Local obstructions to approximating tropical curves in surfaces

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This talk is based on joint work with Erwan Brugallé to appear. For simplicity, we will only consider fan tropical curves in the standard tropical plane $P \subset \mathbb{R}^3$. This tropical plane, P is the tropical hypersurface defined by "$x + y + z + 1$" $= \max\{x, y, z, 0\}$ see Figure 1. In addition throughout we will fix $\mathcal{P} \subset (\mathbb{C}^*)^3$ the complex plane given by the equation $x + y + z + 1 = 0$. We remark that $\mathcal{P}$ is $\mathbb{CP}^2$ minus four lines. Then $\mathcal{P}$ *approximates* P in the sense that $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{P}) = P$. A fan tropical curve C contained in P is a tropical curve with a single vertex which is also the vertex of P. We are concerned with the following question:

Question 1 *Given a fan tropical curve $C \subset P$ does there exist an irreducible algebraic curve $\mathcal{C} \subset \mathcal{P}$ approximating C, i.e.*

$$\lim_{t\to\infty} Log_t(\mathcal{C}) = C$$

and for each edge of C the natural weight from Section 6 of [2] is equal to the weight of the edge of the curve?

It is known that the answer is not always positive. As a first example, Vigeland exhibited families of tropical lines on generic tropical surfaces in $\mathbb{R}^3$ of degree greater than two [6]. By an integer affine linear map these lines in surfaces can be transformed to fan curves in the plane P. From complex geometry it is known that these families of lines on surfaces cannot be approximated. However, until recently it was not known how to forbid these curves based only on the tropical data. In [1] the authors provide some necessary conditions to approximating tropical curves contained in P and an affine hyperplane which can rule out some, but not all, of Vigeland's forbidden curves. In [5] more examples of non-approximable curves in P are given, here the reasons for ruling out the curves come from tropical intersection theory. Moreover, conditions for approximating curves using the Riemann-Hurwitz formula have been given by Brugallé and Mikhalkin.

The above question is also of broader interest than just lifting curves in P. The standard tropical plane is one of the local models for tropical surfaces, for example, all smooth tropical surfaces in $\mathbb{R}^3$ are locally P up to a integer affine transformation. A curve which is not everywhere locally approximable in a surface cannot be globally approximated. However, local approximability still does not imply global approximability. The general conditions to approximating curves in P generalise to other local models of smooth tropical surfaces, in this talk we stick to the tropical plane P purely for simplicity.

To tackle the problem of local approximation we invoke two tools of complex geometry, intersection with the Hessian curve and the adjunction formula. The main tool allowing us to translate to the tropical world is tropical intersection theory from [5]. At the end we return to the case of fan tropical curves $C \subset P$ and contained in an affine plane as considered in [1].

Intersection with the Hessian curve

Given a curve $\mathcal{C} \subset \mathbb{CP}^2$ defined by a homogeneous polynomial $P(x, y, z)$ of degree $d \geq 3$, the Hessian curve $\mathcal{H}_{\mathcal{C}}$ is the zero set of the degree $3(d-2)$ polynomial $\det(\mathrm{Hess}(P))$. If $\mathcal{C}$ does not have a

