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Harmonic Measure on Fractal Sets

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Abstract. Many problems in complex analysis can be reduced to the evaluation of the *universal spectrum*: the supremum of multifractal spectra of harmonic measures for all planar domains. Its exact value is still unknown, with very few estimates available. We start with a brief survey of related problems and available estimates from above. Then we discuss in more detail estimates from below, describing the search for a fractal domain which attains the maximal possible spectrum.

1. Introduction

It became apparent during the last decade that extremal configurations in many important problems in classical complex analysis exhibit complicated fractal structure. This makes such problems more difficult to approach than similar ones where extremal objects are smooth. A striking example is given by coefficient problem for two standard classes of univalent functions S and Σ.

1.1. Coefficient problems for univalent functions. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disc and $\mathbb{D}_- = \{|z| > 1\}$ be its complement. The classes S and Σ are defined by

$$S = \{\phi(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \ \phi \text{ is univalent on } \mathbb{D}\} ,$$

and

$$\Sigma = \{\phi(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots, \ \phi \text{ is univalent on } \mathbb{D}_-\} .$$

Univalent means analytic and injective, the letters S and Σ stand for German *schlicht*. Here and below we use a_n and b_n to denote the Taylor coefficients of functions from S (or $S_b = S \cap \mathcal{L}^\infty$) and Σ correspondingly. A complete description of all possible coefficient sequences (a_n) and (b_n) is perhaps beyond reach. So one asks what are the maximal possible values of individual coefficients, especially when n tends to infinity. The long history behind this question goes back to works of Koebe and Bieberbach.

Class S. It is easy to see that the Koebe function $k(z) = \sum_{n=1}^\infty nz^n$ is in fact a univalent map from the unit disk to the plane with a half-line $(-\infty, 1/4]$ removed. It was conjectured by Bieberbach [8] in 1916 that this function is extremal in the class S, namely that for any function there one has $|a_n| \le n$.

The Bieberbach conjecture was proved by de Branges [17] in 1985 with the help of Loewner evolution [40, 39] which we discuss below. The asymptotical behavior of $\max|a_n|$ was settled much earlier by Littlewood. In 1925 [35] he showed by an elegant argument that $|a_n| \le en$ for any function $\phi \in S$.

Class Σ. The corresponding problem for class Σ appears more difficult, with even the question of asymptotic behavior still wide open.

Bieberbach [7] showed in 1914 using his area theorem that $|b_n| \le 1/\sqrt{n}$. While it is easy to produce examples of functions belonging to Σ with $|b_n| \gtrsim 1/n$, Littlewood showed in [34] that those are not extremal. Moreover it is unclear how to construct an extremal function.

Not just the problem of finding the sharp upper bound for $|b_n|$, but even determining the correct decay rate is extremely difficult. We define

$$\gamma_\phi \ := \ \limsup_{n\to\infty} \frac{\log b_n}{\log n} + 1 \ ,$$

i.e., γ_ϕ is the smallest number γ such that $|b_n| \lesssim n^{\gamma-1}$. We then define $\gamma = \gamma_\Sigma$ as the supremum of γ_ϕ's over all $\phi \in \Sigma$. To find the value of γ one has to solve two problems: prove a sharp estimate from above and construct a function exhibiting the extremal decay rate of coefficients.

The origins of the difficulties for the class Σ were explained by Carleson and Jones [13] in 1992. Define another constant β_ϕ to be the growth rate of lengths of Green's lines $\Gamma_\delta = \phi\left(\{z : |z| = 1+\delta\}\right)$:

$$\beta_\phi \ := \ \limsup_{\delta\to 0} \frac{\log \operatorname{length}(\Gamma_\delta)}{|\log \delta|} \ ,$$

and let $\beta = \beta_\Sigma$ be the supremum of β_ϕ's over all $\phi \in \Sigma$. Define γ_b, β_b, γ_s, and β_s as the corresponding constants for the classes $S_b = S \cap \mathcal{L}^\infty$ and S.

Theorem 1.1 (Carleson & Jones, 1992)**.** *The following holds:*

$$\gamma = \beta = \gamma_b = \beta_b < \gamma_s = \beta_s = 2 \ .$$

The inequalities $\gamma \le \beta$ for all the three pairs are due to Littlewood [35], who used them in the proof that $|a_n| \le en$. The apparent equality was quite unexpected. Indeed, Littlewood's argument was quite transparent and in one place used seemingly irreversible inequality. For a function $\phi(z) = \sum a_k z^k$ in the class S he wrote

$$\begin{aligned} e \operatorname{length}(\Gamma_{1/n}) &\ge \left(1-\frac{1}{n}\right)^{-n} \int_{|z|=1-1/n} |\phi'(z)||dz| \\ &= \int_{|z|=1-1/n} |z^{1-n}\phi'(z)|d\theta \overset{*}{\ge} \left| \int z^{1-n}\phi'(z)d\theta \right| \\ &= \left| \int z^{1-n} \sum_k k a_k z^{k-1} d\theta \right| = \left| \int \sum_k k a_k z^{k-n} d\theta \right| = 2\pi n |a_n| \ . \end{aligned}$$

Essentially the same argument is valid for the other two classes, and it follows immediately that $\gamma_\phi \le \beta_\phi$. Note that to have an identity, one must attain an approximate equality in the triangle inequality marked by (*). Thus $z^{1-n}\phi'(z)$ should have approximately the same argument around the circle. Carleson and Jones achieved this by a small perturbation of ϕ, while preserving change in β_ϕ and γ_ϕ.

