The integral Thom polynomial for $\Sigma^{1,1}$

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Abstract

We give a direct proof that the Poincaré dual class to the singularity locus $\Sigma^{1,1}$ for a map $f : M^n \to V^p$ equals the Pontriagin class $\overline{P}_t(f^*(T(V)) - T(M))$ over the rationals, whenever $p - n = 2t - 1$ is odd. This was first proved by A. Sczűcs in [9] using results from cobordism theory. We also give an alternative proof that uses explicitely Schubert cycles on the grassmannian, in the spirit of René Thom's first calculations of dual classes to singular loci.

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Introduction Let $\xi^n = (E \to X)$ and $\eta^p = (F \to X)$ be two smooth real vector bundles of rank n and p respectively. We will denote by ξ_x the fiber of ξ above x. Set :

$$S^2(\xi, \eta) = \mathrm{HOM}(\xi, \eta) \oplus \mathrm{HOM}(\xi \bigcirc \xi, \eta)$$

where $\xi \bigcirc \xi$ denotes the symmetric tensor product, and denote by $\pi : S^2(\xi, \eta) \to X$ the natural projection. Elements of $S^2(\xi, \eta)$ are pairs (α, β), where $\alpha = \xi_x \to \eta_x$ and $\beta : \xi_x \bigcirc \xi_x \to \eta_x$ are both linear maps and $x = \pi(\alpha, \beta)$. For $i \leq n$, let

$$\Sigma^i(\xi, \eta) = \{(\alpha, \beta) \in S^2(\xi, \eta) \mid \dim(\ker(\alpha)) = i\}$$

and, for $j \leq i$:

$$\Sigma^{i,j}(\xi, \eta) = \{(\alpha, \beta) \in \Sigma^i(\xi, \eta) \mid \dim(\ker(\beta^*)) = j\}$$

where $\beta^* : \ker(\alpha) \to \mathrm{HOM}(\ker(\alpha, \mathrm{coker}(\alpha))$ is naturally deduced from β. If $\varphi : X \to S^2(\xi, \eta)$ is a section of π, then we set

$$\Sigma^i(\varphi) = \varphi^{-1}(\Sigma^i(\xi, \eta) \quad , \quad \Sigma^{i,j}(\varphi) = \varphi^{-1}(\Sigma^{i,j}(\xi, \eta) \quad .$$

The reason for defining Σ^i and $\Sigma^{i,j}$ is that they determine the homonymous Thom-Boardman singularities loci for a smooth map $f : M^n \to V^p$ between smooth varieties, by identifying the second order jet space $J^2(M, N)$ to $S^2(TM, TN)$ (see [7]).

It turns out that the closures $\overline{\Sigma}^i$ and $\overline{\Sigma}^{i,j}$ are stratified spaces, regular enough to carry fundamental classes over the integers mod 2, and sometimes over the integers. By Poincaré duality, there correspond cohomology classes, which can be expressed, according to a theorem of Thom [2], in terms of a universal polynomial in the characteristic classes of ξ and η.

The dual class to a stratified space can be characterized as follows; assume that Σ is a smooth, not necessarily closed, submanifold of codimension k of the smooth manifold X. Assume that there is a class $U_\Sigma \in H^k(X, X \setminus \overline{\Sigma}; \Lambda)$, where Λ is some coefficient ring, such that for each $x \in \Sigma$, if $V_x \subset X$ is a small neighborhood of x homeomorphic to a ball and such that $V_x \cap \Sigma$ is a ball on Σ, then, denoting by $j : (V_x, V_x \cap \Sigma) \to (X, X \setminus \overline{\Sigma})$ the natural inclusion, $j^*(U_\Sigma)$ is a generator of $H^k(V_x, V_x \setminus \Sigma; \Lambda) \simeq \Lambda$. Let $i : (X, \emptyset) \to (X, X \setminus \overline{\Sigma})$ denote the natural inclusion; then $i^*(U_\Sigma)$ deserves to be called the dual class to $\overline{\Sigma}$ in X, because if $\overline{\Sigma}$ carries a fundamental class, then $i^*(U_\Sigma)$ is indeed the Poincaré dual class to $\overline{\Sigma}$. But for the existence of a dual class such as U_Σ, it is not necessary to assume that $\overline{\Sigma}$ carries a fundamental class; it suffices to find a desingularisation $\pi : \tilde{\Sigma} \to X$ of $\overline{\Sigma}$ (that is, a map π that restricted to an open dense subset of $\tilde{\Sigma}$ is an diffeomorphism onto an open, dense subset of $\overline{\Sigma}$) such that the virtual normal bundle of π, that is the formal difference $\pi^*(TX) - T(\tilde{\Sigma})$, is orientable in the ring Λ, and then $i^*(U_\Sigma)$ will be equal to $\pi_!(1)$, where $\pi_! : H^*(\tilde{\Sigma}) \to H^*(X)$ denotes the Gisin homomorphism associated to π.

