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Advances in Mathematics 228 (2011) 1590–1630

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Discrete complex analysis on isoradial graphs

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Received 13 April 2009; accepted 23 June 2011

Available online 16 July 2011

Communicated by N.G. Makarov

Abstract

We study discrete complex analysis and potential theory on a large family of planar graphs, the so-called isoradial ones. Along with discrete analogues of several classical results, we prove uniform convergence of discrete harmonic measures, Green's functions and Poisson kernels to their continuous counterparts. Among other applications, the results can be used to establish universality of the critical Ising and other lattice models.

MSC: 39A12; 52C20; 60G50

Keywords: Discrete harmonic functions; Discrete holomorphic functions; Discrete potential theory; Isoradial graphs; Random walk

1. Introduction

1.1. Motivation

This paper is concerned with discrete versions of complex analysis and potential theory in the complex plane. There are many discretizations of harmonic and holomorphic functions, which

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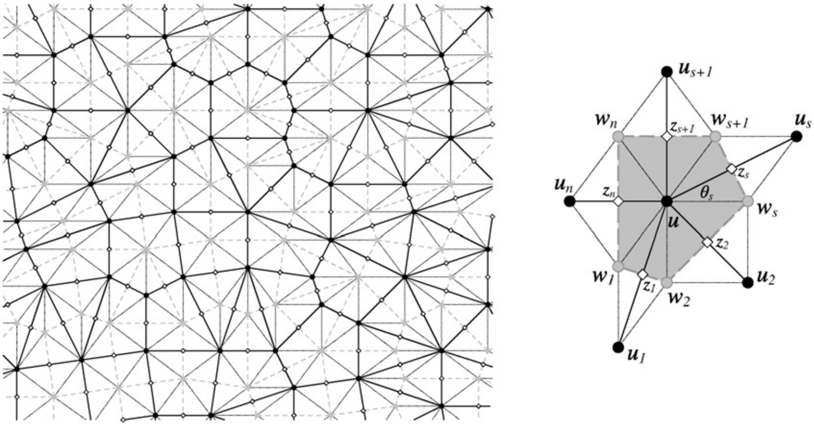
doi:10.1016/j.aim.2011.06.025



(A) (B)

Fig. 1. (A) An isoradial graph Γ (black vertices, solid lines), its dual isoradial graph Γ^* (gray vertices, dashed lines), the corresponding rhombic lattice or quad-graph (vertices $\Lambda = \Gamma \cup \Gamma^*$, thin lines, rhombic faces) and the set $\diamondsuit = \Lambda^*$ of rhombi centers (diamond-shaped points). (B) Local notations near $u \in \Gamma$. The dual face $W(u)$ is shaded.

have a long history. Besides proving discrete analogues of the usual complex analysis theorems, one can ask to which extent discrete objects approximate their continuous counterparts. This can be used to give "discrete" proofs of continuous theorems (see, e.g., [15] for such a proof of the Riemann mapping theorem) or to prove convergence of discrete objects to continuous ones. One of the goals of our paper is to provide tools for establishing convergence of critical 2D lattice models to conformally invariant scaling limits.

There are no "canonical" discretizations of Laplace and Cauchy–Riemann operators, the most studied ones (and perhaps the most convenient) are for the square grid. There are also definitions for other regular lattices, as well as generalizations to larger families of embedded into $\mathbb{C}$ planar graphs (see [22] and references therein).

We will work with isoradial graphs (or, equivalently, rhombic lattices) where all faces can be inscribed into circles of equal radii. Rhombic lattices were introduced by R.J. Duffin [8] in late sixties as (perhaps) the largest family of graphs for which the Cauchy–Riemann operator admits a nice discretization, similar to that for the square lattice. They reappeared recently as isoradial graphs in the work of Ch. Mercat [16] and R. Kenyon [12], as the largest family of graphs where certain 2D statistical mechanical models (notably the Ising and dimer models) preserve some integrability properties. Note that isoradial graphs can be quite irregular – see e.g. Fig. 1(A). It was shown by R. Kenyon and J.-M. Schlenker [13] that many planar graphs admit isoradial embeddings – in fact, there are only two topological obstructions. Also isoradial graphs have a well-defined mesh size δ – the common radius of the circumscribed circles.

It is thus natural to consider this family of graphs in the context of *universality* for 2D models with (conjecturally) conformally invariant scaling limits (as the mesh tends to zero).

The primary goal of our paper is to provide a "toolbox" of discrete versions of continuous results (particularly "hard" estimates) sufficient to perform a passage to the scaling limit. Of particular interest to us is the *critical Ising model*, and this paper starts a series devoted to its *universality* (which means that the scaling limit is independent of the shape of the lattice). See [20,3] for the strategy of our proof, [4] for the convergence of certain discrete holomorphic observables and [21] for the square lattice case.

Our results can also be applied to other lattice models. The uniform convergence of the discrete Poisson kernel (1.3) already implies universality for the loop-erased random walks on isoradial graphs. Namely, our paper together with [14] implies that their trajectories converge to SLE(2) curves (see Section 3.2, especially Remark 3.6, in [14]). There are several other fields where discrete harmonic and discrete holomorphic functions defined on isoradial graphs play essential role and hence where our results may be useful: approximation of conformal maps [2]; discrete integrable systems [1]; and the theory of discrete Riemann surfaces [18].

Local convergence of discrete harmonic (holomorphic) functions to continuous harmonic (holomorphic) functions is a rather simple fact. Moreover, it was shown by Ch. Mercat [17] that each continuous holomorphic function can be approximated by discrete ones. Thus, the discrete theory is close to the continuous theory "locally". Nevertheless, until recently almost nothing was known about the "global" convergence of the functions defined in discrete domains as the solutions of some discrete *boundary value problems* to their continuous counterparts. This setup goes back to the seminal paper by R. Courant, K. Friedrichs and H. Lewy [6], where convergence is established for harmonic functions with smooth Dirichlet boundary conditions in smooth domains, discretized by the square lattice, but not much progress has occurred since. For us it is important to consider discrete domains with possibly very *rough boundaries* and to establish convergence without any regularity assumptions about them. Besides being of independent interest, this is indispensable for establishing convergence to Oded Schramm's SLEs, since the latter curves are fractal.

1.2. Preliminary definitions

The planar graph Γ embedded in $\mathbb{C}$ is called *isoradial* iff each face is inscribed into a circle of a common radius δ. If all circle centers are inside the corresponding faces, then one can naturally embed the dual graph Γ^* in $\mathbb{C}$ isoradially with the same δ, taking the circle centers as vertices of Γ^*. The name *rhombic lattice* is due to the fact that all quadrilateral faces of the corresponding bipartite graph Λ (having the vertex set $\Gamma \cup \Gamma^*$) are rhombi with sides of length δ (see Fig. 1(A)). We will often work with rhombi half-angles, denoted by θ, for which we also require the following mild but indispensable and widely used assumption (see, e.g., [5, pp. 124 and 130], where the similar assumption is called Zlámal's condition):

(♠) the rhombi half-angles are uniformly bounded from 0 and $\frac{1}{2}\pi$ (in other words, all these angles belong to $[\eta, \frac{1}{2}\pi - \eta]$ for some fixed $\eta > 0$), i.e., there are no "too flat" rhombi in Λ.

Note that condition (♠) implies that for each $u_1, u_2 \in \Gamma$ the Euclidean distance $|u_2 - u_1|$ and the combinatorial distance $\delta \cdot d_\Gamma(u_1, u_2)$ (where $d_\Gamma(u_1, u_2)$ is the minimal number of vertices in the path connecting u_1 and u_2 in Γ) are comparable. Below we often use the notation *const* for absolute positive constants that does not depend on the mesh δ or the graph structure but, in principle, *may depend on* η.

The function $H : \Omega^\delta_\Gamma \to \mathbb{R}$ defined on some subset (discrete domain) Ω^δ_Γ of Γ is called *discrete harmonic*, if

$$\sum_{s=1}^{n} \tan\theta_s \cdot \big(H(u_s) - H(u)\big) = 0 \tag{1.1}$$

at all $u \in \Omega_\Gamma^\delta$ where the left-hand side makes sense. Here θ_s denotes the half-angles of the corresponding rhombi, see also Fig. 1(B) for notations. As usual, this definition is closely related to the *random walk on Γ* such that the probability to make the next step from u to u_k is proportional to $\tan\theta_k$. Namely, $\mathrm{RW}(t+1) = \mathrm{RW}(t) + \xi^{(t)}_{\mathrm{RW}(t)}$, where the increments $\xi^{(t)}$ are independent with distributions

$$\mathbf{P}(\xi_u = u_k - u) = \frac{\tan\theta_k}{\sum_{s=1}^n \tan\theta_s} \quad \text{for } k = 1, \ldots, n.$$

Under our assumption all these probabilities are uniformly bounded from 0. Note that the choice of $\tan\theta_s$ as the edge weights in (1.1) gives

$$\mathbf{E}[\operatorname{Re}\xi_u] = \mathbf{E}[\operatorname{Im}\xi_u] = 0 \quad \text{and} \quad \begin{array}{l} \mathbf{E}\big[(\operatorname{Re}\xi_u)^2\big] = \mathbf{E}\big[(\operatorname{Im}\xi_u)^2\big] = T_u, \\ \mathbf{E}[\operatorname{Re}\xi_u \operatorname{Im}\xi_u] = 0, \end{array} \tag{1.2}$$

where $T_u = \delta^2 \cdot \sum_{s=1}^n \sin 2\theta_s / \sum_{s=1}^n \tan\theta_s$ (see Lemma 2.2). Our results may be directly interpreted as the convergence of the hitting probabilities for this random walk. Moreover, condition ($\spadesuit$) implies that quadratic variations satisfy $0 < \mathrm{const} \cdot \delta^2 \leqslant T_u \leqslant 2\delta^2$, and so one can define a proper lazy random walk (or make a time re-parametrization) according to (1.2) so that it converges to standard 2D Brownian motion.

1.3. Main results

Let $\Omega_\Gamma^\delta \subset \Gamma$ be some bounded, simply connected discrete domain and $\operatorname{Int}\Omega_\Gamma^\delta$, $\partial\Omega_\Gamma^\delta$ denote the sets of interior and boundary vertices, respectively (see Section 2.1 for more accurate definitions). For $u \in \operatorname{Int}\Omega_\Gamma^\delta$ and $E \subset \partial\Omega_\Gamma^\delta$ the discrete harmonic measure $\omega^\delta(u; E; \Omega_\Gamma^\delta)$ is the probability of the event that the random walk on Γ starting at u first exits Ω_Γ^δ through E. Equivalently, $\omega^\delta(\,\cdot\,; E; \Omega_\Gamma^\delta)$ is the unique solution of the following discrete Dirichlet boundary value problem:

- $\omega^\delta(\,\cdot\,; E; \Omega_\Gamma^\delta)$ is discrete harmonic everywhere in Ω_Γ^δ;
- $\omega^\delta(a; E; \Omega_\Gamma^\delta) = 1$ for $a \in E$ and $\omega^\delta(a; E; \Omega_\Gamma^\delta) = 0$ for $a \in \partial\Omega_\Gamma^\delta \setminus E$.

We prove *uniform* (with respect to the shape Ω_Γ^δ and the structure of the underlying isoradial graph) *convergence* of the basic objects of the discrete potential theory and their discrete *gradients* (which are discrete holomorphic functions defined on subsets of $\diamondsuit = \Lambda^*$, see Section 2.4 and Definition 3.7 for further details) to continuous counterparts. Namely, we consider

- solution of the discrete Dirichlet problem with continuous boundary values;
- discrete harmonic measure $\omega^\delta(\,\cdot\,; a^\delta b^\delta; \Omega_\Gamma^\delta)$ of boundary arcs $a^\delta b^\delta \subset \partial\Omega_\Gamma^\delta$;
- discrete Green's function $G^\delta_{\Omega_\Gamma^\delta}(\,\cdot\,; v^\delta)$, $v^\delta \in \operatorname{Int}\Omega_\Gamma^\delta$;
- discrete Poisson kernel

$$P^\delta\big(\,\cdot\,; v^\delta; a^\delta; \Omega_\Gamma^\delta\big) := \frac{\omega^\delta(\,\cdot\,; \{a^\delta\}; \Omega_\Gamma^\delta)}{\omega^\delta(v^\delta; \{a^\delta\}; \Omega_\Gamma^\delta)}, \quad a^\delta \in \partial\Omega_\Gamma^\delta, \tag{1.3}$$

normalized at the interior point $v^\delta \in \operatorname{Int}\Omega_\Gamma^\delta$;

- discrete Poisson kernel $P^{\delta}_{o^{\delta}}(\,\cdot\,; a^{\delta}; \Omega^{\delta}_{\Gamma})$, $a^{\delta} \in \partial\Omega^{\delta}_{\Gamma}$, normalized at the boundary point $o^{\delta} \in \partial\Omega^{\delta}_{\Gamma}$ by some analogue of the condition $[\partial_n P](o^{\delta}) = 1$ (we assume that the boundary $\partial\Omega^{\delta}_{\Gamma}$ is "straight" near o^{δ}, see precise definitions in Section 3.4).

1.4. *Organization of the paper*

We begin with the exposition of basic facts concerning discrete harmonic and discrete holomorphic functions on isoradial graphs. The larger part of Section 2 follows [8,16,12,18,2]. Unfortunately, none of these papers contains all the preliminaries that we need. Besides, the basic notation (sign and normalization of the Laplacian, definition of the discrete exponentials and so on) varies from source to source, so for the convenience of the reader we collected all preliminaries in the same place. Note that our notation (e.g., the normalization of discrete Green's functions and the parametrization of discrete exponentials) is chosen to be as close in the limit to the standard continuous objects as possible. Also, we prefer to deal with functions rather than to use the language of forms or cochains [18] which is more adapted for the topologically nontrivial cases.

The main part of our paper is Section 3, where the convergence theorems are proved. The proofs essentially use compactness arguments, so it does not give any estimate for the convergence rate. Thus, as in [21], we derive the "uniform" convergence from the "pointwise" one, using the compactness of the set of bounded simply connected domains in the Carathéodory topology (see Proposition 3.8). The other ingredients are the classical Arzelà–Ascoli theorem, which allows us to choose a convergent subsequence of discrete harmonic functions (see Proposition 3.1) and the weak Beurling-type estimate (Proposition 2.11) which we use in order to identify the boundary values of the limiting harmonic function. We prove C^1-convergence, but stop short of discussing the C^{∞} topology since there is no straightforward definition of the second discrete derivative for functions on isoradial graphs (see Section 2.5). Note however that a way to overcome this difficulty was suggested in [2].

2. Discrete harmonic and holomorphic functions. Basic facts

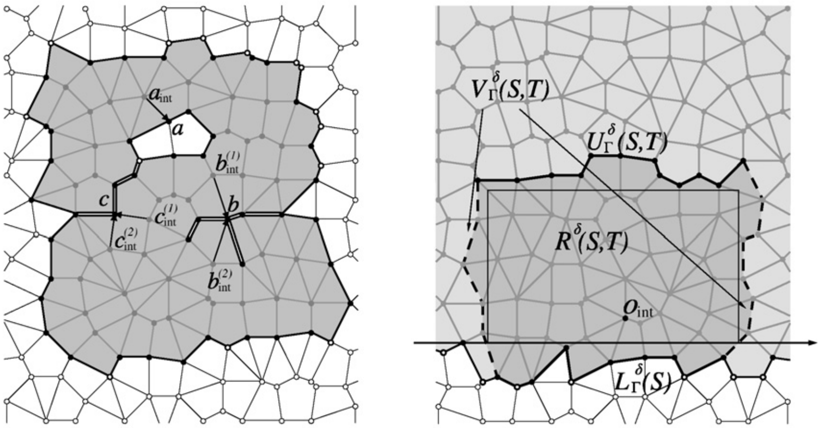
2.1. *Basic definitions. Approximation property*

Let $\Gamma = \Gamma^{\delta}$ be some *infinite* isoradial graph embedded into $\mathbb{C}$ and $V_{\Omega^{\delta}} \subset \Gamma$ be some *connected* subset of vertices (identified with points in $\mathbb{C}$). Let $E_{\Omega^{\delta}}$ be the set of all edges (open intervals in $\mathbb{C}$) incident to $V_{\Omega^{\delta}}$ and $F_{\Omega^{\delta}}$ be the set of all faces (open polygons in $\mathbb{C}$) incident to $E_{\Omega^{\delta}}$.

We call $\boldsymbol{\Omega^{\delta}} := F_{\Omega^{\delta}} \cup E_{\Omega^{\delta}} \cup V_{\Omega^{\delta}} \subset \mathbb{C}$ the **polygonal representation** of a **discrete domain** $\boldsymbol{\Omega^{\delta}_{\Gamma}} := \operatorname{Int}\Omega^{\delta}_{\Gamma} \cup \partial\Omega^{\delta}_{\Gamma}$, where *interior* and *boundary vertices* are defined as

$$\operatorname{Int}\Omega^{\delta}_{\Gamma} := V_{\Omega^{\delta}} \quad \text{and} \quad \partial\Omega^{\delta}_{\Gamma} := \big\{\big(a; (a_{\mathrm{int}}a)\big)\colon a_{\mathrm{int}} \in V_{\Omega^{\delta}},\ (a_{\mathrm{int}}a) \in E_{\Omega^{\delta}},\ a \notin V_{\Omega^{\delta}}\big\},$$

respectively. Further, we say that Ω^{δ}_{Γ} is simply connected, if Ω^{δ} is simply connected. The reason for this definition of $\partial\Omega^{\delta}_{\Gamma}$ is that the same a may serve as several different boundary vertices, if it can be approached from $\operatorname{Int}\Omega^{\delta}_{\Gamma}$ by several edges – see e.g. vertices b and c in Fig. 2(A). However, when no confusion arises, we will often treat $\partial\Omega^{\delta}_{\Gamma}$ as a subset of Γ, not indicating explicitly the corresponding outgoing edges.



(A) (B)

Fig. 2. (A) Discrete domain. The interior vertices are gray, the boundary vertices are black and the outer vertices are white. Both b and c have two interior neighbors, and so we treat, e.g., $(b;\,(b_{\mathrm{int}}^{(1)}b))$ and $(b;\,(b_{\mathrm{int}}^{(2)}b))$ as different elements of $\partial\Omega_\Gamma^\delta$. (B) Discrete half-plane $\mathbb{H}^\delta$ and discrete rectangle $R^\delta(S,T)$. The lower, upper and vertical parts of $\partial R_\Gamma^\delta(S,T)$ are denoted by $L_\Gamma^\delta(S)$, $U_\Gamma^\delta(S,T)$ and $V_\Gamma^\delta(S,T)$, respectively.

Below we often need some natural *discretizations of standard continuous domains* (e.g., discs and rectangles). For an open convex $D\subset\mathbb{C}$ we introduce $D_\Gamma^\delta\subset\Gamma$ and its polygonal representation $D^\delta\subset\mathbb{C}$ by defining $\operatorname{Int}\Omega_\Gamma^\delta=V_{D^\delta}$ as the vertices of the (largest) connected component of Γ lying inside D (see Figs. 2(B) and 3(A)).

Let

$$\mu_\Gamma^\delta(u):=\frac{\delta^2}{2}\sum_{u_s\sim u}\sin 2\theta_s \tag{2.1}$$

be the *weight* of a vertex $u\in\Gamma$, where θ_s are the half-angles of the corresponding rhombi. Note that $\mu_\Gamma^\delta(u)$ is the area of a dual face $W(u)=w_1w_2\dots w_n$ (see Fig. 1(B)).