The identity $\gamma = \beta$ explains the nature of extremal maps ϕ: those maximize the length of Green's lines Γ_δ. For class S the boundary $\partial\Omega$ of the image domain $\Omega = \phi(\mathbb{D})$ may be unbounded, so the Green's lines can be long because of large diameter. This is exactly what happens for the extremal Koebe function. For classes Σ and S_b the situation is different: $\partial\Omega$ is compact. So for the length of Green's lines to be large, they must "wiggle" a lot, and $\partial\Omega$ must be of infinite length (even $\dim_{\mathrm{H}} \partial\Omega > 1$ for $\beta > 0$). This difference explains why the problem for class S is much easier than for classes Σ and S_b. So we know that extremal domains for the latter classes should be *fractal* (self-similar), but there is no understanding of their origin or structure.

1.2. Multifractal analysis of harmonic measure. In [42] Makarov put this problem in a proper perspective, utilizing the language of *multifractal analysis*, an intensively developing interdisciplinary subject on the border between mathematics and physics. The concepts were introduced by Mandelbrot in 1971 in [44, 45]. We use the definitions that appeared in 1986 in a seminal physics paper [22] by Halsey, Jensen, Kadanoff, Procaccia, Shraiman who tried to understand and describe scaling laws of physical measures on different fractals of physical nature (strange attractors, stochastic fractals like DLA, etc.). Multifractal analysis studies different multifractal spectra (which quantitatively describe the sets where certain scaling laws apply to the mass concentration), their interrelation, and connections to other properties of the underlying measure.

There are various definitions of spectra, in our context constructions similar to the *grand ensemble* in statistical mechanics lead to the *integral means spectrum* which for a given function $\phi \in \Sigma$ (or the corresponding domain $\phi(\mathbb{D}_-)$) is defined by

$$\beta_\phi(t) \ := \ \limsup_{r\to 1+} \frac{\log \int_0^{2\pi} |\phi'(re^{i\theta})|^t d\theta}{|\log(r-1)|}, \ t \in \mathbb{R} \ .$$

The *universal integral means spectrum* $B(t)$ is defined as the supremum of $\beta_\phi(t)$ for all $\phi \in \Sigma$. Clearly the constant β is equal to $B(1)$.

Let ω be the harmonic measure, i.e., the image under the map ϕ of the normalized length on the unit circle. Another useful function is the *dimension spectrum* which is defined as the dimension of the set of points, where harmonic measure satisfies a certain power law:

$$f(\alpha) \ := \ \dim\Big\{z : \ \omega\left(B(z,\delta)\right) \approx \delta^\alpha \ , \ \delta \to 0\Big\}, \ \alpha \ge \frac{1}{2} \ .$$

Here dim stands for the Hausdorff or Minkowski dimension, leading to possibly different spectra. Of course, in the general situation there will be many points, where measure behaves differently at different scales, so one has to add lim sup's and lim inf's to the definition above – consult [42] for details. The *universal dimension spectrum* $F(\alpha)$ is defined as the supremum of $f(\alpha)$'s over all $\phi \in \Sigma$. Note that by Beurling's theorem the minimal possible power α for simply connected domains is $1/2$, which corresponds to points at the tips of the inward pointing spikes.

The basic question about dimensional structure of harmonic measure on planar domains was resolved by Makarov [41] in 1985 when he showed that dimension of harmonic measure (i.e., minimal Hausdorff dimension of the Borel support) on simply-connected domains is always one, and Jones and Wolff [26] proved that for multiply connected domains it is always at most one. In the language of spectra Makarov's theorem corresponds to the behavior of $F(\alpha)$ near $\alpha = 1$ and $B(t)$ near $t = 0$, see discussion in [42].

Makarov [42] developed in 1999 the general multifractal framework for harmonic measure. Among other things he showed that Hausdorff and Minkowski versions of universal spectra coincide (while they might differ for individual maps), and that universal integral means and dimension spectra are connected by a Legendre transform:

$$\begin{aligned} B(t) - t + 1 &= \sup_{\alpha>0} \left(F(\alpha) - t\right)/\alpha \ , \\ F(\alpha) &= \inf_{t} \left(t + \alpha(B(t) - t + 1)\right) \ . \end{aligned} \tag{1.1}$$

The same holds for spectra of individual maps, provided the corresponding domains are "nice" fractals. Makarov extended Carleson-Jones fractal approximation from $B(1)$ to $B(t)$, see below. He gave a complete characterization of all functions which can occur as spectra: those are precisely all positive convex functions which are majorated by the universal spectrum and satisfy $\beta(t) - t\beta'(t\pm) \geq -1$. In the same paper Makarov described how the universal spectrum is related to many other problems in the geometric function theory. We will mention several connections later.

On the basis of work of Brennan, Carleson, Jones, Makarov and computer experiments Krätzer [30] in 1996 formulated the

Universal spectrum conjecture 1.
$B(t) = t^2/4$ for $|t| < 2$ and $B(t) = |t| - 1$ for $|t| \geq 2$.

which by the work of Makarov is equivalent to

Universal spectrum conjecture 2. *$F(\alpha) = 2 - 1/\alpha$ for $\alpha \geq 1/2$.*

These conjectures are based on several others, discussed below. Unfortunately, besides numerical, there is not much evidence to support them. All known methods to obtain estimates from above seem to be essentially non-sharp. It is unclear at the moment which approach could lead to the sharp estimates from above. So it becomes even more important to search for extremal configurations in the hope that they will help to understand underlying structure and produce estimates from above as well. In this note we give an exposition of available methods.

1.3. Survey of related problems. Before discussing the values of the universal spectra we would like to briefly mention some of the problems which can be reduced to its study. For an extensive discussion, see [42].

