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# HARMONIC MEASURE AND POLYNOMIAL JULIA SETS

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ABSTRACT. There is a natural conjecture that the universal bounds for the dimension spectrum of harmonic measure are the same for simply connected and for non-simply connected domains in the plane. Because of the close relation to conformal mapping theory, the simply connected case is much better understood, and proving the above statement would give new results concerning the properties of harmonic measure in the general case.

We establish the conjecture in the category of domains bounded by polynomial Julia sets. The idea is to consider the coefficients of the dynamical zeta-function as subharmonic functions on a slice of Teichmüller's space of the polynomial, and then to apply the maximum principle.

## 1. Dimension spectrum of harmonic measure

In this paper we discuss some properties of harmonic measure in the complex plane. For a domain in the Riemann sphere $\Omega \subset \hat{\mathbb{C}}$ and a point $a \in \Omega$, let $\omega = \omega_a$ denote the harmonic measure of $\Omega$ evaluated at $a$. The measure $\omega_a$ can be defined, for instance, as the hitting distribution of a Brownian motion started at $a$: if $e \subset \partial\Omega$, then $\omega_a(e)$ is the probability that a random Brownian path first hits the boundary at a point of $e$.

Much work has been devoted to describing dimensional properties of $\omega$ when the domain is as general as possible. In particular, Jones and Wolff [7] proved that no matter what the domain $\Omega$ is, harmonic measure is concentrated on a Borel set of Hausdorff dimension at most one; in other words,

$$\dim \omega \leq 1 \qquad \text{for all plane domains.} \tag{1.1}$$

We are interested in finding similar (but stronger) universal estimates involving the dimension spectrum of $\omega$.

### 1.1. Universal spectrum.

For every positive $\alpha$, we denote

$$f_\omega^+(\alpha) = \dim\{\alpha_\omega(z) \leq \alpha\},$$

where $\alpha_\omega(z)$ is the lower pointwise dimension of $\omega$:

$$\alpha_\omega(z) = \liminf_{\delta \to 0} \frac{\log \omega B(z,\delta)}{\log \delta}.$$

$B(z,\delta)$ is a general notation for the disc with center $z$ and radius $\delta$.

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The *universal dimension spectrum* is the function

$$\Phi(\alpha) = \sup_{\omega} f_{\omega}^{+}(\alpha), \tag{1.2}$$

where the supremum is taken over harmonic measures of all planar domains.

We would like to compare $\Phi(\alpha)$ with the corresponding spectrum defined for arbitrary *simply connected* domains in the plane:

$$\Phi_{\mathrm{sc}}(\alpha) = \sup\{f_{\omega}^{+}(\alpha) : \ \Omega \text{ is simply connected}\}.$$

Because of the close relation to conformal mapping theory, the simply connected case is much better understood and more information concerning dimension spectrum is available. Harmonic measure of a simply connected domain is the image of the Lebesgue measure under the boundary correspondence given by the Riemann map, and estimates of the Riemann map derivative control the boundary distortion.

For example, an elementary estimate of the integral means of the derivative implies the inequality

$$\Phi_{\mathrm{sc}}(\alpha) \leq \alpha - c(\alpha-1)^2, \qquad (0 < \alpha \leq 2), \tag{1.3}$$

with some positive constant $c$, see (1.14) and Lemma 3 below. This proves the statement

$$\dim \omega = 1 \qquad \text{for simply connected domains.} \tag{1.4}$$

Indeed, from the definition of dimension spectrum it follows that

$$\alpha = \dim \omega \quad \Rightarrow \quad f_{\omega}^{+}(\alpha) = \alpha.$$

On the other hand, by (1.3) we have $f_{\omega}^{+}(\alpha) \neq \alpha$ if $\alpha \neq 1$. The estimate (1.3) is in fact a bit stronger than the dimension result; the relation between (1.3) and (1.4) is basically the same as the relation between the central limit theorem and the law of large numbers in probability theory.

Comparing the statements (1.1) and (1.4), it is natural to ask whether estimates like (1.3) extend to general, non-simply connected domains. We *conjecture* that

$$\Phi(\alpha) \leq \alpha - c(\alpha-1)^2, \qquad (1 \leq \alpha \leq 2), \tag{1.5}$$

which is of course stronger than (1.1). More generally, we state the following

**Conjecture.** *For all $\alpha \geq 1$, we have*

$$\Phi(\alpha) = \Phi_{\mathrm{sc}}(\alpha). \tag{1.6}$$

It is easy to see that (1.6) is false if $\alpha < 1$, for the universal spectrum is trivial in this case,

$$\Phi(\alpha) \ \equiv \ \alpha, \qquad (\alpha \leq 1),$$

but the spectrum $\Phi_{\mathrm{sc}}(\alpha)$ is not, see (1.3). We refer to [13] for further discussion of the universal spectrum and related topics.

The goal of this work is to give some partial justification of the above conjecture.

1.2. **Fractal approximation.** A proof of (1.5) that would be based on traditional methods of potential theory (as, for example, in [7]) seems to be out of reach, let alone a proof of the conjecture. We propose to apply methods of conformal dynamics, and to this end we first restate the conjecture using the idea of fractal approximation.

According to [4] and [13], one can replace the supremum in the definition of the universal dimension spectrum (1.2) with the one taken over harmonic measures on (conformally) self-similar boundaries:

$$\Phi(\alpha) = \sup\{f_\omega^+(\alpha) :\ \partial\Omega \text{ is a conformal Cantor set}\}. \tag{1.7}$$

A set $J \subset \mathbb{C}$ is said to be a *conformal Cantor set* if it is generated by some analytic map of the form

$$F :\ \bigcup_{j=1}^{d} D_j \to D, \tag{1.8}$$

where $\{D_j\}$ is a finite collection of open topological discs such that the closures $\bar{D}_j$ are pairwise disjoint and sit inside a simply connected domain $D$. It is also required that the restriction of $F$ to each $D_j$ be a bijection $D_j \to D$.

If $J = \partial\Omega$ is a conformal Cantor set, then we have

$$f_\omega^+(\alpha) = \sup\{f_\omega(\alpha') :\ \alpha' \leq \alpha\},$$

where

$$f_\omega(\alpha) = \dim\left\{z \in J :\ \lim_{\delta\to 0}\frac{\log \omega B(z,\delta)}{\log\delta} = \alpha\right\}.$$

To prove the conjecture it is therefore sufficient to show that the inequality

$$f_\omega(\alpha) \leq \Phi_{\rm sc}(\alpha), \qquad (\alpha \geq 1)$$

holds for every conformal Cantor set.