Let $\phi:\Omega^\delta\to\mathbb{C}$ be a Lipschitz (i.e., satisfying $|\phi(u_1)-\phi(u_2)|\leqslant C|u_1-u_2|$) function and $\phi^\delta:=\phi|_{\Omega_\Gamma^\delta}$ be its restriction to Ω_Γ^δ. Note that all points in a dual face $W(u)$ are δ-close to its center u. Thus, approximating values of ϕ on $W(u)$ by $\phi(u)$ and taking into account that $\operatorname{Area}(\Omega^\delta\setminus\bigcup_{u\in\operatorname{Int}\Omega_\Gamma^\delta}W(u))\leqslant\delta\cdot\operatorname{Length}(\partial\Omega^\delta)$, we arrive at the simple inequality

$$\Biggl|\sum_{u\in\operatorname{Int}\Omega_\Gamma^\delta}\phi^\delta(u)\mu_\Gamma^\delta(u)-\iint\limits_{\Omega^\delta}\phi(x+iy)\,dx\,dy\Biggr|\leqslant C\delta\cdot\operatorname{Area}\bigl(\Omega^\delta\bigr)+M\delta\cdot\operatorname{Length}\bigl(\partial\Omega^\delta\bigr) \tag{2.2}$$

with the same constant C and $M:=\sup\{|\phi(z)|,\ z\in\Omega^\delta\colon\ \operatorname{dist}(z,\partial\Omega^\delta)\leqslant\delta\}$.

Definition 2.1. Let Ω_Γ^δ be some connected discrete domain and $H:\Omega_\Gamma^\delta\to\mathbb{R}$. We define the **discrete Laplacian** of H at $u\in\operatorname{Int}\Omega_\Gamma^\delta$ by

$$\bigl[\Delta^\delta H\bigr](u):=\frac{1}{\mu_\Gamma^\delta(u)}\sum_{u_s\sim u}\tan\theta_s\cdot\bigl[H(u_s)-H(u)\bigr]$$

(see Fig. 1(B) for notations). We call H **discrete harmonic** in Ω_Γ^δ iff $[\Delta^\delta H](u)=0$ at all interior vertices $u\in\operatorname{Int}\Omega_\Gamma^\delta$.

It is easy to see that discrete harmonic functions satisfy the **maximum principle**:

$$\max_{u\in\Omega_\Gamma^\delta} H(u)=\max_{a\in\partial\Omega_\Gamma^\delta} H(a). \tag{2.3}$$

Further, a simple calculation shows that the **discrete Green's formula**

$$\sum_{u\in\operatorname{Int}\Omega_\Gamma^\delta}\bigl[H\Delta^\delta G-G\Delta^\delta H\bigr](u)\mu_\Gamma^\delta(u)=\sum_{a\in\partial\Omega_\Gamma^\delta}\tan\theta_{a_{\mathrm{int}}a}\cdot\bigl[H(a_{\mathrm{int}})G(a)-H(a)G(a_{\mathrm{int}})\bigr] \tag{2.4}$$

holds true for any two functions $H,G:\Omega_\Gamma^\delta\to\mathbb{R}$. Here and below, for a boundary vertex $(a;(a_{\mathrm{int}}a))$, $\theta_{a_{\mathrm{int}}a}$ denotes the half-angle of the rhombus having $a_{\mathrm{int}}a$ as a diagonal.

Lemma 2.2 *(Approximation property). Let $\phi\in C^3$ be a smooth function defined in the disc $B(u,2\delta)\subset\mathbb{C}$ for some $u\in\Gamma$. Denote by ϕ^δ its restriction to Γ. Then*

(i)

$$\Delta^\delta\phi^\delta\equiv 0,\quad \text{if } \phi \text{ is constant or a linear function},\quad \text{and}$$
$$\Delta^\delta\phi^\delta\equiv\Delta\phi\equiv 2(a+c),\quad \text{if } \phi(x+iy)\equiv ax^2+bxy+cy^2 \text{ is quadratic in } x \text{ and } y.$$

(ii)

$$\bigl|\bigl[\Delta^\delta\phi^\delta\bigr](u)-[\Delta\phi](u)\bigr|\leqslant \mathrm{const}\cdot\delta\cdot\sup_{B(u,2\delta)}\bigl|D^3\phi\bigr|.$$

Proof. We start by enumerating neighbors of u as $u_1,\dots,u_n$ and its neighbors on the dual lattice as $w_1,\dots,w_n$ – see Fig. 1(B). Obviously, $\Delta^\delta\phi^\delta\equiv 0$, if ϕ is a constant. Since

$$\sum_{u_s\sim u}\tan\theta_s\cdot(u_s-u)=-i\sum_{u_s\sim u}(w_{s+1}-w_s)=0,$$

one obtains $\Delta^\delta\phi^\delta\equiv 0$ for linear functions $x=\operatorname{Re}u$ and $y=\operatorname{Im}u$. Similarly,

$$\sum_{u_s\sim u}\tan\theta_s\cdot\bigl(u_s^2-u^2\bigr)=-i\sum_{u_s\sim u}(w_{s+1}-w_s)(u+u_s)=-i\sum_{u_s\sim u}\bigl(w_{s+1}^2-w_s^2\bigr)=0,$$

so $\Delta^\delta\phi^\delta\equiv 0$ for $x^2-y^2=\operatorname{Re}u^2$ and $2xy=\operatorname{Im}u^2$. The result for x^2+y^2 follows from

$$\sum_{u_s\sim u}\tan\theta_s\cdot|u_s-u|^2=2\delta^2\sum_{u_s\sim u}\sin 2\theta_s=4\mu_\Gamma^\delta(u),$$

thus proving (i). Finally, Taylor formula implies (ii). $\square$

2.2. Green's function. Dirichlet problem. Harnack Lemma. Lipschitzness

Definition 2.3. Let $u_0 \in \Gamma$. We call $H = G_\Gamma(\,\cdot\,; u_0) : \Gamma \to \mathbb{R}$ the **free Green's function** iff it satisfies the following:

(i) $[\Delta^\delta H](u) = 0$ for all $u \neq u_0$ and $[\Delta^\delta H](u_0) \cdot \mu_\Gamma^\delta(u_0) = 1$;
(ii) $H(u) = o(|u - u_0|)$ as $|u - u_0| \to \infty$;
(iii) $H(u_0) = \frac{1}{2\pi}(\log\delta - \gamma_{\mathrm{Euler}} - \log 2)$, where γ_{Euler} is the Euler constant.

Remark 2.4. We use a nonstandard normalization at u_0 (usually the additive constant is chosen so that $G(u_0; u_0) = 0$) in order to have convergence to the standard continuous Green's function $\frac{1}{2\pi}\log|u - u_0|$ as the mesh δ goes to zero.

Theorem 2.5 *(Kenyon). There exists a unique Green's function $G_\Gamma(\,\cdot\,; u_0)$. Moreover, it satisfies*

$$G_\Gamma(u; u_0) = \frac{1}{2\pi}\log|u - u_0| + O\left(\frac{\delta^2}{|u - u_0|^2}\right), \quad u \neq u_0, \tag{2.5}$$

uniformly with respect to the shape of the isoradial graph Γ and $u_0 \in \Gamma$.

Proof. This asymptotic form for isoradial graphs was first obtained in [12]. Some small improvements (the correct additive constant and the order of the remainder) were done in [2]. We give a sketch of Kenyon's beautiful proof in Appendix A.1. □

Let Ω_Γ^δ be some *bounded* connected discrete domain. It is well known that for each $f : \partial\Omega_\Gamma^\delta \to \mathbb{R}$ there exists a unique discrete harmonic function H in Ω_Γ^δ such that $H|_{\partial\Omega_\Gamma^\delta} = f$ (e.g., H minimizes the corresponding Dirichlet energy, see [8]). Clearly, H depends on f linearly, and so

$$H(u) = \sum_{a \in \partial\Omega_\Gamma^\delta} \omega^\delta\big(u; \{a\}; \Omega_\Gamma^\delta\big) \cdot f(a)$$

for all $u \in \Omega_\Gamma^\delta$, where $\omega^\delta(u; \,\cdot\,; \Omega_\Gamma^\delta)$ is some probabilistic measure on $\partial\Omega_\Gamma^\delta$ which is called **harmonic measure** at u. It is harmonic as a function of u and has a standard interpretation as the exit probability for the random walk on Γ (the measure of a set $E \subset \partial\Omega_\Gamma^\delta$ is the probability that the random walk started from u exits Ω_Γ^δ through E).

Definition 2.6. For $u_0 \in \operatorname{Int}\Omega_\Gamma^\delta$, we call $H = G_{\Omega_\Gamma^\delta}(\,\cdot\,; u_0)$ the **Green's function in $\boldsymbol{\Omega_\Gamma^\delta}$** iff

(i) $[\Delta^\delta H](u) = 0$ for all interior vertices $u \in \operatorname{Int}\Omega_\Gamma^\delta$ except u_0, and $[\Delta^\delta H](u_0) \cdot \mu_\Gamma^\delta(u_0) = 1$;
(ii) $H = 0$ on the boundary $\partial\Omega_\Gamma^\delta$.

Note that these properties determine $G_{\Omega_\Gamma^\delta}(\cdot\,; u_0)$ uniquely. Namely, $G_{\Omega_\Gamma^\delta} = G_\Gamma - G^*_{\Omega_\Gamma^\delta}$, where

$$G^*_{\Omega_\Gamma^\delta} = G^*_{\Omega_\Gamma^\delta}(\cdot\,; u_0) := \sum_{a\in\partial\Omega_\Gamma^\delta} \omega^\delta\big(\cdot\,; \{a\}; \Omega_\Gamma^\delta\big)\cdot G_\Gamma(a; u_0)$$

is a unique solution of the discrete boundary value problem

$$\Delta^\delta G^*_{\Omega_\Gamma^\delta} = 0 \quad \text{in } \Omega_\Gamma^\delta, \qquad G^*_{\Omega_\Gamma^\delta} = G_\Gamma(\cdot\,; u_0) \quad \text{on } \partial\Omega_\Gamma^\delta.$$

Applying Green's formula (2.4) to $H = \omega^\delta(\cdot\,; \{a\}; \Omega_\Gamma^\delta)$ and $G = G_{\Omega_\Gamma^\delta}(\cdot\,; u_0)$, one obtains

$$\omega^\delta\big(u_0; \{a\}; \Omega_\Gamma^\delta\big) = -\tan\theta_{a_{\mathrm{int}}a}\cdot G_{\Omega_\Gamma^\delta}(a_{\mathrm{int}}; u_0), \quad \text{where } a = \big(a; (a_{\mathrm{int}}a)\big)\in\partial\Omega_\Gamma^\delta. \tag{2.6}$$

It was noted by U. Bücking [2] that, since the remainder in (2.5) is of order $O(\delta^2|u-u_0|^{-2})$, one can directly use R.J. Duffin's ideas [7] in order to derive the Harnack Lemma for discrete harmonic functions.

Recall that $B_\Gamma^\delta(z,r)\subset\Gamma$ denotes the discretization of an open disc $B(z,r)\subset\mathbb{C}$.

Proposition 2.7 *(Discrete Harnack Lemma). Let $u_0\in\Gamma$ and $H : B_\Gamma^\delta(u_0,R)\to\mathbb{R}$ be a nonnegative discrete harmonic function.*

(i) *If $u_1\sim u_0$, then*

$$\big|H(u_1)-H(u_0)\big| \leqslant \mathrm{const}\cdot\frac{\delta H(u_0)}{R}.$$

(ii) *If $u_1,u_2\in B_\Gamma^\delta(u_0,r)\subset \operatorname{Int} B_\Gamma^\delta(u_0,R)$, then*

$$\exp\left[-\mathrm{const}\cdot\frac{r}{R-r}\right] \leqslant \frac{H(u_2)}{H(u_1)} \leqslant \exp\left[\mathrm{const}\cdot\frac{r}{R-r}\right].$$

Remark 2.8. In Section 3.4 we also give a version of the boundary Harnack principle which compares the values of a positive harmonic function in the bulk with its normal derivative on a "straight" part of the boundary (see Proposition 3.19).

Proof. In order to make our presentation complete, we recall briefly the arguments from [7] and [2] in Appendix A.2. □

Corollary 2.9 *(Lipschitzness of discrete harmonic functions). Let H be discrete harmonic in $B_\Gamma^\delta(u_0,R)$ and $u_1,u_2\in B_\Gamma^\delta(u_0,r)\subset\operatorname{Int}B_\Gamma^\delta(u_0,R)$. Then*

$$\big|H(u_2)-H(u_1)\big| \leqslant \mathrm{const}\cdot\frac{M|u_2-u_1|}{R-r}, \quad \textit{where } M = \max_{B_\Gamma^\delta(u_0,R)}\big|H(u)\big|.$$

Proof. By assumption (♠) we can find a path $u_1 = v_0v_1v_2 \dots v_{k-1}v_k = u_2$, connecting u_1 and u_2 inside $B_\Gamma^\delta(u_0, r)$, such that $k \leqslant \mathrm{const} \cdot \delta^{-1}|u_2 - u_1|$. Since $0 \leqslant H + M \leqslant 2M$, applying Harnack's inequality to $H + M$, one gets

$$\bigl|H(u_2) - H(u_1)\bigr| \leqslant \sum_{j=0}^{k-1} \bigl|H(v_{j+1}) - H(v_j)\bigr| \leqslant \mathrm{const} \cdot \frac{|u_2 - u_1|}{\delta} \cdot \frac{\delta M}{R - r}. \qquad \square$$

2.3. Weak Beurling-type estimates

The following simple fact is based on the approximation property (Lemma 2.2) for the discrete Laplacian on isoradial graphs.

Lemma 2.10. *Let $u_0 \in \Gamma$, $r > 0$ and $B_\Gamma^\delta(u_0, r)$ be the discretization of a disc $B(u_0, r)$ (see Fig. 3(A)). Let $a, b \in \partial B_\Gamma^\delta(u_0, r)$ be two boundary vertices such that*

$$\arg(b - u_0) - \arg(a - u_0) \geqslant \frac{1}{4}\pi.$$

Then,

$$\omega^\delta\bigl(u; ab; B_\Gamma^\delta(0, r)\bigr) \geqslant \mathrm{const} > 0 \quad \textit{for all } u \in B_\Gamma^\delta\left(u_0, \frac{1}{2}r\right),$$

where ab denotes the discrete counter clockwise arc from a to b.

Proof. Fix some small $\rho > 0$ and a smooth function $\phi_0 : B(0, 1 + \rho) \to \mathbb{R}$ such that

(ia) $\phi_0(z) \leqslant 1$ for all $z = re^{i\phi}$, $r \in (1 - \rho, 1 + \rho)$, $\phi \in [0, \frac{1}{4}\pi]$;
(ib) $\phi_0(z) \leqslant 0$ for all $z = re^{i\phi}$, $r \in (1 - \rho, 1 + \rho)$, $\phi \in [\frac{1}{4}\pi, 2\pi]$;
(ii) ϕ_0 is subharmonic, moreover $[\Delta\phi_0](\zeta) \geqslant \mathrm{const} > 0$ everywhere in $B(0, 1 + \rho)$;
(iii) $\phi_0(z) \geqslant \mathrm{const} > 0$ for all $z \in B(0, \frac{1}{2} + \rho)$.

For instance, one can take $\phi_0(z) := h(z) - c + d|z|^2$, where h is the (continuous) harmonic measure of the arc $\{\zeta\colon |\zeta| = 1 + \rho\colon \arg\zeta \in [\frac{1}{12}\pi, \frac{1}{6}\pi]\}$; $c > 0$ is chosen so that (ib) and (iii) are fulfilled (c exists, if ρ is small enough); and $d > 0$ is sufficiently small.

Let

$$\phi^\delta(u) := \phi_0\left(\frac{u - u_0}{a - u_0}\right) \quad \text{for } u \in B_\Gamma^\delta(u_0, r).$$

Then, $\phi^\delta \leqslant 1$ on the discrete arc ab and $\phi^\delta \leqslant 0$ on the complementary arc ba.

If δ/r is small enough, then, due to (ii) and Lemma 2.2 (approximation property), ϕ^δ is discrete subharmonic in $B_\Gamma^\delta(u_0, r)$. Using the maximum principle, one obtains

$$\omega^\delta\bigl(u; ab; B_\Gamma^\delta(0, r)\bigr) \geqslant \phi^\delta(u) \geqslant \mathrm{const} > 0 \quad \text{for all } u \in B_\Gamma^\delta\left(0, \frac{1}{2}r\right).$$

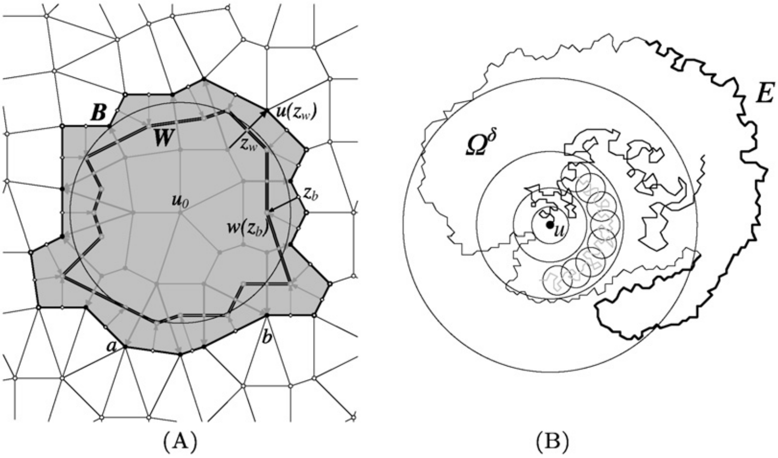


Fig. 3. (A) A discrete disc. The "black" polygonal boundary B and the "white" contour W are shown together with the correspondences $z \mapsto u(z)$, $z \in W_\diamondsuit$, and $z \mapsto w(z)$, $z \in B_\diamondsuit$. (B) The proof of the weak Beurling-type estimate (Proposition 2.11). The probability that the random walk makes a whole turn inside the annulus (and so hits the boundary $\partial\Omega^\delta$) is uniformly bounded from 0 due to Lemma 2.10.

If $\delta/r \geqslant \text{const} > 0$, then the claim is trivial, since the random walk starting at u_0 can reach the discrete arc ab in a uniformly bounded number of steps. $\square$

Let Ω^δ_Γ be some connected discrete domain, $u \in \Omega^\delta_\Gamma$ and $E \subset \partial\Omega^\delta_\Gamma$. We set

$$\operatorname{dist}_{\Omega^\delta_\Gamma}(u; E) := \inf\bigl\{R:\ u \text{ and } E \text{ are connected in } \Omega^\delta_\Gamma \cap B(u, R)\bigr\}.$$

The following proposition is a simple discrete version of the classical Beurling estimate with a (sharp) exponent $1/2$ replaced by some (small) positive β.

Proposition 2.11 *(Weak Beurling-type estimates). There exists an absolute constant $\beta > 0$ such that for any simply connected discrete domain Ω^δ_Γ, interior vertex $u \in \operatorname{Int}\Omega^\delta_\Gamma$ and some part of the boundary $E \subset \partial\Omega^\delta_\Gamma$ one has*

$$\omega^\delta\bigl(u; E; \Omega^\delta_\Gamma\bigr) \leqslant \text{const} \cdot \left[\frac{\operatorname{dist}(u; \partial\Omega^\delta_\Gamma)}{\operatorname{dist}_{\Omega^\delta_\Gamma}(u; E)}\right]^\beta \quad \text{and} \quad \omega^\delta\bigl(u; E; \Omega^\delta_\Gamma\bigr) \leqslant \text{const} \cdot \left[\frac{\operatorname{diam} E}{\operatorname{dist}_{\Omega^\delta_\Gamma}(u; E)}\right]^\beta.$$

Above we set $\operatorname{diam} E := \delta$, *if E consists of a single vertex.*

Proof. The proof is quite standard. Let $d = \operatorname{dist}(u; \partial\Omega^\delta_\Gamma)$ and $r = \operatorname{dist}_{\Omega^\delta_\Gamma}(u; E)$. Recall that $\omega^\delta(u; E; \Omega^\delta_\Gamma)$ is equal to the probability that the random walk starting at u first hits the boundary of Ω^δ_Γ inside E. Using Lemma 2.10 (see Fig. 3(B)), it is easy to show that for each $d \leqslant r' \leqslant \frac{1}{2}r$ the probability to cross the annulus $B(u, 2r') \setminus B(u, r')$ inside Ω^δ_Γ without touching its boundary is bounded above by some absolute constant $p < 1$ that does not depend on r' and the shape of Ω^δ_Γ. Hence,

$$\omega^\delta\big(u; E; \Omega_\Gamma^\delta\big) \leqslant p^{\log_2(r/d)-1} = p^{-1} \cdot (d/r)^{-\log_2 p},$$

so the first estimate holds true with the exponent $\beta = -\log_2 p > 0$.