The Brennan's conjecture. Brennan [11] conjectured that any conformal map $\psi : \Omega \to \mathbb{D}$ satisfies for all positive ϵ

$$\iint_\Omega |\psi'(z)|^{4-\epsilon} dm(z) < \infty \ ,$$

where m is the planar Lebesgue measure. By considering the inverse map, it is easy to see that this conjecture equivalent to $B(-2) = 1$. See the paper [14] of Carleson and Makarov and the Ph.D. thesis [6] of Bertilsson for reformulations and partial results. For the best known upper bounds for $B(-2)$ see recent papers by Shimorin [53] and Hedenmalm, Shimorin [24].

The Hölder domains conjecture. Let the map ϕ be Hölder continuous: $\phi \in S \cap \text{Höl}(\eta)$. Jones and Makarov proved (see [25] and [42, Th. 4.3]) that the Hausdorff dimension of the boundary of the image domain $\Omega = \phi(\mathbb{D})$ satisfies

$$\dim_{\text{H}} \partial\Omega \le 2 - C\,\eta \ ,$$

for some positive constant C. They conjectured that for small values of η the constant C can be taken arbitrarily close to 1.

It turns out that the universal spectrum conjecture suggests an even stronger statement. Indeed, a corollary of Makarov's theory (see [42, 43] by Makarov and Pommerenke) is that the universal spectrum $B_\eta(t)$ for the class $S \cap \text{Höl}(\eta)$ is equal to

$$\begin{aligned} B(t), &\quad t < t_\eta \ , \\ (1-\eta)(t-t_\eta) + B(t_\eta), &\quad t \ge t_\eta \ , \end{aligned}$$

where t_η is such that the tangent to $B(t)$ at $t = t_\eta$ has a slope $1 - \eta$. On the other hand the maximal possible dimension of $\partial\Omega$ is the root of the equation

$$B_\eta(t) = t - 1 \ .$$

After combining these statements and plugging in $B(t) = t^2/4$, an easy calculation then shows that the universal spectrum conjecture for $t \in [0, 2]$ is equivalent to the *Hölder domains conjecture*, which states that the following estimate holds and is sharp for η-Hölder domains:

$$\dim_{\text{H}} \partial\Omega \le 2 - \eta \ .$$

Multiply connected domains. One can define similar spectra for multiply connected domains. Since the class of domains is larger, they are a priori different (e.g., the integral means spectrum cannot be defined or rather is infinite for multiply connected domains when t is negative). However a combination of results of Binder, Makarov, Smirnov [10] and Binder, Jones [9] proves that they coincide whenever both are finite (i.e., B's for $t \geq 0$ and F's for $\alpha \geq 1/2$).

Value distribution of entire functions. There is yet another constant α studied by Littlewood [36], which is the smallest α such that

$$\sup_{p\in\mathcal{P}_n} \int_{\mathbb{D}} \frac{|p'|}{1+|p|^2} dm \leq \text{const}(\epsilon)\, n^{\alpha+\epsilon}, \quad \forall\, \epsilon > 0 \ ,$$

where $\mathcal{P}_n$ is the collection of all polynomials of degree n. The mentioned results together with Eremenko [20] and Beliaev, Smirnov [4] imply that $\alpha = B(1)$. Since α is more difficult to estimate it greatly improves the previously known estimates $1.11 \cdot 10^{-5} < \alpha < 1/2 - 2^{-264}$ from [1, 33].

The constant α plays role in a seemingly unrelated problem in value distribution of entire functions. Under assumption that $\alpha < 1/2$ (proved only later by Lewis and Wu [33]) Littlewood proved in [36] a surprising theorem: for any entire function f of finite order most roots of $f(z) = w$ for any w lie in a small set. This can be quantified in several ways, one particular implication is that for any entire function f of finite order $\rho > 0$ there is a set E such that for any w for sufficiently large R most roots of $f(z) = w$ inside $\{|z| < R\}$ lie in E while

$$\text{Area}(E \cap \{|z| < R\}) \lesssim R^{2-2\rho(1/2-\alpha)} \ .$$

See [36, 4] for an exact formulation.

Universal spectra for other classes of maps. It was shown by Makarov in [42] that universal spectra for many other classes of univalent maps (e.g., Hölder continuous, with bounds on the dimension of the boundary of the image domain, with k-fold symmetry) can be easily obtained from the universal spectrum $B(t)$ for the class Σ. For example, while the universal spectrum for S_b is the same: $B_b(t) = B(t)$, the universal spectrum $B_s(t)$ for the class S satisfies

$$B_s(t) \ = \ \max\left(B(t), 3t-1\right) \ .$$

In particular, one notices immediately that $\gamma_s = B_s(1) = 2$.

This ideology can be applied to an old problem about coefficients of m-fold symmetric univalent functions:

$$\phi(z) = z + a_{m+1} z^{m+1} + a_{2m+1} z^{2m+1} + \dots \ .$$

Szegö conjectured that $|a_n| = O(n^{-1+2/m})$. This conjecture was proved for $m = 1$ by Littlewood [35, Th. 20], for $m = 2$ by Littlewood and Paley [37], for $m = 3$ and (with a logarithmic correction) for $m = 4$ by Levin [32]. On the other hand, Littlewood [34] proved that the conjecture fails for large m.

