establish many multiple coincidence formulas (*mehrfache Coinzidenzen*), with a rigour and effectiveness that are not lesser than their modern analog, to be found for example in [3], although not as general.

In terms of cohomology, if X denotes some space of configurations of geometric objects (for example, the points on a surface, the space of conics), a condition x can be represented as the cohomology class that is Poincaré dual to the fundamental homology class of a cycle Ω_x on X. Then the class $x \cdot y$ is dual to $\Omega_x \cap \Omega_y$, provided that this two cycles are in general position. The formulas proved by Schubert on the basic elements correspond to the calculation of the cohomology ring of the complex projective space $\mathbb{P}^3$, the grassmannian $\mathcal{G}$ of lines in the projective space, and finally the space $\mathcal{PS}$ (*Punkt un Strahl*), whose elements are pairs consisting of a line in space and a point on the line.

To give a line in $\mathbb{P}^3$ is equivalent to give a 2 dimensional vector subspace of $\mathbb{C}^4$; with this point of view we see that there is a natural vector bundle η of rank 2 on $\mathcal{G}$, called the tautological bundle, which consists of pairs (α, v), where $\alpha \in \mathcal{G}$ (regarded as a 2 dimensional subspace of $\mathbb{C}^4$) and $v \in \alpha$. In fact, the space $\mathcal{PS}$ in nothing but the projective bundle associated to η.

Note that Schubert did not introduce symbols to denote the spaces $\mathbb{P}^3$, $\mathcal{G}$ and $\mathcal{PS}$, since somehow they constitute the ambient universe. We shall denote by $\check{\mathbb{P}}^3$ the dual space of $\mathbb{P}^3$, that is the space of projective 2 planes in $\mathbb{P}^3$. We will assume some familiarity with characteristic classes of vector bundles.

2 Formulas for the basic configuration spaces

We introduce symbols which denote geometric objects in the various basic configuration spaces $\mathbb{P}^3$, $\check{\mathbb{P}}^3$, $\mathcal{G}$ and $\mathcal{PS}$. The same symbols will denote basic conditions imposed on the basic objects. The sets of basic objects that are thus defined generate the homology of the spaces; in the case of $\mathbb{P}^3$ and $\mathcal{G}$ they even provide a minimal cell decomposition, that is a cell decomposition such that each cell represents a homology class, and this homology classes are a set of free additive generators of the homology. By expressing their intersections in terms of basic conditions, the cohomology ring of these spaces will be determined explicitely.

Of course, we will use the same notation as Schubert, which is based on the german names of the various objects. It is therefore useful to recall some german words :

Punkt :	point
Gerade :	line
Ebene :	plane
Strahl:	litterally : ray; here it will denote most of the time the lines lying in a given plane going through a given point in the plane, that is a pencil of lines. Sometimes this word is synonym of line, like in *Punkt und Strahl*
Fläche :	surface.

Note that, lacking a more precise word, we shall use *condition* for the german *Bedingung* to denote a requirement imposed to some geometric objects.

We shall work with the cohomology ring of spaces, but the Chow ring could be used as well.

When a formula is numbered, the number is the same as in [7].

2.1 The complex projective space $\mathbb{P}^3$

The basic conditions that can be put on the points of space are :

Notation	Condition
p	the point must lye in a given plane
p_g	the point must lye on a given line
P	the point itself is given

The following relations are easily verified :

$$1)\ \ p^2 = p_g \quad , \quad 2)\ \ p^3 = p \cdot p_g \quad , \quad 3)\ \ p \cdot p_g = P \quad , \quad 4)\ \ p^3 = P \quad .$$

As an example, the pedantic geometric interpretation of the first formula goes as follows : let e_1, $e_2 \subset \mathbb{P}^3$ be two planes and

$$\Omega_{e_i} = \{P \in \mathbb{P}^3 \mid P \in e_i\} \quad , \quad i = 1, 2 \quad .$$

Then p^2 denotes the points in $\Omega_{e_1} \cap \Omega_{e_2}$ when e_1 and e_2 are in general position, that is when they intersect in a line g; the set of points that must be on a line has been denoted by p_g.

Let us now interpretate this formulas in cohomology. Denote by $t \in H^2(\mathbb{P}^3, \mathbb{Z})$ the dual class of the cycle constituted by the points in a plane of $\mathbb{P}^3$; then t^2 is the dual to the cycle constitued by the points on a line, and t^3 is dual to the 0 cycle constituted by a single point.

If we choose a flag $p \in g \subset e$, denoting by Ω_p, Ω_g and Ω_e the corresponding sets, then $\Omega_p \subset \Omega_g \subset \Omega_e \subset \mathbb{P}^3$ is a minimal cell decomposition of $\mathbb{P}^3$.

The case of $\check{\mathbb{P}}^3$, the space of 2 planes in $\mathbb{P}^3$, can be treated in a similar way :

Notation	Condition
e	the plane must go through a given point
e_g	the plane must contain a given line
E	the plane itself is given

We have the formulas :

$$5)\ \ e^2 = e_g \quad , \quad 6)\ \ e^3 = e \cdot e_g \quad , \quad 7)\ \ e \cdot e_g = E \quad , \quad 8)\ \ e^3 = E \quad .$$

2.2 The grassmannian $\mathcal{G}$ of lines in $\mathbb{P}^3$

Here are the basic conditions :

Notation	Condition	Dimension
g	the line must cut a given line	3
g_e	the line must lie in a given plane	2
g_p	the line must go through a given point	2
g_s	the line must belong to a given pencil	1
G	the line itself is given	0