To prove the second estimate, let us fix any vertex $e \in E$. By definition of $d = \mathrm{dist}_{\Omega_\Gamma^\delta}(u; E)$, it's clear that E and u_0 are disconnected in $\Omega_\Gamma^\delta \cap B(e, \frac{1}{2}d)$ (otherwise u_0 and E would be for sure connected in $\Omega_\Gamma^\delta \cap B(u_0, d)$). Now one can mimic the arguments given above for annuli $B(e, 2r') \setminus B(e, r')$ with $\mathrm{diam}\, E \leqslant r' \leqslant \frac{1}{4}d$. $\square$

2.4. *Discrete holomorphic functions. Definitions*

Above we discussed the theory of discrete harmonic functions defined on the isoradial graph Γ (or, in a similar manner, on its dual Γ^*). Now, following [7,16,12], we introduce the notion of *discrete holomorphic* functions. These are defined either on vertices $\Lambda = \Gamma \cup \Gamma^*$ of the rhombic lattice, or on the set $\diamondsuit = \Lambda^*$ of the rhombi centers. Note that, in contrast to similar Γ and Γ^*, Λ and $\diamondsuit$ have essentially different combinatorial properties, so we obtain two essentially different definitions. As it will be shown in Section 2.5, the first class (holomorphic functions defined on Λ) can be thought as couples of harmonic functions and their harmonic conjugates, while the second (holomorphic functions defined on $\diamondsuit$) consists of gradients of harmonic functions. We are mostly interested in the second class, but start with some preliminaries concerning functions defined on Λ.

Definition 2.12. Let $z \in \diamondsuit$ be a center of the rhombus $u^-w^-u^+w^+$, where $u^\pm \in \Gamma$ and $w^\pm \in \Gamma^*$ are listed in counter clockwise order. Let a function H be defined on some part of Λ including $u^\pm$, $w^\pm$. We define its discrete derivatives $\partial^\delta H$, $\bar\partial^\delta H$ at z as

$$\big[\partial^\delta H\big](z) := \frac{1}{2}\left[\frac{H(u^+) - H(u^-)}{u^+ - u^-} + \frac{H(w^+) - H(w^-)}{w^+ - w^-}\right],$$
$$\big[\bar\partial^\delta H\big](z) := \frac{1}{2}\left[\frac{H(u^+) - H(u^-)}{\overline{u^+ - u^-}} + \frac{H(w^+) - H(w^-)}{\overline{w^+ - w^-}}\right].$$

We use the same notations, if H is defined on Γ (or Γ^*) only, formally setting $H|_{\Gamma^*} := 0$ (or $H|_\Gamma := 0$, respectively). We call H **discrete holomorphic at** z iff $[\bar\partial^\delta H](z) = 0$, which is equivalent to say that

$$2\big[\partial^\delta (H|_\Gamma)\big](z) = \frac{H(u^+) - H(u^-)}{u^+ - u^-} = \frac{H(w^+) - H(w^-)}{w^+ - w^-} = 2\big[\partial^\delta (H|_{\Gamma^*})\big](z). \tag{2.7}$$

These difference operators naturally discretize the standard differential operators $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\bar\partial h = \frac{1}{2}(h'_x + ih'_y)$. In particular, ∂^δ and $\bar\partial^\delta$ have approximation properties similar to those in Lemma 2.2. Namely,

$$\big|\big[\partial^\delta \phi|_\Lambda\big](z) - (\partial\phi)(z)\big|,\ \big|\big[\bar\partial^\delta \phi|_\Lambda\big](z) - (\bar\partial\phi)(z)\big| = O\big(\delta^2\big)$$

for smooth functions ϕ.

Further, for $z \in \diamondsuit$, let θ_z denote the half-angle of the corresponding rhombus $u^-w^-u^+w^+$ along the diagonal u^-u^+, so that

$$w^+ - w^- = i\tan\theta_z \cdot \big(u^+ - u^-\big).$$

We define the *weight* of z by

$$\mu_\diamondsuit^\delta(z) := \mathrm{Area}\big(u^-w^-u^+w^+\big) = \delta^2 \sin 2\theta_z.$$

Also, for $v \in \Gamma$ and, in the same way, for $v \in \Gamma^*$, we set (cf. (2.1))

$$\mu_\Lambda^\delta(v) := \frac{1}{4}\sum_{z_s \sim v} \mu_\diamondsuit^\delta(z_s) = \frac{\mu_\Gamma^\delta(v)}{2}.$$

Clearly, formulas similar to (2.2) are fulfilled for ϕ's defined on subsets of $\diamondsuit$ or Λ. It is easy to check that Definition 2.12 may be rewritten in the following form:

$$\big[\partial^\delta H\big](z) = \frac{1}{4\mu_\diamondsuit^\delta(z)} \sum_{v=u^\pm,w^\pm} \overline{\mu_{zv}} H(v), \qquad \big[\bar\partial^\delta H\big](z) = \frac{1}{4\mu_\diamondsuit^\delta(z)} \sum_{v=u^\pm,w^\pm} \mu_{zv} H(v),$$

where the weights μ_{zv} are given by

$$\mu_{zu^\pm} := 2\tan\theta_z \cdot \big(u^\pm - z\big) = i \cdot \big(w^\mp - w^\pm\big),$$
$$\mu_{zw^\pm} := 2\cot\theta_z \cdot \big(w^\pm - z\big) = i \cdot \big(u^\pm - u^\mp\big).$$

The difference operators ∂^δ and $\bar\partial^\delta$ given above map functions defined on Λ to functions on $\diamondsuit$. Further, we introduce their formal adjoint $-(\partial^\delta)^*$, $-(\bar\partial^\delta)^*$, also denoted by $\bar\partial^\delta$ and ∂^δ, respectively, to keep the notation short. Note that no confusion arises since the latter operators, vice versa, map functions defined on $\diamondsuit$ to functions on Λ.

Definition 2.13. Let a function F be defined on some subset of $\diamondsuit$. For $v \in \Lambda$, we set

$$\big[\bar\partial^\delta F\big](v) := -\frac{1}{4\mu_\Lambda^\delta(v)} \sum_{z_s\sim v} \mu_{z_s v} F(z_s) \quad \text{and} \quad \big[\partial^\delta F\big](v) := -\frac{1}{4\mu_\Lambda^\delta(v)} \sum_{z_s\sim v} \overline{\mu_{z_s v}} F(z_s),$$

if the right-hand sides make sense. We call F **discrete holomorphic at v** iff $[\bar\partial^\delta F](v) = 0$.

These definitions are natural discretization of the formulas

$$(\bar\partial\phi)(v) \approx \frac{\iint_{W(v)} (\bar\partial\phi)(x+iy)\,dx\,dy}{\mathrm{Area}(W(v))} = -\frac{i}{2\,\mathrm{Area}(W(v))} \oint_{\partial W(v)} \phi(\zeta)\,d\zeta,$$

$$(\partial\phi)(v) \approx \frac{\iint_{W(v)} (\partial\phi)(x+iy)\,dx\,dy}{\mathrm{Area}(W(v))} = \frac{i}{2\,\mathrm{Area}(W(v))} \oint_{\partial W(v)} \phi(\zeta)\,d\bar\zeta,$$

where $W(v)$ denotes the corresponding dual face (e.g., see Fig. 1(B), if $v = u \in \Gamma$). For constant and linear ϕ's, these discretizations give the true answers, thus

$$\bigl|\bigl[\bar\partial^\delta \phi|_\diamond\bigr](v) - (\bar\partial\phi)(v)\bigr|,\ \bigl|\bigl[\partial^\delta \phi|_\diamond\bigr](v) - (\partial\phi)(v)\bigr| = O(\delta)$$

for all smooth functions ϕ. Note that, in general, one cannot replace $O(\delta)$ by $O(\delta^2)$.

2.5. *Factorization of Δ^δ. Basic properties of discrete holomorphic functions*

The following factorization of Δ^δ was noted in [16] and [12]:

Proposition 2.14. *For functions H defined on subsets of Λ the following is fulfilled:*

$$\bigl[\Delta^\delta H\bigr](u) = 4\bigl[\partial^\delta \bar\partial^\delta H\bigr](u) = 4\bigl[\bar\partial^\delta \partial^\delta H\bigr](u)$$

at all vertices $u \in \Lambda$ where the right-hand side makes sense.

Proof. Straightforward computations give (see Fig. 1(B) for notations)

$$\bigl[\bar\partial^\delta \partial^\delta H\bigr](u) = \frac{1}{8\mu_\Lambda^\delta(u)} \sum_{s=1}^{k} \bigl[\tan\theta_s \cdot \bigl[H(u_s) - H(u)\bigr] - i \cdot \bigl[H(w_{s+1}) - H(w_s)\bigr]\bigr] = \frac{[\Delta^\delta H](u)}{4}$$

and similarly for $[\partial^\delta \bar\partial^\delta H](u)$. □

In the lemmas below we list basic properties of discrete holomorphic functions coming from this factorization of Δ^δ. We often omit the word "discrete" (e.g., writing "holomorphic on $\diamond$" instead of "discrete holomorphic on $\diamond$") for short.

Lemma 2.15.

(i) *Let a function H be defined on some subset of Λ. If H is holomorphic on Λ, then H is harmonic on both Γ and Γ^*, i.e. both components $H|_\Gamma$, $H|_{\Gamma^*}$ are complex-valued harmonic functions.*

(ii) *Conversely, in simply connected domains, H is (complex-valued) harmonic on Γ iff there exists a (complex-valued) harmonic on Γ^* function $\tilde H$ such that $H + i\tilde H$ is holomorphic on Λ. $\tilde H$ is called* ***discrete harmonic conjugate*** *to H and is defined uniquely up to an additive constant. Moreover, $\tilde H$ is real-valued, if H is real-valued.*

Proof. (i) The claim easily follows by writing $\Delta^\delta H = 4\partial^\delta \bar\partial^\delta H = 0$.

(ii) For any $u \in \Gamma$ and $z_s \in \diamond$, $z_s \sim u$ (see Fig. 1(B) for notations), the holomorphicity condition at z_s defines the increments $\tilde H(w_{s+1}) - \tilde H(w_s)$ uniquely. These increments are locally consistent, i.e. their sum around u is zero, iff $[\Delta^\delta H](u) = 0$. In simply connected domains, the local consistency directly implies the global one. □

Due to Lemma 2.15, each holomorphic on Λ function is a couple of a complex-valued harmonic function $H|_\Gamma$ and its harmonic conjugate $H|_{\Gamma^*}$. Since the real part of $H|_\Gamma$ depends only on the imaginary part of $H|_{\Gamma^*}$ (and vice versa), both functions

$$\mathcal{B}H := \operatorname{Re} H|_\Gamma + i \operatorname{Im} H|_{\Gamma^*} \quad \text{and} \quad \mathcal{W}H := i \operatorname{Im} H|_\Gamma + \operatorname{Re} H|_{\Gamma^*} \tag{2.8}$$

are still holomorphic on Λ and completely independent of each other. Thus, to avoid a "doubling of information", at least unless some boundary conditions are specified, it is natural to consider (as many authors do) only those H, which are purely real on Γ (black vertices of Λ) and purely imaginary on Γ^* (white vertices of Λ), or vice versa.

Lemma 2.16.

(i) *Let H be a (complex-valued) harmonic function defined on some subset of Γ or Γ^*. Then its derivative $F = \partial^\delta H$ is holomorphic on $\diamondsuit$ (recall that, defining $\partial^\delta H$, we formally set $H|_{\Gamma^*} := 0$ or $H|_\Gamma := 0$, respectively). The same holds true, if H is a holomorphic function defined on some subset of Λ.*

(ii) *Conversely, in simply connected domains, if F is holomorphic on $\diamondsuit$, then there exists a holomorphic on Λ function H (which we call **discrete primitive** $\int^\delta F(z)\, d^\delta z$) such that $\partial^\delta H = F$. Its complex-valued harmonic components $H|_\Gamma$ and $H|_{\Gamma^*}$ are defined uniquely up to (different) additive constants by*

$$H\big(v^+\big) - H\big(v^-\big) := F(z) \cdot \big(v^+ - v^-\big), \qquad z = \frac{1}{2}\big(v^- + v^+\big),$$

where $v^\pm \in \Gamma$ or $v^\pm \in \Gamma^*$ are neighbors of $z \in \diamondsuit$.

Proof. (i) The claim easily follows by writing $\bar\partial^\delta F = \bar\partial^\delta \partial^\delta H = \frac{1}{4}\Delta^\delta H = 0$.

(ii) Since we are looking for holomorphic H's, it's necessary and sufficient to have $\partial^\delta(H|_\Gamma) = \partial^\delta(H|_{\Gamma^*}) = \frac{1}{2}F$ (see (2.7)). Thus, the increments $H(v^+) - H(v^-)$ are defined uniquely. For any $u \in \Lambda$, the condition $[\bar\partial^\delta F](u) = 0$ guarantees that these increments are locally consistent (i.e., their sum around u is zero). In simply connected domains, this implies the global consistency as well. $\square$

Due to Lemma 2.16, there is a correspondence between holomorphic on $\diamondsuit$ functions and their primitives, which are complex-valued harmonic functions on Γ (and, in the same way, on Γ^*). Since the latter space is naturally split on purely real and purely imaginary functions, the same should take place for functions, holomorphic on $\diamondsuit$.

Definition 2.17. Let $z \in \diamondsuit$ be the center of the rhombus $u^- w^- u^+ w^+$, where $u^\pm \in \Gamma$ and $w^\pm \in \Gamma^*$, and F be a complex-valued function defined at z. We set

$$[\mathcal{B}F](z) := \operatorname{Proj}\big[F(z); \overline{u^+ - u^-}\,\big] \quad \text{and} \quad [\mathcal{W}F](z) := \operatorname{Proj}\big[F(z); \overline{w^+ - w^-}\,\big],$$

where

$$\operatorname{Proj}[F; \xi] := \operatorname{Re}\left(F \frac{\bar\xi}{|\xi|}\right) \frac{\xi}{|\xi|} = \frac{F + \overline{F}\xi^2}{2|\xi|^2}$$

denotes the orthogonal projection of F onto the line $\xi\mathbb{R}$. Note that $|\mathcal{B}F|, |\mathcal{W}F| \leqslant |F|$ and $F = \mathcal{B}F + \mathcal{W}F$, since $u^+ - u^- \perp w^+ - w^-$.

Remark 2.18. Let $F=\partial^\delta H$, where H is purely real on Γ and purely imaginary on Γ^*, or, vice versa, $\operatorname{Re} H|_\Gamma=0$ and $\operatorname{Im} H|_{\Gamma^*}=0$. Then, $F=\mathcal{B}F$ or $F=\mathcal{W}F$, respectively.

The next lemma shows that, exactly as it happens for holomorphic on Λ functions, each holomorphic on $\diamondsuit$ function F consists of two completely independent halves: $\mathcal{B}F$ and $\mathcal{W}F$, the first coming as a gradient of a real-valued harmonic on Γ function and the second as a gradient of a real-valued harmonic on Γ^* function.

Lemma 2.19. *A function F is holomorphic on some subset of $\diamondsuit$ if and only if both projections $\mathcal{B}F$ and $\mathcal{W}F$ are holomorphic on this subset. Moreover, in this case,*

$$\mathcal{B}F=\partial^\delta\biggl[\mathcal{B}\biggl[\int^\delta F(z)\,d^\delta z\biggr]\biggr] \quad \text{and} \quad \mathcal{W}F=\partial^\delta\biggl[\mathcal{W}\biggl[\int^\delta F(z)\,d^\delta z\biggr]\biggr],$$

where $H=\int^\delta F(z)\,d^\delta z$ is any (local) primitive of F and $\mathcal{B}H$, $\mathcal{W}H$ are given by (2.8).

Proof. It is easy to check that

$$\begin{aligned}
\bar\partial^\delta[\mathcal{B}F]&=\operatorname{Re}\bigl[\bar\partial^\delta F\bigr] \quad \text{and} \quad \bar\partial^\delta[\mathcal{W}F]=i\operatorname{Im}\bigl[\bar\partial^\delta F\bigr] \quad \text{on } \Gamma,\\
\bar\partial^\delta[\mathcal{B}F]&=i\operatorname{Im}\bigl[\bar\partial^\delta F\bigr] \quad \text{and} \quad \bar\partial^\delta[\mathcal{W}F]=\operatorname{Re}\bigl[\bar\partial^\delta F\bigr] \quad \text{on } \Gamma^*,
\end{aligned}$$

thus F is holomorphic iff both $\mathcal{B}F$ and $\mathcal{W}F$ are holomorphic. In this case, the primitive H is locally well defined (up to additive constants), $F=\partial^\delta H=\partial^\delta[\mathcal{B}H]+\partial^\delta[\mathcal{W}H]$, and so $\mathcal{B}F=\partial^\delta[\mathcal{B}H]$, $\mathcal{W}F=\partial^\delta[\mathcal{W}H]$ (see (2.8) and Remark 2.18). □

It is worthwhile to note that there exists a natural **averaging operator m^δ**, which maps functions defined on Λ to functions on $\diamondsuit$. Namely, m^δ is given by

$$\bigl[m^\delta H\bigr](z):=\frac{1}{4}\bigl[H\bigl(u^-\bigr)+H\bigl(w^-\bigr)+H\bigl(u^+\bigr)+H\bigl(w^+\bigr)\bigr], \quad z\in\diamondsuit, \tag{2.9}$$

where, as above, $u^\pm\in\Gamma$ and $w^\pm\in\Gamma^*$ denote neighbors of $z\in\diamondsuit$.

Lemma 2.20. *Let H be holomorphic on (some part of) Λ. Then the averaged function $m^\delta H$ is holomorphic on $\diamondsuit$ at all $u\in\Lambda$, where the expression $[\bar\partial^\delta m^\delta H](u)$ makes sense.*

Proof. The condition $[\bar\partial^\delta H](z_s)=0$ (see Fig. 1(B) for notations) implies

$$\bigl[m^\delta H\bigr](z_s)=\frac{H(u)}{2}+\frac{H(w_{s+1})(w_{s+1}-u)-H(w_s)(w_s-u)}{2(w_{s+1}-w_s)}.$$

Summing the terms $(w_{s+1}-w_s)[m^\delta H](z_s)$ around u, one arrives at $[\bar\partial^\delta m^\delta H](u)=0$. □

Below we will also need the averaging operator m^δ (adjoint to (2.9)) which, conversely, maps functions defined on $\diamondsuit$ to functions on Λ:

$$\big[m^\delta F\big](v) := \frac{1}{4\mu_\Lambda^\delta(v)} \sum_{v\sim z_s\in\diamondsuit} \mu_\diamondsuit^\delta(z_s)F(z_s), \quad v\in\Lambda. \tag{2.10}$$

Unfortunately, there are two *unpleasant facts* that make discrete complex analysis on rhombic lattices more complicated than the standard continuous theory and even than the square lattice discretization:

- One cannot (pointwise) multiply discrete holomorphic functions: the product FG is *not* necessary holomorphic if both F and G are holomorphic.
- One cannot differentiate discrete holomorphic functions infinitely many times. Moreover, we don't know any "local" discretizations of ∂ that map holomorphic functions on Λ or $\diamondsuit$ to holomorphic functions defined on the same set (Λ or $\diamondsuit$). One cannot use natural combinations of ∂^δ and m^δ since both $\partial^\delta F$ and $m^\delta F$ are *not* necessary exact holomorphic on Λ, if F is holomorphic on $\diamondsuit$.

The first obstacle (multiplication) exists in all discrete theories. Concerning the second, note that in our case there is some "nonlocal" discrete differentiation (so-called dual integration, see [8] and [18]). Also in two particular cases the local differentiation leads to holomorphic function again: for the classical definition on the square grid (since in this case both Λ and $\diamondsuit$ are square grids, see the book by J. Lelong-Ferrand [15]) and for some particular definition on the triangular lattice (see [9]).

2.6. The Cauchy kernel. The Cauchy formula. Lipschitzness

The following asymptotic form of the discrete Cauchy kernel is due to R. Kenyon.

