# Poincaré Series, Pressure, and Periodic Points

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**Abstract**

We study Poincaré series and pressure as functions of multipliers of repelling periodic orbits for rational functions satisfying the summability condition.

## 1 Introduction

**Non-uniform hyperbolicity.** Global attractors or repellers of unstable systems often display a very complicated fractal structure which in turn is crucial in understanding underlying dynamics. Julia set of rational function with hyperbolic periodic points is a prototype example of global holomorphic repeller. Even in the simplest quadratic case, Julia sets display a striking complexity. One of the main objectives of the paper is to explore fractal parameters of rational Julia sets through thermodynamical formalism. In the theory of iterations, equilibrium measures for conformal potentials can be used to characterize metric properties of invariant sets. This method has its origins in [2, 9] and led further to a systematic study of the so-called conformal (Sullivan-Patterson) measures, [10, 3]. Our technical arguments are based on non-uniformly hyperbolicity of [4].

Suppose that $F$ is a rational function without parabolic periodic points. Denote by Crit the set of all critical points $\{c \in \hat{\mathbb{C}} : F'(c) = 0\}$ of $F$. We say that $F$ satisfies the *summability condition* with an exponent $\alpha$ if for every critical point $c \in \mathrm{Crit} \cap J$, there exists a positive integer $n(c)$ such that,

$$\sum_{n=n(c)}^{\infty} |(F^n)'(c)|^{-\alpha} < \infty$$

and $F$ does not have parabolic cycles.

For simplicity, we assume that there is not any critical point which belongs to another critical orbit. Otherwise all theorems remain valid with the following amendment: a "block" of critical points

$$F : \quad c_1 \mapsto \ldots \mapsto c_2 \mapsto \ldots \quad \ldots \mapsto c_k \ , \tag{1}$$

of multiplicities $\mu_1, \mu_2, \ldots, \mu_k$ enters the statements as if it is a single critical point of multiplicity $\prod \mu_j$.

It is believed that the class of rational functions which satisfy the summability condition with any exponent $\alpha > 0$ is generic in the measure theoretical sense in the space of all rational maps of a given degree, see [8]. Generic systems are usually distinguished by some form of hyperbolicity. A system can become hyperbolic if one considers only small pieces of the phase space and a high iterate of the map on each piece. If it is possible to find such pieces almost everywhere, we say that the system induces hyperbolicity or is non-uniformly hyperbolic with respect to a given measure. This approach originates from the work of Jakobson on the abundance of absolutely continuous invariant measures in unimodal maps as well as Benedicks and Carleson's work on the Hénon attractor. In holomorphic dynamics a similar strategy was followed in [4] to study non-uniform hyperbolicity with respect to the so-called conformal measures. Conformal or *Sullivan-Patterson* measures are dynamical analogues of Hausdorff measures and capture important (hyperbolic) features of the underlying dynamics. If is $F$ a rational map with the Julia set $J$ then a Borel measure $\nu$ supported on $J$ is called *conformal with an exponent* $p$ (or $p$*-conformal*) if for every Borel set $A$ on which $F$ is injective one has

$$\nu(F(B)) = \int_B |F'(z)|^p \, d\nu \; .$$

As observed in [10], the set of pairs $(p, \nu)$ with $p$-conformal measure $\nu$ is compact (in the weak-$*$ topology). Hence, there exists a conformal measure with the *minimal exponent*

$$\delta_{conf} \; := \; \inf\{p : \exists \text{ a } p\text{-conformal measure on } J.\}$$

The minimal exponent $\delta_{conf}$ is also called a *conformal dimension* of $J$.

In [4] it is proven that if

$$\sup_{c \in \mathrm{Crit} \cap J} \sum_{n=n_c}^{\infty} n \cdot |(F^n(c))'|^{-\alpha} \; < \; \infty \; ,$$

holds then a strong version of non-uniform hyperbolicity is true, namely, there exists a unique absolutely continuous invariant measure $\sigma$ with respect to a unique $\delta_{conf}$-conformal measure with $\delta_{conf} = \mathrm{HDim}(J)$. Additionally, $\sigma$ is mixing and has a positive Lyapunov exponent.

**Poincaré series.** Patterson-Sullivan's construction of conformal measures is based on Poincaré series, [10]. This construction was further studied in [4]. In the current paper we will focus on the relations between Poincaré series and fractal geometry of Julia sets.