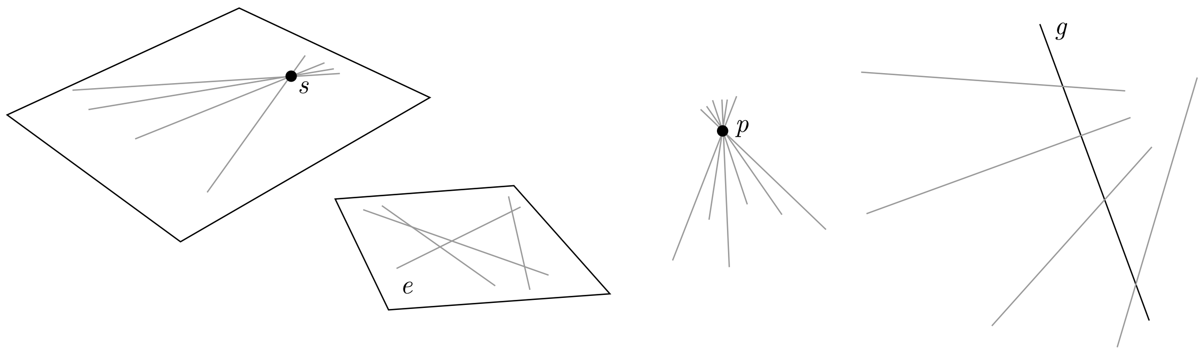

Figure 2: Schubert cycles Ω_s, Ω_e Ω_p, Ω_g

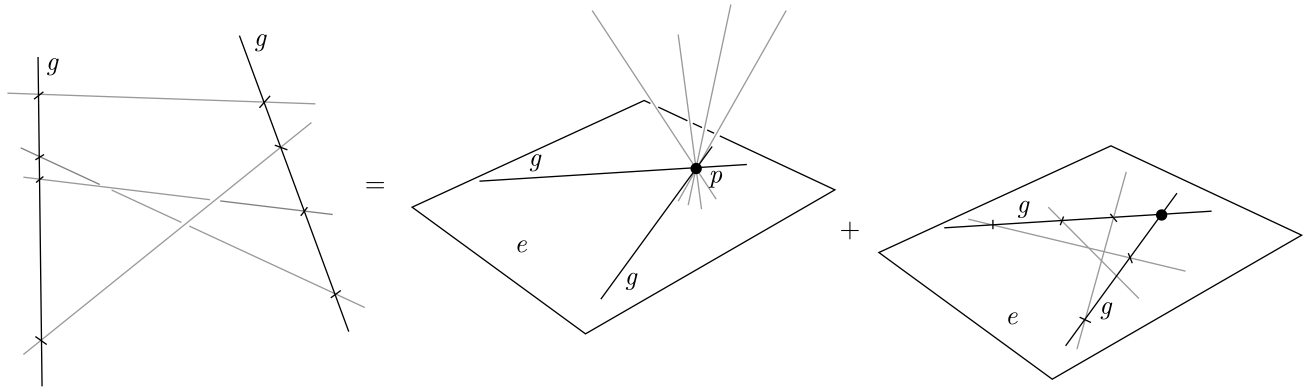

Figure 3: How to see that $g^2 = g_p + g_e$

Choose a flag $P \in g \subset e \subset \mathbb{P}^3$ and denote by Ω_g, Ω_e, Ω_p, Ω_s, $\Omega_G = \{G\}$ the sets of lines satisfying conditions g, g_e, g_p, g_s and G respectively. We have a diagram of inclusions :

$$\begin{array}{ccccccc} & & & \Omega_p & & & \\ & & \nearrow & & \searrow & & \\ \Omega_G & \longrightarrow \Omega_s & & & & \Omega_g & \longrightarrow \mathcal{G} \\ & & \searrow & & \nearrow & & \\ & & & \Omega_e & & & \end{array}$$

and the $\Omega_\bullet$ are the cells of a minimal cell decomposition of $\mathcal{G}$ (see [6, § 6]). These cells are called *Schubert cycles*.

Now we will compute all the possible products of the basic conditions g, g_p, g_e, g_s. In order to express g^2 in terms of basic conditions, we suppose that the two given lines g and g' intersect in a point P; by taking e to be the plane of g and g', we have :

$$\Omega_g \cap \Omega_{g'} = \Omega_p \cup \Omega_e$$

(see figure 3) and from this Schubert deduces, by invoking the principle of conservation of the number, since g and g' are not in general position, that :

$$9) \quad g^2 = g_p + g_e \quad .$$

We will justify this formula in two ways : first, by expressing the calculations in cohomology. Secondly, by showing that Ω_g and $\Omega_{g'}$ intersect transversally, outside the locus Ω_s of lines in e through P, which is of lower dimension; this justifies Schubert's procedure : in spite of the fact that the situation is degenerated because g and g' are in a same plane, the intersection of Ω_g and $\Omega_{g'}$ has the right dimension 2 and there is no multiplicity to be taken into account.

Note that the linear group $\mathcal{G}\ell(4,\mathbb{C})$ acts transitively on $\mathcal{G}$. Therefore it can be used to put cycles in general position : if we choose generic lines, points or planes, the corresponding Schubert cycles will be transversal. It is straghtforward to check the following formulas :

$$10)\ \ g\cdot g_p = g_s \quad , \quad 11)\ \ g\cdot g_e = g_s$$
$$12)\ \ g\cdot g_s = G \quad , \quad 13)\ \ g_p\cdot g_e = 0$$

By multiplying 9) by g and using 10) and 11) we get :

$$14)\ \ g^3 = g\cdot g_p + g\cdot g_e \quad , \quad 15)\ \ g^3 = 2\cdot g_s$$

By multiplying by g again :

$$16)\ \ g^4 = 2\cdot g\cdot g_s = 2\cdot g^2\cdot g_e = 2\cdot g^2\cdot g_p = 2\cdot g_p^2 = 2\cdot g_e^2 = 2\cdot G \quad .$$

Note that the formula $g^4 = 2\cdot G$ tells us that there are 2 lines that cut four given lines in general position. This is a first and very often quoted example of application of Schubert calculus to enumerative geometry (more on this in § 2.2.2).

2.2.1 The cohomology ring of $\mathcal{G}$

Let us consider the grassmannian $\mathcal{G}$ as the space of 2 dimensional vector subspaces of $\mathbb{C}^4$. Let $\eta = (E \xrightarrow{\pi} \mathcal{G})$ be the tautological vector bundle of rank 2 :

$$E = \left\{(\alpha, v)\in \mathcal{G}\times\mathbb{C}^4 \mid v\in\alpha\right\} \quad , \quad \pi(\alpha, v) = \alpha \quad .$$

Let $c_i(\eta)\in H^{2i}(\mathcal{G},\mathbb{Z})$, $i = 1, 2$, be the Chern classes of η, and $s_i(\eta)\in H^{2i}(\mathcal{G},\mathbb{Z})$, $i = 1,\ldots,4$, the Segre classes (see for example [2]). They are bound by the relation :

$$(1 + c_1(\eta) + c_2(\eta))\cdot(1 + s_1(\eta) + s_2(\eta) + s_3(\eta) + s_4(\eta)) = 1 \quad .$$

Denote by $\mathbb{I}^n$ the trivial bundle of rank n, with an unspecified basis. Since $\eta\subset\mathbb{I}^4$, we can set $\eta' = \mathbb{I}^4/\eta$, and then $c(\eta') = s(\eta)$.

Let now x_1 et x_2 be formal variables and let $y_1, y_2\in\mathbb{Z}[x_1, x_2]$ be defined by the relation :

$$(1 + x_1 + x_2)\cdot(1 + y_1 + y_2 + y_3 + y_4) = 1$$

which amounts to set :

$$y_1 = -x_1 \quad , \quad y_2 = x_1^2 - x_2 \quad , \quad y_3 = 2x_1x_2 - x_1^3 \quad , \quad y_4 = x_1^4 - x_2^2 + 3x_1^2x_2$$

as one can easily check. It can be shown (see [8, proposition page 69]) that the ring homomorphism :

$$\mathbb{Z}[x_1, x_2]\to H^*(\mathcal{G},\mathbb{Z}) \quad , \quad x_i\mapsto c_i(\eta)$$

induces a ring isomorphism :

$$\mathbb{Z}[x_1, x_2]/I(y_3, y_4)\xrightarrow{\simeq} H^*(\mathcal{G},\mathbb{Z})$$

where $I(y_3, y_4)$ denotes the ideal generated by y_3 et y_4. It follows that $H^*(\mathcal{G},\mathbb{Z})$ is generated as a group by :

$$c_1 \quad , \quad c_1^2 \quad , \quad c_2 \quad , \quad c_1c_2 \quad , \quad c_2^2$$

and the ring structure is determined by the relations $2c_1c_2 - c_1^3 = 0$, $c_1^4 - c_2^2 + 3c_1^2c_2 = 0$, whence $2c_1^2c_2 - c_1^4 = 0$ and $c_1^2c_2 = c_2^2$.