Theorem 2.21 *(Kenyon). Let $z_0\in\diamondsuit$. There exists a unique function $F=K(\,\cdot\,;z_0):\Lambda\to\mathbb{C}$ such that*

(i) $[\bar\partial^\delta F](z)=0$ *for all* $z\neq z_0$ *and* $[\bar\partial^\delta F](z_0)\cdot\mu_\diamondsuit^\delta(z_0)=1$*;*
(ii) $|F(u)|\to 0$ *as* $|u-z_0|\to\infty$*.*

Moreover, the following asymptotics hold:

$$K(u;z_0)=\frac{2}{\pi}\mathrm{Proj}\bigg[\frac{1}{u-z_0};\overline{u_0^+-u_0^-}\bigg]+O\bigg(\frac{\delta}{|u-z_0|^2}\bigg),\quad u\in\Gamma;$$
$$K(w;z_0)=\frac{2}{\pi}\mathrm{Proj}\bigg[\frac{1}{w-z_0};\overline{w_0^+-w_0^-}\bigg]+O\bigg(\frac{\delta}{|w-z_0|^2}\bigg),\quad w\in\Gamma^*,$$

where $u_0^\pm\in\Gamma$ and $w_0^\pm\in\Gamma^*$ are the black and white neighbors of z_0, respectively.

Proof. We give a short sketch of Kenyon's arguments [12] in Appendix A.1. □

Let Ω^δ_Γ be a bounded simply connected discrete domain (see Figs. 2(A), 3(A)). Denote by $B = u_0u_1u_2\dots u_n$, $u_s \in \Gamma$, its closed polyline boundary, enumerated in counter clockwise order. Denote by $W = w_0w_1w_2\dots w_m$, $w_s \in \Gamma^*$, the closed polyline path (enumerated in counter clockwise order) passing through the centers of all faces touching B from inside. For functions G defined on $B_\diamondsuit := \diamondsuit \cap B$ and $W_\diamondsuit := \diamondsuit \cap W$, we introduce "discrete contour integrals"

$$\oint\limits_B^\delta G(z)\,d^\delta z := \sum_{s=0}^{n-1} G\biggl(\frac{1}{2}(u_{s+1}+u_s)\biggr)\cdot(u_{s+1}-u_s),$$

$$\oint\limits_W^\delta G(z)\,d^\delta z := \sum_{s=0}^{m-1} G\biggl(\frac{1}{2}(w_{s+1}+w_s)\biggr)\cdot(w_{s+1}-w_s).$$

We also set $\Omega^\delta_\Lambda := \Lambda \cap \Omega^\delta$,

$$\Omega^\delta_\diamondsuit := \diamondsuit \cap \Omega^\delta, \qquad \overline{\Omega}{}^\delta_\diamondsuit := \Omega^\delta \cup B_\diamondsuit \quad \text{and} \quad \operatorname{Int}\Omega^\delta_\diamondsuit := \Omega^\delta_\diamondsuit \setminus W_\diamondsuit,$$

where Ω^δ denotes the polygonal representation of Ω^δ_Γ.

Proposition 2.22 *(Cauchy formula). Let $F : \overline{\Omega}{}^\delta_\diamondsuit \to \mathbb{C}$ be a discrete holomorphic function, i.e., $[\bar\partial^\delta F](v) = 0$ for all $v \in \Omega^\delta_\Lambda$. Then, for any $z_0 \in \operatorname{Int}\Omega^\delta_\diamondsuit$,*

$$F(z_0) = \frac{1}{4i}\Biggl[\oint\limits_B^\delta K\bigl(w(z);z_0\bigr)F(z)\,d^\delta z + \oint\limits_W^\delta K\bigl(u(z);z_0\bigr)F(z)\,d^\delta z\Biggr],$$

where $w(z) \in W_{\Gamma^*} := \Gamma^* \cap W$ denotes the nearest "white" vertex to $z \in B_\diamondsuit$, and $u(z) \in B_\Gamma := \Gamma \cap B$ denotes the nearest "black" vertex to $z \in W_\diamondsuit$ (see Fig. 3(A)*).*

Proof. By definitions of the discrete Cauchy kernel K and the operator $\bar\partial^\delta$, one has

$$4F(z_0) = \sum\sum_{z\in\Omega^\delta_\diamondsuit,\, z\sim v,\, v\in\overline{\Omega}{}^\delta_\Lambda} F(z)\mu_{zv}K(v;z_0) = \sum\sum_{v\in\overline{\Omega}{}^\delta_\Lambda,\, v\sim z,\, z\in\Omega^\delta_\diamondsuit} K(v;z_0)\mu_{zv}F(z),$$

where $\overline{\Omega}{}^\delta_\Lambda := \Omega^\delta_\Lambda \cup B_\Gamma$. Since $\sum_{v\sim z,\, z\in\overline{\Omega}{}^\delta_\diamondsuit} \mu_{zv}F(z) = 0$ for all $v \in \Omega^\delta_\Lambda$, this gives

$$\begin{aligned} 4F(z_0) &= \sum_{v\in B_\Gamma,\, v\sim z,\, z\in\Omega^\delta_\diamondsuit} K(v;z_0)\mu_{zv}F(z) - \sum_{v\in\Omega^\delta_\diamondsuit,\, v\sim z,\, z\in B_\diamondsuit} K(v;z_0)\mu_{zv}F(z) \\ &= \sum_{z\in W_\diamondsuit} K\bigl(u(z);z_0\bigr)F(z)\mu_{zu(z)} - \sum_{z\in B_\diamondsuit} K\bigl(w(z);z_0\bigr)F(z)\mu_{zw(z)}. \end{aligned}$$

Both sums coincide with the discrete contour integrals defined above. □

The Cauchy formula may be nicely rewritten in the asymptotic form for both components $\mathcal{B}F$ and $\mathcal{W}F$ of a holomorphic function F separately. Recall that these components are completely independent of each other (see Lemma 2.19).

Corollary 2.23 *(Asymptotic Cauchy formula). Let $F : \overline{\Omega}^\delta_\diamondsuit \to \mathbb{C}$ be a discrete holomorphic function, $z_0 \in \operatorname{Int}\Omega^\delta_\diamondsuit$ and $u_0^\pm \in \Gamma$, $w_0^\pm \in \Gamma^*$ be its neighboring vertices. Then*

$$[\mathcal{B}F](z_0) = \operatorname{Proj}\left[\frac{1}{2\pi i}\left(\oint\limits_B^{\delta} \frac{[\mathcal{B}F](z)}{z-z_0}\,d^\delta z + \oint\limits_W^{\delta} \frac{[\mathcal{B}F](z)}{z-z_0}\,d^\delta z\right); \overline{u_0^+ - u_0^-}\right] + O\left(\frac{\delta M L}{d^2}\right),$$

where $d = \operatorname{dist}(z_0, W)$, $M = \max_{z \in B_\diamondsuit \cup W_\diamondsuit} |F(z)|$ and $L = \operatorname{Length}(B) + \operatorname{Length}(W)$. The same formula holds true for $\mathcal{W}F$, if one replaces $u_0^+ - u_0^-$ by $w_0^+ - w_0^-$.

Proof. We plug Kenyon's asymptotics (Theorem 2.21) into Proposition 2.22:

if $z \in W_\diamondsuit$, then $[\mathcal{B}F](z)\,d^\delta z/4i \in \mathbb{R}$, and so

$$K\big(u(z); z_0\big) \cdot \frac{[\mathcal{B}F](z)\,d^\delta z}{4i} = \operatorname{Proj}\left[\frac{[\mathcal{B}F](z)\,d^\delta z}{2\pi i(z-z_0)}; \overline{u_0^+ - u_0^-}\right] + O\left(\frac{\delta M |d^\delta z|}{d^2}\right);$$

if $z \in B_\diamondsuit$, then $[\mathcal{B}F](z)\,d^\delta z/4i \in i\mathbb{R}$, and so, again,

$$K\big(w(z); z_0\big) \cdot \frac{[\mathcal{B}F](z)\,d^\delta z}{4i} = \operatorname{Proj}\left[\frac{[\mathcal{B}F](z)\,d^\delta z}{2\pi i(z-z_0)}; \overline{u_0^+ - u_0^-}\right] + O\left(\frac{\delta M |d^\delta z|}{d^2}\right),$$

since $w_0^+ - w_0^- \perp u_0^+ - u_0^-$. The claim follows by summing along B and W. □

Finally, the Cauchy formula implies Lipschitzness of discrete holomorphic functions. Since $\mathcal{B}F$ and $\mathcal{W}F$ are independent of each other, this should be valid for both components separately. On the other hand, the phase of $[\mathcal{B}F](z)$ depends only on the direction of the edge u^-u^+ passing through z, so one cannot expect that $[\mathcal{B}F](z_1)$ and $[\mathcal{B}F](z_2)$ are close in the usual sense, if z_1 and z_2 are close. Thus, we firstly use the operator m^δ defined by (2.10) and average our function around vertices $v \in \Lambda$.

Proposition 2.24 *(Lipschitzness of discrete holomorphic functions). Let $u \in \Gamma$ and let F be discrete holomorphic in $\overline{B}^\delta_\diamondsuit(u, R)$. Then, for all $z_s \sim u$, $z_s \in \diamondsuit$ (see Fig.* 1(B) *for notations),*

$$\big|[\mathcal{B}F](z_s) - \operatorname{Proj}\big[2\big[m^\delta(\mathcal{B}F)\big](u); \overline{u_s - u}\,\big]\big| \leqslant \text{const} \cdot \frac{M\delta}{R}, \quad \text{where } M = \max_{\overline{B}^\delta_\diamondsuit(u,R)} \big|F(z)\big|.$$

The same formula holds true for $\mathcal{W}F$, if one replaces $u_s - u$ by $w_{s+1} - w_s$. Furthermore, if $v_1, v_2 \in B^\delta_\Lambda(u, r)$, $r < R$, then

$$\big|\big[m^\delta F\big](v_2) - \big[m^\delta F\big](v_1)\big| \leqslant \text{const} \cdot \frac{M|v_2 - v_1|}{R - r}.$$

Proof. Let B and W be the same discrete contours as above (see Fig. 3(A)), note that their lengths are bounded by $\mathrm{const}\cdot R$. Applying Corollary 2.23 for all $z_s \sim u$ and taking into account that $|(z-z_s)^{-1}-(z-u)^{-1}| \leqslant \mathrm{const}\cdot \delta/R^2$, one obtains

$$[\mathcal{B}F](z_s) = \mathrm{Proj}[A; \overline{u_s-u}\,] + O\left(\frac{M\delta}{R}\right), \qquad A := \frac{1}{2\pi i}\left(\oint\limits_B^{\delta} \frac{[\mathcal{B}F](z)}{z-u}\,d^\delta z + \oint\limits_W^{\delta} \frac{[\mathcal{B}F](z)}{z-u}\,d^\delta z\right).$$

Due to the identity

$$\begin{aligned}\frac{1}{4\mu_\Lambda^\delta(u)}\sum_{z_s\sim u}\mu_\diamondsuit^\delta(z_s)\mathrm{Proj}[A;\overline{u_s-u}\,] &= \frac{1}{4\mu_\Lambda^\delta(u)}\sum_{z_s\sim u}\delta^2\sin 2\theta_s\cdot\frac{A+e^{-2i\arg(u_s-u)}\overline{A}}{2}\\ &= \frac{A}{2}+\frac{\delta^2\overline{A}}{16i\mu_\Lambda^\delta(u)}\sum_{u_s\sim u}\left(e^{-2i\arg(w_s-u)}-e^{-2i\arg(w_{s+1}-u)}\right)\\ &= \frac{A}{2},\end{aligned}$$

it gives

$$\left[m^\delta(\mathcal{B}F)\right](u) = \frac{A}{2} + O\left(\frac{M\delta}{R}\right).$$

In particular, $|[\mathcal{B}F](z_s) - \mathrm{Proj}[2[m^\delta(\mathcal{B}F)](u); \overline{u_s-u}\,]| \leqslant \mathrm{const}\cdot M\delta/R$. The proof for $\mathcal{W}F$ goes exactly in the same way, since $e^{-2i\arg(w_{s+1}-w_s)} = -e^{-2i\arg(u_s-u)}$. Moreover, using the same calculations for $[m^\delta F](w_s)$, one obtains

$$\left|\left[m^\delta(\mathcal{B}F)\right](w_s) - \left[m^\delta(\mathcal{B}F)\right](u)\right|,\ \left|\left[m^\delta(\mathcal{W}F)\right](w_s) - \left[m^\delta(\mathcal{W}F)\right](u)\right| \leqslant \mathrm{const}\cdot\frac{M\delta}{R},$$

so the same estimate holds true for the function $m^\delta F = m^\delta(\mathcal{B}F) + m^\delta(\mathcal{W}F)$.

Summing these inequalities along the path connecting v_1 and v_2 inside $B_\Gamma^\delta(u,r)$ (due to condition (♠), there is a path of length $\leqslant \mathrm{const}\cdot\delta^{-1}|v_2-v_1|$), one immediately arrives at the estimate for $|[m^\delta F](v_2) - [m^\delta F](v_1)|$. □

3. Convergence theorems

3.1. Precompactness in the C^1-topology

In the continuous setup, each uniformly bounded family of harmonic functions (defined in some common domain Ω) is precompact in the C^∞-topology. Using Corollary 2.9 and Proposition 2.24, it is easy to prove the analogue of this statement for discrete harmonic functions.

Below we widely use the following convention. Let a function H^δ be defined in a discrete domain $\Omega_\Gamma^\delta \subset \Gamma^\delta$. Then, H^δ can be thought of as defined in its polygonal representation $\Omega^\delta \subset \mathbb{C}$ by some standard continuation procedure, say, linear on edges and harmonic inside faces. Note that this continuation is bounded in Ω^δ, if H^δ is bounded on Ω_Γ^δ, and Lipschitz in Ω^δ, if H^δ is Lipschitz on Ω^δ (with the same constants).

Proposition 3.1. *Let $H^{\delta_j}:\Omega_\Gamma^{\delta_j}\to\mathbb{R}$ be (real-valued) discrete harmonic functions defined in discrete domains $\Omega_\Gamma^{\delta_j}\subset\Gamma^{\delta_j}$ with $\delta_j\to 0$. Let $\Omega\subset\bigcup_{n=1}^{+\infty}\bigcap_{j=n}^{+\infty}\Omega^{\delta_j}\subset\mathbb{C}$ be some continuous domain. If H^{δ_j} are uniformly bounded on Ω, i.e.*

$$\max_{u\in\Omega_\Gamma^{\delta_j}\cap\Omega}\big|H^{\delta_j}(u)\big|\leqslant M<+\infty\quad \textit{for all } j,$$

then there exists a subsequence $\delta_{j_k}\to 0$ (which we denote by δ_k for short) and two functions $h:\Omega\to\mathbb{R}$, $f:\Omega\to\mathbb{C}$ such that (we denote by "$\rightrightarrows$" uniform convergence)

$$H^{\delta_k}\rightrightarrows h\quad \textit{uniformly on compact subsets } K\subset\Omega$$

and

$$\frac{H^{\delta_k}(u_k^+)-H^{\delta_k}(u_k^-)}{|u_k^+-u_k^-|}\rightrightarrows \operatorname{Re}\left[f(u)\cdot\frac{u_k^+-u_k^-}{|u_k^+-u_k^-|}\right], \tag{3.1}$$

if $u_k^\pm\in\Gamma^{\delta_k}$, $u_k^+\sim u_k^-$ and $u_k^\pm\to u\in K\subset\Omega$ as $k\to\infty$. Moreover, the limit function h, $|h|\leqslant M$, is harmonic in Ω and $f=h'_x-ih'_y=2\partial h$ is analytic in Ω.

Remark 3.2. In other words, the discrete gradients of H^δ defined by the left-hand side of (3.1) converge to ∇h. Looking at the edge $u_k^-u_k^+$ one sees only the discrete directional derivative of H^δ along the unit vector $\tau_k:=(u_k^+-u_k^-)/|u_k^+-u_k^-|$ which converges to $\langle\nabla h(u),\tau_k\rangle=\operatorname{Re}[2\partial h(u)\cdot\tau_k]$.

Proof. Due to the uniform Lipschitzness of bounded discrete harmonic functions (see Corollary 2.9) and the Arzelà–Ascoli theorem, the sequence $\{H^{\delta_j}\}$ is precompact in the uniform topology on any compact subset $K\subset\Omega$. Moreover, their discrete derivatives (defined for $z\in\Omega_\diamondsuit^{\delta_j}$)

$$F^{\delta_j}(z):=\big[\partial^{\delta_j}H^{\delta_j}\big](z)=\frac{H^{\delta_j}(u_j^+(z))-H^{\delta_j}(u_j^-(z))}{u_j^+(z)-u_j^-(z)},\quad z\sim u_j^\pm(z)\in\Gamma^{\delta_j},$$

are discrete holomorphic and uniformly bounded on any compact subset $K\subset\Omega$. Then, due to Proposition 2.24 and the Arzelà–Ascoli theorem, the sequence of averaged functions $m^{\delta_j}F^{\delta_j}$ (defined on $\Omega_\Gamma^{\delta_j}$ by (2.10)) is precompact in the uniform topology on any compact subset of Ω. Thus, for some subsequence $\delta_k\to 0$, one has

$$H^{\delta_k}\rightrightarrows h\quad\text{and}\quad 2m^{\delta_k}F^{\delta_k}\rightrightarrows f$$

uniformly on compact subsets of Ω. Moreover, due to Proposition 2.24, it also gives

$$\big|F^{\delta_k}(z)-\operatorname{Proj}\big[f(z);\overline{u_k^+(z)-u_k^-(z)}\big]\big|\rightrightarrows 0\quad\text{uniformly on compact subsets of }\Omega.$$

It is easy to see that h is harmonic. Indeed, let $\phi:\Omega\to\mathbb{R}$ be an arbitrary $C_0^\infty(\Omega)$ test function (i.e., $\phi\in C^\infty$ and $\operatorname{supp}\phi\subset\Omega$). Denote by h^δ, ϕ^δ and $(\Delta\phi)^\delta$ the restrictions of h, ϕ and $\Delta\phi$ onto the lattice Γ^δ. The approximation properties ((2.2) and Lemma 2.2) and discrete integration by parts give

$$\langle h, \Delta\phi\rangle_\Omega = \lim_{\delta=\delta_k\to 0}\sum_{u\in\Omega_\Gamma^\delta} h^\delta(u)(\Delta\phi)^\delta(u)\mu_\Gamma^\delta(u) = \lim_{\delta=\delta_k\to 0}\sum_{u\in\Omega_\Gamma^\delta} h^\delta(u)\big[\Delta^\delta\phi^\delta\big](u)\mu_\Gamma^\delta(u)$$
$$= \lim_{\delta=\delta_k\to 0}\sum_{u\in\Omega_\Gamma^\delta} H^\delta(u)\big[\Delta^\delta\phi^\delta\big](u)\mu_\Gamma^\delta(u) = \lim_{\delta=\delta_k\to 0}\sum_{u\in\Omega_\Gamma^\delta}\big[\Delta^\delta H^\delta\big](u)\phi^\delta(u)\mu_\Gamma^\delta(u) = 0.$$

Furthermore, for any path $[u_0u_n]^\delta = u_0u_1\ldots u_n$, $u_{s+1}\sim u_s$, $u_s\in\Gamma^\delta$, one has

$$H^\delta(u_n) - H^\delta(u_0) = \int\limits_{[u_0u_n]^\delta}^{\delta} F^\delta(z)\,d^\delta z = \sum_{s=0}^{n-1} F^\delta\left(\frac{1}{2}(u_{s+1}+u_s)\right)\cdot(u_{s+1}-u_s).$$

Taking appropriate discrete approximations of segments $[uv]\subset\Omega$ (recall that rhombi angles are bounded from 0 and π, so one may find polyline approximations with uniformly bounded lengths) and passing to the limit as $\delta=\delta_k\to 0$, one obtains

$$h(v)-h(u) = \int\limits_{[uv]}\mathrm{Re}\big[f(z)\,dz\big] = \mathrm{Re}\Bigg[\int\limits_{[uv]} f(z)\,dz\Bigg] \quad \text{for all segments } [uv]\subset\Omega.$$

It gives $\alpha h_x'(u)+\beta h_y'(u) = \mathrm{Re}[(\alpha+i\beta)f(u)]$ for all $u\in\Omega$ and $\alpha,\beta\in\mathbb{R}$, so $f = 2\partial h$. □

As an illustration of what directly follows from basic facts collected in Section 2, we give a proof of the most classical convergence result for solutions of the Dirichlet boundary value problem, when a *single* domain $\Omega\subset\mathbb{C}$ bounded by Jordan curves is approximated by discrete ones, "growing from inside". Later, in Theorem 3.10, we will prove the *uniform* (w.r.t. Ω) version of the same result for simply connected Ω's.