If φ is generic, i.e. it satisfies some transversality conditions, then the dual classes to $\overline{\Sigma}^{i,j}(\xi,\eta)$ and $\overline{\Sigma}^{i}(\xi,\eta)$ transpose to the dual classes to $\overline{\Sigma}^{i,j}(\varphi)$ and $\overline{\Sigma}^{i}(\varphi)$ respectively.

In [6] I have indicated how to compute the universal Poincaré dual class to $\overline{\Sigma}^{i}(\xi,\eta)$ and $\overline{\Sigma}^{i,j}(\xi,\eta)$ in the cohomology over the integers, which is shown to exist whenever $p-n$ and i are even, and j is divisible by 4. In this approach, there is no restriction on the presence of higher order singularities, but the existence of a dual class over the integers for $\Sigma^{i,j}$ depends on the existence of a dual class over the integers for Σ^{i}. Not so in the approach of [9, Lemma 3], where it is shown, in the case of morphisms associated to smooth maps, that when $p-n = 2t-1$, the dual class to $\Sigma^{1,1}$ with rational coefficients equals the Pontriagin class $\overline{P}_t(\eta - \xi)$; in this case higher order singularities play no role, because they are of larger codimension than $\Sigma^{1,1}$ and so they do not interfere with it. The proof given uses heavy results from cobordism theory, for which no geometric version seems to be available.

In [3], the case of contact class singularities for maps between manifolds of equal dimension is considered.

Since [9, Lemma3] is used in subsequent papers (for example [1]) to provide a "geometric approach" to immersions of manifolds in $\mathbb{R}^n$, I thought it appropriate to provide a direct, geometric calculation of the dual class to $\Sigma^{1,1}$ over the integers.

The main ingredient is a desingularisation of the closure of Σ^1, which is not the usual one, but its twin sister : instead of considering the space of pairs (ℓ, α), where ℓ is a line in a fiber ξ_x of ξ, $\alpha : \xi_x \to \eta_x$ is linear and $\alpha \mid \ell = 0$, we consider the space of pairs (h, α), where now h is a $(n-1)$-dimensional subspace of a fiber η_x of η, such that $\mathrm{Im}(\alpha) \subset h$. The gain is that if n is odd and p even, the space of $(n-1)$-subspaces of a fiber of η is an orientable manifold.

In the last §, we show how to calculate this dual class directly on the grassmannian, using Schubert cycles, in the spirit of Thom's first calculations [10, page 48].

Notation and preliminaries Let us denote by $q^{1,1} = q^{1,1}(\xi,\eta)$ the dual class to $\overline{\Sigma}^{1,1}(\xi,\eta)$ over the integers; if it exists, it lives in $H^{2(r+1)}(S^2(\xi,\eta),\mathbb{Z}) \simeq H^{2(r+1)}(X,\mathbb{Z})$. Denote by $q^{1,1}_{\mathbb{Q}}$ and $q^{1,1}_{\mathbb{Z}_2}$ the rational and mod 2 reductions respectively of $q^{1,1}$. If γ is another real vector bundle, then $q^{1,1}(\xi,\eta) = q^{1,1}(\xi\oplus\gamma,\eta\oplus\gamma)$, same over the rationals or mod 2.