Remark. *In [8, proposition page 69], it is asserted that* $H^*(\mathcal{G}) \simeq \mathbb{Z}[c_1, c_2]/I(\{s_j, j > 2\})$ *where* s_j *are defined for all positive* j *by the relations :*

$$(1 + c_1 + c_2)(1 + s_1 + s_2 + \cdots + s_j + \cdots) = 1$$

holding in the graded ring $\mathbb{Z}[c_1, c_2]$*. But it is easy to see that* $s_j \in I(s_1, \ldots, s_{j-1})$*, and so*

$$I(\{s_j, j > 2\}) = I(s_3, s_4) \quad .$$

We will express the Poincaré duals of the various Schubert cells in terms of the Chern and Segre classes of η. Here are the results :

Symbolic notation	–	g	g_p	g_e	g_s	G
Cycle	$\mathcal{G}$	Ω_g	Ω_p	Ω_e	Ω_s	Ω_G
Dual class	1	s_1	s_2	c_2	$s_1 c_2$	$c_2^2 = s_2^2$

To do so, let v_i, $i = 1, \ldots, 4$ be a basis of $\mathbb{C}^4$; we will denote by $\langle v_{i_1}, \ldots, v_{i_k} \rangle$ the space generated by $v_{i_1}, \ldots, v_{i_k}$. The conditions defining Schubert cycles will be expressed using the flag :

$$P = \langle v_1 \rangle \subset g = \langle v_1, v_2 \rangle \subset e = \langle v_1, v_2, v_3 \rangle \subset \mathbb{C}^4$$

$\boxed{\Omega_g}$

Consider the bundle morphism $\varphi_g : \eta \to \mathbb{I}^4/\langle v_1, v_2 \rangle$ induced by the natural inclusion of η into $\mathbb{I}^4$. Recalling that the line g is the projective space associated to the vector space $\langle v_1, v_2 \rangle$, we see that

$$\Omega_g = \Sigma(\varphi_g)$$

where $\Sigma(\varphi_g) \subset \mathcal{G}$ denotes the singular locus of φ_g, that is the set of lines $\ell \in \mathcal{G}$ such that the restriction of φ_g to the fiber above ℓ is not injective. If we consider the morphism $\Lambda^2(\varphi_g) : \Lambda^2(\eta) \to \Lambda^2(\mathbb{I}^4/\langle v_1, v_2 \rangle)$ as a section of $(\Lambda^2(\eta))^* \otimes \Lambda^2(\mathbb{I}^4/\langle v_1, v_2 \rangle) \simeq (\Lambda^2(\eta))^*$, the set of zeros of this section identifies with $\Sigma(\varphi_g)$, and therefore its dual class is $c_1(\Lambda^2(\eta))^* = -c_1(\eta) = s_1(\eta)$.

$\boxed{\Omega_e}$

Consider the natural bundle morphism $\varphi_e : \eta \to \mathbb{I}^4/\langle v_1, v_2, v_3 \rangle$, which corresponds to a section σ of $\eta^* \otimes \mathbb{I}^4/\langle v_1, v_2, v_3 \rangle$. Since Ω_e is the set of zeros of this section, its dual class is $c_2(\eta^*) = c_2(\eta)$.

$\boxed{\Omega_p}$

Here we take the natural morphism $\varphi_p : \langle v_1 \rangle \to \mathbb{I}^4/\eta$, that can be seen as a section of $\mathbb{I}^4/\eta$; its zeros constitute Ω_p, therefore the dual class is $c_2(\mathbb{I}^4/\eta) = s_2(\eta)$.

$\boxed{\Omega_s \text{ et } \Omega_G}$

Let e' be the projective plane corresponding to $\langle v_1, v_2, v_4 \rangle$ and g' the projective line corresponding to $\langle v_1, v_4 \rangle$. Notice that $\Omega_s = \Omega_{g'} \cap \Omega_e$ and $\Omega_G = \Omega_e \cap \Omega_{e'}$, these intersections being transversal. It follows that the dual classes are $s_1 c_2$ and c_2^2 respectively.

For example, we can recover formula 9) by observing that $s_1^2 = c_1^2 = (c_1^2 - c_2) + c_2 = s_2 + c_2$.
Also $s_1^4 = s_1(-c_1^3) = s_1(-2c_1c_2) = 2c_1^2c_2 = 2c_2^2$ shows that $g^4 = 2G$.
The other formulas can be recovered in a similar way.

2.2.2 Justification of 9) using the principle of conservation of the number

In order to introduce local coordinates on $\mathcal{G}$, we choose a vector subspace $\alpha_0 \subset \mathbb{C}^4$ of dimension 2 and a supplementary vector subspace α'. Denote by $\mathrm{Hom}(\alpha_0, \alpha')$ the space of linear maps from α_0 to α'. Define $\varphi : \mathrm{Hom}(\alpha_0, \alpha') \to \mathcal{G}$ by associating to $A \in \mathrm{Hom}(\alpha_0, \alpha')$ its graph; it is a bijection on the open subset

$$U_{\alpha_0, \alpha'} = \{\beta \in \mathcal{G} \mid \beta \cap \alpha' = \{0\}\} \quad .$$

It can be verified that this defines a smooth atlas on $\mathcal{G}$; we shall denote by ℓ_A the projective line corresponding to $A \in \mathrm{Hom}(\alpha_0, \alpha')$.

Lemma. *Let $A, B \in Hom(\alpha_0, \alpha')$ and assume that $\ell_A \in \Omega_{\ell_B}$, so that there exists a vectorial line $\ell_0 \subset \alpha_0$ such that $A|\ell_0 = B|\ell_0$.*

Then ℓ_A is a regular point of Ω_{ℓ_B} if and only if $A \neq B$, and if so :

$$T(\Omega_{\ell_B})_{\ell_A} = \{\overline{A} \in Hom(\alpha_0, \alpha') \mid \overline{A}|\ell_0 : \ell_0 \to \alpha'/Im(A-B) \text{ is zero } \}$$

Proof: Instead of describing Ω_{ℓ_B} near ℓ_A, it is easier to work in the space $\mathrm{Hom}(\alpha_0, \alpha') \times \mathrm{Hom}(\ell_0, \ell')$, where ℓ' is a vectorial line supplementary to ℓ_0 in α_0. Denote by $i_{\ell_0} : \ell_0 \subset \alpha_0$ the inclusion, and by $p : \mathrm{Hom}(\alpha_0, \alpha') \times \mathrm{Hom}(\ell_0, \ell' \to \mathrm{Hom}(\alpha_0, \alpha'))$ the projection; the equation

$$(A' - B) \circ (i_{\ell_0} + \lambda) = 0 \quad , \quad A' \in \mathrm{Hom}(\alpha_0, \alpha')\,,\, \lambda \in \mathrm{Hom}(\lambda_0, \lambda')$$

defines a subset $\tilde{\Omega}$ which is in bijection through p with $\Omega_B \cap U_{\alpha_0,\alpha'}$, except above $A' = B$. If we take the derivative of this equation at $A' = A$ we find :

$$\overline{A} \circ i_{\ell_0} + (A - B) \circ \overline{\lambda} = 0$$

where overlined symbols denote tangent vectors; if $A \neq B$, $\mathrm{Ker}(A - B) = \ell_0$ and so

$$\exists\, \overline{\lambda} \text{ such that } \overline{A} \circ i_{\ell_0} + (A - B) = 0 \iff \overline{A} \circ i_{\ell_0} : \ell_0 \to \alpha'/\mathrm{Im}(A - B) \text{ is zero}$$

q.e.d.

Proposition. *Let ℓ_{B_1} and ℓ_{B_2} be two distinct lines, meeting in a point $P_{1,2}$. Then $\Omega_{\ell_{B_1}}$ and $\Omega_{\ell_{B_2}}$ intersect transversally except on the set of lines through $P_{1,2}$ that lie in the plane through ℓ_{B_1} and ℓ_{B_2}.*

Proof: Let $\ell_A \in \Omega_{\ell_{B_1}} \cap \Omega_{\ell_{B_2}}$. Assume first that ℓ_A goes through $P_{1,2}$, and therefore is not in the plane through ℓ_{B_1} and ℓ_{B_2}. Let $\ell_{1,2} \subset \alpha_0$ be the vectorial line corresponding to $P_{1,2}$, that is such that $B_1|\ell_{1,2} = B_2|\ell_{1,2} = A|\ell_{1,2}$. It follows from the lemma that

$$T(\Omega_{\ell_{B_1}})_A \cap T(\Omega_{\ell_{B_2}})_A = \{\overline{A} \mid \overline{A}|\ell_{1,2} : \ell_{1,2} \to \alpha'/\mathrm{Im}(A - B_i) \text{ is zero },\ i = 1, 2\}$$

Since $A - B_1$ and $A - B_2$ have the same kernel $\ell_{1,2}$, if they also had the same image we would have :

$$A - B_1 = \lambda(A - B_2)$$

where λ is a scalar, and $\lambda \neq 1$, otherwise $B_1 = B_2$. It would follow that

$$A = \frac{1}{1-\lambda} B_1 - \frac{\lambda}{1-\lambda} B_2$$

and ℓ_A would lie in the plane through ℓ_{B_1} and ℓ_{B_2}, a contradiction. Therefore the two conditions that $\overline{A}|\ell_0 \to \alpha'/\mathrm{Im}(A - (B_i)$ should vanish, for $i = 1, 2$, are independent, and therefore transversality holds.