We can now state our main result. We say that $J$ is a *polynomial conformal Cantor set* if the map $F$ in (1.8) extends to a polynomial of degree $d$. In other words, $J$ is the usual Julia set of a polynomial such that the orbits of all critical points escape to infinity.

**Theorem A.** *If $\omega$ is harmonic measure on a polynomial conformal Cantor set, then*

$$f_\omega(\alpha) \leq \Phi_{\rm sc}(\alpha), \qquad (\alpha \geq 1). \tag{1.9}$$

We believe that a "polynomial" version of (1.7) should be true, i.e. to compute the universal spectrum it should be enough to only consider polynomial conformal Cantor sets. The conjecture would then follow from Theorem A. In this respect, let us mention that the dimension results (1.4) and (1.1) were first discovered for polynomial Julia sets, see [15]. Also compare [12] and [20].

1.3. **Pressure function.** For a polynomial $F$, let $\Omega_F$ denote the basin of attraction to infinity,

$$\Omega_F = \{z : F^n z \to \infty\},$$

so that $J_F = \partial\Omega_F$ is the Julia set of $F$. The harmonic measure $\omega_\infty$ of $\Omega_F$ is the measure of maximal entropy with respect to $F$. We will apply some standard technique of ergodic theory to rewrite (1.9) in a more convenient form.

The *pressure function* of a polynomial $F$ of degree $d$ is defined by the formula

$$P_F(t) = \limsup_{n\to\infty} \frac{1}{n} \log_d \sum_{z\in F^{-n}z_0} |F_n'(z)|^{-t}, \tag{1.10}$$

where $F_n'$ denotes the derivative of the $n$-th iterate of $F$, and $z_0 \in \Omega_F$ is some point not in the orbit of the critical set. The limit (1.10) does not depend on the choice of $z_0$. The following two assertions are well known.

**Lemma 1.** *If $J = J_F$ is a polynomial conformal Cantor set, then*

$$f_\omega(\alpha) = \inf_{t\geq 0} \,[t + \alpha P_F(t)], \qquad (\alpha \geq 1). \tag{1.11}$$

**Lemma 2.** *If $F$ is a polynomial with connected Julia set, then*

$$P_F(t) = \beta(t) - t + 1,$$

*where $\beta(t) = \beta_{\Omega_F}(t)$ is the integral means spectrum of $\Omega_F$.*

By definition, the integral means spectrum $\beta_\Omega(t)$ of a simply connected domain $\Omega$ is the function

$$\beta_\Omega(t) = \limsup_{r\to 1} \frac{\log \int_{|z|=r} |\varphi'(z)|^t \, |dz|}{|\log(1-r)|}, \qquad (t \in \mathbb{R}),$$

where $\varphi$ is a Riemann map taking the unit disc onto $\Omega$.

We define the *universal integral means spectrum*

$$B(t) = B_{sc}(t) = \sup_\Omega \beta_\Omega(t),$$

by taking supremum over domains containing $\infty$. The following fact was established in [13].

**Lemma 3.** *If we denote*

$$\Pi(t) = B(t) - t + 1, \tag{1.12}$$

*then*

$$\Phi_{\rm sc}(\alpha) = \inf_{t\geq 0} \,[t + \alpha\Pi(t)], \qquad (\alpha \geq 1). \tag{1.13}$$

The reason for (1.13) to be valid is that relations similar to (1.11) hold for all domains with self-similar boundaries, and by "fractal approximation" the same is true on the level of universal bounds.

Let us mention at this point that by Lemma 3, the inequality (1.3) we discussed earlier is a consequence of the well-known estimate

$$B(t) \leq Ct^2, \qquad (|t| \leq 1). \tag{1.14}$$

From Lemma 2 and (1.12), it follows that

$$J_F \quad \text{is connected} \quad \Rightarrow \quad P_F(t) \leq \Pi(t). \tag{1.15}$$

We will extend the latter inequality to Julia sets of polynomials with al critical points escaping to infinity and show that if $t \geq 0$, then

$$J_F \quad \text{is a polynomial conformal Cantor set} \quad \Rightarrow \quad P_F(t) \leq \Pi(t). \tag{1.16}$$

This will complete the proof of Theorem A: we obtain (1.9) from Lemma 1 and Lemma 3 by applying the Legendre transform to both sides of the inequality in (1.16).

1.4. **Two results in polynomial dynamics.** The verification of (1.16) follows a natural strategy. Given a polynomial $F$ with all critical points escaping to infinity, we use a construction due to Branner and Hubbard [2] to embed $F$ in a holomorphic polynomial family

$$\lambda \mapsto F_\lambda, \qquad \lambda \in \mathbb{D} := \{|\lambda| < 1\},$$

so that the boundary values of the family exist as polynomials with connected Julia set. Using a subharmonicity argument, one can then extend the bound (1.15) of the pressure function from the boundary circle to the unit disc.

We recall the Branner-Hubbard construction in Section 2; see also [16] for an interpretation in terms of Teichmüller's spaces. In the case of quadratic polynomials, we can simply take

$$F_\lambda(z) = z^2 + c(\lambda),$$

where $\lambda \mapsto c(\lambda)$ is a universal covering map of the complement $\mathbb{C} \setminus \mathcal{M}$ of the Mandelbrot set $\mathcal{M}$.

It is important that almost all limit polynomials have "nice" ergodic properties. For instance, it is known from [6] and [23] that almost every point on the boundary of the Mandelbrot set is a Collet-Eckmann polynomial. The following weaker statement, which goes back to Douady [5] in the quadratic case, will be sufficient for our argument. (The method of [23] can actually be extended to give "topological" Collet-Eckmann condition in Theorem B.)