Let $\mathcal{G}_n(\mathbb{R}^N)$ be the grassmannian of n-dimensional vector spaces in $\mathbb{R}^N$ and denote by ζ its tautologial bundle, of rank n. It can be shown (see [4, *Problem* 15-C]) that the torsion elements of $H^*(\mathcal{G}_n(\mathbb{R}^N),\mathbb{Z})$ are of order 2 and that the torsion subgroup has a supplementary free subgroup, which is generated by monomials in the Pontriagin classes $P_1(\zeta),\dots,P_{[n/2]}(\zeta)$. Therefore, an element of $H^*(\mathcal{G}_n(\mathbb{R}^N),\mathbb{Z})$ is determined by its mod 2 reduction, which can be written as a polynomial in the Stiefel-Whitney classes $w_1(\zeta),\dots,w_n(\zeta)$, and its rational reduction, which can be written as a polynomial in $P_1(\zeta),\dots,P_{[n/2]}(\zeta)$.

For any bundle $\xi^n = (E \to X)$ of rank n, $e(\xi)$ willd denote its Euler class, first obstruction to construct a non-vanishing section of ξ, which is a cohomology class with coefficients in the integers twisted by the orientation of ξ. Then, $e(\xi)^2 \in H^{2n}(X,\mathbb{Z})$, the ordinary cohomology with integers coefficients, and if $n = 2m$ is even, then $P_m(\xi) = e(\xi)^2$.

The grassmannian $\mathcal{G}_m(\mathbb{R}^{m+n})$ is orientable if and only if n is even; if $m = 2k$ and $n = 2\ell$, then $P_k^\ell(\zeta) \in H^{4k\ell}(\mathcal{G},\mathbb{Z})$ is the generator of the top cohomology group. To see this, take the sections $\sigma_1,\dots,\sigma_{2\ell}$ of ζ^{2k} defined by the orthogonal projection on the fibers of ζ of the vectors $e_{2k+1},\dots e_{2k+2\ell}$ of the standard basis of $\mathbb{R}^{2k+2\ell}$. Their common zeroes is the plane spanned by $e_1\dots,e_{2k}$ and it follows that $e(\zeta)^{2\ell} = P_k(\zeta)^\ell$ is the dual class to a point in $\mathcal{G}$.

We will denote by $\overline{P}_i(\xi)$ the Pontriagin classes of the virtual bundle $-\xi$. In the case of the tautologcal bundle ζ^m over $\mathcal{G}_m(\mathbb{R}^{m+n})$, $-\zeta$ can be represented by the orthogonal bundle ζ' to ζ in $\mathbb{R}^{m+n}$, of rank n, so $\overline{P}_i(\zeta) = P_i(\zeta')$. If $m = 2k$, $n = 2\ell$, using the diffeomorphism from $\mathcal{G}_{2k}(\mathbb{R}^{2k+2\ell})$ to $\mathcal{G}_{2\ell}(\mathbb{R}^{2k+2\ell})$ sending a plane to its orthogonal, one sees that the top cohomology class of $\mathcal{G}_{2k}(\mathbb{R}^{2k+2\ell})$ is also equal to $\overline{P}_\ell(\zeta)^\ell$.

If $\eta = (F \to X)$ is a vector bundle, denote by $\pi_{\mathcal{G}} : \mathcal{G}_k(\eta) \to X$ the associated bundle with fibers the grassmannian of k-planes in the fibers of η.

The main statement and its proof Our aim is the following proposition :

Proposition *Assume that ξ and η are real vector bundles of rank n and $n+2t-1$ respectively. Then $\overline{\Sigma}^{1,1}(\xi,\eta)$ admits a dual class $q^{1,1}$ in the cohomology* *with integer coefficients which is determined by :*

$$q_{\mathbb{Q}}^{1,1} = P_t(\eta - \xi) \quad , \quad q_{\mathbb{Z}_2}^{1,1} = (w_{2t}^2 + w_{2t-1}w_{2t+1})(\eta - \xi)$$

Proof: Using the fact that $q^{1,1}(\xi, \eta) = q^{1,1}(\xi \oplus \gamma, \eta \oplus \gamma)$ for any bundle γ, we are reduced to the case when $\xi = \mathcal{O}^{2M+1}$, where $\mathcal{O}$ denotes the trivial bundle, and η is of rank $2M + 2t$. By taking the classifying map of η, we are reduced to the case where X is a grassmannian of $2(M+t)$-planes in some high enough dimensional euclidian space. This shows that $q^{1,1}$ is determined by $q_{\mathbb{Q}}^{1,1}$ and by $q_{\mathbb{Z}_2}^{1,1}$.