If ℓ_A lies in the plane through ℓ_{B_1} and ℓ_{B_2}, but does not go through $P_{1,2}$, let $P_1 = \ell_A \cap \ell_{B_1}$ and $P_2 = \ell_A \cap \ell_{B_2}$ and let $\ell_1, \ell_2 \subset \alpha_0$ be the vectorial lines such that :

$$(A - B_1)|\ell_1 = 0 \quad , \quad (A - B_2)|\ell_2 = 0 \quad .$$

Then :

$$T(\Omega_{\ell_{B_1}})_A \cap T(\Omega_{\ell_{B_2}})_A = \{\overline{A} \mid \overline{A}|\ell_i : \ell_i \to \alpha'/\mathrm{Im}(A - B_i) \text{ is zero },\ i = 1, 2\}$$

and since $\ell_1 \neq \ell_2$, these two conditions are independent, and transversality follows.

q.e.d.

Let $P, Q, R, S \in \mathbb{P}^3$ be 4 points, not in a same plane, and such that 3 among them are never aligned. Then, if we take the four lines $\ell_{P,Q}$ through P and Q, $\ell_{Q,R}$, $\ell_{R,S}$ and $\ell_{S,P}$, one sees that the corresponding Schubert cycles $\Omega_{\ell_{\bullet,\bullet}}$ intersect transversally in the two lines $\ell_{P,R}$ and $\ell_{Q,S}$. Indeed, the intersection of two of the cycles is transveral according to the proposition, and the transversality of the remaining intersections

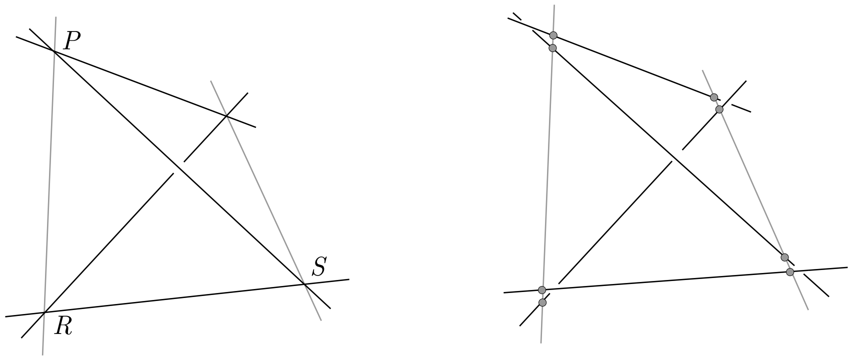

Figure 4: The 2 lines that cut 4 given lines

is elementary (for example : the intersection of the set of lines lying in the plane through P, Q, S and the set of lines in the plane through Q, R, S).

The enumerative problem of finding the number of lines cutting 4 given lines is often cited as an example to illustrate Schubert methods in enumerative geometry (see [5]). What we have shown justifies to resource to the moderately degenerate case where each of the 4 given lines meets another one, and since the intersections of the 4 cycles are transversal, there are no multiplicities to take into account. This last fact can be perceived with some imagination, by moving a little the 4 black lines in figure 4, and seeing that near each gray line there is only one solution.

On the other hand, here is a degenerate situation that has been pointed out to me by my colleague Alexandre Gabard. On a smooth quadric surface in $\mathbb{P}^3$ there are two sytems of lines, which correspond to the horizontal and the vertical lines respectively if one identifies the quadric with $\mathbb{P}^1 \times \mathbb{P}^1$. If one takes 4 lines of one system, they might seem to be in general position in the space ; however, any line of the other system cuts the four given lines : we are in a very degenerate situation, with an infinite number of solutions.

2.3 The space $\mathcal{PS}$ of points on a line of $\mathbb{P}^3$

Recall that the space $\mathcal{PS}$ is the set of pairs consisting of a point on a line of $\mathbb{P}^3$. In order to express conditions on elements of this space, we shall use symbols of the form xy, where x is y symbol expressing a condition on points and y is a symbol expressing a condition on lines. Thus the symbol pg denotes the pairs consisting of a point on a line, where the point must lie on a given plane, and the line must cut a given line; if we denote by Ω_{pg} the set of these pairs, and yet by g a line and by e a plane, we have :

$$\Omega_{pg} = \{(\ell, Q) \in \mathcal{PS} \mid \ell \cap g \neq \emptyset \quad , \quad Q \in e\} \quad .$$

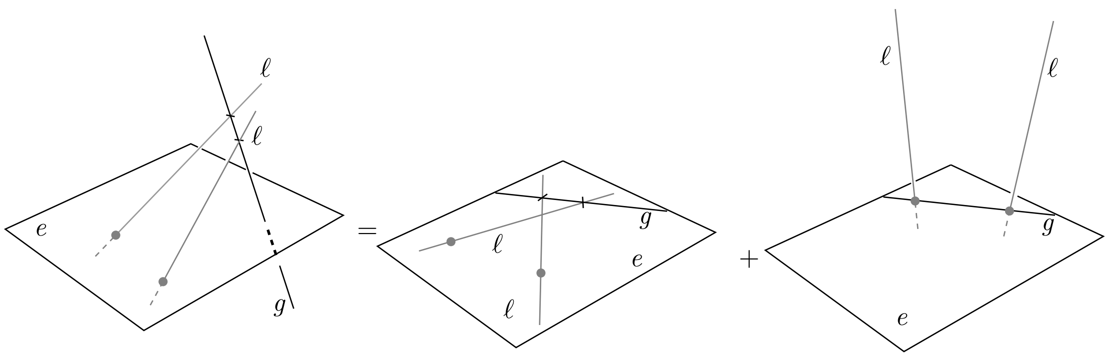

Figure 5: How to see that $pg = g_e + p_g$

By the principle of conservation of the number, we can take the line g in the plane e, in which case :

$$\Omega_{pg} = \{(\ell, Q) \in \mathcal{PS} \mid Q \in g\} \cup \{(\ell, Q) \in \mathcal{PS} \mid \ell \subset e\}$$

(see figure 5) and so the formula of [7, page 25] follows [2]

$$\text{I)} \quad pg = p_g + g_e = p^2 + g_e \quad .$$

This is a fundamental formula, in the sense that any other formula in $\mathcal{PS}$ will follow from this one and from the formulas that we have already shown to hold in $\mathbb{P}^3$ and $\mathcal{G}$; the reason is explained in the next § .