**Theorem B.** *Let $F$ be a polynomial with all critical points escaping to infinity, and let $\{F_\lambda\}$ be its Branner-Hubbard family. Then the following is true for almost every point $\zeta \in \partial\mathbb{D}$. For every $z \in \mathbb{C}$, there exists a limit*

$$F_\zeta(z) = \lim_{r \to 1-} F_{r\zeta}(z),$$

*and $F_\zeta$ is a polynomial with connected Julia set and all cycles repelling.*

This theorem will be used in combination with another technical result. If we consider the pressure as a function on the parameter space of a Branner-Hubbard family, then it is not immediately clear how to apply the maximum principle because there are poles in the sum

$$\sum_{z \in F^{-n} z_0} |F_n'(z)|^{-t}$$

of the definition (1.10). A way out of this difficulty will be to work with a version of the pressure function that involves multipliers of periodic points. Let us denote

$$Z_n(F,t) = \sum_{a \in \mathrm{Fix}(F^n)} |F_n'(a)|^{-t},$$

see [21] for the connection with dynamical zeta-function. It is well known that if $F$ is a *hyperbolic* polynomial, then we have

$$P_F(t) = \lim_{n\to\infty} \frac{1}{n} \log_d Z_n(F,t). \tag{1.17}$$

**Theorem C.** *If a polynomial $F$ of degree $d$ has connected Julia set and has no non-repelling cycles, then*

$$P_F(t) \;\geq\; \limsup_{n\to\infty} \frac{1}{n} \log_d Z_n(F,t).$$

1.5. **Proof of Theorem A (assuming Theorems B and C).** As we mentioned, it is sufficient to show that if $F$ is a polynomial with all critical points escaping to infinity and if $t \geq 0$, then

$$P_F(t) \leq \Pi(t). \tag{1.18}$$

Let $\{F_\lambda\}$ be the Branner-Hubbard family with $F_0 = F$. Consider the functions

$$s_n(\lambda) = \frac{1}{n} \log_d Z_n(F_\lambda, t), \qquad (\lambda \in \mathbb{D}).$$

Since all periodic points of each polynomial $F_\lambda$ are repelling, the functions $s_n$ are *uniformly bounded.* (This is the only place where we use $t \geq 0$.)

For every $n$, the correspondence $\lambda \to \mathrm{Fix}(F_\lambda^n)$ is a multi-valued holomorphic function with branching points corresponding to polynomials with parabolic cycles. There are no such polynomials in the case under consideration, and so every periodic point $a_\nu \in \mathrm{Fix}(F^n)$ determines a single-valued function

$$\lambda \mapsto a_\nu(\lambda) \in \mathrm{Fix}(F_\lambda^n), \qquad a_\nu(0) = a_\nu.$$

It follows that the functions $s_n$ are *subharmonic* in the unit disc: we have

$$s_n = \frac{1}{n} \,\log_d \sum_\nu h_\nu \bar{h}_\nu,$$

where

$$h_\nu(\lambda) = [(F_\lambda^n)'(a_\nu(\lambda))]^{t/2}$$

are holomorphic functions, and therefore

$$\Delta s_n = \mathrm{const}\; \frac{\sum |h_\nu|^2 \sum |\partial h_\nu|^2 - |\sum \bar{h}_\nu \partial h_\nu|^2}{(\sum |h_\nu|^2)^2} \geq 0.$$

We should note that the subharmonicity of pressure-like quantities is a well-known general principle; see [1] for a beautiful application to quasiconformal maps.

Let us also define the values

$$s_n(\zeta) = \frac{1}{n} \log_d Z_n(F_\zeta, t)$$

for all boundary points $\zeta \in \partial\mathbb{D}$ satisfying the conclusion of Theorem B. The set of such $\zeta$'s has full Lebesgue measure; the polynomials $F_\zeta$ have no non-repelling cycles and their Julia sets are connected. It is clear that $s_n(\zeta)$ is a radial limit of the function $s_n(\lambda)$ wherever $F_\zeta$ is a radial limit of the polynomial family $F_\lambda$; in particular, this is true for almost all $\zeta \in \partial\mathbb{D}$. Since the functions $s_n(\lambda)$ are bounded and subharmonic, we have

$$s_n(0) \leq \frac{1}{2\pi}\int_{\partial\mathbb{D}} s_n(\zeta)|d\zeta|. \tag{1.19}$$

One the other hand, applying Theorem C and (1.16), we obtain the inequalities

$$\lim_{n\to\infty} \tilde{s}_n(\zeta) \leq P_{F_\zeta}(t) \ \leq \Pi(t), \tag{1.20}$$

where

$$\tilde{s}_n(\zeta) = \sup_{k\geq n} s_k(\zeta).$$

Combining (1.19) and (1.20), we prove (1.18):

$$\begin{aligned} P_F(t) \ &= \ \lim_{n\to\infty} s_n(0) \ \leq \ \lim_{n\to\infty} \frac{1}{2\pi}\int \tilde{s}_n(\zeta)\ |d\zeta| \\ &= \ \frac{1}{2\pi}\int \lim_{n\to\infty} \tilde{s}_n(\zeta)\ |d\zeta| \ \leq \ \Pi(t), \end{aligned}$$

where the first equality is by (1.17), and the second one follows from Lebesgue's convergence theorem. □

The rest of the paper contains the proofs of Theorem B and Theorem C. Both proofs depend on the work of Jan Kiwi [8], [9]. We refer to [3] and [17] for general facts concerning polynomial dynamics.

## 2. Branner-Hubbard families

In this section we briefly recall the Branner-Hubbard construction of wringing complex structures, see [2], and then derive Theorem B from a result of Kiwi. We will use the half plane

$$\mathbb{H} = \{\gamma = \alpha + i\beta :\ \alpha > 0,\ \beta \in \mathbb{R}\},$$

as a parameter space for Branner-Hubbard families $\{F_\gamma\}$. The map

$$\lambda(\gamma) = \frac{\gamma - 1}{\gamma + 1}$$

transforms this parameter space into $\mathbb{D}$, the case we considered in the first section.

### 2.1. **Wringing complex structures.**

Let $\Gamma$ denote the subgroup of $GL(2,\mathbb{R})$ formed by matrices

$$\gamma = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \qquad \text{with} \quad \alpha > 0,$$

which we identify with complex numbers

$$\gamma = \alpha + i\beta \in \mathbb{H}.$$

$\Gamma$ acts on the Riemann sphere $\hat{\mathbb{C}}$ as a group of quasiconformal homeomorphisms

$$A_\gamma(z) = z|z|^{\gamma-1}, \qquad (0 \mapsto 0,\ \infty \mapsto \infty). \tag{2.1}$$

The Beltrami coefficient of $A_\gamma$ is

$$\mu_\gamma(z) = \lambda(\gamma)\frac{z}{\bar{z}},$$

and the corresponding Beltrami field $\mathcal{E}_\gamma$ of infinitesimal ellipses is invariant with respect to the transformation

$$F_0 : z \mapsto z^d.$$

Let $\mathcal{P} = \mathcal{P}_d$ denote the space of polynomials of degree $d$, and let $\mathcal{S}$ denote the subspace of $\mathcal{P}$ which consists of polynomials such that the orbits of all critical points escape to infinity. We also use the notation $\mathcal{P}_*$ and $\mathcal{S}_*$ for the corresponding spaces of monic centered polynomials. Clearly, $\mathcal{P}_* \cong \mathbb{C}^{d-1}$, and if we identify equivalent polynomials (two polynomials are equivalent if they are conformally conjugate), then

$$\mathcal{S}/\sim\ \cong\ \mathcal{S}_*/\sim\ \cong\ \mathcal{S}_*/\mathbb{Z}_{d-1},$$

where $\mathbb{Z}_{d-1}$ acts according to the formula $F(z) \mapsto \bar{\eta}F(\eta z)$ with $F \in \mathcal{S}_*$ and $\eta^{d-1} = 1$.