Let $\pi_{\mathcal{G}} : \mathcal{G}_{2M}(\eta^{2(M+t)}) \to S^2(\mathcal{O}^N, \eta)$ be the Grassmann bundle associated to $\pi^*(\eta)$, where $\pi : S^2(\mathcal{O}^N, \eta) \to X$ is the natural projection, and denote by η_{2M} the vector bundle of rank $2M$ on $\mathcal{G}_{2M}(\eta)$ defined as the tautological bundle on the fibers of π. Since η_{2M} is a subbundle of $\pi^*(\eta)$, we can define the vector bundle η'_{2t} of rank $2t$ as the quotient $\eta'_{2t} = \pi^*(\eta)/\eta_{2M}$. An element of $\mathcal{G}_{2M}(\eta^{2(M+t)})$ can be written (α, β, p), where $(\alpha, \beta) \in S^2(\mathcal{O}^N, \eta)$, $p \in \mathcal{G}_{2M}(\eta_x)$, $x = \pi(\alpha, \beta)$. Set :

$$\tilde{\Sigma}^1 = \left\{ ((\alpha, \beta), p) \in \mathcal{G}_{2M}(\eta^{2(M+t)}) \mid \mathrm{Im}(\alpha) \subset p \right\}$$

If we define the section $\sigma^1 : \mathcal{G}_{2M}(\eta^{2(M+t)}) \to \mathrm{HOM}(\mathcal{O}^{2M+1}, \eta'_{2t})$ by sending (α, β, p) to the composition

$$\mathcal{O}^{2M+1} \xrightarrow{\alpha} \eta_x \to \left(\eta'_{2t}\right)_x = \eta_x / p$$

then $\tilde{\Sigma}^1 = Z(\sigma^1)$, where $Z(\bullet)$ denotes the set of zeroes of a section $\bullet$ of some vector bundle. Since σ^1 is transversal to the zero section, we have :

$$\delta_{\tilde{\Sigma}^1, \mathcal{G}_{2M}(\eta)} = e(\eta'_{2t})^{2M+1}$$

where δ denotes the dual class and $e(\bullet)$ is the Euler class, both with twisted coefficients.

Since $\Sigma^2(\mathcal{O}^N, \eta) \subset S^2(\mathcal{O}^N, \eta)$ is of codimension $4t + 2$ and $\Sigma^{1,1}(\mathcal{O}^N, \eta)$ of codimension $4t$, removing $\Sigma^2(\mathcal{O}^N, \eta)$ from $S^2(\mathcal{O}^N, \eta)$ will not affect the calculations, but will make them a lot easier. So now we shall work in $S^2(\mathcal{O}^N, \eta) \setminus \Sigma^2(\mathcal{O}^N, \eta)$. Then, we have a line bundle $K = \ker(\alpha)$ over $\tilde{\Sigma}$. Set

$$\tilde{\Sigma}^{1,1} = \left\{ ((\alpha, \beta), p) \in \tilde{\Sigma}^1 \mid \tilde{\beta} : K \bigcirc K \to \eta'_{2t} \text{ is zero } \right\}$$

where $\tilde{\beta}$ is the composition :

$$K \bigcirc K \subset \mathcal{O} \bigcirc \mathcal{O} \xrightarrow{\beta} \pi^*(\eta) \to \eta'_{2t} \quad .$$

If we define the section σ^2 of $\mathrm{HOM}(K \bigcirc K, \eta'_{2t})$ by $\sigma^2(\alpha,\beta,p) = \tilde{\beta}$, then $\tilde{\Sigma}^{1,1} = Z(\sigma^2)$. Clearly σ^2 is transversal to the zero section, and since $K \bigcirc K$ is a trivial line bundle (remember, we are working over the reals) :

$$\delta_{\tilde{\Sigma}^{1,1},\tilde{\Sigma}^1} = e(\eta'_{2t})$$

and therefore

$$\delta_{\tilde{\Sigma}^{1,1},\mathcal{G}_{2M}(\eta)} = \delta_{\tilde{\Sigma}^{1,1},\tilde{\Sigma}^1} \cdot \delta_{\tilde{\Sigma}^1,\mathcal{G}_{2M}(\eta)} = e(\eta')^{2M+2}$$