Let us show some other formulas in $\mathcal{PS}$ anyway. By multiplying I) by p, then by g we get :

$$p^2 g = pp_g + pg_e = p^3 + pg_e$$
$$pg_e + pg_p = pg^2 = p_g g + g_e g = p_g g + g_s = p^2 g + g_s$$

and by adding the far left and far right expressions of these two lines :

$$\text{II)} \quad pg_p = p^3 + g_s$$

and similarly one obtains (see [7, page 26]) :

$$\text{III)} \quad pg_s = p^2 g_p = G + p^3 g = G + p^2 g_e \quad .$$

Justification of I) **using cohomology** If we regard $\mathcal{G}$ as the space of 2 dimensional vector subspaces of $\mathbb{C}^4$, $\mathcal{PS}$ is the projective bundle associated to the tautological bundle η of rank 2. The tautological line bundle $\gamma = (F \xrightarrow{\pi} \mathcal{PS})$ on $\mathcal{PS}$ can be defined as

$$F = \left\{(\alpha, \ell, v) \in \mathcal{G} \times \mathbb{P}^3 \times \mathbb{C}^4 \mid \ell \subset \alpha \, , \, v \in \ell\right\} \quad , \quad \pi(\alpha, \ell, v) = (\alpha, \ell) \quad .$$

Set $s = c_1(\gamma)$. Notice that $H^*(\mathcal{PS})$ is a $H^*(\mathcal{G})$-module via the homomorphism induced by the natural projection $p : \mathcal{PS} \to \mathcal{G}$. We know that (see [8, theorem page 62]) the ring homomorphism

$$H^*(\mathcal{G})[s] \to H^*(\mathcal{PS}) \quad , \quad s \mapsto c_1(\gamma)$$

induces an isomorphism

$$H^*(\mathcal{G})[s]/I(s^2 - sc_1(\eta) + c_2(\eta)) \xrightarrow{\simeq} H^*(\mathcal{PS})$$

where $I(s^2 - sc_1(\eta) + c_2(\eta))$ denotes the ideal generated by the polynomial $s^2 - sc_1(\eta) + c_2(\eta)$, which is nothing else than $c_2(p^*(\eta)/\gamma)$ once we substitute s by $c_1(\gamma)$; it vanishes because $p^*(\gamma)/\gamma$ is of rank 1.

Let us allow to denote by the same symbol a condition on the basic elements as well as the Poincaré dual to the cycle defined by this condition. For example, using § 2.2.1, we shall write $g = s_1(\eta) = -c_1(\eta)$; we will also write s for $c_1(\gamma)$, so that $p = -s$.

It follows from the very definition of the symbols that $pg = (-s)(-c_1(\eta))$, and by § 2.2.1 $g_e = c_2(\eta)$. On $H^*(\mathcal{PS})$ we have the relation

$$sc_1(\eta) = s^2 + c_2(\eta)$$

that can also be written

$$pg = p^2 + g_e$$

which is formula I). Therefore this formula is exactly the relation by which $H^*(\mathcal{G})[s]$ has to be divided in order to obtain $H^*(\mathcal{PS})$.

[2] There is a misprint in [7, page 25] : the formula shown there is $pg = p_g + g$

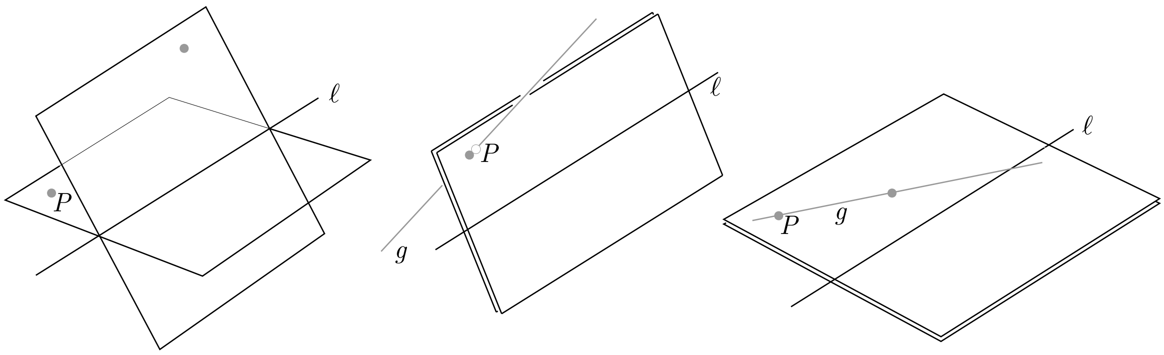

Figure 6: The space Y and conditions ε and g

3 Coincidence formulas

If $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bi-degree (p, q), the restriction of its equation to the diagonal is of degree $p+q$, therefore X cuts this diagonal in $p+q$ points, counted with multiplicity. We can reformulate this remark by saying that X is a one parameter family of pairs (P, Q) of points on the line; if there are q pairs in the family with a given first point P, and p pairs with a given second point Q, then there are $p+q$ pairs of the form (P, P). This is the *Principle of correspondence* that has been stated and proved by Chasles [1, Lemme I, page 1175].

We will generalize this formula according to [7, pages 42 and following], using the same notation and its fruitful ambiguity. Consider pairs of points in the projective space and imagine that when two points of a pair come to coincide, the line joining the two points has a well defined limit. Let us call p and q the two points of a pair, and g the line joining them; denote by ε the condition that p and q are infinitely near, but still determine the line joining them.

Assume that a one parameter system X of such pairs of points is given. Note that if we still denote by p the number of pairs $(P, Q) \in X$ such that P lies in a given plane (a condition that we also denoted by p), and by q the number of pairs $(P, Q) \in X$ such that Q lies in a given plane (a condition that we also denote by q), then X is of bi-degree (p, q).

Now we take a line ℓ and consider pairs of planes through ℓ, such that the first plane contains P, the second contains Q. These pairs constitute a curve Y in the space of pairs of planes through ℓ, that can be identified to $\mathbb{P}^1 \times \mathbb{P}^1$; Y is also of bi-degree (p, q). It follows from Chasles *Principle of correspondence* that there are

$$p+q$$

single planes that contain a pair of points in X. Among these, first we have ε of them that arise because they contain a pair of coinciding points of X, secondly we have those planes who contain a line g joining two distinct points of a pair in X; the latter is equivalent to say that the line g must cut the line ℓ (this kind of conditions on lines has been denoted by g, same notation as the line g joining P and Q !); see figure 6. Therefore we have

$$(\spadesuit) \qquad \varepsilon = p + q - g \quad .$$

This formula proves to be useful to establish enumerative formulas concerning special positions of lines with respect to a surface, among others. For example, Schubert uses it to show that a generic surface of degree n possesses

$$\frac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30)$$

lines that are tangent at four distint points (see [7, page 237, formula 21)]. We will follow Schubert's procedure to establish this formula in our third and last example of the next paragraph.

First, let us justify formula ($\spadesuit$). Consider the space $\mathbb{P}^3 \tilde{\times} \mathbb{P}^3$ obtained by blowing up the diagonal Δ in $\mathbb{P}^3 \times \mathbb{P}^3$. The map that associates to $(P, Q) \in \mathbb{P}^3 \times \mathbb{P}^3 \setminus \Delta$ the line through P and Q extends to a map

$\varphi : \mathbb{P}^3 \tilde{\times} \mathbb{P}^3 \to \mathcal{G}$, such that

$$\varphi^*(\Lambda^2(\eta)) = \gamma^* \otimes (\mathcal{O}(1)_1 \otimes \mathcal{O}(1)_2)$$

where γ denotes the line bundle associated to the blown up diagonal, $\mathcal{O}(1)_i$ the pull-back by the projection of $\mathbb{P}^3 \tilde{\times} \mathbb{P}^3$ on the i-th factor of $\mathbb{P}^3 \times \mathbb{P}^3$ of the vector bundle of homogeneous 1-forms on $\mathbb{P}^3$.