Makarov proved [42] that the universal spectrum $B^{[m]}(t)$ for m-fold symmetric functions satisfies

$$B^{[m]}(t) = \max\left\{B(t), \left(1+\frac{2}{m}\right)t - 1\right\} .$$

Particularly the growth rate of coefficients is given by

$$\begin{aligned} &2/m - 1, \quad m \le 2/B(1) ,\\ &B(1) - 1, \quad m \ge 2/B(1) . \end{aligned}$$

This theorem together with Carleson and Jones conjecture suggests that Szegö conjecture holds for $k \le 8$ and fails for $k \ge 9$. The previously known estimates for $B(1)$ show that Szegö conjecture holds for $k = 1, 2, 3, 4$, and fails for $k \ge 12$. Our improved estimate $B(1) > 0.23$ (see Theorem 2.4 below) implies that conjecture is indeed wrong for $k \ge 9$.

1.4. Estimating universal spectra. The known results about universal spectra use variety of approaches to produce estimates from above and below. At present the estimates from above are rather far from being sharp, and it is unclear which methods can possibly give exact results. In the hope to gain understanding we concentrate in the next sections on estimates from below, that is on constructing (fractal) maps with large spectra. There is also hope that eventually the universal spectrum will be evaluated exactly by showing that it is equal to the spectrum of some particular "fractal" map, for which it can be calculated (cf. discussion of fractal approximation below).

Before we pass to fractal examples, we sketch the situation with estimates from above, using $B(1)$ as an example. See also Problems 6.5, 6.7, and 6.8 from the Hayman's problem list [23] and the survey paper [51] and books [49, 50] by Pommerenke.

Conjectural value of $\gamma = \gamma_b = B(1)$ is 1/4, but existing estimates are quite far. The first result in this direction is due to Bieberbach [7] who in 1914 used his area theorem to prove that $\gamma \le 1/2$. Littlewood, Paley, and Levin proved aforementioned estimates on $|a_n|$ for k-fold symmetric functions for $k = 1, 2, 3, 4$. Clunie and Pommerenke in [16] proved that $\gamma \le 1/2 - 1/300$ and $\gamma_b \le 1/2 - \epsilon$ for some $\epsilon > 0$. They used a differential inequality on $\int |\phi'(r\xi)|^\delta$ for a fixed small δ. Carleson and Jones [13] established that $\gamma = \gamma_b$ and used Marcinkiewicz integrals to prove $\gamma < 0.49755$. This estimate was improved by Makarov and Pommerenke [43] to $\gamma < 0.4886$ and then by Grinshpan and Pommerenke [21] to $\gamma < 0.4884$. The best current estimate is due to Hedenmalm and Shimorin [24] who quite recently proved that $B(1) < 0.46$.

2. Searching for extremal fractals

It is clear that extremal domains should be fractal. There are several standard classes of fractals that one can study. For most of them the *fractal approximation* holds. This means that the supremum of spectra over this particular class of fractals is equal to the universal spectrum. These results can help to understand the nature of extremal domains, but it is not clear if one can get any *upper* bound in this way. Another problem is that it is extremely difficult to work with harmonic measure on fractals because the radial behavior of conformal map depends on $\arg z$ in a highly non-regular way. We will argue that solution to this problem might lie in considering random fractals, when averaging over many maps makes behavior of ϕ' statistically the same for all values of $\arg z$. Below we give a short overview of fractals and methods that were used in the search of lower bounds.

2.1. Lacunary series. The first estimate from below is due to Littlewood [34] who disproved for large m the Szegö conjecture about coefficients of m-fold symmetric functions: using lacunary series he constructed an explicit function with $|a_n| > A(m)n^{-1+a/\log m}$ for infinitely many n, where A is a universal constant. Much later Clunie [15] used the same technique for class Σ and constructed a function with $|b_n| > n^{0.002-1}$ for infinitely many n. Similar technique was used by Pommerenke [47, 48], see the discussion below.

The method consisted of writing a specific Taylor series convergent in $\mathbb{D}$ and using argument principle to check that the resulting function is a schlicht map. It turns out that such series describe maps to fractal domains. Since it is much easier to construct analytic functions (rather than univalent ones) it is interesting whether more advanced univalence criteria can be used to obtain interesting examples.

2.2. Geometric snowflakes. Canonical geometric construction, called *snowflake*, was introduced by von Koch [28, 29] as an example of a nowhere differentiable curve. We start with a "building block" – a polygon $P = P_0$. The construction proceeds in the following fashion: to obtain P_{n+1}, a part of each side of P_n is replaced by a scaled copy of P. In the limit a fractal called snowflake is obtained, which we identify with a conformal map of $\mathbb{D}_-$ to its complement. Carleson and Jones proved that to find the value of β it is enough to study snowflakes.

Let $\Sigma_{\text{snowflake}}$ be the class of conformal mappings whose image domain is a snowflake, and set $\beta_{\text{snowflake}} = \sup \beta_\phi$, where the supremum is taken over all snowflakes $\phi \in \Sigma_{\text{snowflake}}$. Then

Theorem 2.1 (Fractal approximation, Carleson & Jones, 1992)**.**

$$\beta_{\text{snowflake}} = \beta \ .$$

Makarov developed their machinery to extend the result to the multifractal spectra. In [42, Th. 5.1] he gives a complete proof in the multiply connected situation (when one works with Cantor sets rather than von Koch snowflakes), and outlines it in the simply connected case. Again, $F_{\text{snowflake}}(\alpha)$ and $B_{\text{snowflake}}(t)$ are defined as suprema of $f_\phi(\alpha)$ and $\beta_\phi(t)$ over $\phi \in \Sigma_{\text{snowflake}}$:

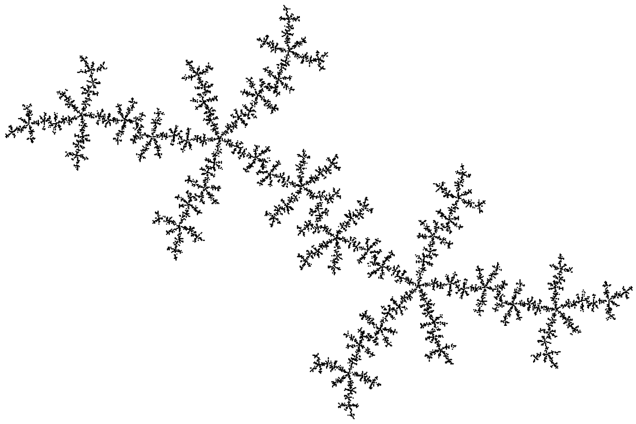

FIGURE 1. Julia set for $z^2 - 0.56 + 0.664i$

Theorem 2.2 (Fractal approximation, Makarov, 1999).

$$F_{\text{snowflake}}(\alpha) = F(\alpha) ,$$
$$B_{\text{snowflake}}(t) = B(t) .$$

Fractal approximation tells us that it is enough to study harmonic measure on snowflakes. Construction of the snowflake is geometric, so it is easy to control dimensions, but estimating harmonic measure is much harder.

2.3. Julia sets. Harmonic measure arises in a natural way for Julia sets of polynomials. If $p(z)$ is a polynomial, we denote by $\mathcal{F}_\infty$ its domain of attraction to infinity, that is the set of z such that iterates $p(p(\dots p(z)\dots))$ tend to infinity. The Julia set of p is then the boundary of $\mathcal{F}_\infty$. It was demonstrated by Brolin [12] that harmonic measure on $\mathcal{F}_\infty$ is balanced (has constant Jacobian under mapping by p) and by Lyubich [38] that it maximizes entropy. Similarly multifractal spectra have dynamical meaning. For example the integral means spectrum is related to the thermodynamical pressure:

$$\beta(t) - t + 1 \ = \ \sup \left\{ I(\mu) - t \int \log p' d\mu \right\} \Big/ \log \deg p \ ,$$

where the supremum is taken over all invariant measures μ and $I(\mu)$ denotes entropy, see [42] and the references therein. This provides more tools to analyze harmonic measure, for example establishing its dimension in this particular case is easier and has more intuitive reasons, than in general case – compare [46] of Manning to Makarov's [41] treatment of the general situation.

Carleson and Jones [13] studied numerically β for domains of attraction to infinity for quadratic polynomials $f(z) = z^2 + c$, and obtained non-rigorous estimate $\beta \approx 0.24$ for $c = -0.560 + 0.6640i$. The Figure 1 shows the corresponding Julia set. Based on this computer experiment and on analogy with conformal field theory they conjectured that $B(1) = 1/4$.

Recently Binder and Jones [9] proved fractal approximation by Julia sets. Together with theorem by Binder, Makarov, and Smirnov [10] it implies that $B(t) = B_{mc}(t),\ t \geq 0$, where B_{mc} is the (a priori larger) universal spectrum for multiply connected domains. It is conjectured by Jones that there is a fractal approximation by quadratic polynomials. If true the universal spectrum will probably be attained by the Mandelbrot set.

Despite this progress, it is still unclear whether one can employ Julia sets to estimate the universal spectra – rigorous dimension estimates are very hard in this class of fractals.

2.4. Conformal snowflakes. We would like to introduce a new class of random conformal snowflakes. This class is interesting because fractal approximation holds, while estimates of the spectra reduce to (much simpler) eigenvalue estimates for integral equations. Also it appears that even simple building blocks lead to snowflakes with rather large spectrum. We start with a deterministic construction, which is related to those used by Littlewood and Pommerenke.

Denote by Σ' the class of univalent maps of $\mathbb{D}_- = \{|z| > 1\}$ into itself, preserving infinity. Fix an integer $k \geq 2$. We define the Koebe k-root transform of $\phi \in \Sigma'$ by $K_k\phi(z) = \sqrt[k]{\phi(z^k)} \in \Sigma'$. The first generation of the snowflake is given by some function $\Phi_0 = \phi \in \Sigma'$. Let $\Phi_n(z) = K_{k^n}\phi(z)$. The nth approximation to the snowflake is given by $f_n = \Phi_0(\Phi_1(\dots\Phi_n(z)\dots))$. We define *conformal snowflake* as the limit $f = \lim f_n$. Let $\psi = \phi^{-1}$ and $g_n = f_n^{-1}$. It is easy to check that

$$f_{n+1}(z) = \phi\left(\sqrt[k]{f_n(z^k)}\right) ,$$

$$g_{n+1}(z) = \sqrt[k]{g_n\left(\psi(z)^k\right)} .$$

Therefore the limit map $g = \lim g_n$ satisfies

$$g(z)^k = g(\psi(z)^k) .$$

So g semi-conjugates dynamical systems $z \mapsto z^k$ and $z \mapsto \psi(z)^k$ on $\mathbb{D}_-$, and the resulting snowflake is a Julia set of ψ^k acting on $\mathbb{D}_-$ (i.e., the attractor of inverse iterates). Because construction is based on iterated conformal maps, harmonic measure is easier to handle than in the case of geometric snowflakes, and even polynomial Julia sets.

It turns out that there is a fractal approximation for conformal snowflakes:

Theorem 2.3 (Fractal approximation)**.** *Let $B_{csf}(t)$ be the universal integral means spectrum for conformal snowflakes, then*

$$B_{csf}(t) = B(t) .$$

The proof is quite similar to the proof of fractal approximation for snowflakes due to Carleson and Jones. We sketch the proof for the case $t = 1$, the complete proof appears in [2]. Let us choose a function ϕ such that it has a long Green's line with potential $1/k$, namely length$\,(\Gamma_{1/k}(\phi)) \approx k^\beta$, with $\beta = B(1)$.