Proposition 3.3. *Let $\Omega\subset\mathbb{C}$ be a (possibly not simply connected) continuous domain, bounded by a finite number of closed nonintersecting Jordan curves, $\partial\Omega = J_1\cup\cdots\cup J_n$, and $g: J^r\to\mathbb{R}$ be a continuous function defined in some closed r-neighborhood J^r of $\partial\Omega$. Let a sequence of discrete domains $\Omega_\Gamma^{\delta_j}\subset\Gamma^{\delta_j}$, $\delta_j\to 0$, approximate Ω so that*

$$\Omega\setminus J^r\subset\Omega^{\delta_1}\subset\Omega^{\delta_2}\subset\cdots\subset\Omega \quad \textit{and} \quad \bigcup_{j=1}^{+\infty}\Omega^{\delta_j} = \Omega.$$

Let H^{δ_j} denote the discrete harmonic continuation of g from $\partial\Omega_\Gamma^{\delta_j}\subset J^r$ into $\Omega_\Gamma^{\delta_j}$ and h be the continuous harmonic continuation of g from J into Ω. Then,

$$H^{\delta_j}\rightrightarrows h \quad \textit{uniformly on compact subsets } K\subset\Omega.$$

Moreover, discrete gradients (3.1) *of functions H^{δ_j} uniformly converge to ∇h.*

Proof. Since J^r is compact, g is bounded by some constant $M := \max_{z\in J^r}|g(z)|$ and uniformly continuous on J^r. Set

$$\nu(\rho) := \max_{z,w\in J^r:\ |z-w|\leqslant\rho} \bigl|g(z)-g(w)\bigr| \to 0 \quad \text{as } \rho\to 0.$$

By the maximum principle, all H^{δ_j} are uniformly bounded in Ω. Then, Proposition 3.1 allows one to extract a subsequence H^{δ_k} which converges to some harmonic function H (and the gradients of H^{δ_k} converge to ∇H). Thus, it is sufficient to prove that each subsequential limit coincides with h, i.e. to identify the boundary values of H.

Let $z = z^{\delta_k} \in \Omega_\Gamma^{\delta_k} \subset \Gamma^{\delta_k}$, $w\in\partial\Omega$ be (one of) the closest to z points on $\partial\Omega$, and $d := |z-w|$. Since $H^\delta = g$ on $\partial\Omega_\Gamma^\delta$, for any $\delta=\delta_k$ and $\rho\geqslant 2d$, one has

$$\begin{aligned}\bigl|H^\delta(z)-g(w)\bigr| &\leqslant \max\bigl\{\bigl|H^\delta(u)-g(w)\bigr|,\ u\in\partial\Omega_\Gamma^\delta\cap B(z,\rho)\bigr\}\cdot\omega^\delta\bigl(z;\partial\Omega_\Gamma^\delta\cap B(z,\rho);\Omega_\Gamma^\delta\bigr)\\ &\quad+\max\bigl\{\bigl|H^\delta(u)-g(w)\bigr|,\ u\in\partial\Omega_\Gamma^\delta\setminus B(z,\rho)\bigr\}\cdot\omega^\delta\bigl(z;\partial\Omega_\Gamma^\delta\setminus B(z,\rho);\Omega_\Gamma^\delta\bigr)\\ &\leqslant \nu(2\rho)+2M\cdot\mathrm{const}\cdot(2d/\rho)^\beta,\end{aligned}$$

where we have used $\mathrm{dist}(z;\partial\Omega_\Gamma^\delta)\leqslant d+2\delta\leqslant 2d$ and the weak Beurling-type estimate (Proposition 2.11) for the second discrete harmonic measure. Choosing $\rho(d)$ so that $\nu(2\rho(d))\cdot\rho(d)^\beta = d^\beta$ and passing to the limit as $\delta\to 0$, we obtain the estimate

$$\bigl|H(z)-g(w)\bigr| = O\bigl(\nu\bigl(2\rho(d)\bigr)\bigr)\to 0 \quad \text{as } d=|z-w|\to 0.$$

Thus, boundary values of H coincide with those of h, hence $H=h$ in Ω. □

3.2. *Carathéodory topology and uniform C^1-convergence*

Below we need some standard concepts of geometric function theory (see [19, Chapters 1, 2]).

Let Ω be a simply connected domain. A *crosscut* C of Ω is an open Jordan arc in Ω such that $\overline{C} = C\cup\{a,b\}$ with $a,b\in\partial\Omega$. A *prime end* of Ω is an equivalence class of sequences (*null-chains*) (C_n) of prime ends such that $\overline{C}_n\cap\overline{C}_{n+1}=\emptyset$, C_n separates C_0 from C_{n+1} and $\operatorname{diam} C_n\to 0$ as $n\to\infty$ (null chains (C_n), $(\tilde{C})_n$ are equivalent iff for all sufficiently large m there exists n such that C_m separates $\tilde{C}_0$ from $\tilde{C}_n$ and $\tilde{C}_m$ separates C_0 from C_n).

Let $P(\Omega)$ denote the set of all prime ends of Ω and let $\phi:\Omega\to\mathbb{D}$ be a conformal map. Then (see Theorem 2.15 in [19]) ϕ induces the natural bijection between $P(\Omega)$ and the unit circle $\mathbb{T}=\partial\mathbb{D}$.

Let $u_0\in\mathbb{C}$ be given and $\Omega_n,\Omega\subset\mathbb{C}$, be simply connected domains $\neq\mathbb{C}$ with $u_0\in\Omega_n,\Omega$. We say that $\Omega_n\to\Omega$ as $n\to\infty$ *in the sense of kernel convergence with respect to* u_0 iff

(i) some neighborhood of every $z\in\Omega$ lies in Ω_n for large enough n;
(ii) for every $a\in\partial\Omega$ there exist $a_n\in\partial\Omega_n$ such that $a_n\to a$ as $n\to\infty$.

Let $\phi_k:\Omega_k\to\mathbb{D}$, $\phi:\Omega\to\mathbb{D}$ be the Riemann uniformization maps normalized at u_0 (i.e., $\phi(u_0)=0$ and $\phi'(u_0)>0$). Then (see Theorem 1.8 in [19])

$$\Omega_k\to\Omega \quad \text{w.r.t. } u_0 \quad \Leftrightarrow \quad \phi_k^{-1}\rightrightarrows\phi^{-1} \quad \text{uniformly on compact subsets of } \mathbb{D}.$$

Using the Koebe distortion theorem (see Section 1.3 in [19]), it is easy to see that

(a) $\phi_k \rightrightarrows \phi$ as $k \to \infty$ uniformly on compact subsets $K \subset \Omega$;
(b) for any Ω_1, Ω_2 such that $u_0 \in \Omega_1 \subset \Omega_2 \neq \mathbb{C}$, the set of all simply connected domains $\{\Omega\colon \Omega_1 \subset \Omega \subset \Omega_2\}$ is compact in the topology of kernel convergence w.r.t. u_0.

Definition 3.4. Let $\Omega = (\Omega; v, \ldots; a, b, \ldots)$ be a simply connected bounded domain with several (possibly none) marked interior points $v, \ldots \in \operatorname{Int}\Omega$ and prime ends (boundary points) $a, b, \ldots \in P(\Omega)$ (we admit coincident points, say, $a = b$) and let $u \in \Omega$. We write

$$(\Omega_k; u_k) = (\Omega_k; u_k, v_k, \ldots; a_k, b_k, \ldots) \xrightarrow{\text{Cara}} (\Omega; u) = (\Omega; u, v, \ldots; a, b, \ldots) \quad \text{as } k \to \infty,$$

iff the domains Ω_k are uniformly bounded, $u_k \to u$, $\Omega_k \to \Omega$ in the sense of kernel convergence w.r.t. u and $\phi_k(v_k) \to \phi(v)$, $\ldots$, $\phi_k(a_k) \to \phi(a)$, $\ldots$, where $\phi_k : \Omega_k \to \mathbb{D}$, $\phi : \Omega \to \mathbb{D}$ are the Riemann uniformization maps normalized at u.

Remark 3.5. Since $v \in \Omega$, one has $|\phi(v)| < 1$. Thus, $\phi_k(v_k) \to \phi(v)$ implies $v_k \to v$. Moreover, one can equivalently use the point v instead of u in the definition given above.

Definition 3.6. Let Ω be a simply connected bounded domain, $u, v, \ldots \in \Omega$ and $r > 0$. We say that the inner points $u, v, \ldots$ are **jointly r-inside Ω** iff $B(u, r), B(v, r), \ldots \subset \Omega$ and there are paths $L_{uv}, \ldots$ connecting these points r-inside Ω (i.e., $\operatorname{dist}(L_{uv}, \partial\Omega), \ldots \geqslant r$). In other words, $u, v, \ldots$ belong to the same connected component of the r-interior of Ω.

Note that for each $0 < r < R$ there exists some $C(r, R)$ such that, if $\Omega \subset B(0, R)$ and $u, v, \ldots$ are jointly r-inside Ω, then

$$\bigl|\phi(v)\bigr|, \ldots \leqslant C(r, R) < 1, \tag{3.2}$$

where $\phi : \Omega \to \mathbb{D}$ is the Riemann uniformization map normalized at u. Indeed, considering the standard plane metric, one concludes that the extremal distance (see, e.g., Chapter IV in [11]) from L_{uv} to $\partial\Omega$ in $\Omega \setminus L_{uv}$ is not less than $r/\pi R^2$. Thus, the conformal modulus of the annulus $\mathbb{D} \setminus \phi(L_{uv})$ is bounded below by some $\operatorname{const}(r, R) > 0$. Since $\phi(u) = 0$, (3.2) holds true.

Now we formulate a general framework for the theorems below. Suppose that some harmonic function (e.g., harmonic measure, Green's function, Poisson kernel, etc.)

$$h(\cdot\,; \Omega) = h(\cdot\,, v, \ldots; a, b, \ldots; \Omega) : \Omega \to \mathbb{R}$$

is associated with each (continuous) domain $\Omega = (\Omega; v, \ldots; a, b, \ldots)$.

Similarly, let $\Omega^\delta_\Gamma = (\Omega^\delta_\Gamma; v^\delta, \ldots; a^\delta, b^\delta, \ldots)$ denote a simply connected bounded discrete domain with several marked vertices $v^\delta, \ldots \in \operatorname{Int}\Omega^\delta_\Gamma$ and $a^\delta, b^\delta, \ldots \in \partial\Omega^\delta_\Gamma$ and

$$H^\delta\bigl(\cdot\,; \Omega^\delta_\Gamma\bigr) = H^\delta\bigl(\cdot\,, v^\delta, \ldots; a^\delta, b^\delta, \ldots; \Omega^\delta_\Gamma\bigr) : \Omega^\delta_\Gamma \to \mathbb{R}$$

be some discrete harmonic function associated with this configuration. The idea of Proposition 3.8 is to use the compactness argument again, now for the set of all simply-connected domains. Recall that $\Omega^\delta \subset \mathbb{C}$ denotes the polygonal representation of Ω^δ_Γ.

Definition 3.7. We say that $\boldsymbol{H^\delta}$ **are uniformly** $\boldsymbol{C^1}$**-close to** $\boldsymbol{h}$ **inside** $\boldsymbol{\Omega^\delta}$, iff for all $0 < r < R$ there exist $\varepsilon(\delta) = \varepsilon(\delta, r, R)$, $\tilde{\varepsilon}(\delta) = \tilde{\varepsilon}(\delta, r, R) \to 0$ as $\delta \to 0$ such that for all discrete domains $\Omega^\delta_\Gamma = (\Omega^\delta_\Gamma; v^\delta, \dots; a^\delta, b^\delta, \dots)$ and $u^\delta \in \operatorname{Int}\Omega^\delta_\Gamma$ the following holds true:

If $\Omega^\delta \subset B(0, R)$ and $u^\delta, v^\delta, \dots$ are jointly r-inside Ω^δ, then

$$\left|H^\delta\left(u^\delta, v^\delta, \dots; a^\delta, b^\delta, \dots; \Omega^\delta_\Gamma\right) - h\left(u^\delta, v^\delta, \dots; a^\delta, b^\delta, \dots; \Omega^\delta\right)\right| \leqslant \varepsilon(\delta) \tag{3.3}$$

and, for all $u^\delta_s \sim u^\delta$, $u^\delta_s \in \Omega^\delta_\Gamma$,

$$\left|\frac{H^\delta(u^\delta_s; \Omega^\delta_\Gamma) - H^\delta(u^\delta; \Omega^\delta_\Gamma)}{|u^\delta_s - u^\delta|} - \operatorname{Re}\left[2\partial h\left(u^\delta; \Omega^\delta\right) \cdot \frac{u^\delta_s - u^\delta}{|u^\delta_s - u^\delta|}\right]\right| \leqslant \tilde{\varepsilon}(\delta), \tag{3.4}$$

where $2\partial h = h'_x - i h'_y$.

Proposition 3.8. *Let* (a) $H^\delta \to h$ *"pointwise" as* $\delta \to 0$*, i.e.,*

$$H^\delta\left(u^\delta; \Omega^\delta_\Gamma\right) \to h(u; \Omega), \quad \textit{if } \left(\Omega^\delta; u^\delta\right) \xrightarrow{\text{Cara}} (\Omega; u) \textit{ as } \delta \to 0; \tag{3.5}$$

and (b) h *be Carathéodory-stable, i.e.,*

$$h(u_k; \Omega_k) \to h(u; \Omega), \quad \textit{if } (\Omega_k; u_k) \xrightarrow{\text{Cara}} (\Omega; u) \textit{ as } k \to \infty. \tag{3.6}$$

Then functions H^δ *are uniformly* C^1*-close to* h *inside* Ω^δ *(see Definition* 3.7*).*

Remark 3.9. Typically, if one is able to prove (a) using the "toolbox" developed in Section 2, then the same reasoning applied in the continuous setup would lead to (b), since all these tools are just discrete versions of classical facts from complex analysis.

Proof. Suppose (3.3) does not hold true, i.e.,

$$\left|H^\delta\left(u^\delta; \Omega^\delta_\Gamma\right) - h\left(u^\delta; \Omega^\delta\right)\right| \geqslant \varepsilon_0 > 0$$

for some sequence $(\Omega^\delta_\Gamma; u^\delta)$, $\delta = \delta_j \to 0$, such that $B(u^\delta, r) \subset \Omega^\delta \subset B(0, R)$. Taking a subsequence, one may assume that $u^\delta \to u$ for some $u \in B(0, R)$. The set of all simply connected domains $\Omega : B(u, \frac{1}{2}r) \subset \Omega \subset B(0, R)$ is compact in the Carathéodory topology. Thus, taking a subsequence again, one may assume that

$$\left(\Omega^\delta; u^\delta, v^\delta, \dots; a^\delta, b^\delta, \dots\right) \xrightarrow{\text{Cara}} (\Omega; u, v, \dots; a, b, \dots) \quad \text{as } \delta = \delta_k \to 0$$

(note that the marked points $v^\delta, \dots$ cannot reach the boundary due to (3.2)). Then, (a) the pointwise convergence $H^\delta \to h$ and (b) the Carathéodory stability of h easily give a contradiction. Indeed, both

$$\text{(a)} \quad H^\delta\left(u^\delta; \Omega^\delta_\Gamma\right) \to h(u; \Omega) \quad \text{and} \quad \text{(b)} \quad h\left(u^\delta; \Omega^\delta\right) \to h(u; \Omega) \quad \text{as } \delta = \delta_k \to 0.$$

In view of Proposition 3.1, the proof for discrete gradients goes by the same way. Assume (3.4) does not hold for some sequence of discrete domains. As above, one may take a subsequence $\delta=\delta_k$ such that $(\Omega^\delta;u^\delta)\xrightarrow{\mathrm{Cara}}(\Omega,u)$. Note that (b) directly implies

$$h\big(\cdot\,;\Omega^\delta\big)\rightrightarrows h(\cdot\,;\Omega)\quad\text{uniformly on } \overline{B}\left(u,\frac{1}{2}r\right) \text{ as } \delta=\delta_k\to 0.$$

Indeed, $(\Omega^\delta;\tilde{u})\xrightarrow{\mathrm{Cara}}(\Omega;\tilde{u})$ for all $\tilde{u}\in\overline{B}(u,\frac{1}{2}r)$. If $|h(\tilde{u}^\delta;\Omega^\delta)-h(\tilde{u}^\delta;\Omega)|\geqslant\varepsilon_0>0$ for some $\tilde{u}^\delta$ and all $\delta=\delta_k$, then, taking a subsequence $\delta=\delta_m$ so that $\tilde{u}^\delta\to\tilde{u}\in\overline{B}(u,\frac{1}{2}r)$, one obtains a contradiction with $h(\tilde{u}^\delta;\Omega^\delta)\to h(\tilde{u};\Omega)$ and $h(\tilde{u}^\delta;\Omega)\to h(\tilde{u};\Omega)$.

The uniform estimate $|H^\delta(\cdot\,;\Omega^\delta_\Gamma)-h(\cdot\,;\Omega^\delta)|\leqslant\varepsilon(\delta)\to 0$ (see above) gives

$$H^\delta\big(\cdot\,;\Omega^\delta_\Gamma\big)\rightrightarrows h(\cdot\,;\Omega)\quad\text{uniformly on } \overline{B}\left(u,\frac{1}{2}r\right) \text{ as } \delta=\delta_k\to 0.$$

In particular, all $H^{\delta_k}(\cdot\,;\Omega^{\delta_k}_\Gamma)$ are uniformly bounded in $\overline{B}(u,\frac{1}{2}r)$. Thus, using Proposition 3.1, one can find a subsequence $\delta=\delta_m$ such that the discrete derivatives of H^{δ_m} converge (as defined by (3.4)) to $f=2\partial h(\cdot\,;\Omega)$ which gives a contradiction. □

3.3. Basic uniform convergence theorems

We start with a uniform (w.r.t. Ω) version (Theorem 3.10) of Proposition 3.3 for simply-connected domains. It immediately gives the uniform convergence for the discrete Green's functions (Corollary 3.11). Then, we prove very similar Theorem 3.12 devoted to the discrete harmonic measure of boundary arcs. The last result, Theorem 3.13 devoted to the discrete Poisson kernel $P^\delta(\cdot\,;v^\delta;a^\delta;\Omega^\delta_\Gamma)$ (see (1.3)), needs more technicalities, essentially because of the unboundedness of P^δ near a^δ.

Let $g:\overline{B}(0,R)\to\mathbb{R}$ be a continuous function. Then, for a simply connected domain $\Omega\subset B(0,R)$, let $h_g(\cdot\,;\Omega):\Omega\to\mathbb{R}$ denote a unique solution of the Dirichlet boundary value problem

$$\Delta h_g(\cdot\,;\Omega)=0\quad\text{inside } \Omega,\qquad h_g(\cdot\,;\Omega)=g\quad\text{on } \partial\Omega.$$

This is the classical result that the solution h_g exists for any simply connected Ω (see, e.g., §III.5, §III.6 and Corollary 6.2 in [11]). Note that this also follows from the proof of Theorem 3.10, where h_g naturally appears as a limit of discrete approximations.