To see that φ has these properties, we can use the Plücker imbedding ψ of the grassmannian $\mathcal{G}$ into $\mathbb{P}^5$, defined as follows. If $g \in \mathcal{G}$, choose distinct $P, Q \in g$; if $P = [x_1, \dots, x_4]$, $Q = [y_1, \dots, y_4]$, et $x = (x_1, \dots, x_4)$, $y = (y_1, \dots, y_4)$, set

$$\psi(g) = [x \wedge y] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) \simeq \mathbb{P}^5 \quad .$$

It can be checked that ψ is well defined and that it is an imbedding, whose image is

$$\left\{[P \wedge Q] \in \mathbb{P}(\Lambda^2(\mathbb{C}^4)) \mid P \wedge Q \neq 0\right\} \quad ,$$

and this image can be identified to $\mathcal{G}$. Note that the pull-back by ψ of the vector bundle $\mathcal{O}(1)_{\mathbb{P}(\Lambda^2(\mathbb{C}^4))}$ of homogeneous 1-forms on $\mathbb{P}(\Lambda^2(\mathbb{C}^4))$ is naturally isomorphic to $\Lambda^2(\eta^*)$. Consider the map

$$\Phi : \mathbb{C}^4 \times \mathbb{C}^4 \to \Lambda^2(\mathbb{C}^4) \quad , \quad (x, y) \mapsto x \wedge y \quad .$$

Its derivative with respect to x, at a point point (y, y), $y \neq 0$, can be writtent $v \mapsto v \wedge y$, and its kernel is the line supporting y. It follows from this that Φ induces a morphism

$$\varphi : \mathbb{P}^3 \tilde{\times} \mathbb{P}^3 \to \mathcal{G} \quad \text{with} \quad \varphi^*(\Lambda^2(\eta^*)) \simeq \gamma^* \otimes (\mathcal{O}(1)_1 \otimes \mathcal{O}(1)_2) \quad .$$

With an additional effort, one can even show that $\varphi^*(\eta) = (\gamma^* \otimes \mathcal{O}_1(-1)) \oplus \mathcal{O}_2(-1)$ but we won't use it. Recall now from § 2.2.1 that the dual class to Ω_g is $s_1(\eta) = -c_1(\eta)$; since

$$\varphi^*(-c_1(\eta)) = c_1(\gamma^* \otimes \mathcal{O}(1)_1 \otimes \mathcal{O}(1)_2)$$

setting $t_i = c_1(\mathcal{O}(1)_i$, $i = 1, 2$, $\varepsilon = c_1(\gamma)$, we get :

$$\varphi^*(-c_1(\eta)) = t_1 + t_2 - \varepsilon \quad .$$

In order to recover Schubert's coincidence formula (♠), we must observe that in this context the condition g, that is the condition that the line through a pair (P, Q) cuts a given line, corresponds to $\varphi^*(s_1(\eta)) = \varphi^*(-c_1(\eta))$; and the conditions p and q correspond to t_1 and t_2 respectively, that is the dual class to a hyperplane in the first factor $\mathbb{P}^3$, respectively the second.

3.1 Coincidences of intersections of a line and a surface

We will present three examples of computations using the coincidence formula. The first one will be justified also using cohomology, the other two will be treated only the Schubert's way.

Let $F \subset \mathbb{P}^3$ be a smooth surface of degree n; following [7, page 229], we denote by p_1, p_2, $\dots p_n$ the points of intersection of a line g with F. Denote by ε_2 the condition that 2 of these points coincide. Then it follows from the coincidence formula (♠) that

♣ $$\varepsilon_2 = p_1 + p_2 - g \quad .$$

As usual, the same symbol p is used to express a condition (to be in a plane) and its recipient (a point).

3.2 First example: the class of a curve

Let's multiply formula ♣ by g_s :

$$\varepsilon_2 g_s = p_1 g_s + p_2 g_s - G$$

using formula III) :

$$\varepsilon_2 g_s = G + p_1^3 g + G + p_2^3 g - G = G$$

because $p^3 = 0$ (the generic intersection of 3 planes and a surface is empty). It remains to interpretate the symbol G in this context : it represents the pairs of distinct points on the intersection of a fixed line and the surface; there are $n(n-1)$ such pairs. Thus we recover the formula for the class (i.e. degree of the dual) of a plane curve of degree n; indeed, $\varepsilon_2 g_s$ represents the lines tangent to the surface belonging to a given pencil, which is the same as the lines in a plane, passing through a given point, tangent to the plane curve obtained as intersection of the surface with the plane.

Justification using cohomology Denote by $F\tilde{\times}F$ the space obtained by blowing up the diagonal in $F \times F$. We have $F\tilde{\times}F \subset \mathbb{P}^3\tilde{\times}\mathbb{P}^3$, and would like to express its dual class. The following result will help us.

Let X be a smooth variety and A, $Y \subset X$ smooth subvarieties that intersect nicely, that is such that $A \cap Y$ is smooth, and that for all $x \in A \cap Y$:

$$TA_x \cap TY_x = T(A \cap Y)_x \quad .$$

Then we have an exact sequence of vector bundles :

$$0 \to T(A \cap Y) \to TA|_{A\cap Y} \oplus TY|_{A\cap Y} \to TX|_{A\cap Y} \to E \to 0$$

where E is defined by the sequence itself; it is called the *excess bundle* and k will denote its rank. Note that $k = 0$ if and only if A and Y intersect transversally.

Proposition. *Let X be a smooth variety, A, $Y \subset X$ smoth subvarieties that intersect nicely. Then, denoting by :*

$\delta_{U,V}$	*the dual class to U in V*
$\tilde{X}$	*the blowing up of X along Y*
$\tilde{A}$	*the strict transform of A*
ε	*the dual class to the exceptional divisor in $\tilde{X}$*
$p : \tilde{X} \to X$	*the projection of the blowing up*
$j : \tilde{Y} \subset \tilde{X}$	*the natural inclusion*

we have :

$$\delta_{\tilde{A},\tilde{X}} = p^*(\delta_{A,X}) - j_!\left((p|\tilde{Y})^*(\delta_{A\cap Y,Y}) \cdot \underbrace{\sum_{i=0}^{k-1}(-1)^i\varepsilon^i c_{k-i-1}(E)}_{=c_{k-1}(E/\gamma)}\right)$$

It is a special case of [2, Theorem 6.7].

As an application, consider the subvarieties $F \times F$ and Δ of $\mathbb{P}^3 \times \mathbb{P}^3$. In this case, the excess bundle identifies to the normal bundle of F in $\mathbb{P}^3$, that is $\mathcal{O}(n)_\Delta$, and $\delta_{F,\mathbb{P}^3} = nt$, therefore $\delta_{F\times F,\mathbb{P}^3\times\mathbb{P}^3} = nt_1 \cdot nt_2 = n^2t_1t_2$. It follows that

$$(\heartsuit) \qquad \delta_{F\tilde{\times}F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3} = n^2t_1t_2 - nt\varepsilon$$

where t denotes indistinctly t_1 or t_2, since $\varepsilon t_1 = \varepsilon t_2$.

The formula $\heartsuit$ can also be proved in Schubert's spirit as follows. Denote by $p : \mathbb{P}^3\tilde{\times}\mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}^3$ the projection of the blowing up and set $\tilde{\Delta}_F = \tilde{\Delta} \cap^{-1} (F \times F \cap \Delta)$. Then :

$$p^{-1}(F \times F) = (F\tilde{\times}F) \cup \tilde{\Delta}_F \quad .$$

Taking the dual classes, we see that

$$p^*(\delta_{F\times F,\mathbb{P}^3\times\mathbb{P}^3}) = \delta_{F\tilde{\times}F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3} + \delta_{\tilde{\Delta}_F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3}$$

and

$$\delta_{\tilde{\Delta}_F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3} = \delta_{\tilde{\Delta}_F,\tilde{\Delta}} \cdot \delta_{\tilde{\Delta},\mathbb{P}^3\tilde{\times}\mathbb{P}^3} = (nt) \cdot \varepsilon$$

whence the formula $\heartsuit$.