Then for $\Phi_j = \sqrt[k^j]{\phi(z^{k^j})}$ the Green's line with potential $1/k^j$ has length $\approx k^\beta$. One can argue that the length of Green's line for f_n is the product of the lengths of Green's lines for Φ_j's, since those oscillate on different scales:

$$\text{length}\,(\Gamma_{1/k^n}(\Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_n)) \approx \prod_{j=0}^{n} \text{length}\,(\Gamma_{1/k^j}(\Phi_j)) \approx k^{n\beta} \ ,$$

and it follows that the specific snowflake we constructed almost attains the universal β.

As we noted above Pommerenke used a similar construction in [47, 48] to produce maps with large coefficients. Let

$$\phi_k(z) = z \left(\frac{1-\lambda}{1-\lambda z^{mq^k}} \right)^{2/mq^k} \ ,$$

where λ and q are parameters. He studied functions f_k defined recursively by $f_k(z) = f_{k-1}(\phi_k(z))$. Using this construction he first found functions from S_b and Σ with $|a_n|, |b_n| > \text{const}\, n^{0.139-1}$, and then improved the estimate to $|a_n|, |b_n| > \text{const}\, n^{0.17-1}$. Later Kayumov [27] used this technique to prove that $B(t) > t^2/5$ for $0 < t < 2/5$.

2.5. Random conformal snowflakes. Conformal snowflakes are easier to work with than Julia sets or geometric snowflakes. However they share the same problem: behavior of f' depends on symbolic dynamics of the $\arg z$. To solve this problem we introduce a random rotation on every step:

$$g_{n+1}(z) = \sqrt[k]{g_n\left(\psi(e^{i\theta_n} z)^k\right)} \ , \tag{2.1}$$

where θ_n are independent random variables uniformly distributed in $[0, 2\pi[$. Capacity estimates show that there exist a limiting random conformal map $g = g_\infty$, and sending $n \to \infty$ we obtain the stationarity of g under the random transformation (2.1):

$$g(z) = \sqrt[k]{g(\psi(e^{i\theta} z)^k)} \ , \tag{2.2}$$

where θ is uniformly distributed in $[0, 2\pi[$, and equality should be understood in the sense of random maps having the same distribution. Using (2.2) one can write a similar equation for the derivative g', and also integral equations (depending on the building block and k) for the expectations like $\mathbb{E}|g'|^t$. This reduces the determination of the spectrum of a random conformal snowflake to the evaluation of the spectral radius of a particular integral operator (3.3) on the half-line. While its exact value seems beyond reach for the time being, one can obtain decent estimates. As an example, we prove in [2] the following

Theorem 2.4. *There is a particular snowflake with $\beta(1) > 0.23$.*

This snowflake is generated by a simple slit map. Figures 2 and 3 show its third generation and the blow up of its boundary with three Green's lines.

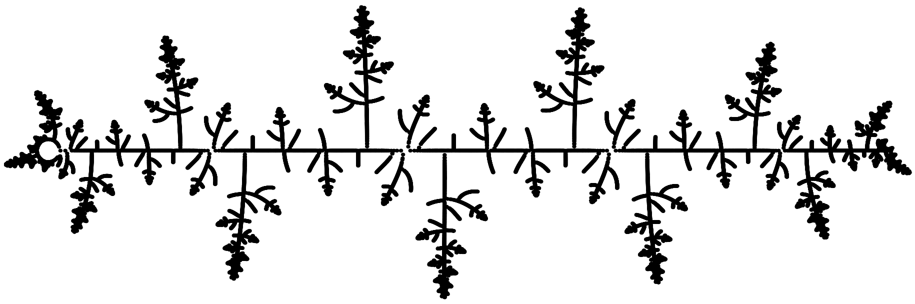

Figure 2. Random conformal snowflake from Theorem 2.4

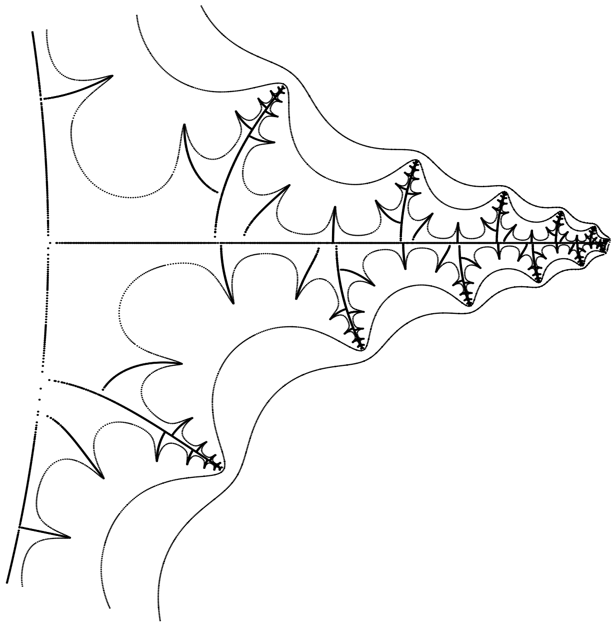

Figure 3. Blow up of the boundary of the random conformal snowflake from Theorem 2.4 with three Green's lines

The general theory of random conformal snowflakes is developed in [2, 3]. In particular the fractal approximation Theorem 2.3 extends to the random conformal snowflakes. Since the building blocks can be taken smooth and relate to the spectra in a simple way, we hope that eventually one might be able to develop some kind of a variational principle, which together with the fractal approximation might yield estimates from above.