Similarly, for a discrete simply-connected domain Ω^δ, let $H^\delta_g=H^\delta_g(\cdot\,;\Omega^\delta_\Gamma)$ be a unique solution of the discrete Dirichlet problem

$$\Delta^\delta H^\delta_g=0\quad\text{in } \Omega^\delta_\Gamma,\qquad H^\delta_g=g\quad\text{on } \partial\Omega^\delta_\Gamma.$$

Theorem 3.10. *For any continuous $g:\overline{B}(0,R)\to\mathbb{R}$, the functions H^δ_g are uniformly C^1-close inside Ω^δ (in the sense of Definition* 3.7*) to h^δ_g. Moreover, the estimates* (3.3) *and* (3.4) *are also uniform in*

$$g\in\mathcal{G}_R(M,\nu):=\Big\{g\colon \max_{|z|\in\overline{B}(0,R)}\big|g(z)\big|\leqslant M,\ \max_{|z-w|\leqslant\rho}\big|g(z)-g(w)\big|\leqslant\nu(\rho)\Big\},$$

if both $M < +\infty$ and the modulus of continuity $\nu(\rho) \to 0$ as $\rho \to 0$ are fixed. In other words, there exist $\varepsilon(\delta)$, $\tilde{\varepsilon}(\delta) \to 0$ as $\delta \to 0$ (which may depend on r, R, M, ν) such that (3.3), (3.4) *are fulfilled for any $g \in \mathcal{G}_R(M, \nu)$ and any simply connected $\Omega \subset B(0, R)$.*

Proof. Let g be fixed. It is sufficient to verify both assumptions (a) and (b) in Proposition 3.8. In fact, (a) was already essentially verified in the proof of Proposition 3.3.

Indeed, H_g^δ are uniformly bounded in Ω by a constant M, and so Proposition 3.1 allows one to extract a convergent subsequence $H_g^{\delta_k} \rightrightarrows H$. Thus, it is sufficient to prove that each subsequential limit H coincides with g on $\partial\Omega$.

Let $z = z^{\delta_k} \in \Omega_\Gamma^{\delta_k} \subset \Gamma^{\delta_k}$, $w \in \partial\Omega$ be (one of) the closest to z points on $\partial\Omega$, and $d := |z - w|$. Due to the geometric description of the kernel convergence, there is a sequence of points $w^\delta \in \partial\Omega^\delta$ approximating w as $\delta \to 0$. Thus, one still has $\operatorname{dist}(z; \partial\Omega_\Gamma^\delta) \leqslant 2d$ for δ small enough, and the proof finishes exactly as before.

As it was pointed out in Remark 3.9, the Carathéodory stability of h_g follows from the same reasonings applied in the continuous setup. Namely, one can always find a subsequence of the uniformly bounded harmonic functions $h_g(\cdot\,; \Omega_k)$ uniformly converging on compact subsets of Ω together with their gradients. Then, exactly as above, the classical Beurling estimate implies that $h = g$ on $\partial\Omega$, and so each subsequential limit coincides with $h_g(\cdot\,; \Omega)$.

Finally, for $g \in \mathcal{G}_R(M, \nu)$, let $\varepsilon(\delta; g)$ and $\tilde{\varepsilon}(\delta; g)$ denote the best possible bounds in (3.3) and (3.4), respectively. Due to the (both, discrete and continuous) maximum principles and the Harnack inequalities for harmonic functions, one sees that

$$\bigl|\varepsilon(\delta; g_1) - \varepsilon(\delta; g_2)\bigr| \leqslant 2\|g_1 - g_2\|_C \quad \text{and} \quad \bigl|\tilde{\varepsilon}(\delta; g_1) - \tilde{\varepsilon}(\delta; g_2)\bigr| \leqslant \text{const} \cdot \frac{\|g_1 - g_2\|_C}{r},$$

where $\|g\|_C := \max_{z \in \overline{B}(0,R)} |g(z)|$ is the standard sup-norm in the space $C(\overline{B}(0, R))$. Thus, $\varepsilon(\delta; \cdot)$ and $\tilde{\varepsilon}(\delta; \cdot)$ are uniformly (in δ) continuous (as functions of g) on the set $\mathcal{G}_R(M, \nu)$. Since $\varepsilon(\delta; g), \tilde{\varepsilon}(\delta; g) \to 0$ for any fixed $g \in \mathcal{G}_R(M, \nu)$, this implies

$$\max_{g \in \mathcal{G}_R(M,\nu)} \varepsilon_g(\delta), \max_{g \in \mathcal{G}_R(M,\nu)} \tilde{\varepsilon}_g(\delta) \to 0 \quad \text{as } \delta \to 0,$$

due to the compactness of the set $\mathcal{G}_R(M, \nu) \subset C(\overline{B}(0, R))$. $\square$

Let Ω_Γ^δ be some bounded simply connected discrete domain. Recall that the discrete Green's function $G_{\Omega_\Gamma^\delta}(\cdot\,; v^\delta)$, $v^\delta \in \operatorname{Int}\Omega_\Gamma^\delta$ can be written as

$$G_{\Omega_\Gamma^\delta}\bigl(\cdot\,; v^\delta\bigr) = G_\Gamma\bigl(\cdot\,; v^\delta\bigr) - G^*_{\Omega_\Gamma^\delta}\bigl(\cdot\,; v^\delta\bigr),$$

where $G^*_{\Omega_\Gamma^\delta} = G^*_{\Omega_\Gamma^\delta}(\cdot\,; v^\delta) : \Omega_\Gamma^\delta \to \mathbb{R}$ is a solution of the discrete Dirichlet problem

$$\Delta^\delta G^*_{\Omega_\Gamma^\delta} = 0 \quad \text{in } \Omega_\Gamma^\delta, \qquad G^*_{\Omega_\Gamma^\delta} = G_\Gamma \quad \text{on } \partial\Omega_\Gamma^\delta.$$

Theorem 2.5 claims uniform C^1-convergence of the free Green's function G_Γ to its continuous counterpart $G_{\mathbb{C}}(u; v) := \frac{1}{2\pi} \log|u - v|$ with an error $O(\delta^2|u - v|^{-2})$ for the functions and so

$O(\delta|u-v|^{-2})$ for the gradients. Let $G^*_\Omega = G^*_\Omega(\cdot\,;v):\Omega\to\mathbb{R}$ denote a solution of the corresponding continuous Dirichlet problem

$$\Delta G^*_\Omega = 0 \quad \text{in } \Omega, \qquad G^*_\Omega = G_{\mathbb{C}} \quad \text{on } \partial\Omega.$$

Corollary 3.11. *The discrete harmonic functions $G^*_{\Omega^\delta_\Gamma}(\cdot\,;v^\delta)$ are uniformly C^1-close inside Ω^δ (in the sense of Definition* 3.7*) to their continuous counterparts $G^*_{\Omega^\delta}(\cdot\,;v^\delta)$.*

Proof. Let $g(u;v) := \frac{1}{2\pi}\max\{\log|u-v|, \log r\}$. Note that all the functions $g(\cdot\,;v)$ are uniformly bounded and equicontinuous. Let $\Delta^\delta \tilde{G}^*_{\Omega^\delta_\Gamma} = \tilde{G}^*_{\Omega^\delta_\Gamma}(\cdot\,;v^\delta)$ denote a solution of the discrete Dirichlet problem

$$\Delta^\delta \tilde{G}^*_{\Omega^\delta_\Gamma} = 0 \quad \text{in } \Omega^\delta_\Gamma, \qquad \tilde{G}^*_{\Omega^\delta_\Gamma} = g = G_{\mathbb{C}} \quad \text{on } \partial\Omega.$$

Due to Theorem 3.10, the functions $\tilde{G}^*_{\Omega^\delta_\Gamma}$ are uniformly C^1-close to $G^*_{\Omega^\delta}$ inside Ω^δ. On the other hand, since $B(v^\delta, r)\subset\Omega^\delta$, one has

$$\big|G^*_{\Omega^\delta_\Gamma} - \tilde{G}^*_{\Omega^\delta_\Gamma}\big| \leqslant \text{const}\cdot\delta^2/r^2 \quad \text{on } \partial\Omega^\delta_\Gamma.$$

Then, the maximum principle and the discrete Harnack estimate (Corollary 2.9) guarantees that $G^*_{\Omega^\delta_\Gamma}$ are uniformly C^1-close to $\tilde{G}^*_{\Omega^\delta_\Gamma}$ inside Ω^δ. $\square$

Theorem 3.12. *The discrete harmonic measures $\omega^\delta(\cdot\,;b^\delta a^\delta;\Omega^\delta_\Gamma)$ are uniformly C^1-close inside Ω^δ (in the sense of Definition* 3.7*) to their continuous counterparts $\omega(\cdot\,;b^\delta a^\delta;\Omega^\delta)$.*

Proof. By conformal invariance, the continuous harmonic measure is Carathéodory stable, so the second assumption in Proposition 3.8 holds true. Thus, it is sufficient to prove pointwise convergence (3.5) (see also Remark 3.9).

Let $(\Omega^\delta; u^\delta; a^\delta, b^\delta) \xrightarrow{\text{Cara}} (\Omega; u; a, b)$. The functions $0\leqslant\omega^\delta(\cdot\,;b^\delta a^\delta;\Omega^\delta_\Gamma)\leqslant 1$ are uniformly bounded in Ω. Due to Proposition 3.1, one can find a subsequence $\delta_k\to 0$ such that

$$\omega^{\delta_k}\big(\cdot\,;b^{\delta_k}a^{\delta_k};\Omega^{\delta_k}_\Gamma\big) \rightrightarrows H$$

uniformly on compact subsets of Ω, where $H:\Omega\to\mathbb{R}$ is some harmonic function. It is sufficient to prove that $H(u)=\omega(u;ba;\Omega)$ for each subsequential limit.

Let $z^\delta\in\operatorname{Int}\Omega^\delta_\Gamma$. The weak Beurling-type estimate (see Proposition 2.11) gives

$$0\leqslant\omega^\delta\big(z^\delta;b^\delta a^\delta;\Omega^\delta_\Gamma\big)\leqslant\text{const}\cdot\left[\frac{\operatorname{dist}(z^\delta;\partial\Omega^\delta_\Gamma)}{\operatorname{dist}_{\Omega^\delta_\Gamma}(z^\delta;b^\delta a^\delta)}\right]^\beta$$

uniformly as $\delta\to 0$. Passing to the limit as $\delta=\delta_k\to 0$, one obtains

$$0\leqslant H(z)\leqslant\text{const}\cdot\left[\frac{\operatorname{dist}(z;\partial\Omega)}{\operatorname{dist}_\Omega(z;ba)}\right]^\beta \quad \text{for all } z\in\operatorname{Int}\Omega.$$

Therefore, $H \equiv 0$ on the boundary arc $ab \subset P(\Omega)$. Similar arguments give $H \equiv 1$ on the arc $ba \subset P(\Omega)$. Hence, $H = \omega(\,\cdot\,; ba; \Omega)$ and, in particular, $H(u) = \omega(u; ba; \Omega)$. $\square$

Let Ω^δ_Γ be a simply connected discrete domain, $a^\delta \in \partial\Omega^\delta_\Gamma$ and $v^\delta \in \operatorname{Int}\Omega^\delta_\Gamma$. We call

$$P^\delta = P^\delta\big(\,\cdot\,; v^\delta; a^\delta; \Omega^\delta_\Gamma\big) : \Omega^\delta_\Gamma \to \mathbb{R}$$

the **discrete Poisson kernel normalized at v^δ**, if

$$\Delta^\delta P^\delta = 0 \quad \text{in } \Omega^\delta_\Gamma, \qquad P^\delta = 0 \quad \text{on } \partial\Omega^\delta_\Gamma \setminus \{a^\delta\}, \quad \text{and} \quad P^\delta\big(v^\delta\big) = 1.$$

Note that the function P^δ is uniquely defined by these conditions (see (1.3)) and $P^\delta \geqslant 0$.

In the continuous setup, let Ω be a simply connected domain, $a \in P(\Omega)$ be some prime end and $v \in \operatorname{Int}\Omega$. Let $P = P(\,\cdot\,; v; a; \Omega)$ denote a solution of the boundary value problem

$$\Delta P = 0 \quad \text{in } \Omega, \qquad P = 0 \quad \text{on } \partial\Omega \setminus \{a\}, \qquad P \geqslant 0, \quad \text{and} \quad P(v) = 1$$

(note that P is uniquely defined by these conditions for any simply connected domain Ω as the conformal image of the standard Poisson kernel defined in the unit disc $\mathbb{D}$).

Theorem 3.13. *The discrete Poisson kernels $P^\delta(\,\cdot\,; v^\delta; a^\delta; \Omega^\delta_\Gamma)$ are uniformly C^1-close inside Ω^δ (in the sense of Definition* 3.7*) to their continuous counterparts $P(\,\cdot\,; v; a^\delta; \Omega^\delta)$.*

Proof. The continuous Poisson kernel $P(\,\cdot\,; v, a, \Omega)$ is Carathéodory stable due to its conformal invariant definition, so (3.6) holds true. Thus, it is sufficient to prove pointwise convergence (3.5) (see also Remark 3.9).

Let $(\Omega^\delta; u^\delta, v^\delta; a^\delta) \xrightarrow{\text{Cara}} (\Omega; u, v; a)$. Recall that $v^\delta \to v$ and $B(v, r) \subset \Omega^\delta$ for some $r > 0$, if δ is small enough. It follows from $P(v^\delta) = 1$ and the discrete Harnack Lemma (Proposition 2.7(ii)) that P^δ are uniformly bounded on each compact subset of Ω. Then, due to Proposition 3.1, one can find a subsequence $\delta_k \to 0$ such that

$$P^{\delta_k}\big(\,\cdot\,; v^{\delta_k}; a^{\delta_k}; \Omega^{\delta_k}_\Gamma\big) \rightrightarrows H$$

uniformly on compact subsets of Ω, where $H \geqslant 0$ is some harmonic function in Ω. It is sufficient to prove that $H(u) = P(u; v; a; \Omega)$ for each subsequential limit H.

Let $d > 0$ be small enough. Then, there exists a crosscut $\gamma^a_d \subset B(a_d, \frac{1}{2}d)$ in Ω separating a and u, v (see Fig. 4). Moreover, one may assume that $u, v \notin B(a_d, 4d)$ and u, v belong to the same component of $\Omega \setminus \overline{B(a_d, 4d)}$. For sufficiently small δ let

$$L_d \subset \big\{z\colon |z - a_d| = d\big\} \cap \Omega^\delta$$

be an arc separating v^δ and a^δ in Ω^δ (we take the arc closest to v^δ, see Fig. 4). Let Ω^δ_d denote the connected component of $\Omega^\delta \setminus L_d$ containing v^δ. Since $\Omega^\delta \xrightarrow{\text{Cara}} \Omega$ w.r.t. v and $v^\delta \to v$, one has

$$\omega\big(v^\delta; L_d; \Omega^\delta_d\big) \geqslant \text{const}(d) > 0 \tag{3.7}$$

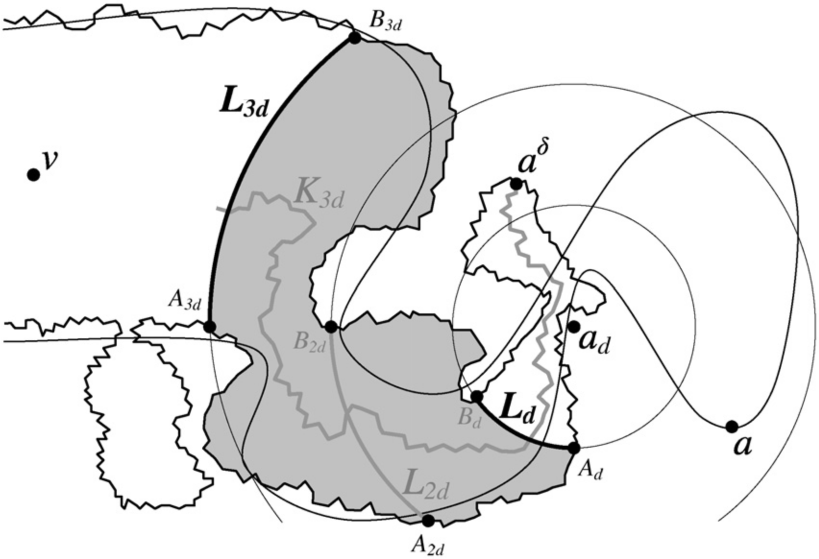


Fig. 4. Parts of the continuous domain Ω and some discrete domain Ω^δ close to Ω in the Carathéodory topology. Marked boundary points $a^\delta \in \partial\Omega^\delta$ and $a \in P(\Omega)$ are close to each other in this topology since the corresponding small cross-cuts near a_d are close. L_d denotes the closest to v arc in $\{z\colon |z-a_d| = d\} \cap \Omega^\delta$ which separates v and a^δ. The quadrilateral R_d^{3d} is shaded.

(here and below constants const(d) do not depend on δ). Similarly, let Ω_{3d}^δ be the connected component of $\Omega^\delta \setminus L_{3d}$ containing u, v. Denote

$$M_{3d}^\delta = \max\{P^\delta(z^\delta),\ z^\delta \in \Omega_{3d}^\delta \cap \Gamma^\delta\}.$$

Since the function P^δ is discrete harmonic, one has

$$M_{3d}^\delta = P^\delta(z_0^\delta) \leqslant P^\delta(z_1^\delta) \leqslant P^\delta(z_2^\delta) \leqslant \cdots$$

for some nearest-neighbor path $K_{3d}^\delta = \{z_0^\delta \sim z_1^\delta \sim z_2^\delta \sim \cdots,\ z_s^\delta \in \Omega_\Gamma^\delta\}$, starting at some $z_0^\delta \in \Omega_{3d}^\delta$. Since $P^\delta|_{\partial\Omega_\Gamma^\delta \setminus \{a^\delta\}} = 0$, the unique possibility for this path to end is a^δ.

Using (3.7), it is not hard to conclude (see Lemma 3.14 below) that the following holds true for the *continuous* harmonic measures:

$$\omega(v^\delta; K_{3d}; \Omega^\delta \setminus K_{3d}) \geqslant \omega(v^\delta; K_{3d} \cap \Omega_d^\delta; \Omega_d^\delta \setminus K_{3d}) \geqslant \text{const} \cdot \omega(v^\delta; L_d; \Omega_d^\delta) \geqslant \text{const}(d),$$

where K_{3d} is the corresponding polyline starting at z_0^δ and ending at a^δ. Applying Theorem 3.12 with $\varepsilon = \frac{1}{2}\text{const}(d)$, one obtains the same inequality

$$\omega^\delta(v^\delta; K_{3d}^\delta; \Omega_\Gamma^\delta \setminus K_{3d}^\delta) \geqslant \text{const}(d) > 0$$

for *discrete* harmonic measures *uniformly* as $\delta \to 0$ (with smaller $\mathrm{const}(d)$). Recall that $P^\delta(v^\delta) = 1$ by definition and $P^\delta(v) \geqslant M^\delta_{3d}$ along the path M^δ_{3d}. Thus,

$$M^\delta_{3d} \leqslant \mathrm{const}(d), \quad \text{if } \delta \text{ is small enough.}$$

Finally, let $z^\delta \in \Gamma^\delta \cap \Omega^\delta_{3d}$ be such that $|z^\delta - a_d| > 3d$. The weak Beurling-type estimate immediately gives

$$P^\delta\big(z^\delta\big) \leqslant \mathrm{const} \cdot \left[\frac{\mathrm{dist}(z^\delta; \partial\Omega^\delta)}{\mathrm{dist}(z^\delta; L_{3d})}\right]^\beta \cdot M^\delta_{3d} \leqslant \mathrm{const}(d) \cdot \left[\frac{\mathrm{dist}(z^\delta; \partial\Omega^\delta)}{|z^\delta - a_d| - 3d}\right]^\beta.$$

Passing to the limit as $\delta = \delta_k \to 0$, one obtains

$$H(z) \leqslant \mathrm{const}(d) \cdot \left[\frac{\mathrm{dist}(z; \partial\Omega)}{|z - a_d| - 3d}\right]^\beta \quad \text{for all } z \in \Omega_{3d} \text{ such that } |z - a_d| > 3d.$$

Thus, $H = 0$ on $\partial\Omega \setminus \{a\}$. Since $H \geqslant 0$ and $H(v) = 1$, this gives $H = P(\,\cdot\,; v; a; \Omega)$. $\square$

Lemma 3.14. *Let $\Omega \subset \mathbb{C}$ be some simply connected domain, $v \in \mathrm{Int}\,\Omega$ and $a \in P(\Omega)$. Let $L_d \subset \{z\colon\ |z - a_d| = d\} \cap \Omega$ be the arc separating v and a that is closest to v, and Ω_d be the connected component of $\Omega \setminus L_d$ containing v. Let K_{3d} be some path connecting L_{3d} and L_d inside the conformal quadrilateral $R^{3d}_d = \Omega_d \setminus \overline{\Omega_{3d}}$ (see Fig. 4). Then*

$$\omega(v; K_{3d}; \Omega_d \setminus K_{3d}) \geqslant \mathrm{const} \cdot \omega(v; L_d; \Omega_d)$$

for some absolute positive constant.