In particular, taking $n = 1$, i.e. F is a plane, we obtain that

$$\delta_{\varphi^{-1}(\Omega_e)} = t_1t_2 - t\varepsilon \quad .$$

Now $g_s = gg_e$, hence $\varphi^*(g_s) = \varphi^*(g)\varphi^*(g_e) = (t_1 + t_2 - \varepsilon)(t_1t_2 - t\varepsilon)$. In order to calculate $\varepsilon_2 g_s$ we must multiply $\varepsilon \cdot \varphi^*(g_s)$ by $\delta_{F\tilde{\times}F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3}$ and evaluate this class on $\mathbb{P}^3\tilde{\times}\mathbb{P}^3$, which amounts to evaluate $\delta_{F\tilde{\times}F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3} \cdot \varphi^*(g_s)$ on $\tilde{\Delta}$; but

$$\langle\delta_{F\tilde{\times}F,\mathbb{P}^3\tilde{\times}\mathbb{P}^3} \cdot \varphi^*(g_s), \tilde{\Delta}\rangle = \langle(n^2t_1t_2 - nt\varepsilon)(t_1t_2 - t\varepsilon)(t_1 + t_2 - \varepsilon), \tilde{\Delta}\rangle = \langle(n^2t^2 - nt\varepsilon)(t^2 - t\varepsilon)(2t - \varepsilon), \tilde{\Delta}\rangle$$

and since $t^4 = 0$, $(n^2t^2 - nt\varepsilon)(t^2 - t\varepsilon)(2t - \varepsilon) = t^2(-n\varepsilon^3 + \varepsilon^2(n^2t + 3nt))$. Instead of evaluating on $\tilde{\Delta}$, we can apply the integration over the fibers of π (or Gysin homomorphism) $\pi_!$ and evaluate on $\mathbb{P}^3$, where $\pi : \tilde{\Delta} \to \mathbb{P}^3$ is the natural projection, that is the projection of the projective bundle associated to $T\mathbb{P}^3$; we have the following formulas :

$$\pi_!(\varepsilon^2) = 1 \quad , \quad \pi_!(\varepsilon^3) = c_1(T\mathbb{P}^3) = 4t \quad ,$$

either by the very definition of Segre classe given in [2, § 3.1]), or using [8, theorem page 62], and so

$$\langle t^2(-n\varepsilon + \varepsilon^2(n^2t + 3nt)), \tilde{\Delta}\rangle = \langle t^3(-4n + n^2 + 3n), \mathbb{P}^3\rangle = n(n-1) \quad .$$

3.3 Second example: the number of bitangent lines to a plane curve

This example, as well as the next and last one, will be treated the Schubert's way, without any further justification (see [7, page 229]).

Let $F \subset \mathbb{P}^3$ be a smooth surface and denote by ε_{22} the condition that a line is tangent at two distinct points of F. This condition says that, among the points $p_1, \ldots, p_n$, intersection of the line with F, two pairs coincide, say p_1, p_2 and p_3, p_4. It follows from the coincidence formula (♠) that :

$$2 \cdot \varepsilon_{22} = (p_1 + p_2 - g)(p_3 + p_4 - g)$$

where the coefficient 2 is due to the fact that the roles of (p_1, p_2) et (p_3, p_4) can be exchanged on a bitangent line; and so

$$2 \cdot \varepsilon_{22} = p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 - p_1g - p_2g - p_3g - p_4g + \underbrace{g^2}_{=g_e+g_p} \quad .$$

The symbols p_ip_j, $i \neq j$, all have the same meaning, and also the symbols gp_i; therefore we can write :

$$2 \cdot \varepsilon_{22} = 4p_1p_3 - 4p_1g + g_e + g_p \quad .$$

Now we multiply by g_e : $\varepsilon_{22}g_e$ denotes the lines bitangent to the surface, lying in a given plane; they are therefore the bitangent lines to the curve obtained by intersecting the surface with the plane. We have :

$$2 \cdot \varepsilon_{22}g_e = 4p_1p_3g_e - 4p_1gg_e + g_e^2 + \underbrace{g_pg_e}_{=0} = 4p_1p_3g_e - 4p_1g_s + G \overset{\text{(by III))}}{=} 4p_1p_3g_e - 4p_1^3g - 3G \quad .$$

Let us calculate $p_1p_3g_e$. In fact, we are working in

$$\left(F\tilde{\times}F\right) \times \left(F\tilde{\times}F\right) \times \mathcal{G}$$

and a generic element of this set can be represented by $((P_1, P_2), (Q_1, Q_2), g)$, with $P_i, Q_i \in F \cap g$. The condition p_i requires that P_i lies in a plane e_i, $i = 1, 3$, and g_e requires that g lies in a plane e. Now $e \cap e_i$ cuts F in n points, $i = 1, 3$; therefore the line in the configurations satisfying p_1p_3g is determined by one of the n^2 pairs of points, P_1 on $e \cap e_1 \cap F$ and P_3 on $e \cap e_3 \cap F$; for such a choice of P_1 and P_3, we still can choose P_2 et P_4 among the $n-2$ remaining points on the line. Therefore there are

$$n^2(n-2)(n-3)$$

possible configurations. In order to determine G, notice that for a given line g, there is a total of $n(n-1)(n-2)(n-3)$ pairs of distinct points in $g \cap F$. Finally, $p_1^3 = 0$. So we get :

$$2\varepsilon_{22}g_e = 4n^2(n-2)(n-3) - 3n(n-1)(n-2)(n-3)$$

and therefore

$$\varepsilon_{22}g_e = \frac{1}{2}n(n-2)(n-3)(n+3) \quad .$$

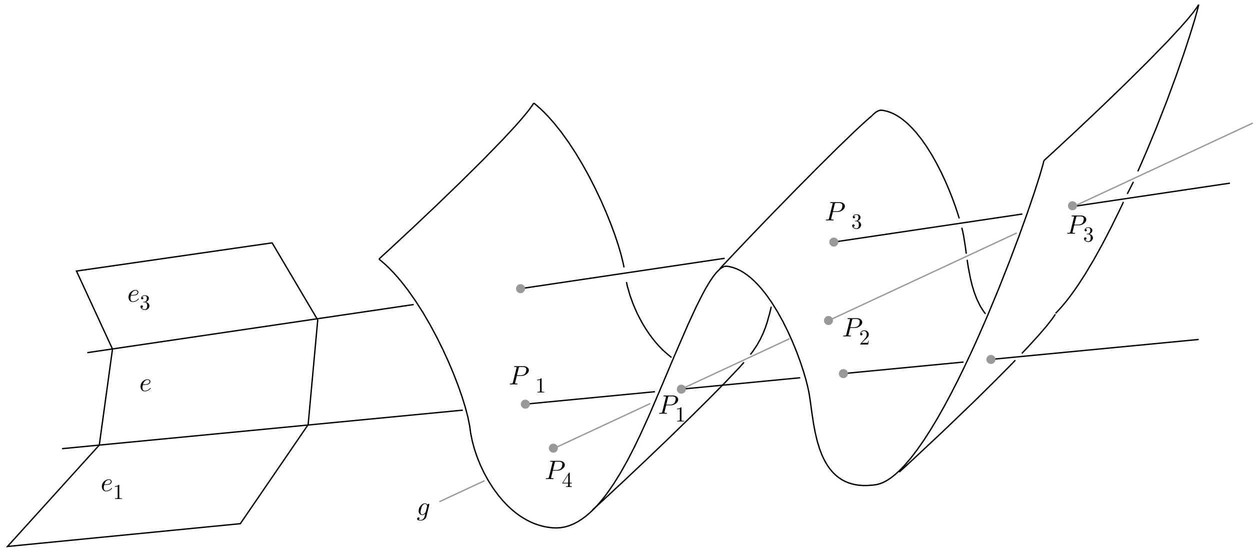

Figure 7: How to calculate $p_1 p_3 g_e$

3.4 Third example: the number of lines tangent to a surface at four distinct points

Denote by ε_{2222} the condition that a line is tangent to a surface F of degree n at four distinct points; that amounts to say that among the n points of intersection of the line with the surface, 4 pairs come to coincide, say (p_1, p_2), (p_3, p_4), (p_5, p_6) and (p_7, p_8). It follows from the fact that the grassmannian $\mathcal{G}$ of lines in $\mathbb{P}^3$ is of dimension 4 that for a generic surface there is a finite number of such quadritangent lines. We shall compute their number in a similar way as for the second example; it corresponds to formula 21) in [7, pages 232 through 237]. We will label some equations with letters, which have no analogue in [7].