We call a point $z$ admissible if it does not belong to $\bigcup_{i=0}^{\infty} F^n(\mathrm{Crit})$. Take an admissible point $z$ and assume that $F$ has no elliptic Fatou components and $J \neq \hat{\mathbb{C}}$. We define the Poincaré series by

$$\mathbf{\Sigma}_\delta(z) \; := \; \sum_{n=1}^{\infty} \; \sum_{y \in F^{-n}z} \left|(F^n)'(y)\right|^{-\delta} \; .$$

The series converges for every $\delta > \delta_{Poin}(z)$ and the minimal such $\delta_{Poin}(z)$ is called the Poincaré exponent (of $F$ at the point $z$). By the standard distortion considerations, if $\mathcal{F}$ is a component of the Fatou set, then for all admissible $z \in \mathcal{F}$ Poincaré exponents coincide, so we set $\delta_{Poin}(\mathcal{F}) := \delta_{Poin}(z)$. We define the *Poincaré exponent* by

$$\delta_{Poin}(J) \;:=\; \max \; \{\delta_{Poin}(\mathcal{F})\}\,,$$

Should $J = \hat{\mathbb{C}}$ then by the definition $\delta_{Poin}(J) = 2$. By Theorem 1 of [4]. if a rational function $F$ satisfies the summability condition with an exponent $\alpha < \frac{\delta_{Poin}(J)}{\mu_{max}+\delta_{Poin}(J)}$ then

$$\delta_{Poin}(z) \;= \delta_{Poin}(c) \;=\; \delta_{Poin}(J) \;=\; \inf\left\{\delta_{Poin}(x) : \, x \in \hat{\mathbb{C}}\right\} \,. \tag{2}$$

for every $z$ which is at a positive distance to the critical orbits and every $c \in \mathrm{Crit} \cap J$ of the maximal multiplicity.[1] Moreover, the Poincaré series with the critical exponent $\delta_{Poin}(J)$ diverges for every point $z \in \hat{\mathbb{C}}$.

In the current paper we will prove a statistical version of Theorem 1 of [4].

**Theorem 1** *If a rational function $F$ satisfies the summability condition with an exponent $\alpha < \frac{\delta_{Poin}(J)}{\mu_{max}+\delta_{Poin}(J)}$ then for almost all $z \in \hat{\mathbb{C}}$ in the sense of Lebesgue measure,*

$$\delta_{Poin}(z) \;= \inf\left\{\delta_{Poin}(x) : \, x \in \hat{\mathbb{C}}\right\} \,.$$

Let $\mathrm{HDim}(J)$ and $\mathrm{MDim}(J)$ stand for Hausdorff and Minkowski dimensions of $J$, respectively, and the hyperbolic dimension of a Julia set $J_c$ is defined as

$$\mathrm{HypDim}(J) = \sup_{K \subset J} \mathrm{HDim}(K)\,,$$

where the supremum is taken over all hyperbolic subsets of $J$. In [3] it was proven that $\delta_{conf} = \mathrm{HypDim}(J)$ for every rational map $F$. If a rational function $F$ satisfies the *summability condition* with an exponent $\alpha < \frac{p}{\mu_{max}+p}$ where $p$ is any (e.g., maximal) of the quantities in the formula below, then according to [4],

$$\delta_{conf}(J) = \delta_{Poin}(J) = \delta_{Whit}(J) = \mathrm{MDim}(J) = \mathrm{HDim}(J). \tag{3}$$

Let $\mathcal{P}_n$ be a set of all periodic points of period $n$ in the Julia set $J$. We define the Poincaré series on periodic points as

$$\mathbf{\Sigma}_\delta^{\mathrm{Per}} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathcal{P}_n} \left|(F^n)'(y)\right|^{-\delta}$$

and the corresponding Poincaré exponent as

$$\delta_{Per} \;:=\; \inf\{\delta : \, \mathbf{\Sigma}_\delta^{\mathrm{Per}} \;<\; \infty\}\,.$$

[1]The equality $\delta_{Poin}(J) = \inf\{\delta_{Poin}(x) : \, x \in \hat{\mathbb{C}}\}$ can be regarded as an alternative definition of the Poincaré exponent when $J = \hat{\mathbb{C}}$.