Given a polynomial $F \in \mathcal{S}$, there exists a conformal map, the extended Böttcher function,

$$\phi : \Omega^* \to \Delta^*$$

satisfying

$$F_0 \circ \phi = \phi \circ F,$$

where $\Omega^*$ is an open $F$-invariant set of full area measure in $\hat{\mathbb{C}}$ and $\Delta^*$ is an open $F_0$-invariant set of full measure in the exterior unit disc $\Delta = \{|z| > 1\}$. It follows that the Beltrami fields $\phi^{-1}\mathcal{E}_\gamma$ are defined almost everywhere in $\hat{\mathbb{C}}$, and the corresponding family of quasiconformal homeomorphisms

$$R(\gamma, F) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \qquad (0, 1, \infty) \mapsto (0, 1, \infty),$$

is holomorphic in $\gamma$. It is shown in [2] that the equivalence class $[\gamma F] \in \mathcal{S}/\sim$ of the polynomial

$$\gamma F := R(\gamma, F) \circ F \circ R(\gamma, F)^{-1}$$

depends only on $[F]$ and $\gamma$, and that the map

$$(\gamma, [F]) \mapsto [\gamma F]$$

is a group action on $\mathcal{S}/\sim$.

If $\mathcal{O} \subset \mathcal{S}_*/\mathbb{Z}_{d-1}$ is an orbit of $\Gamma$, then each polynomial $F_1 \in \mathcal{S}_*$ with $[F_1] \in \mathcal{O}$ provides a uniformization of the orbit:

$$\gamma \in \mathbb{H} \quad \mapsto \quad [\gamma F_1] \in \mathcal{O}.$$

This map lifts to a map $\mathbb{H} \to \mathcal{S}_*$, which we call the *Branner-Hubbard family of* $F_1$, and for which we use the notation

$$\gamma \mapsto F_\gamma \quad \text{or} \quad \{F_\gamma\}_{\gamma \in \mathbb{H}}.$$

Note that because of the group action structure,

$$\{F_{\gamma\gamma_0}\}_{\gamma\in\Gamma} \quad \text{is the Branner-Hubbard family of } F_{\gamma_0}. \tag{2.2}$$

The monic centered polynomials $F_\gamma$ depend analytically on $\gamma$. In fact, we have

$$F_\gamma = R_\gamma \circ F_1 \circ R_\gamma^{-1},$$

for some holomorphic family $\{R_\gamma\}$ of quasiconformal automorphisms of $\hat{\mathbb{C}}$. We also have the following equation:

$$\phi_\gamma := A_\gamma \circ \phi_1 \circ R_\gamma^{-1}, \qquad (\text{near } \infty), \tag{2.3}$$

where $\phi_\gamma$ is the Böttcher functions of $F_\gamma$ satisfying $\phi_\gamma(z) \sim z$ at infinity.

The following lemma describes the boundary behavior of the family $\{F_\gamma\}$.

**Lemma 1.** *Consider a Branner-Hubbard family of $F_1 \in \mathcal{S}$*

$$F_\gamma(z) = z^d + a_{d-2}(\gamma) z^{d-2} + \cdots + a_0(\gamma), \quad (\gamma \in \mathbb{H}).$$

*The functions $a_j(\gamma)$, $Im\gamma = \beta$ have finite angular limits $a_j(i\beta)$ for almost all $\beta \in \partial\mathbb{H}$. The limit polynomials*

$$F_{i\beta}(z) = z^d + a_{d-2}(i\beta) z^{d-2} + \cdots + a_0(i\beta)$$

*have connected Julia sets.*

*Proof:* For a polynomial $F$, let $G_F(\cdot)$ denote the Green's function of the Julia set $J_F$ with pole of infinity, and let $m(F)$ be the maximal *escape rate* of the critical set:

$$m(F) \;=\; \max\{G_F(c):\; c \in \mathrm{Crit}(F)\}.$$

We will write $G_\gamma$ for the Green's function of $F_\gamma$.

To prove the first statement, we need the following result of Branner and Hubbard [2]:

$$\forall \varrho > 0, \quad \text{the set} \quad \{F \in \mathcal{P}_*:\; m(F) \le \varrho\} \quad \text{is compact.} \tag{2.4}$$

From (2.1) and (2.3), it follows that

$$G_\gamma \circ R_\gamma = \alpha G_1,$$

where $\alpha$ is the real part of $\gamma$. Since $R_\gamma$ sends the critical set of $F_1$ onto the critical set of $F_\gamma$, we have

$$m(F_\gamma) = \alpha m(F_1). \tag{2.5}$$

Applying (2.4), we see that the coefficients $a_j(\gamma)$ are uniformly bounded in the strip $\{0 < \alpha < 1\}$, and so the existence of angular limits follows from Fatou's theorem.

The second statement of the lemma follows from (2.5) and the well-known fact that the function $m(F): \mathcal{P}_* \to \mathbb{R}$ is continuous. □

2.2. **Proof of Theorem B.** The rest of the argument follows Kiwi's approach in [8].