The random conformal snowflakes can be considered as Julia sets of random sequences of schlicht maps. One can similarly study the spectra for more traditional Julia sets of random sequences of polynomials. Unfortunately, after some technical difficulties one arrives at integral equations which are rather hard to work with.

2.6. Schramm-Loewner Evolutions. A very interesting class of random "conformal" fractals was recently introduced by Schramm [52]. The whole plane *Schramm-Loewner Evolution* with parameter $\kappa \geq 0$, or SLE_κ, is defined as the solution of the Loewner equation (cf. [40, 39])

$$\partial_\tau g_\tau(z) = -g_\tau(z)\frac{g_\tau(z)+\xi_\tau}{g_\tau(z)-\xi_\tau} \ , \tag{2.3}$$

where the driving force is given by $\xi_\tau = \exp(i\sqrt{\kappa}B_\tau)$ with B_τ being the standard one-dimensional Brownian motion. The initial condition is

$$\lim_{\tau\to-\infty} e^\tau g_\tau(z) = z \ .$$

This equation describes the evolution of random univalent maps g_τ from $\mathbb{C}\setminus H_\tau$ onto $\mathbb{D}_-$. One calls SLE_κ this family of random maps, as well as the family of random hulls H_τ and inverse maps $f_\tau = g_\tau^{-1}$. See Lawler's book [31] for the proof of existence and basic properties.

The traces of the Schramm-Loewner evolutions are the only possible conformally invariant scaling limits of cluster perimeters in critical lattice models. As such the values of their spectra were (non-rigorously) predicted by the physicist Duplantier [18, 19] by means of Conformal Field Theory and Quantum Gravity arguments:

Theorem 2.5 (CFT prediction, Duplantier, 2000)**.** *The $f(\alpha)$ spectrum for the bulk of SLE_κ is equal to*

$$f(\alpha) = \alpha - \frac{(25-c)(\alpha-1)^2}{12(2\alpha-1)} \ ,$$

where c is the central charge which is related to κ by

$$c = \frac{(6-\kappa)(6-16/\kappa)}{4} \ .$$

The prediction should be understood as the "mean" or the "almost sure" value of the spectra.

Below we sketch a rigorous proof of the Duplantier's prediction, given by us in [2, 5]. As in the case of conformal snowflakes, stationarity implies that expressions like $\mathbb{E}|f'(z)|^t$ satisfy certain equations. This time the equation turns out to be a heat equation (3.1) with variable coefficients, and asymptotics of solutions can be evaluated exactly.

The maximal value of such spectra is attained for $\kappa = 4$:

$$f(\alpha) = \frac{3}{2} - \frac{1}{4\alpha-2}, \quad \kappa = 4 \ ,$$

which gives for example $\beta(1) = 3 - 2\sqrt{2} \approx 0.17$. So SLE does not have a large spectrum, but at present it is perhaps the only fractal where the spectra can be written exactly.

In hope of obtaining large spectrum it is natural to generalize SLE, considering other driving forces. In our derivations the Markov property plays essential role, so the first logical choice would be to consider Lévy processes. One can apply the same technique as in the case of SLE and reduce the problem of finding the spectrum to the analysis of a particular integro-differential equation, but at present we do not have good rigorous estimates of its spectral radius. On the other hand, numerical experiments by us and by Kim and Meyer suggest that Loewner Evolution driven by Cauchy process has a large spectrum. In view of Theorems 2.4 and 2.5 there is certainly no fractal approximation by SLE's, but one can argue that a fractal approximation principle could hold in the class of "Lévy-Loewner Evolutions."

3. Estimates of spectra for random fractals

For random fractals it is very natural to study the *mean spectrum*, i.e., behavior of $\mathbb{E}|f'(z)|^t$ instead of $|f'(z)|^t$. When available, correlation estimates can be used to show that the mean spectrum is attained by almost every realization of the fractal. Moreover, one can show using Makarov's fractal approximation theorem that the universal spectrum is greater than the mean spectrum for any class of fractals, so if we are looking for the estimates from below it suffices.

Random models that we mentioned above have some kind of stationarity. This means that $\mathbb{E}|f'(z)|^t$ is invariant with respect to some random transformation which implies that it is a solution of a particular equation. Usually it is much easier to analyze the asymptotic behavior of solutions rather than average local behavior of conformal maps. Below we describe how to apply these ideas in the case of SLE and random conformal snowflakes.

3.1. Exact solutions for SLE. Let $f_\tau : \mathbb{D}_- \to H_\tau$ be the whole plane SLE_κ. Then $e^{-\tau} f_\tau$ has the same distribution as f_0 (see [31] for the proof). One can check that $F(z) = \mathbb{E}\,[e^{-t\tau}|f'_\tau(z)|^t]$ is a t-covariant martingale with respect to the filtration generated by the driving force B_s, $s < \tau$. This implies that $F(z) = F(r, \theta)$ solves the second-order PDE:

$$t\left(\frac{r^4 + 4r^2(1 - r\cos\theta) - 1}{(r^2 - 2r\cos\theta + 1)^2} - 1\right)F + \\ + \frac{r(r^2-1)}{r^2 - 2r\cos\theta + 1}F_r - \frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1}F_\theta + \frac{\kappa}{2}F_{\theta\theta} = 0\ . \tag{3.1}$$

Here the first term is contributed by t-covariance, the second and the third form the derivative in the direction of the Loewner flow (with constant driving force), whereas the forth term is the generator of the driving force – the Brownian motion.