which is now a cohomology class with integer coefficients. Since $e(\eta'_{2t})^2 = P_t(\eta')$, we have :

$$\delta_{\tilde{\Sigma}^{1,1},\mathcal{G}_{2M}(\eta)} = P_t(\eta'_{2t})^{M+1}$$

and we have that $q^{1,1} = \pi_!(P_t(\eta'_{2t})^{M+1})$. The calculation of $\pi_!$ can be done with rational coefficients and Pontriagin classes in a similar way as it was done in [7, § 2] for the complex case. We have :

$$P_t(\eta'_{2t})^{M+1} = \underbrace{\begin{vmatrix} P_t(\eta') & P_{t-1}(\eta') & \dots \\ 0 & \ddots & \ddots \\ & 0 & P_t(\eta') \end{vmatrix}}_{M+1} = \underbrace{\begin{vmatrix} \overline{P}_{M+1}(\eta') & \overline{P}_M(\eta') & \dots \\ & \ddots & \\ & \dots & \overline{P}_{M+1}(\eta') \end{vmatrix}}_{t}$$

$$\overset{\pi_!}{\longmapsto} \underbrace{\begin{vmatrix} \overline{P}_1(\eta) & \dots & \\ & \ddots & \\ & 0 & \overline{P}_1(\eta) \end{vmatrix}}_{t} = P_t(\eta) = q^{1,1}_{\mathbb{Q}}$$

The calculation of $q^{1,1}_{\mathbb{Z}_2}$ was done in [7].

A proof using Schubert cycles An alternative approach of the formula for $q^{1,1}$ is by working directly with Schubert cycles of the grassmannian, that we will now present, in the spirit of [10, page 46, 48].

Let $\xi = (E \to X)$ and $\eta = (F \to X)$ be again two smooth real vector bundles of rank respectively n and $n + 2t - 1$. Now we take a bundle γ such that $\eta \oplus \gamma$ is the trivial bundle $\mathcal{O}^{2N+2t-1}$, and replace ξ by $\xi \oplus \gamma$, which will be of rank $2N$, and we will denote it again by ξ.

Let $\theta_\xi : X \to \mathcal{G}_{2N}(\mathbb{R}^{2K+1})$, for some large K, be a classifying map for ξ. If $\zeta = (E(\zeta) \to \mathcal{G}_{2N}(\mathbb{R}^{2K+1}))$ denotes the tautological bundle, then we have a commutative diagram :

$$\begin{array}{ccc} E & \stackrel{\psi_\xi}{\longrightarrow} & E(\mathcal{G}) \\ \downarrow & & \downarrow \\ X & \stackrel{\theta_\xi}{\longrightarrow} & \mathcal{G} \end{array}$$

where ψ_ξ is a linear isomorphism restricted to each fiber. Let $\varphi : X \to S^2(\xi, \mathcal{O}^{2N+2t-1})$ be a generic section, for which we want to compute the dual class to $\overline{\Sigma}^{1,1}(\varphi)$. For each $x \in X$, $\phi(x) = (\alpha_x, \beta_x)$, where $\alpha_x : \xi_x \to \mathbb{R}^{2N+2t-1}$ and $\beta_x : \xi_x \bigcirc \xi_x \to \mathbb{R}^{2N+2t-1}$ are linear maps. We can associate to φ (actually, to the component of φ in $\mathrm{HOM}(\xi, \mathcal{O}^{2N+2t-1})$) a map :

$$\theta_\varphi : X \to \mathcal{G}_{2N}(\mathbb{R}^{2K+1} \times \mathbb{R}^{2N+2t-1}) = \mathcal{G}_{2N}(\mathbb{R}^{2N+2K+2t}) \quad ,$$
$$x \mapsto \text{ graph of } \alpha_x \circ (\psi_\xi \mid \xi_x)^{-1} : \theta_\xi(x) \to \mathbb{R}^{2N+2t-1} \; .$$

Note that θ_ξ and θ_φ are homotopic.