Proof. Note that $\omega(v; L_d; \Omega_d) \leqslant \omega(v; L_d; \Omega_d \setminus K_{3d}) + \omega(v; K_{3d}; \Omega_d \setminus K_{3d})$. Thus, it is sufficient to prove that

$$\omega(v; L_d; \Omega_d \setminus K_{3d}) \leqslant \mathrm{const} \cdot \omega(v; K_{3d}; \Omega_d \setminus K_{3d}).$$

Furthermore, monotonicity arguments give

$$\omega(v; L_d; \Omega_d \setminus K_{3d}) \leqslant \int_{L_{2d}} \omega\big(v; |dz|; \Omega_{2d} \setminus K_{3d}\big) \cdot \omega\big(z; L_d \cup L_{3d}; R^{3d}_d \setminus K_{3d}\big)$$

and, in a similar manner,

$$\omega(v; K_{3d}; \Omega_d \setminus K_{3d}) \geqslant \int_{L_{2d}} \omega\big(v; |dz|; \Omega_{2d} \setminus K_{3d}\big) \cdot \omega\big(z; K_{3d}; R^{3d}_d \setminus K_{3d}\big).$$

Let $L_d = A_d B_d$ and so on (see Fig. 4). Applying monotonicity arguments once more, one sees

$$\omega\big(z; K_{3d}; R^{3d}_d \setminus K_{3d}\big) \geqslant \min\big\{\omega\big(z; A_{3d}A_d; R^{3d}_d\big), \omega\big(z; B_d B_{3d}; R^{3d}_d\big)\big\}.$$

Thus, it is sufficient to prove that

$$\omega\big(z;\, A_dB_d \cup B_{3d}A_{3d};\, R_d^{3d}\big) \leqslant \mathrm{const}\cdot \min\big\{\omega\big(z;\, A_{3d}A_d;\, R_d^{3d}\big), \omega\big(z;\, B_dB_{3d};\, R_d^{3d}\big)\big\}$$

for all $z \in L_{2d}$. Due to the conformal invariance of harmonic measure, the last estimate follows from the uniform bounds on the extremal distances (conformal modulii of quadrilaterals)

$$\lambda_{A_{nd}B_{nd}B_{md}A_{md}}(A_{nd}A_{md};\, B_{md}B_{nd}) \geqslant \frac{1}{2\pi}\log\frac{m}{n}, \quad 1 \leqslant n < m \leqslant 3. \qquad \square$$

3.4. Boundary Harnack principle and normalization on a "straight" part of the boundary

Recall that $\mathbb{H}^\delta$ denotes the polygonal representation of a half-plane $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$ discretization (i.e., the union of all faces, edges and vertices that intersect $\mathbb{H}$, see Fig. 2(B)). As for bounded domains, denote by $\omega^\delta(u^\delta; \{x^\delta\}; \mathbb{H}^\delta_\Gamma)$ the probability of the event that the random walk starting at $u^\delta \in \mathbb{H}^\delta_\Gamma$ first hits the boundary $\partial\mathbb{H}^\delta_\Gamma$ at a vertex $x^\delta \in \partial\mathbb{H}^\delta_\Gamma$. It is easy to see (e.g., using the unboundedness of the free Green's function (2.5) or Proposition 2.11) that

$$\sum_{x^\delta \in \partial\mathbb{H}^\delta_\Gamma} \omega^\delta\big(u^\delta; \big\{x^\delta\big\}; \mathbb{H}^\delta_\Gamma\big) = 1.$$

Let

$$\Im^\delta\big(u^\delta\big) = \operatorname{Im} u^\delta - \sum_{x^\delta \in \partial\mathbb{H}^\delta_\Gamma} \operatorname{Im} x^\delta \cdot \omega^\delta\big(u^\delta; \big\{x^\delta\big\}; \mathbb{C}^\delta_+\big) \quad \text{for } u^\delta \in \mathbb{H}^\delta_\Gamma. \tag{3.8}$$

The function $\Im^\delta$ is discrete harmonic in $\mathbb{H}^\delta_\Gamma$, $\Im^\delta = 0$ on $\partial\mathbb{H}^\delta_\Gamma$ and $|\Im^\delta(u^\delta) - \operatorname{Im} u^\delta| \leqslant 2\delta$ for all $u^\delta \in \mathbb{H}^\delta_\Gamma$ (note that these conditions define $\Im^\delta$ uniquely). In particular, if $\operatorname{Im} u^\delta \in [3\delta, 5\delta]$, then $\Im^\delta(u^\delta) \asymp \delta$ (here and below we write

$$f \asymp g \quad \text{iff} \quad \mathrm{const}_1 \cdot f \leqslant g \leqslant \mathrm{const}_2 \cdot g$$

for some positive absolute constants). Since $\Im^\delta \geqslant 0$ is discrete harmonic, this implies

$$\Im^\delta\big(x^\delta_{\mathrm{int}}\big) \asymp \delta \quad \text{for all } x^\delta = \big(x^\delta; \big(x^\delta_{\mathrm{int}}x^\delta\big)\big) \in \partial\mathbb{H}^\delta_\Gamma.$$

Below we say that a discrete domain Ω^δ has a **"straight" boundary** near $x^\delta \in \partial\Omega^\delta$, if Ω^δ and $\mathbb{H}^\delta$ coincide near x^δ (certainly, it's more natural to include not only $\mathbb{H}^\delta$ itself but all discrete half-planes into the definition but $\mathbb{H}^\delta$ will be sufficient for our purposes).

Definition 3.15. For a function H defined in a domain Ω^δ having a "straight" boundary near x^δ we define the value of its (inner) normal derivative at x^δ as

$$\big[\partial^\delta_n H\big]\big(x^\delta\big) := \frac{H(x^\delta_{\mathrm{int}}) - H(x^\delta)}{\Im^\delta(x^\delta_{\mathrm{int}})}. \tag{3.9}$$

Remark 3.16. In other words, we use the value $\Im^\delta(x^\delta_{\mathrm{int}})$ as a natural normalization constant, so that $[\partial^\delta_n \Im^\delta](x^\delta) = 1$. Note that, if $H(x^\delta) = 0$, then $[\partial^\delta_n H](x^\delta) \asymp \delta^{-1} H(x^\delta_{\mathrm{int}})$.

Below we need some rough estimates for the discrete harmonic measure in rectangles. Let $R(s,t) := (-s;s) \times (0;t) \subset \mathbb{C}$ be an open rectangle, $o^\delta \in \partial \mathbb{H}^\delta_\Gamma$ denote the closest to 0 boundary vertex, $R^\delta_\Gamma = R^\delta_\Gamma(s,t) \subset \Gamma$ be the discretization of $R(s,t)$, and

$$\begin{aligned} L^\delta_\Gamma = L^\delta_\Gamma(s) &:= \{v^\delta \in \partial R^\delta_\Gamma(s,t)\colon\ \operatorname{Im} v^\delta \leqslant 0\}, \\ U^\delta_\Gamma = U^\delta_\Gamma(s,t) &:= \{v^\delta \in \partial R^\delta_\Gamma(s,t)\colon\ \operatorname{Im} v^\delta \geqslant t\}, \\ V^\delta_\Gamma = V^\delta_\Gamma(s,t) &:= \{v^\delta \in \partial R^\delta_\Gamma(s,t)\colon\ |\operatorname{Re} v^\delta| \geqslant s\} \end{aligned}$$

be the lower, upper and vertical parts of the boundary $\partial R^\delta_\Gamma(s,t)$ (see Fig. 2(B)).

Lemma 3.17. *Let $t \geqslant 2\delta$ and $s \geqslant 2t$. Then*

$$\omega^\delta\big(o^\delta_{\mathrm{int}}; U^\delta_\Gamma; R^\delta_\Gamma(s,t)\big) \asymp \delta/t \quad \textit{and} \quad \omega^\delta\big(o^\delta_{\mathrm{int}}; V^\delta_\Gamma; R^\delta_\Gamma(s,t)\big) \leqslant \mathrm{const} \cdot \delta t/s^2.$$

Remark 3.18. The last estimate is very rough but sufficient for us. Standard arguments similar to the proof of Proposition 2.11 easily give an exponential bound.

Proof. We consider two harmonic polynomials

$$h_1(x+iy) = \frac{y+2\delta}{t+2\delta} \quad \text{and} \quad h_2(x+iy) := \frac{y}{t+2\delta} - \frac{x^2 + (y+2\delta)(t+2\delta-y)}{s^2}.$$

Their restrictions on Γ are discrete harmonic due to Lemma 2.2(i), and

$$\begin{aligned} &h_1(x+iy) \geqslant 1 \geqslant h_2(x+iy), \quad \text{if } y \in [t; t+2\delta], \\ &h_1(x+iy) \geqslant 0 \geqslant h_2(x+iy), \quad \text{if } y \in [-2\delta; 0], \\ &h_1(x+iy) \geqslant 0 \geqslant h_2(x+iy), \quad \text{if } y \in [-2\delta, t+2\delta] \text{ and } |x| \in [s; s+2\delta]. \end{aligned}$$

Thus, $h_1(v^\delta) \geqslant \omega^\delta(v^\delta; U^\delta_\Gamma; R^\delta_\Gamma) \geqslant h_2(v^\delta)$ for all $v^\delta \in \partial R^\delta_\Gamma$, and so, by the maximum principle, for all $v^\delta \in R^\delta_\Gamma$. In particular, if $t \geqslant 5\delta$ (the case $t \leqslant 5\delta$ is trivial), then

$$\frac{7\delta}{t+2\delta} \geqslant \omega^\delta\big(v^\delta; U^\delta_\Gamma; R^\delta_\Gamma\big) \geqslant \frac{3\delta}{t+2\delta} - \frac{5\delta t}{s^2} \geqslant \frac{\delta}{t+2\delta} \quad \text{for } v^\delta \in [-2\delta; 2\delta] \times [3\delta; 5\delta],$$

because of $5t(t+2\delta) \leqslant 7t^2 \leqslant 2s^2$. Since ω^δ is discrete harmonic and nonnegative, we obtain $\omega^\delta \asymp \delta/t$ everywhere near o^δ. The upper bound for $\omega^\delta(o^\delta; V^\delta_\Gamma; R^\delta_\Gamma)$ follows by the consideration of the quadratic harmonic polynomial

$$h_3(x+iy) = \frac{x^2 + (y+2\delta)(t+2\delta-y)}{s^2}$$

which is nonnegative on $L^\delta_\Gamma \cup U^\delta_\Gamma$ and not less than 1 on V^δ_Γ. $\square$

Proposition 3.19 *(Boundary Harnack principle). Let $t \geqslant \delta$, H be a nonnegative discrete harmonic function in a discrete rectangle $R^\delta(2t,2t)$, o^δ be the boundary vertex closest to 0, and c^δ denote the inner vertex closest to the point $c = it$. If $H = 0$ everywhere on the lower boundary $L^\delta(2t)$, then the double-side estimate*

$$\left[\partial_n^\delta H\right]\left(o^\delta\right) \asymp \frac{H(c^\delta)}{t}$$

holds true with some constants independent of δ and t.

Proof. Recall that $[\partial_n^\delta H](o^\delta) \asymp \delta^{-1} \cdot H(o_{\mathrm{int}}^\delta)$ (see Remark 3.16). Let $t \geqslant 4\delta$ (the case $t \leqslant 4\delta$ is trivial). It follows from discrete Harnack Lemma (Proposition 2.7(ii)) that the values of H on $U_\Gamma^\delta(t, \frac{1}{2}t)$ are uniformly comparable with $H(c^\delta)$. Then,

$$H\left(o_{\mathrm{int}}^\delta\right) \geqslant \omega^\delta\left(o_{\mathrm{int}}^\delta; U_\Gamma^\delta; R_\Gamma^\delta\left(t, \frac{1}{2}t\right)\right) \cdot \mathrm{const} \cdot H\left(c^\delta\right) \geqslant \mathrm{const} \cdot \delta/t \cdot H\left(c^\delta\right).$$

Further, note that $M := \max_{v \in R_\Gamma^\delta(t, \frac{1}{2}t)} H(v) \leqslant \mathrm{const} \cdot H(c^\delta)$. Indeed, by the maximum principle, $H \geqslant M$ holds true along some nearest-neighbor path K running from $\partial R^\delta(t, \frac{1}{2}t)$ to $U^\delta(2t,2t)$ or $V^\delta(2t,2t)$ (this path cannot end on $L^\delta(2t)$ since $H = 0$ there). Arguing as in the proof of Proposition 2.11, it is easy to see that the probability that the random walk starting at c^δ hits K before $\partial R^\delta(2t,2t)$ is bounded below by some absolute constant, so $H(c^\delta) \geqslant \mathrm{const} \cdot M$. Then, Lemma 3.17 gives

$$H\left(o_{\mathrm{int}}^\delta\right) \leqslant \omega^\delta\left(o_{\mathrm{int}}^\delta; U_\Gamma^\delta \cup V_\Gamma^\delta; R_\Gamma^\delta\left(t, \frac{1}{2}t\right)\right) \cdot M \leqslant \mathrm{const} \cdot \delta/t \cdot H\left(c^\delta\right). \qquad \square$$

From now on, we consider only discrete domains $(\Omega_\Gamma^\delta; a^\delta)$ such that

$$R_\Gamma^\delta(T,T) \subset \Omega_\Gamma^\delta, \qquad L_\Gamma^\delta(T) \subset \partial\Omega_\Gamma^\delta, \quad \text{and} \quad a^\delta \in \partial\Omega_\Gamma^\delta \setminus L_\Gamma^\delta(T) \tag{3.10}$$

for some $T > 0$. Note that all continuous domains $(\Omega; a)$ appearing as Carathéodory limits of these $(\Omega^\delta; a^\delta)$ satisfy

$$(-T;T) \times (0;T) \subset \Omega, \qquad [-T;T] \subset \partial\Omega \quad \text{and} \quad a \in \partial\Omega \setminus (-T;T). \tag{3.11}$$

For a domain $(\Omega; a)$ satisfying (3.11), we define the continuous Poisson kernel $P_0 = P_0(\,\cdot\,; a; \Omega)$ normalized at 0 as the unique solution of the boundary value problem

$$\Delta P_0 = 0 \quad \text{in } \Omega, \qquad P_0 = 0 \quad \text{on } \partial\Omega \setminus \{a\}, \qquad P_0 \geqslant 0, \quad \text{and} \quad [\partial_n P_0](0) = 1,$$

where $[\partial_n P_0](0) = [\partial_y P_0](0)$ denotes the (inner) normal derivative of P_0 at 0.

For a discrete domain $(\Omega_\Gamma^\delta; a^\delta)$ satisfying (3.10), we call $P_{o^\delta}^\delta = P_{o^\delta}^\delta(\,\cdot\,; a^\delta; \Omega_\Gamma^\delta)$ the **discrete Poisson kernel normalized at o^δ**, if

$$\Delta^\delta P_{o^\delta}^\delta = 0 \quad \text{in } \Omega_\Gamma^\delta, \qquad P_{o^\delta}^\delta = 0 \quad \text{on } \partial\Omega_\Gamma^\delta \setminus \{a\}, \quad \text{and} \quad \left[\partial_n^\delta P_{o^\delta}^\delta\right]\left(o^\delta\right) = 1,$$

where the discrete normal derivative ∂_n^δ is given by (3.9). Note that $P^\delta_{o^\delta}$ is uniquely defined by these conditions, namely

$$P^\delta_{o^\delta}\big(\,\cdot\,;a^\delta;\Omega^\delta_\Gamma\big)=\omega^\delta\big(\,\cdot\,;a^\delta;\Omega^\delta_\Gamma\big)\cdot\frac{\Im^\delta(o^\delta_{\mathrm{int}})}{\omega^\delta(o^\delta_{\mathrm{int}};a^\delta;\Omega^\delta_\Gamma)}.$$

Theorem 3.20. *The discrete Poisson kernels $P^\delta_{o^\delta}(\,\cdot\,;a^\delta;\Omega^\delta_\Gamma)$ defined for the class of discrete domains satisfying* (3.10) *with some $T>0$ are uniformly C^1-close inside Ω^δ (in the sense of Definition* 3.7*) to the continuous Poisson kernels $P_0(\,\cdot\,;a^\delta;\tilde\Omega^\delta)$, where $\tilde\Omega^\delta$ denotes the modified polygonal representation of the discrete domain Ω^δ_Γ with the "straight" part of the boundary $L^\delta(T)\subset\partial\Omega^\delta$ replaced by the straight segment $[-T,T]$. The rate of the uniform convergence may depend on T.*

Proof. The continuous Poisson kernel P_0 is Carathéodory stable, so (3.6) holds true. Thus, it is sufficient to check (3.5).

Let $(\Omega^\delta;u^\delta;a^\delta)\xrightarrow{\text{Cara}}(\Omega;u;a)$ and c^δ denote the vertex closest to the point $\frac12 iT$. Due to the boundary Harnack principle (Proposition 3.19), the values $P^\delta_{o^\delta}(c^\delta)$ are uniformly bounded by some constant (depending on T). Hence, $P^\delta_{o^\delta}(\,\cdot\,;a^\delta;\Omega^\delta_\Gamma)$ are uniformly (w.r.t. δ) bounded on each compact subset of Ω because of the discrete Harnack Lemma (Proposition 2.7). Then, due to Proposition 3.1, one can take a subsequence $\delta_k\to0$ so that

$$P^{\delta_k}_{o^\delta}\big(\,\cdot\,;a^{\delta_k};\Omega^{\delta_k}_\Gamma\big)\rightrightarrows H$$

uniformly on compact subsets of Ω, where $H\geqslant0$ is some harmonic in Ω function. We need to prove that $H(u)=P_0(u;a;\Omega)$ for each subsequential limit H.

Repeating the arguments given in the proof of Theorem 3.13, one obtains that, first, for each $r>0$ the functions $P^\delta_{o^\delta}$ are uniformly bounded *everywhere* in Ω^δ_Γ away from a^δ (in particular, everywhere in the smaller rectangle $R^\delta_\Gamma(\frac12T,\frac12T)$) and, second, $H=0$ on $\partial\Omega\setminus\{a\}$. Therefore, due to $H\geqslant0$,

$$H=\mu P_0(\,\cdot\,;a;\Omega)\quad\text{for some }\mu\geqslant0.$$

Now one needs to prove that $\mu=1$. Let

$$Q^\delta(\,\cdot\,):=P^\delta_{o^\delta}\big(\,\cdot\,;a^\delta;\Omega^\delta_\Gamma\big)-\Im^\delta(\,\cdot\,).$$

By definition, the function Q^δ is discrete harmonic in $R^\delta_\Gamma(T,T)$, $Q^\delta=0$ on the lower boundary $L^\delta_\Gamma(T)$, $Q^\delta(o^\delta_{\mathrm{int}})=0$ and

$$Q^\delta(v)\rightrightarrows\mu P_0(v;a;\Omega)-\operatorname{Im}v\quad\text{as }\delta=\delta_k\to0$$

uniformly on compact subsets of $R(T,T)$. Since $P_0|_{(-T,T)}=0$ and $[\partial_nP_0](0)=1$, one has

$$P_0(x+iy)=y+O\big(xy+y^2\big)\quad\text{for }x+iy\in\left[-\frac12T;\frac12T\right]\times\left[0;\frac12T\right].$$

Thus, for any fixed $T \gg s \gg t > 0$, the following hold true:

$$Q^\delta \rightrightarrows (\mu-1)t + O(st) \quad \text{as } \delta \to 0 \text{ uniformly on } U_\Gamma^\delta(t,s),$$
$$Q^\delta = 0 \quad \text{on } L_\Gamma^\delta(s) \quad \text{and} \quad \left|Q^\delta\right| \leqslant \text{const} \quad \text{on } V_\Gamma^\delta(s,t).$$

Then, the normalization $Q^\delta(o_{\text{int}}^\delta) = 0$ and Lemma 3.17 give

$$\left|(\mu-1)t + O(st) + o_{\delta\to 0}(1)\right| \cdot \delta/t \leqslant \text{const} \cdot \delta t/s^2 \quad \text{as } \delta \to 0.$$

So, for any s and t, one has $|\mu - 1| \leqslant \text{const} \cdot (s + t/s^2)$. Setting $t := s^3$ and passing to the limit as $s \to 0$, one arrives at $\mu = 1$. □

Acknowledgments

We would like to thank Vincent Beffara for many helpful comments. Some parts of this paper were written at the MFO, Oberwolfach (during the Oberwolfach–Leibniz Fellowship of the first author), and at the IHÉS, Bures-sur-Yvette. The authors are grateful to the Institutes for the hospitality.