It follows from the coincidence formula (♠) that

$$4!\varepsilon_{2222} = (p_1 + p_2 - g)(p_3 + p_4 - g)(p_5 + p_6 - g)(p_7 + p_8 - g)$$

where the coefficient 4! is due to the fact that the roles of the 4 pairs of points that come to coincide, (p_1, p_2), (p_3, p_4), (p_5, p_6) and (p_7, p_8), can be permuted.

The symbols $p_i p_j$, $i \neq j$ all express the same condition, therefore we deduce :

(a)
$$\begin{aligned} 4!\varepsilon_{2,2,2,2} =& 2^4 p_1 p_2 p_3 p_4 - 2^3 \cdot 4 g p_1 p_2 p_3 + 2^2 \binom{4}{2} g^2 p_1 p_2 - 2 \binom{4}{3} g^3 p_1 + g^4 \\ =& 16 p_1 p_2 p_3 p_4 - 32 g p_1 p_2 p_3 + 24 g^2 p_1 p_2 - 8 g^3 p_1 + g^4 \end{aligned}$$

(Ther is a misprint in [7, page 234, line -7] : g_4 is written instead of g^4.)

It follows from formula I) that :

(b)
$$gp = p_g + g_e \implies g p_1 p_2 p_3 = p_1^2 p_2 p_3 + g_e p_1 p_2$$

Also :

(c)
$$g^2 \overset{9)}{=} g_p + g_e \implies g^2 p_1 p_2 = \underbrace{g_p p_1 p_2}_{=G} + g_e p_1 p_2$$

and

(d)
$$g^3 p_1 \overset{9)}{=} (g_e + g_p) g p_1 \overset{10) \text{ et } 11)}{=} 2 g_s p_1 = 2G$$

therefore we can use (b), (c) and (d) in (a) and find :

(e) $$4!e_{2222} = 16p_1p_2p_3p_4 - 32p_1^2p_2p_3 - 8g_ep_1p_2 + 10G$$

We compute separately each of the terms appearing in (e); we shall work in

$$\left(F\tilde{\times}F\right) \times \left(F\tilde{\times}F\right) \times \left(F\tilde{\times}F\right) \times \left(F\tilde{\times}F\right) \times \mathcal{G}$$

1. $\boxed{G}$

   This represents the number of 8-uples of distinct points $P_1, \ldots, P_8$ that can be choosen in the intersection of a generic line with the surface F of degree n, that is :

   $$G = n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$$

2. $\boxed{g_ep_1p_2}$

   In the second example, working with 2 pairs of points, we found the formula $g_ep_1p_3 = n^2(n-2)(n-3)$.

   In the present case, we still have to choose 4 points among the remaining $n-4$ points in the intersection of a line with the surface. Therefore we find :

   $$g_ep_1p_2 = n^2(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$$

3. $\boxed{p_1^2p_2p_3}$

   Let's call e_1, e_1' the planes expressing condition p_1^2, so that P_1 will be among the n points of the intersection $F \cap e_1 \cap e_1'$. Let e_2 and e_3 be the planes expressing conditions p_2 and p_3 respectively, and let ℓ be the line containing $P_1, \ldots, P_8$.

   The point P_2 must lye on the cone over the curve $F \cap e_2$ with vertex P_1 , that we shall denote by $C_{P_1}(F \cap e_2)$; it is of degree n. The point P_3 must lye on $C_{P_1}(F \cap e_2)$ and on the curve $F \cap e_3$, which is of degree n. For a given P_1, there are n^2 possible choices for ℓ, P_2 et P_3; we must discard the choices corresponding to the n points of the intesection $e_2 \cap e_3 \cap F$, because otherwise we would have $P_2 = P_3$. We are left then with $n^2 - n = n(n-1)$ solutions.

   Since there are n possible choices for P_1, we find $n^2(n-1)$ possibilities for P_1, P_2, P_3. We still have to choose $P_4, \ldots P_8$ among the remaining $(n-3)$ points on $\ell \cap F$; therefore we get :

   $$p_1^2p_2p_3 = n^2(n-1)(n-3)(n-4)(n-5)(n-6)(n-7)$$

4. $\boxed{p_1p_2p_3p_4}$

   Suppose first that we only have the four points $P_1, \ldots, P_4$. Let e_1, e_2, e_3, e_4 be the planes expressing respectively conditions p_1, p_2, p_3, p_4. Let's find the degree of the ruled surface F' consisting of the lines ℓ touching $e_2 \cap F$, $e_3 \cap F$, $e_3 \cap F$; we have to compute the intersection of F' with a generic line, which amounts to compute $gp_2p_3p_4 = (g_e + p_2^2)p_3p_4$. We know by the previous formulas that $g_ep_2p_3 = n^2(n-2)$ and $p_2^2p_3p_4 = n^2(n-1)$, therefore F' is of degree $n^2(2n-3)$. By Bézout's theorem[3], the intersection $F \cap F' \cap e_1$ consists of $n^3(2n-3)$ points, among which, according to 3) above :

   - $n^2(n-1)$ are in e_2
   - $n^2(n-1)$ are in e_3
   - $n^2(n-1)$ are in e_4

   and the remaining $n^3(2n-3) - 3n^2(n-1) = n^2(2n^2 - 6n + 3)$ constitute $p_1p_2p_3p_4$. If we take now in account the possible choices for $P_5, \ldots, P_8$ among the remaining $n-4$ points, we get :

   $$p_1p_2p_3p_4 = n^2(2n^2 - 6n + 3)(n-4)(n-5)(n-6)(n-7)$$

[3]Schubert deduces Bézout's theorem from the coincidence formula ([7, page 46]).

Remark. *Following [7, page 236], let us note that for $n = 3$, if we count only the possible choices for P_1 through P_4, we found the well known fact that there are 27 lines that cut a cuiuc surface in four four points, which implies that the lines lye entirely in the cubic surface. This is a degenerate case of quadrisecant lines; however, it can be shown that there are no multiplicities to be taken in account, by using the fact that along a line $\ell \subset F$, the tangent plane to F at a point $x \in \ell$ varies with x.*

It remains to substitute the above formulas in (e) :

$$4!\varepsilon_{2222} = 16n^2(2n^2 - 6n + 3)(n-4)(n-5)(n-6)(n-7) - 32n^2(n-1)(n-3)(n-4)(n-5)(n-6)(n-7) \\ - 8n^2(n-2)(n-3)(n-4)(n-5)(n-6)(n-7) + 10 = n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)$$

and it follows that

$$\varepsilon_{2222} = \frac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3 + 6n^2 + 7n - 30)$$

which is the number of lines tangent to F at four distinct points.

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