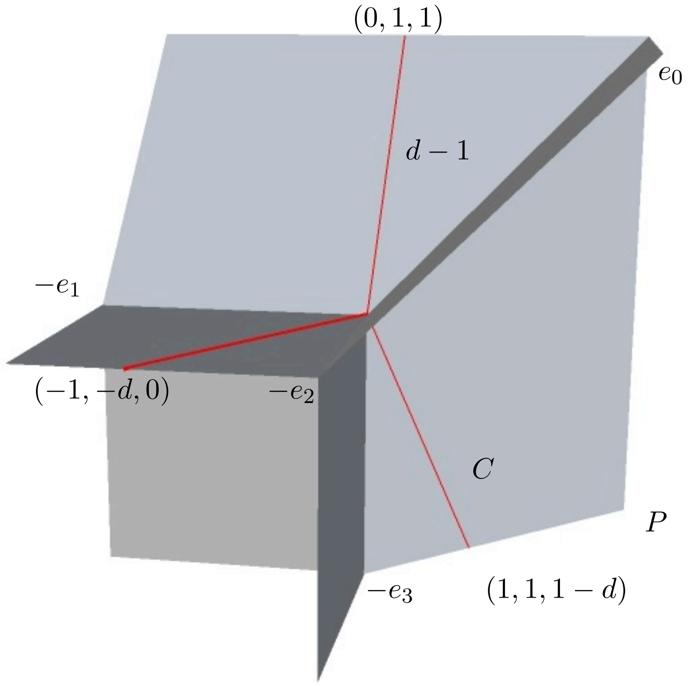

Figure 1: The tropical hyperplane $P \subset \mathbb{R}^3$ containing a trivalent fan tropical curve C of degree d.

component which is a line then $\mathcal{C}$ and $\mathcal{H}_{\mathcal{C}}$ intersect in $3d(d-2)$ points counted with multiplicity. Suppose $[1:0:0] \in \mathcal{C} \cap \mathcal{H}_{\mathcal{C}}$, the Newton polytope for $\mathcal{C}$ with respect to coordinates x and y gives a lower bound for the multiplicity of the intersection at $[1:0:0]$. As there is a duality between the fan tropical curve and Newton polytope this bound can be expressed in terms of some unbounded rays of the tropical curve. As mentioned above, $\mathcal{P}$ can be viewed as $\mathbb{CP}^2$ minus four lines, the intersections of which yield exactly six points. For each of these six points we are able to extract from the tropical curve C a lower bound on the intersection of $\mathcal{C}$ and $\mathcal{H}_{\mathcal{C}}$ where $\mathcal{C}$ is a potential approximation of C. The sum of the six multiplicities must be less than $3d(d-2)$. We translate this condition to the level of the tropical curve to obtain an inequality with terms involving the degree and tropical self intersection of the curve along with the weights of edges, but the full formula is too technical to be included here. By applying this formula we are able to forbid all members of Vigeland's families of lines on surfaces of degree greater than two.

The adjunction formula

If C is approximated by a smooth embedded curve $\mathcal{C}$ then we may also ask about the genus of a parameterisation of $\mathcal{C}$. Using the tropical intersection product given in [5] it is possible to translate the classical adjunction formula to the following formula, where C^2 is the tropical self intersection, $deg(C)$ is the projective degree of C and w_E denotes the weight of an edge $E \subset C$.

Theorem 1 *If $C \subset P$ is approximated by an embedded irreducible curve $\mathcal{C} \subset \mathcal{P}$ which is parameterised by $f : \mathcal{S} \longrightarrow \mathcal{P}$ then*

$$g(\mathcal{S}) \leq C^2 + deg(C) - \sum_{E_i \subset C} w_{E_i} + 2.$$

In particular, if the right hand side is negative then C is not approximated by any irreducible curve.

Classification of fan curves in $P \cap H$

Now we return to the situation considered in [1]. Here H will denote an affine hyperplane, and the tropical fan curve C is will be contained in $P \cap H$. As mentioned above Bogart and Katz provide necessary conditions to approximating such curves. After strengthening their conditions and constructing some curves we obtain a complete classification of such tropical curves. Once again the main tool allowing us to extend the conditions is the tropical intersection product.

Theorem 2 *An tropical curve $C \subset P \cap H$ of degree d is approximable by a reduced and irreducible complex algebraic curve $\mathcal{C} \in \mathcal{P}$ if and only if C is one of the following:*

1. *C is the stable intersection of P and H (see [4] or [3]);*
2. *C is the curve depicted in Figure 1 up to symmetry of P.*

In case (2) when $d = 1$ the curve is an affine line (not trivalent) which bisects two faces of P. In general, the curve from case (2) is unique in each degree, moreover it is real, rational and has $d+1$ punctures. Note that $C^2 = 0$ in case (1), and $C^2 = -1$ in case (2).

References

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