Let :

$$F^1 = \left\{ H \in \mathcal{G}_{2N}(\mathbb{R}^{2N+2K+2t}) \mid \dim(H \cap (\mathbb{R}^{2K+1} \times \{0\}) \geq 1 \right\} \quad ;$$

then, because $\ker(\alpha_x \circ (\psi_\xi \mid \xi_x)^{-1}) = H \cap (\mathbb{R}^{2K+1} \times \{0\})$, where $H = \theta_\varphi(x)$, we have $\overline{\Sigma}^1(\varphi) = \theta_\varphi^{-1}(F^1)$.

The dual class to $\Sigma^{1,1}(\varphi)$ in $\Sigma^1(\varphi)$ equals $e((K \bigcirc K)^* \otimes Q) = e(Q)$, where K is the kernel bundle : $K = \{v \in \xi_x \mid \alpha_x(v) = 0\}$, and Q is the cokernel bundle, defined similarly. These bundles are defined only outside $\Sigma^i(\varphi)$, $i \geq 2$, but, as before, we can neglect these sets, because they will note interfere with the dual class to $\Sigma^{1,1}(\varphi)$.

We will now calculate $e(Q)$ directly on the grassmannian. If $\theta_\varphi(x) = H \in F^1$ and $\dim(H \cap (\mathbb{R}^{2K+1} \times \{0\}) = 1$, then its projection on the factor $\{0\} \times \mathbb{R}^{2N+2t-1}$ is of dimension $2N-1$ and constitutes the image of α_x. Its orthogonal in $\{0\} \times \mathbb{R}^{2N+2t-1}$ is of dimension $2t$; let us call $Q_{\mathcal{G}}$ the bundle of all the orthogonal subspaces corresponding to the $\theta_\varphi(x)$, $x \in \Sigma^1(\varphi)$. In order to determine a cycle representing $e(Q_{\mathcal{G}})$, let $v \in \{0\} \times \mathbb{R}^{2N+2t-1} \setminus \{0\}$; the orthogonal projection of v on $Q_{\mathcal{G}}$ is a section of this bundle, and it vanishes exactly when $Q_{\mathcal{G}}$ is orthogonal to v, which is equivalent to say that v is in the projection of H on the factor $\{0\} \times \mathbb{R}^{2N+2t-1}$, or that $\dim\left(H \cap (\{0\} \times \mathbb{R}^{2K+1} \oplus v)\right) \geq 2$. If we define the Schubert cycle

$$F^{1,1} = \left\{ H \in \mathcal{G}_{2N}(\mathbb{R}^{2K+2N+2t}) \mid \dim\left(H \cap (\mathbb{R}^{2K+1} \times \{0\} \oplus v)\right) \geq 2 \right\}$$

the pull-back by θ_ξ of the dual class to $F^{1,1}$ in $\mathcal{G}_{2N}(\mathbb{R}^{2K+2N+2t})$ will be the dual class to $\Sigma^{1,1}$ in Σ^1. It remains to determine this dual class.

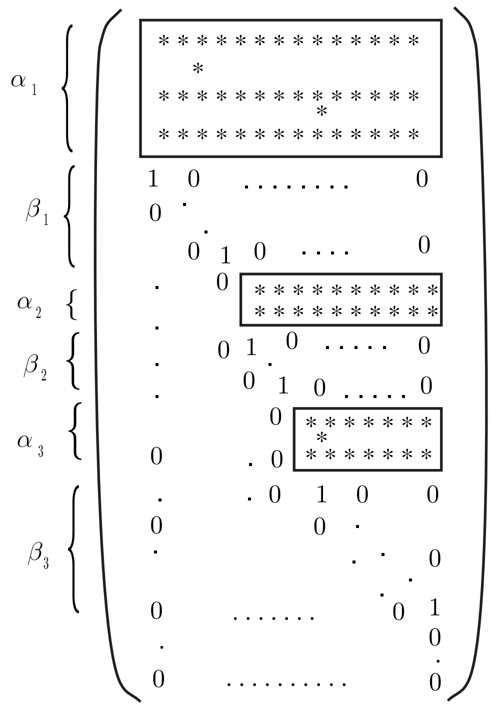

Figure 1: Representation of the Schubert cycle corresponing to the sequence $(\alpha_1, \beta_1, \dots)$

If we take v to be the $(2K+2)$-th vector of the natural basis of $\mathbb{R}^{2N+2K+2t}$, then

$$F^{1,1} = \left\{ H \in \mathcal{G}_{2N}(\mathbb{R}^{2N+2K+2t}) \mid \dim \left(H \cap (\mathbb{R}^{2K+2} \times \{0\}) \right) \geq 2 \right\}$$

In fact, these cycles have been identified as being characteristic cycles over the integers by L.S. Pontriagin in [5, page 218], who denoted them by $\chi_{p,q}$, with $p = 2K$, $q = 2t$ for $F^{1,1}$. Therefore their dual classes deserve to be called Pontriagin classes.