Let $F_1$ be a monic centered polynomial of degree $d$ with all critical points escaping to infinity, and let $\{F_\gamma\}$ be the corresponding Branner-Hubbard family. For simplicity, we will assume that $F_1$ (and therefore every polynomial in the family) is such that the critical points are simple and their orbits are disjoint. The proof in the general case requires only minor technical modifications. We say that the polynomial $F_\gamma$ is *visible* if for each critical point $c_j$, $(1 \le j \le d-1)$, there are precisely two external rays terminating at $c_j$. In this case, let

$$\Theta_j = \{\theta_j^-, \theta_j^+\} \subset S^1 \equiv \mathbb{R}/\mathbb{Z}$$

be the set of the external arguments. The collection of the sets $\Theta_j$,

$$\Theta(F_\gamma) = \{\Theta_1, \ldots \Theta_{d-1}\},$$

is called the *critical portrait* of $F_\gamma$. Every critical portrait determines a partition of $S^1$ into $d$ sets of length $1/d$ each. Consider the map $T : S^1 \to S^1$,

$$T: \ \theta \mapsto d\theta \ (\text{mod } 1).$$

The portrait is said to be *periodic* if with respect to this partition, the $T$-itinerary of one of the point $\theta_j^\pm$ is periodic.

The action of the subgroup of $\Gamma$ formed by diagonal matrices determines a flow

$$t \mapsto F_{t\alpha+i\beta}, \qquad (t>0), \tag{2.6}$$

on the Branner-Hubbard family. This flow preserves visibility (or invisibility) of polynomials. The flow (2.6) also preserves the critical portraits of visible polynomials. For $\beta = 0$, these assertions follow from the fact that by (2.1) and (2.3), the homeomorphism $A_\gamma$ with $\gamma$ real throws the hedgehog of $F_1$ onto the hedgehog of $F_\gamma$. (See [11] regarding *hedgehogs* and disconnected Julia sets.) On the other hand, one can assume $\beta = 0$ without loss of generality, by just choosing a different uniformization (2.2) of the Branner-Hubbard family.

Let us parametrize the orbits of the flow (2.6) by real numbers $\beta$. It is easy to see that only countably many orbits contain invisible polynomials.

**Lemma 2.** *The critical portraits are aperiodic for almost all $\beta$'s.*

*Proof:* Using the group action structure, see (2.2), it is sufficient to show that if $F_1$ is a visible polynomial, then there is a number $\varepsilon > 0$ such that for almost every $\beta \in (-\varepsilon, \varepsilon)$, the critical portrait of $F_{1+i\beta}$ is aperiodic.

Let $\theta_j^\pm \in S^1$ be the external angles and $g_j$ the escape rates of the critical points of $F_1$. It is clear from (2.1) and (2.3) that for small $\beta$'s, the polynomials $F_{1+i\beta}$ are visible and that their external angles $\theta_j^\pm(\beta)$ satisfy the equation

$$\theta_j^\pm(\beta) = \theta_j^\pm + \beta g_j. \tag{2.7}$$

We fix $j$ and a positive integer $p$, and consider the set $E \subset (-\varepsilon, \varepsilon)$ of $\beta$'s such that the itinerary of the point

$$\vartheta = T\theta_j^\pm(\beta)$$

is periodic with period $p$. Let $L(\beta)$ denote the element of the partition of $S^1$ corresponding to $F_{1+i\beta}$ such that $\vartheta \in L(\beta)$. If $\varepsilon$ is small enough, we can find an interval $I \subset S^1$ such that

$$I \cap \left[ \bigcup_{\beta \in E} L(\beta) \right] \ = \ \emptyset.$$

The periodicity of the itineraries implies

$$\forall n, \qquad T^{np}\vartheta \notin I. \tag{2.8}$$

By Poincare's recurrence theorem, the set of $\vartheta$'s satisfying (2.8) has Lebesgue measure zero , and by (2.7) the same is true for the set $E$. $\square$

We can now complete the proof of Theorem B by referring to the following result of Kiwi [8]:

*If a sequence of visible polynomials with the same aperiodic critical portrait tends to a polynomial, then the latter has no non-repelling cycles.*

## 3. Periodic cycles

In this section we prove Theorem C. The proof is preceeded by a few technical lemmas. For the rest of the section we consider only polynomials $F$ with connected Julia sets and *all periodic cycles repelling.* We also assume that the critical points $c_j$ of $F$ are simple and non-preperiodic (the proof in the general case is similar).

3.1. **Multiplicity of the kneading map.** A point $b \in J_F$ is a *cut point* if there are at least two external radii landing at $b$. Let $G$ be a finite, forward invariant set which consists of cut points. For a point $z \in J$, which is not in the grand orbit $\mathcal{O}(G)$ of $G$, we denote by $P(z)$ the component of $J \setminus G$ containing $z$. Depending on the context, we use the same notation $P(z)$ for the corresponding *unbounded puzzle piece*, i.e. the component of the complement of external rays landing at $G$, see [18]. Let us number the pieces of the $G$-partition as $P_1, P_2, \dots, P_N$. The *kneading map*

$$\mathrm{knead}_G:\ J \setminus \mathcal{O}(G) \to \{1, \dots, N\}^{\mathbb{Z}+}$$

is the function $z \ \mapsto\ \{i_0(z), i_1(z), \dots\}$, where $F^\nu(z) \in P_{i_\nu(z)}$.

**Lemma 1.** *For any $\varepsilon > 0$, there exists a finite, forward invariant set $G$ such that if $n > n_0(F, \varepsilon)$ and if $a \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$, then*

$$\#\ \{a' \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G):\ \mathrm{knead}_G(a) = \mathrm{knead}_G(a')\}\ \le\ e^{\varepsilon n}.$$

*Proof:* Given $\varepsilon > 0$, we choose a large number $m = m(\varepsilon)$ to be specified later. For simplicity of notation, let us assume that the fibers of the critical points have pairwise disjoint orbits, in which case there is a finite, forward invariant set $\tilde{G}$ such that the sets

$$\bigcup_j \tilde{P}(c_j) \quad \text{and} \quad \bigcup_j \bigcup_{k=1}^{m} F^{-k} c_j \quad \text{are disjoint,} \tag{3.1}$$

where $\tilde{P}(\cdot)$ denote the $\tilde{G}$-pieces. (For an explanation of this fact and for the definition of *fibers*, see Appendix at the end of the section.) Replacing each critical piece $\tilde{P}(c_j)$ with components of the $F^{-1}\tilde{G}$-partition, we obtain a new puzzle. Let $G$ denote the corresponding set, i.e.

$$G \ = \ \tilde{G} \bigcup_j [F^{-1}\tilde{G} \cap \tilde{P}(c_j)].$$

The $G$-partition has the following (modified) Markov property (cf. [19], Section 7): *each critical puzzle piece maps onto the corresponding critical value piece by a 2-fold branched covering, while every non-critical piece maps univalently onto a "union" of puzzle pieces.*

A sequence $\{i_0, \dots, i_{n-1}\}$ is called a *Markov cycle* if

$$P_{i_1} \subset FP_{i_0},\ P_{i_2} \subset FP_{i_1},\ \dots,\ P_{i_0} \subset FP_{i_{n-1}}.$$

By (3.1), the number of critical indices in such a sequence does not exceed $n/m$. To each periodic point $z \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$ there corresponds the Markov cycle $\{i_0(z), \dots, i_{n-1}(z)\}$, and the Markov cycles of two periodic points are equal if and only if the points have the same $G$-kneading. Thus it remains to show that the number of points $z \in \mathrm{Fix}(F^n) \setminus \mathcal{O}(G)$ with the same Markov cycle $\{i_0, \dots, i_{n-1}\}$ does not exceed $C2^{n/m}$, where $C$ is a constant independent of $n$. This would give an explicit formula for $m = m(\varepsilon)$.