For such an equation it appears possible to analyze exactly the behavior of solutions as $r \to 1+$. Applying formally Frobenius theory one can obtain the local solution near the singular "growth" point $(\theta, r) = (0, 1)$, which, e.g., for $t \le t_* = 3(4+\kappa)^2/(32\kappa)$ has the form

$$(r-1)^{-\beta} \cdot ((r-1)^2 + \theta^2)^\gamma \ , \tag{3.2}$$

for

$$\begin{aligned} \beta \ = \ \beta(t,\kappa) \ &= -t + \frac{(4+\kappa)^2 - (4+\kappa)\sqrt{(4+\kappa)^2 - 8\kappa t}}{4\kappa} \ , \\ \gamma \ = \ \gamma(t,\kappa) \ &= \frac{4+\kappa - \sqrt{(4+\kappa)^2 - 8\kappa t}}{2\kappa} \ . \end{aligned}$$

Tweaking the formula (3.2) one constructs global sub- and super-solutions of the PDE (3.1) which behave as $(r-1)^{-\beta}$ when $r \to 1+$. So by the maximum principle any solution has such asymptotics. So for $t \le t_*$ the mean spectrum $\beta_*(t)$ is equal to $\beta(t)$. It is easy to see that mean spectrum is a convex function bounded by the universal spectrum. The latter is equal to $t-1$ for $t \ge 2$ and since $\beta_*'(t_*-) = 1$, one easily infers that $\beta_*(t) = \beta(t_*) + t - t_*$ for $t > t_*$. The derived spectrum $\beta_*(t)$ is the Legendre transform (1.1) of the Duplantier's prediction for $f(\alpha)$. Details of the proof appear in [2, 5].

Our reasoning applies to the case of Loewner Evolution driven by a Lévy process with generator A. The function $F(z)$ satisfies the same equation (3.1), with the term $\frac{\kappa}{2}F_{\theta\theta}$ substituted by AF. We are not able to perform a rigorous analysis of the resulting equations yet, but this direction of investigations seems rather promising.

3.2. Estimates for snowflakes. Let f be a random conformal snowflake as defined in Section 2.5. Construction of f_n is such that it seems impossible to deduce an equation for $\mathbb{E}|f'|^t$, which seems to be the main obstacle to the exact determination of the corresponding spectra.

We work with the inverse function g instead. The spectrum $\beta(t)$ of the snowflake is roughly speaking the smallest b such that

$$\int_1 (r-1)^{b-1} \int_0^{2\pi} |f'(re^{i\theta})|^t d\theta dr \ < \ \infty \ .$$

In terms of the inverse function g it means that we should study the integrability of $|g'|^{2-t}(|g|-1)^{b-1}$ near $r = 1+$. The latter is comparable to $|g'/g|^{2-t} \log^{b-1}|g|$, for whose expectations we can derive an integral equation. Set

$$F(z) = F(|z|) = \mathbb{E}\left[|g'(z)/g(z)|^{2-t} \log^{b-1}|g(z)|\right] \ ,$$

by the presence of rotation in (2.2) the function F depends on $|z|$ only.

The mean spectrum of a snowflake is the minimal b such that F is integrable near $1+$. Using stationarity of g, namely plugging in instead of g the right-hand side of (2.2), we write

$$F(r) = \mathbb{E}\left[|g'(r)/g(r)|^{2-t}\log^{b-1}|g(r)|\right]$$
$$= \mathbb{E}\left[\left|\frac{g'(\psi(re^{i\theta})^k)\psi'(re^{i\theta})\psi(re^{i\theta})^{k-1}}{g(\psi(re^{i\theta})^k)}\right|^{2-t}\left(\frac{\log|g(\psi(re^{i\theta})^k)|}{k}\right)^{b-1}\right] ,$$

where θ has a uniform distribution in $[0, 2\pi[$. The right-hand side can be rewritten as to separate the expectation with respect to the (independent) distributions of g and θ:

$$\int_0^{2\pi}\mathbb{E}_g\left[\left|\frac{g'(\psi(re^{i\theta})^k)}{g(\psi(re^{i\theta})^k)}\right|^{2-t}\log^{b-1}|g(\psi(re^{i\theta})^k)|\right]\frac{|\psi'(re^{i\theta})\psi(re^{i\theta})^{k-1}|^{2-t}}{k^{b-1}}\frac{d\theta}{2\pi} .$$

By the definition of F the expectation under the integral is equal to $F(\psi(re^{i\theta})^k)$, hence F satisfies the integral equation

$$F(r) = k^{1-b}\int_0^{2\pi}F(\psi(re^{i\theta})^k)\cdot|\psi(re^{i\theta})^{k-1}\psi'(re^{i\theta})|^{2-t}\frac{d\theta}{2\pi} ,$$

and we are searching for the value of b when it ceases to be integrable near $1+$. Thus finding β is reduced to evaluation of the spectral radius in $\mathcal{L}^1$ of the integral operator Q:

$$(Qf)(r) := \int_0^{2\pi}f(|\psi(re^{i\theta})|^k)\cdot|\psi(re^{i\theta})^{k-1}\psi'(re^{i\theta})|^{2-t}\frac{d\theta}{2\pi} . \tag{3.3}$$

It does not seem possible to find the spectral radius exactly in terms of ϕ and k, but one can write good estimates by majoration or approximation. In this way we prove Theorem 2.4 by showing that $\beta(1) > 0.23$ for a snowflake generated by a simple slit map (it maps $\mathbb{D}_-$ onto $\mathbb{D}_-$ with a straight slit of length 73) and $k = 13$, see Figures 2 and 3.

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