Anyway, we want to show now that the dual class to $F^{1,1}$ in $\mathcal{G}_{2N}(\mathbb{R}^{2N+2K+2t})$ equals $\overline{P}_t(\zeta)$, so that $\theta^*_\varphi\big(\overline{P}_t(\zeta)\big) = P_t(\eta - \xi)$ as it should, where now the Pontriagin classes are defined as in [4] in terms of Chern classes of the compexified of a vector bundle, which in turn will be identified as obstructions to the construction of frames.

Let us work with the grassmannian $\mathcal{G}_m(\mathbb{R}^{m+n})$. Consider the $(m+n) \times m$ matrix of figure 1, where $\alpha = (\alpha_1, \dots, \alpha_r)$, $\beta = (\beta_1, \dots, \beta_r)$ are any sequences

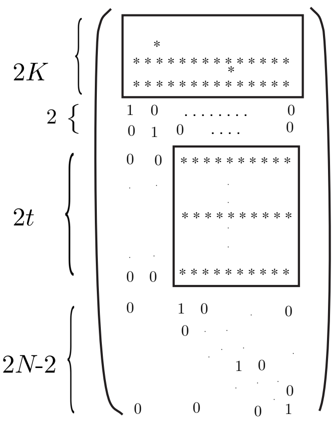

Figure 2: The Schubert cycle $F^{1,1}$

such that $\alpha_1 \geq 0$, $\beta_1 \geq 1$, $\alpha_i, \beta_i \geq 1$ for $i \geq 2$, $\sum_{i=1}^{r} \alpha_i \leq n$, $\sum_{i=1}^{r} \beta_i = m$, and the symbols $*$ represent any real numbers. Call such a matrix of type (α, β). Let $\Omega_{\alpha,\beta}$ denote the subset of $\mathcal{G}_m^{m+n}$ consisting of planes admitting a basis $v_1, \ldots, v_m \in \mathbb{R}^{m+n}$ such that the $(m+n) \times m$ matrix with $v_1, \ldots, v_m$ as column vectors is of type (α, β). Clearly, $\Omega_{\alpha,\beta} \simeq \mathbb{R}^{d(\alpha,\beta)}$, where $d(\alpha, \beta) = \alpha_1 \cdot \beta_1 + (\alpha_1 + \alpha_2) \cdot \beta_2 + \cdots + (\alpha_1 + \cdots + \alpha_r) \cdot \beta_r$; the plane corresponding to $0 \in \mathbb{R}^{d(\alpha,\beta)}$ is called the central element of the cell. It is shown in [4] that the closures $\overline{\Omega}_{\alpha,\beta}$, called Schubert cells or cycles (although Schubert [8] didn't explicitely care about cells nor cycles), constitute a cellular decomposition of the grassmannian. Note that a different basis of $\mathbb{R}^{m+n}$ than the natural basis that we have used, would provide a different decomposition, where the corresponding cycles would be homologous.

Our set $F^{1,1} \subset \mathcal{G}_{2N}(\mathbb{R}^{2N+2K+2t})$ is the Schubert cycle corresponding to the sequences $\alpha = (2K, 2t)$, $\beta = (2, 2N-2)$ (see figure 2). Let us work now in the grassmannian $\mathcal{G}_{2k}(\mathbb{R}^{2k+2\ell})$. We want to identify the class $P_t(\zeta^{2k})$ first. The boundary map for the cellular homology associated to the Schubert cells decomposition can be written explicitely, see for example [5, page 193], and from there one can see that the Schubert cells corresponding to sequences (α, β) such that the α_i and β_i are all even integers (that we can call even cells) constitute a basis for a complement to the torsion part of homology with integer coefficients, that is a basis for the homology with rational coefficient, and the algebraic dual will be a basis for the cohomology with rational coefficients. Consider the cell Ω corresponding to the sequences $(0, 2), (2k-2t, 2t)$; this is the only even