To this end, let us inductively define puzzle pieces $\Pi(k) \subset P_{i_k}$, $(0 \le k \le n-1)$, as follows:

$$\Pi(n-1) = P_{i_{n-1}}, \quad \Pi(k) = F_{i_k}^{-1}\Pi(k+1),$$

where $F_{i_k}^{-1}$ denotes the preimage under the map

$$F : P_{i_k} \to FP_{i_k} \supset \Pi(k+1).$$

It is clear that the puzzle piece $\Pi(0)$ contains all periodic points with the given Markov cycle.

To bound the number of $n$-periodic points in $\Pi(0)$, we consider the sets

$$\pi\Pi(k) \subset S^1$$

defined as the intersection of $\Pi(k)$ with the "circle at infinity". Each set $\pi\Pi(k)$ consists of finitely many open arcs. The map

$$T : \theta \to d\theta \ (\mathrm{mod}\ 1)$$

takes $\pi\Pi(k)$ onto $\pi\Pi(k+1)$ homeomorphically if the index $i_k$ is non-critical, and as a two-fold cover if $i_k$ is critical. Let $C$ be a constant such that each set $\pi P_j$ has at most $C$ components. It follows that the number of arcs in $\pi\Pi(0)$ is at most $C2^{n/m}$. It is also clear that each arc in $\pi\Pi(0)$ has at most one $T$-periodic point of period $n$. □

3.2. **Cycles with close orbits.** We will now use Lemma 1 to prove the following estimate for polynomials without indifferent periodic points. We don't know if the estimate is true for general polynomials.

**Lemma 2.** *For any $\varepsilon > 0$, there exists a positive number $\rho = \rho(F, \varepsilon)$ such that if $n > n_0(F, \varepsilon)$ and if $a \in \mathrm{Fix}(F^n)$, then*

$$\#\ \{a' \in \mathrm{Fix}(F^n) :\ \forall i,\ |F^i(a) - F^i(a')| \le \rho\} \ \le\ e^{\varepsilon n}.$$

*Proof:* Given $\varepsilon > 0$, we find a finite, invariant set $G$ according to Lemma 1. The argument is based on the notion of the *sector map* $\tau$ associated with $G$. For each $b \in G$, the external rays at $b$ divide the plane into sectors. Since we assumed that $b$ was not a critical point, the polynomial $F$ is a local diffeomorphism identifying sectors $S$ at $b$ with sectors $\tau S$ at $F(b)$:

$$F(S \cap U) \subset \tau S,$$

where $U$ is some small neighborhood of $b$. Denote by $C$ the total number of sectors (considering all points of $G$), and fix a number $m = m(\varepsilon) \gg C$. It follows that if $z$ is sufficiently close to the set $G$, then the initial kneading segment of length $m$ is determined up to $C$ choices by the sector map.

Let us now choose $\rho > 0$ so small that if $|z - z'| < \rho$, then either the points $z$ and $z'$ are in the same component of $J \setminus G$, or they are both so close to the set $G$ that we have the situation described in the previous sentence. If the orbits of two periodic points $a$ and $a'$ are $\rho$-close, then their kneading sequences of length $n \gg m$ coincide except for at most $n/m$ segments of length $m$, for which we have $C^{n/m}$

choices. Combining this computation with the estimate of Lemma 1, we complete the proof. □

Lemma 2 will be used in conjunction with the following statement, which is a special case of Mañe's lemma [14].

**Lemma 3.** *Given $\rho > 0$, there is a positive number $\delta = \delta(F, \rho)$ such that if $n \geq 0$, and if $D$ is a domain such that $F^n$ maps $D$ univalently onto a disk $B$ of radius $2\delta$, then*

$$\operatorname{diam}\ (F^n|_D)^{-1}(\frac{1}{2}B)\ <\ \rho. \tag{3.2}$$

Here and below, the notation $\frac{1}{2}B$ means concentric disc of radius half the radius of $B$.

### 3.3. **Good and bad cycles.**

Let $\delta > 0$. In lack of a better name, we say that a cycle $A \in \mathrm{Cycle}(F, n)$ is *$\delta$-good* if there is a periodic point $a \in A$ and a topological disc $D$ containing $a$ such that the restriction of $F^n$ to $D$ is univalent and $F^n(D) = B(a, \delta)$. Otherwise, we say that the cycle is *$\delta$-bad.* The next lemma states that for polynomials with all cycles repelling, most of the cycles are "good".

**Lemma 4.** *For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(F, \varepsilon)$ such that if $n > n(F, \varepsilon)$, then*

$$\#\ \{\ \delta\text{-bad } n\text{-cycles}\}\ \leq\ e^{\varepsilon n}.$$

*Proof:* Fix a large number $m = m(F, \varepsilon)$ to be specified later. For simplicity, we will assume that the orbits of the critical points are pairwise disjoint, and so there is a number $\rho = \rho(F, m)$ such that if $c \neq c'$ are two critical points, then

$$\operatorname{dist}(c, F^k(c')) > 10\rho, \qquad (0 \leq k \leq m). \tag{3.3}$$

We can take $\rho$ small enough so that the estimate of Lemma 2 is valid for a given $\varepsilon$. Finally, we choose $\delta > 0$ satisfying the following two conditions:

- for all $x \in J$ and $k \in [0, m]$, each component of the set $F^{-k}B(x, \delta)$ has diameter less than $\rho$;
- the conclusion (3.2) of Mañe's lemma holds.

Let us estimate the number of $\delta$-bad cycles.