This research was supported by the Swiss N.S.F., by the European Research Council AG CONFRA, by EU RTN CODY and, on the final stage, by the Chebyshev Laboratory (Department of Mathematics and Mechanics, Saint-Petersburg State University) under the grant 11.G34.31.0026 of the Government of the Russian Federation. The first author was partly funded by P. Deligne's 2004 Balzan prize in Mathematics and by the grant MK-7656.2010.1.

Appendix A

A.1. Kenyon's asymptotics for the Green's function and the Cauchy kernel

Below we give a sketch of R. Kenyon's [12] arguments. See also [2].

Proof of Theorem 2.5. Following J. Ferrand [10], R. Kenyon [12] and Ch. Mercat [18], we introduce **discrete exponentials**

$$e(\lambda, u; u_0) := \prod_{j=0}^{2k-1} \frac{1 + \frac{\lambda}{2}(u_k - u_{k-1})}{1 - \frac{\lambda}{2}(u_k - u_{k-1})}, \tag{A.1}$$

where $P_{u_0u} = u_0u_1u_2\ldots u_{2k-1}u_{2k}$ is a path from u_0 to $u_{2k} = u$ on the corresponding rhombic lattice (thus, $u_{2j} \in \Gamma$ and $u_{2j-1} \in \Gamma^*$ for all j). We prefer the parametrization which is closest to the continuous case, so that $e(\lambda, u; u_0) \to \exp[\lambda(u - u_0)]$ as $\delta \to 0$. It's easy to see that this definition does not depend on the choice of the path. Since the angles of rhombi are bounded from 0 and π, one can choose P_{u_0u} so that the following condition holds:

for all j either (a) $|\arg(u_{j+1} - u_j) - \arg(u - u_0)| < \frac{\pi}{2}$ or (b) u_j and u_{j+2} are opposite vertices of some rhombus and $|\arg(u_{j+2} - u_j) - \arg(u - u_0)| < \frac{\pi}{2}$. In particular, $\arg(u_{j+1} - u_j) - \arg(u - u_0) \in (-\pi, \pi)$ for all j.

Define (see R. Kenyon [12] and A. Bobenko, Ch. Mercat and Yu. Suris [1])

$$\tilde{G}_\Gamma(u;u_0)=\frac{1}{8\pi^2 i}\int\limits_C \frac{\log\lambda}{\lambda}\,e(\lambda,u;u_0)\,d\lambda, \tag{A.2}$$

where C is a curve which runs counter clockwise around the disc of (large) radius R from the angle $\arg\overline{(u-u_0)}-\pi$ to $\arg\overline{(u-u_0)}+\pi$, then along the segment $e^{i\arg\overline{(u-u_0)}}[-R,-r]$, then clockwise around the disc of (small) radius r and then back along the same segment (the integral does not depend on the log branch, since $e(0,u;u_0)=e(\infty,u;u_0)=1$).

This function is discrete harmonic away from u_0 since all discrete exponentials are harmonic (as functions of u) and one can use the same contour of integration for all $u_s\sim u$. Furthermore, $\tilde{G}(u_0;u_0)=0$ and, by straightforward computation,

$$\tilde{G}_\Gamma(u_s;u_0)=\frac{\theta_s\cot\theta_s}{\pi},\quad \text{if } u_s\sim u,\quad \text{so}\quad \Delta^\delta\tilde{G}_\Gamma(u_0;u_0)\cdot\mu_\Gamma^\delta(u_0)=1.$$

Rotating and scaling the plane, one may assume that $\arg(u-u_0)=0$ and $\delta=1$. It's easy to see that the contribution of intermediate $\lambda=-t<0$ to the integral in (A.2) is exponentially small. Indeed, in case (a) one has

$$\left|\frac{1-\frac{t}{2}e^{i\beta_j}}{1+\frac{t}{2}e^{i\beta_j}}\right|^2=1-\frac{8t\cos\beta_j}{t^2+4t\cos\beta_j+4}\leqslant\exp\left[-\frac{8t\cos\beta_j}{(t+2)^2}\right],$$

where $\beta_j=\arg(u_{j+1}-u_j)$ and $\cos\beta_j=\operatorname{Re}(u_{j+1}-u_j)>0$. Similarly, in case (b),

$$\left|\frac{(1-\frac{t}{2}e^{i\beta_j})(1-\frac{t}{2}e^{i\beta_{j+1}})}{(1+\frac{t}{2}e^{i\beta_j})(1+\frac{t}{2}e^{i\beta_{j+1}})}\right|^2\leqslant\exp\left[-\frac{8t(\cos\beta_j+\cos\beta_{j+1})}{(t+2)^2}\right],$$

due to $\cos\beta_j+\cos\beta_{j+1}=\operatorname{Re}(u_{j+2}-u_j)>0$. Thus,

$$\left|e(-t,u;u_0)\right|\leqslant\exp\left[-\frac{4t(u-u_0)}{(t+2)^2}\right]$$

and the asymptotics of (A.2) as $|u-u_0|\to\infty$ are determined by the asymptotics of $e(\lambda,u;u_0)$ near 0 and ∞. Some version of the Laplace method (see [12] and [2]) gives

$$\tilde{G}_\Gamma(u;u_0)=\frac{1}{2\pi}\log|u-u_0|+\frac{\gamma_{\text{Euler}}+\log 2}{2\pi}+O\big(|u-u_0|^{-2}\big),$$

where the remainder is of order $|u-u_0|^{-2}$ due to

$$\frac{d^2}{d\lambda^2}\log\big(e(\lambda,u;u_0)\big)\big|_{\lambda=0}=\frac{d^2}{d\lambda^2}\log\big(e(\lambda,u;u_0)\big)\big|_{\lambda=\infty}=0.$$

The uniqueness of $G(\cdot\,;u_0)$ (and $\operatorname{Im}G=0$) easily follows by the Harnack inequality (Corollary 2.9). Indeed, $G:=G_1(\cdot\,;u_0)-G_2(\cdot\,;u_0)$ would be discrete harmonic everywhere on Γ and $\max_{|u|\leqslant R}|H(u)|/R\to 0$ as $R\to\infty$, so $G(u)\equiv G(u_0)=0$. $\square$

Proof of Theorem 2.21. As in Theorem 2.5, $K(\cdot\,;z_0)$ can be explicitly constructed using (modified) discrete exponentials. Similarly to (A.1), denote

$$e\big(\lambda,u_0^{\pm};z_0\big):=\frac{1}{(1-\frac{\lambda}{2}(u_0^{\pm}-w_0^{-}))(1-\frac{\lambda}{2}(u_0^{\pm}-w_0^{+}))} \tag{A.3}$$

for the "black" vertices $u_0^{\pm}\in\Gamma$ of the rhombus $u_0^-w_0^-u_0^+w_0^+$ centered at z_0 and, by induction,

$$\frac{e(\lambda,w;z_0)}{e(\lambda,u;z_0)}:=\frac{1+\frac{\lambda}{2}(w-u)}{1-\frac{\lambda}{2}(w-u)}\quad\text{for all } w\sim u,\ u\in\Gamma,\ w\in\Gamma^*.$$

Then, all $e(\lambda,\,\cdot\,;z_0)$ are well defined and discrete holomorphic on Λ. Let (see [12])

$$K(v;z_0)=\frac{1}{\pi}\int\limits_{-\overline{(v-z_0)}\infty}^{0} e(\lambda,v;z_0)\,d\lambda,$$

where the integral being, say, along the ray $\arg\zeta=\arg\overline{(v-z_0)}\pm\pi$ (taking the path from z_0 to u as in the proof of Theorem 2.5, one guarantees that all poles of $e(\cdot\,,v;z_0)$ are in $|\arg\lambda-\arg(v-z_0)|<\pi$). Then $K(\cdot\,;z_0)$ is holomorphic everywhere except z_0. Straightforward calculations give

$$\mu_{z_0u_0^{\pm}}\cdot K\big(u_0^{\pm};z_0\big)=\frac{4\theta_{z_0}}{\pi},\quad\text{and}\quad \mu_{z_0w_0^{\pm}}\cdot K\big(w_0^{\pm};z_0\big)=\frac{4\theta_{z_0}^*}{\pi}=\frac{4(\frac{1}{2}\pi-\theta_{z_0})}{\pi},$$

so $[\bar\partial^\delta K](z_0;z_0)\cdot\mu_\diamondsuit^\delta(z_0)=1$. Scaling the plane, one may assume that $\delta=1$. As in Theorem 2.5, the integrand is exponentially small for intermediate λ. One has

$$e(\lambda,v;z_0)=\exp\big[\lambda(v-z_0)+O\big(|\lambda|^2|v-z_0|\big)\big],\quad \lambda\to 0,$$

and

$$e(\lambda,v;z_0)=\frac{4\bar\tau^2}{\lambda^2}\cdot\exp\bigg[\frac{4\overline{(v-z_0)}}{\lambda}+O\bigg(\frac{|v-z_0|}{|\lambda|^2}\bigg)\bigg],\quad \lambda\to\infty,$$

where $\tau=e^{i\arg(u_0^+-u_0^-)}$, if $v\in\Gamma$, and $\tau=e^{i\arg(w_0^+-w_0^-)}$, if $v\in\Gamma^*$ ($\bar\tau^2$ comes from the first factors (A.3) of $e(\lambda,v;z_0)$). Summarizing, one arrives at

$$K(u;z_0)=\frac{1}{\pi}\bigg[\frac{1}{v-z_0}+\frac{\bar\tau^2}{\overline{v-z_0}}\bigg]+O\bigg(\frac{1}{|v-z_0|^2}\bigg)=\frac{2}{\pi}\mathrm{Proj}\bigg[\frac{1}{v-z_0};\bar\tau\bigg]+O\bigg(\frac{1}{|v-z_0|^2}\bigg).$$

Finally, $K(\cdot\,,z_0)$ is unique due to Corollary 2.9. □

A.2. Proof of the discrete Harnack Lemma

Below we recall the modification of R.J. Duffin's arguments [7] given by U. Bücking [2]. For the next it is important that the remainder in (2.5) is of order $\delta^2|u-u_0|^{-2}$ (and not just $\delta|u-u_0|^{-1}$).

Proposition A.1. *Let $u_0 \in \Gamma$ and $R \geqslant \delta$. Then*

$$\omega^\delta\big(u_0; \{a\}; B_\Gamma^\delta(u_0, R)\big) \asymp \frac{\delta}{R} \quad \textit{for all } a \in \partial B_\Gamma^\delta(u_0, R),$$

i.e., $\mathrm{const}_1 \cdot \delta/R \leqslant \omega^\delta(u_0; \{a\}; \Omega_\Gamma^\delta) \leqslant \mathrm{const}_2 \cdot \delta/R$ *for some positive absolute constants.*

Proof. One has $R \leqslant |a-u_0| \leqslant R+2\delta$ for all $a \in \partial B_\Gamma^\delta(u_0,R)$. Therefore, (2.5) gives

$$\left| G_{B_\Gamma^\delta(u_0,R)}(u;u_0) - \frac{1}{2\pi}\log\frac{|u-u_0|}{R}\right| \leqslant \frac{\delta}{\pi R} + \mathrm{const}\cdot\left(\frac{\delta^2}{|u-u_0|^2} + \frac{\delta^2}{R^2}\right)$$

for all $u \neq u_0$. In particular, if R/δ is large enough, then

$$\left|G_{B_\Gamma^\delta(u_0,R)}(u;u_0)\right| \asymp \frac{\delta}{R} \quad \text{for all } u \in B_\Gamma^\delta(u_0,R)\text{: } R-5\delta \leqslant |u-u_0| \leqslant R-3\delta.$$

Since Green's function is discrete harmonic and nonpositive near $\partial B_\Gamma^\delta(u_0,R)$, the same holds true for all a_{int}: $(a;a_{\mathrm{int}}) \in \partial B_\Gamma^\delta(u_0,R)$. In view of (2.6), this gives the result for sufficiently large R/δ. For small (i.e. comparable to δ) radii R the claim is trivial, since the random walk can reach a starting from u_0 in a finite number of steps. □

Proposition A.2 *(Mean value property). Let $H : B_\Gamma^\delta(u_0,R) \to \mathbb{R}$ be a nonnegative discrete harmonic function. Then*

$$\left| H(u_0) - \frac{1}{\pi R^2}\sum_{u\in \mathrm{Int}\, B_\Gamma^\delta(u_0,R)} H(u)\mu_\Gamma^\delta(u)\right| \leqslant \mathrm{const}\cdot\frac{\delta H(u_0)}{R}.$$

Proof. Let

$$F(u) := G_\Gamma(u;u_0) - \frac{\log R}{2\pi} + \frac{R^2-|u-u_0|^2}{4\pi R^2}, \quad u \in B_\Gamma^\delta = B_\Gamma^\delta(u_0,R).$$

Note that $[\Delta^\delta F](u) = -(\pi R^2)^{-1}$ for all $u \neq u_0$ (see Lemma 2.2(i)). Using (2.5), it is easy to see that $F(u) = O(\delta^2/R^2)$, if $|u-R| \leqslant \mathrm{const}\cdot\delta$. The discrete Green's formula (2.4) applied to H and F gives

$$H(u_0) - \frac{1}{\pi R^2}\sum_{u\in\mathrm{Int}\, B_\Gamma^\delta} H(u)\mu_\Gamma^\delta(u) = \sum_{a\in\partial B_\Gamma^\delta} \tan\theta_{aa_{\mathrm{int}}}\cdot\big[H(a_{\mathrm{int}})F^\pm(a) - H(a)F^\pm(a_{\mathrm{int}})\big],$$

where the functions $F^{\pm} = F \pm \text{const} \cdot \delta^2/R^2$ are positive/negative, respectively, near $\partial B_\Gamma^\delta(u_0, R)$. Using $H \geqslant 0$, one obtains

$$-\text{const} \cdot \frac{\delta^2}{R^2} \sum_{a \in \partial B_\Gamma^\delta} H(a) \leqslant H(u_0) - \frac{1}{\pi R^2} \sum_{u \in \operatorname{Int} B_\Gamma^\delta} H(u)\mu_\Gamma^\delta(u) \leqslant \text{const} \cdot \frac{\delta^2}{R^2} \sum_{a \in \partial B_\Gamma^\delta} H(a_{\text{int}}).$$

Both sums are comparable to $\delta H(u_0)/R$ due to Proposition A.1. $\square$

Proof of Proposition 2.7. (i) Applying the mean value property for $B_\Gamma^\delta(u_0, R)$ and $B_\Gamma^\delta(u_1, R - 2\delta)$ and taking into account that $H(u_1) \leqslant \text{const} \cdot H(u_0)$, one obtains

$$\pi R^2 H(u_0) - \pi (R - 2\delta)^2 H(u_1) = \sum_{u \in B_\Gamma^\delta(u_0, R) \setminus B_\Gamma^\delta(u_1, R-2\delta)} H(u)\mu_\Gamma^\delta(u) + O\big(\delta R H(u_0)\big).$$

Proposition A.1 gives

$$\sum_{u \in B_\Gamma^\delta(u_0, R) \setminus B_\Gamma^\delta(u_1, R-2\delta)} H(u)\mu_\Gamma^\delta(u) \asymp \sum_{u \in B_\Gamma^\delta(u_0, R) \setminus B_\Gamma^\delta(u_1, R-2\delta)} \delta^2 H(u) = O\big(\delta R H(u_0)\big),$$

so $R^2 \cdot (H(u_1) - H(u_0)) = O(\delta R H(u_0))$.

(ii) Let $u_1 = v_0 v_1 v_2 \ldots v_{k-1} v_k = u_2$ be some path connecting u_1 and u_2 inside $B_\Gamma^\delta(u_0, r)$ (one can choose this path so that $k \leqslant \text{const} \cdot \delta^{-1} r$). Since $B_\Gamma^\delta(v_j, R - r) \subset B_\Gamma^\delta(u_0, R)$,

$$\frac{H(u_2)}{H(u_1)} = \prod_{j=0}^{k-1} \frac{H(v_{j+1})}{H(v_j)} \leqslant \left[1 + \text{const} \cdot \frac{\delta}{R - r}\right]^{\text{const} \cdot r/\delta} \leqslant \exp\left[\text{const} \cdot \frac{r}{R - r}\right]. \qquad \square$$

References

[1] A. Bobenko, Ch. Mercat, Yu. Suris, Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function, J. Reine Angew. Math. 583 (2005) 117–161.

[2] U. Bücking, Approximation of conformal mappings by circle patterns, Geom. Dedicata 137 (2008) 163–197.

[3] D. Chelkak, S. Smirnov, Universality and conformal invariance in the Ising model, in: Oberwolfach Report No. 25/2008 (Stochastic Analysis Workshop).

[4] D. Chelkak, S. Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables, Invent. Math., in press; preprint arXiv:0910.2045, 2009.

[5] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, Stud. Math. Appl., vol. 4, North-Holland Publishing Co., Amsterdam, New York, Oxford, 1978.

[6] R. Courant, K. Friedrichs, H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik, Math. Ann. 100 (1928) 32–74.

[7] R.J. Duffin, Discrete potential theory, Duke Math. J. 20 (1953) 233–251.

[8] R.J. Duffin, Potential theory on a rhombic lattice, J. Combin. Theory 5 (1968) 258–272.

[9] I.A. Dynnikov, S.P. Novikov, Geometry of the triangle equation on two-manifolds, Mosc. Math. J. 3 (2) (2003) 419–438.

[10] J. Ferrand, Fonctions préharmoniques et fonctions préholomorphes, Bull. Sci. Math., 2nd Ser. 68 (1944) 152–180.

[11] J.B. Garnett, D.E. Marshall, Harmonic Measure, New Math. Monogr. Ser., vol. 2, Cambridge University Press, New York, 2005.

[12] R. Kenyon, The Laplacian and Dirac operators on critical planar graphs, Invent. Math. 150 (2002) 409–439.

[13] R. Kenyon, J.-M. Schlenker, Rhombic embeddings of planar quad-graphs, Trans. Amer. Math. Soc. 357 (2004) 3443–3458.
[14] G.F. Lawler, O. Schramm, W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, Ann. Probab. 32 (1B) (2004) 939–995.
[15] J. Lelong-Ferrand, Représentation conforme et transformations à intégrale de Dirichlet bornée, Gauthier–Villars, Paris, 1955.
[16] Ch. Mercat, Discrete Riemann surfaces and the Ising model, Comm. Math. Phys. 218 (1) (2001) 177–216.
[17] Ch. Mercat, Discrete polynomials and discrete holomorphic approximation, arXiv:math-ph/0206041.
[18] Ch. Mercat, Discrete Riemann surfaces, in: Handbook of Teichmüller Theory, vol. I, European Mathematical Society (EMS), Zürich, 2007, pp. 541–575.
[19] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Ser. Comprehen. Stud. in Math., vol. 299, Springer-Verlag, Berlin, 1992.
[20] S. Smirnov, Towards conformal invariance of 2D lattice models, in: Proceedings of the International Congress of Mathematicians (ICM), vol. II: Invited Lectures, Madrid, Spain, August 22–30, 2006, European Mathematical Society (EMS), Zürich, 2006, pp. 1421–1451.
[21] S. Smirnov, Conformal invariance in random cluster models. I. Holomorphic spin structures in the Ising model, Ann. of Math. (2) 172 (2010) 101–133.
[22] S. Smirnov, Discrete complex analysis and probability, in: Rajendra Bhatia, et al. (Eds.), Proceedings of the International Congress of Mathematicians (ICM), vol. I: Plenary Lectures, Hyderabad, India, August 19–27, 2010, World Scientific, New Delhi, 2010.