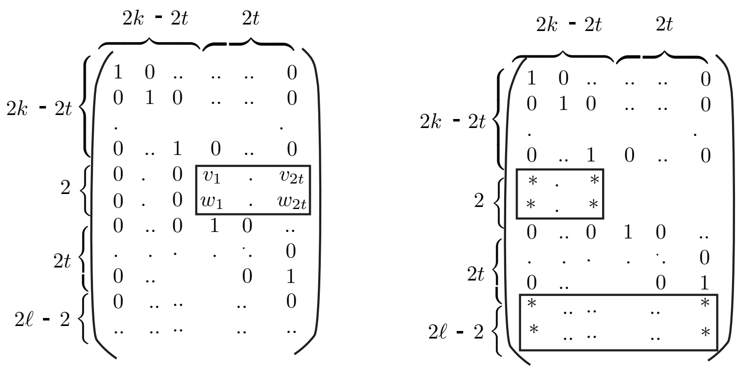

Figure 3: The cycle $\Omega_{(0,2),(2k-2t,2t)}$ and its dual $\Omega_{(2\ell-2,2),(2t,2k-2t)}$

cell of dimension not exceeding $4t$ that does not contain all the basis vectors $e_1, \ldots, e_{2k-2t}, e_{2k-2t+1}$; it will therefore be the support for the first obstruction to the construction of a $2k-2t+1$-frame on the bundle $\zeta \otimes_{\mathbb{R}} \mathbb{C}$, the complexified of ζ.

On the planes of Ω, we have a $(2k-2t)$-frame consituted by $e_1, \ldots, e_{2k-2t}$ and can define an additional vector by $v + i \cdot w$, where $v = (0, \ldots, 0, v_1, \ldots, v_{2t})$ and $w = (0, \ldots, 0, w_1, \ldots, w_{2t})$ are the rows $2k-2t+1$ and $2k-2t+2$ respectively of the matrix representation of the planes of Ω, and $i = \sqrt{-1}$ (see figure 3). Clearly $v+i \cdot w$ is linearly independent from $e_1, \ldots, e_{2k-2t}$, except for $v = w = 0$. It follows that the obstruction class to the construction of a $(2k-2t+1)$-frame associates to Ω the degree of the map $\mathbb{R}^t \times \mathbb{R}^t \to \mathbb{C}^t$, $(v, w) \mapsto v + iw$. Any orientation of $\mathbb{R}^t$ will induce the same product orientation on $\mathbb{R}^t \times \mathbb{R}^t$; but on $\mathbb{C}^t$ we must take the orientation induced by the complex structure. Since our map reads $(v_1, \ldots, v_t, w_1, \ldots, w_t) \mapsto (v_1, w_1, \ldots, v_t, w_t)$, its degree is $(-1)^t$. Since by definition $P_t(\zeta) = (-1)^t c_{2t}(\zeta \otimes \mathbb{C})$, we see that in the rational cohomology, $P_t(\zeta)$ is the class that sends Ω to 1 and all other $4t$-dimensional even cells to 0. It is the Poincaré dual of the cell corresponding to the sequences $(2\ell-2, 2), (2t, 2k-2t)$, because if we write $\Omega^*_{(2\ell-2,2),(2t,2k-2t)}$ for the cell corresponding to the sequences $(2\ell-2, 2), (2t, 2k-2t)$, but defined with respect to the natural basis written in the reversed order, we see that $\Omega_{(0,2),(2k-2t,2t)} \cap \Omega^*_{(2\ell-2,2),(2t,2k-2t)}$ consists in the central element of both Ω and Ω^*, and the intersection is transversal.

When passing from the grassmannian $\mathcal{G}_{2k}(\mathbb{R}^{2k+2\ell})$ to the grassmannian $\mathcal{G}_{2\ell}(\mathbb{R}^{2k+2\ell})$ by associating to each plane its orthogonal, the roles of α and β are exchanged and they must be taken in the reverse order (provided that neither α nor β

contain zeroes), which implies that the dual class to $\Omega_{(2k-2t,2t),(2,2\ell-2)}$ is $\overline{P}_t(\zeta)$, as we wanted.

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