Fix a cover $\mathcal{B}$ of the Julia set with discs of radius $2\delta$. Clearly, we can assume that the concentric discs of radius $\delta$ still cover $J$, and that the multiplicity of the covering is bounded by some absolute constant $M$. Let $n \gg m$. For each periodic point $a \in \mathrm{Fix}(F^n)$, we select a disc $B(a) \in \mathcal{B}$ with $a \in \frac{1}{2}B(a)$. For $i > 0$, let $B_{-i}(a)$ denote the component of the $F^{-i}B(a)$ containing the point $F^{n-i}(a)$ and define $j(a)$ to be the smallest positive integer such that $B_{-j}(a)$ contains a critical point, which we denote by $c(a)$. Note that if $j(a) > n$, then the cycle of $a$ is $\delta$-good.

We need some further notation. Given $a \in \mathrm{Fix}(F^n)$, we define inductively a sequence of positive integers $j_1, j_2, \ldots$ and a sequence of points $a_1 = a, a_2, \ldots$ in the orbit of $a$ as follows :

$$j_k = j(a_k), \qquad a_{k+1} = F^{n-j_k} a_k.$$

The main observation is that

$$j_k + j_{k+1} > m. \tag{3.4}$$

Indeed, if $j_k + j_{k+1} \le m$, then both $j_k$ and $j_{k+1}$ are $\le m$. By construction, we have diam $B_{-j_k}(a_k) < \rho$, and so

$$|a_{k+1} - c(a_k)| \; < \; \rho.$$

On the other hand, the disc $B(a_{k+1})$ contains the $j_{k+1}$-th iterate of the critical point $c(a_{k+1})$, and therefore

$$|a_{k+1} - F^{j_{k+1}} c(a_{k+1})| \; < \; 4\delta \; < \; 8\rho.$$

Combining the two inequalities, we get a contradiction with (3.3).

Define the *schedule* of $a$ to be a finite sequence

$$\mathrm{Sch}(a) = \{j_1(a), j_2(a), \dots, j_l(a)\},$$

where $l$ is the minimal number such that

$$j_1 + \dots + j_l > n.$$

By (3.4), we have

$$l \; \le \; 3n/m. \tag{3.5}$$

We also consider the corresponding sequence of discs in the cover $\mathcal{B}$, and the corresponding sequence of critical points:

$$\mathcal{B}(a) = \{B(a_1), \dots, B(a_l)\}, \qquad \mathcal{C}(a) = \{c(a_1), \dots, c(a_l)\}.$$

As we mentioned, for $\delta$-bad cycles we have all $j_k \le n$, and therefore

$$n \le \sum_{k=1}^{l} j_k \; \le \; 2n. \tag{3.6}$$

The lemma now follows from the three observations below.

(i) *The number of sequences* $\{j_1, \dots, j_l\}$ *satisfying* (3.4) *and* (3.6) *is* $\lesssim m^{4n/m}$.

Indeed, consider the numbers $j_1, (j_1 + j_2), \dots$ as points of the interval $[1, 2n]$. Subdivide the interval into $(2n)/m$ segments of length $m$. Clearly, there are at most two points in each segment, and there are less than $m^2$ choices to select at most two points in any particular segment.

(ii) *Consider all periodic points* $a \in \mathrm{Fix}(F^n)$ *with a given schedule. Then the number of distinct sequences* $\mathcal{B}(a)$ *and* $\mathcal{C}(a)$ *does not exceed* $(dM)^{3n/m}$ *and* $d^{3n/m}$ *respectively. Here* $d = \deg F$.

This follows from (3.5) and the fact that the disc $B(a_k)$ must contain the $j_k$-th iterate of a critical point, so the number of such discs is less than $dM$.

(iii) *If two periodic points* $a$ *and* $a'$ *have identical schedules and identical sequences* $\mathcal{B}(a) = \mathcal{B}(a')$ *and* $\mathcal{C}(a) = \mathcal{C}(a')$, *then the orbits of* $a$ *and* $a'$ *are* $\rho$*-close:*

$$\forall i, \quad |F^i(a_1) - F^i(a_1')| \le \rho.$$

To see this, let $\{j_1, \dots, j_l\}$ be the schedule and let $B = B(a_1) = B(a_1')$. By construction, the components of $F^{-j_1}B$ containing the points $F^{n-j_1}a_1$ and $F^{n-j_1}a_1'$ must coincide because both contain the critical point $c(a_1) = c(a_1')$. It follows that if $n - j_1 < i \le n$, then the $i$-th iterates of $a_1$ and $a_1'$ belong to the same component of the corresponding preimage of $B$, and this component is mapped univalenly onto $B$. Since $a_1$ and $a_1'$ are in $\frac{1}{2}B$, we can apply Lemma 3 to conclude that the iterates of $a_1$ and $a_1'$ are $\rho$-close. Repeat this argument for all discs $B(a_k)$, $k \le l$.

From (iii) and Lemma 2, it now follows that the number of $n$-periodic points with a given schedule and given $\mathcal{B}$- and $\mathcal{C}$-sequences is $\lesssim e^{\varepsilon n}$. On the other hand, by (i) and (ii), the number of possible sequences and schedules satisfying (3.6) is also $\lesssim e^{\varepsilon n}$, provided that $m = m(\varepsilon)$ is so large that $m^{-1}\log m \ll \varepsilon$. Thus the number of bad cycles is $\lesssim e^{2\varepsilon n}$. □

3.4. **Proof of Theorem C.** Let $F$ be a polynomial with all cycles repelling. Given small $\varepsilon$, we choose $\rho = \rho(F,\varepsilon)$ according to Lemma 1, so that the number of $\rho$-close $n$-cycles is $\lesssim e^{\varepsilon n}$. Then we choose a positive number $\delta$ such that

- all but $e^{\varepsilon n}$ $n$-cycles are $4\delta$-good, see Lemma 4;
- the conclusion (3.2) of Mañe's lemma holds.

Fix $n \gg 1$. In each good $n$-cycle, we pick a point $a$ such that $F^n$ maps some domain $D_a \ni a$ onto $B(a, 4\delta)$ univalently. Let $I$ denote the set of the points that we have picked, and let $II$ denote the set of all periodic points in the bad cycles. Then we have

$$\begin{aligned} Z_n(F,t) &= \sum_{a\in \mathrm{Fix}(F^n)} |F_n'(a)|^{-t} \\ &= n\sum_{a\in I} |F_n'(a)|^{-t} + \sum_{a\in II} |F_n'(a)|^{-t} \\ &\le n\sum_{a\in I} |F_n'(a)|^{-t} + ne^{\varepsilon n}. \end{aligned}$$

To estimate the sum over $I$, cover the Julia set with $\lesssim \delta^{-2}$ discs $B$ of radius $2\delta$. In each $B$, fix a point $z_B \notin J$ so that the points $z_B$ are distinct. Finally, to each $a \in I$ assign one of the discs $B = B(a)$ such that $a \in \frac{1}{2}B(a)$. Note that $B(a) \subset B(a, 3\delta)$.

Let $z_a$ denote the preimage of $z_{B(a)}$ under the map

$$F^n :\ D_a \to B(a,4\delta) \supset B(a).$$

Since $F^n$ takes both $a$ and $z_a$ inside $B(a,3\delta)$, by Koebe's lemma we have

$$|F_n'(a)| \asymp |F_n'(z_a)|.$$

Note that if $z_a = z_{a'}$ for some $a, a' \in I$, then the orbits of $a$ and $a'$ are $\rho$-close. Indeed, for $B = B(a) = B(a')$, we have

$$B \subset B(a,4\delta) \cap B(a',4\delta),$$

and therefore $F^n$ maps some domain univalently onto $B$ with both $a$ and $a'$ in the preimage of $\frac{1}{2}B$, and so we can apply (3.2).

It follows that the number of points $a$ such that $z_a$ is a given point of $F^{-n}z_B$ is at most $e^{\varepsilon n}$. We have

$$\sum_{a\in I} |F_n'(a)|^{-t} \lesssim e^{\varepsilon n} \sum_B \sum_{z\in F^{-n}(z_B)} |F_n'(a)|^{-t}.$$

Since for each $z_B$, we have

$$P_F(t) = \limsup_{n\to\infty} \frac{1}{n} \log_d \sum_{z\in F^{-n}(z_B)} |F_n'(z)|^{-t},$$

the theorem follows.

3.5. **Appendix: Fibers.** Let $F$ be a polynomial with all cycles repelling, and let $z \in J_F$. Following Kiwi [8], consider a sequence of partitions corresponding to the sets

$$G_l(F,z) := \{b \in J \setminus \mathcal{O}(z) : \ b \text{ is a cut point}, \ \ F^l b \text{ is periodic of period } \le l\}.$$

Let $P_l(\cdot)$ denote the $G_l(F,z)$-pieces. The connected compact set

$$X(z) \ = \ \bigcap_l \bar{P}_l(z) \ \subset \ J$$

is called the *fiber* of $z$. (We use the term from a paper of Schleicher [22], see also [10].) The fibers satisfy the equation

$$FX(z) = X(Fz).$$

It is also clear that if two points $z_1$ and $z_2$ have infinite orbits, or if they land on the same cycle, then the fibers $X(z_1)$ and $X(z_2)$ are disjoint or coincide.

Our proof of Lemma 1 was based on the following fact mentioned in the proof of Lemma 13.3 of Kiwi's thesis [8]. To make this section self-contained, we reproduce his argument. We will denote by $P_l'(\cdot)$ the puzzle pieces corresponding to $F^{-1}G_l$.

**Lemma 5.** *If $z$ has an infinite orbit, then the fiber of $z$ is wandering.*

*Proof:* (i) Let us first show that if $z$ is a periodic point, then $X(z) = \{z\}$. Since $G_l(F^p,z) \subset G_{lp}(F,z)$, then fibers of $F^p$ contain fibers of $F$, and so by replacing $F$ with an iterate, we can assume that $z$ is fixed. For the same reason we can assume that the landing rays at $z$ are all fixed. The latter implies

$$b \in \mathrm{Fix}(F) \cap \bar{P}_2(z) \quad \Rightarrow \quad \text{the rays landing at } b \text{ are fixed.} \tag{3.7}$$

Indeed, suppose $b$ is not a landing point of some fixed ray. Then $b \in G_1$, and $P_1(z)$ is contained in some sector $S$ at $b$. We have $FP_2(z) \subset P_1(z) \subset S$. Taking some point in $P_2(z)$ close to $b$, we see that $\tau S = S$, where $\tau$ is the sector map, and so the rays at $b$ have to be fixed.

Let $k-1$ be the number of critical points in the fiber $X(z)$. For $l \gg 1$, the map $P_l(z) \to P_l'(z)$ extends to a polynomial-like map $g$ of degree $k$. Observe that $X(z) \subset J_g$, and since the critical points of $g$ belong to $X(z) = gX(z)$, the Julia set $J_g$ of $g$ is connected. It remains to show that $k = 1$. (This would give $X(z) \subset J_g = \{z\}$.) The fixed points of $g$ belong to the set $\mathrm{Fix}(F) \cap \bar{P}_2(z)$. By (3.7), for each fixed point of $g$ there is an $F$-invariant (and therefore $g$-invariant) arc in $J^c \subset J_g^c$ tending to the fixed point.

Let $Q$ be a degree $k$ polynomial which is conjugate to $g$. It follows that there are $Q$-invariant arcs in $\mathbb{C} \setminus J_Q$ tending to each of $k$ fixed points of $Q$. Applying the Riemann map, we get $k$ arc tending to $k$ distinct points on the unit circle invariant with respect to the map $\zeta \mapsto \zeta^k$. A contradiction.

(ii) Suppose now that $z$ is not preperiodic. Replacing $F$ with some iterate, we can reduce the problem to showing that

$$X(Fz) = X(z) \quad \Rightarrow \quad z \in \mathrm{Fix}(F).$$

Suppose $X(Fz) = X(z)$. Then for every $l$, we have a map $F : P_l(z) \to P_l'(z)$, which extends to a polynomial-like map with Julia set contained in $\bar{P}_l(z)$. It follows that

$$\forall l, \quad \bar{P}_l(z) \cap \mathrm{Fix}(F) \ \neq \ \emptyset,$$

and therefore $X(z)$ contains at least one fixed point $b$. Since the partition $G_l(z, F)$ is finer than $G_l(b, F)$, by (i) we have $X(z) \subset X(b) = \{b\}$. $\square$

If the fibers of the critical points have pairwise disjoint orbits, then from Lemma 5 it follows that the statement (3.1) holds for puzzle pieces $\tilde{P}(c_j) = P_l(c_j)$ with $l$ sufficiently large. This is precisely the fact that we used earlier.

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