**Theorem 2** *Suppose that $F$ is a rational function which satisfies the summability condition with an exponent $\alpha < \frac{\delta_{Poin}(J)}{\mu_{max}+\delta_{Poin}(J)}$. Then*

$$\inf_{c\in \mathrm{Crit}\cap J} \delta_{Poin}(c) = \delta_{Per}(J) \; .$$

**Pressure function.** Let $\nu$ be an ergodic $F$-invariant measure. Denote by $h_\mu(F)$ the $\nu$-entropy of $F$. The standard pressure function $P(t)$ is defined as

$$P(t) = \sup_{\nu}\{h_\nu(F) - t\int \log|F'|d\nu\} \; ,$$

where the supremum is taken over all $F$-invariant and ergodic measures $\nu$. F. Przytycki proved in [6] (Theorem A2.9.) that $P(t) = \inf\{P(z,t);\; z \in \hat{\mathbb{C}}\}$, where

$$P(z,t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{y\in F^{-n}(z)} |(F^n)'(y)|^{-t} \; .$$

The function $P(z,t)$ is equal to $P(t)$ for $z$ outside of an exceptional set of Hausdorff dimension 0 and the smallest zero of $P(t)$ is equal to $\mathrm{HypDim}(J)$. We recall that $P(\cdot)$ is strictly decreasing and convex function for every $t \in [0, \mathrm{HypDim}(J)]$, see [6].

A pressure function $H(t)$ on periodic points is given by

$$H(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in \mathcal{P}_n} |(F^n)'(z)|^{-t} \; .$$

Let $\mathcal{L}_n(\eta)$ be a set of all periodic points $z \in \mathcal{P}_n$ such that a branch $F^{-n}$ which fixes $z$ is well defined on the ball $B_\eta(z)$. The periodic points from $\mathcal{L}_\eta$ are 'visible' from the scale $\eta > 0$ and the corresponding pressure function $H_\eta(t)$ is defined by

$$H_\eta(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{z\in \mathcal{L}_n(\eta)} |(F^n)'(z)|^{-t} \; .$$

**Theorem 3** *Suppose that $F$ is a rational function which satisfies the summability condition with an exponent $\alpha < \frac{\delta_{Poin}(J)}{\mu_{max}+\delta_{Poin}(J)}$. Then there exists $\eta > 0$ so that*

$$P(t) = H(t) = H_\eta(t)$$

*for every $t \in [0, \mathrm{HypDim}(J)]$.*

Observe that the smallest zero of $P(t)$ is a number at which the Poincaré series starts to diverge exponentially and is by definition smaller and equal to $\delta_{Poin}(J)$. Therefore, the equality of these two numbers is a reflection of some underlying hyperbolicity of the system. The estimate $P(t) \geq H(t)$ for all polynomials was obtained recently in [1]. Further development related to Thereom 3 can be found in [7].

**Conventions.** Many properties will take into account $\mu_{max} = \max_{c \in \mathrm{Crit} \cap J} \mu(c)$ – the maximal multiplicity of critical points in the Julia set (calculated as in 1, if there are any critical orbit relations).

If the Julia set is not the whole sphere, we use the usual Euclidean metric on the plane, changing the coordinates by a Möbius transformation so that $\infty$ belongs to a periodic Fatou component, and doing all the reasoning on a large compact containing the Julia set. Alternatively (and also when $J = \hat{\mathbb{C}}$) one can use the spherical metric. For simplicity and convenience of the reader we will write all the distortion estimates for the planar metric, when Köbe distortion theorem has a more familiar formulation. The estimates remain valid in the case of spherical metric, with an appropriate version of Köbe distortion theorem (which differs only by a multiplicative constant, since we work with the scales smaller than some very small $R$).

Another general convention is following: we call $F^{-n}(z), \ldots, z$ a sequence of preimages of $z$ by $F$ if for every $1 \leq j \leq n$,

$$F(F^{-j})(z) = F^{-j+1}(z) \ .$$

We will write $A \lesssim B$ whenever $A \leq CB$ with some absolute (but depending on the equation) constant $C$. If $A \leq CB$ and $B \leq CA$ then we write $A \asymp B$. We adopt the convention that $\sum_n (\omega_n)^{-\infty} < \infty$ means that the sequence $\omega_n$ tends to zero as $n \to \infty$.

## 2 Almost everywhere convergence of the Poincaré series

**Shrinking neighborhoods.** To control the distortion, we will use the method of shrinking neighborhoods. Suppose that $\sum_{n=1}^{\infty} \delta_n < 1/2$ and $\delta_n > 0$ for every positive integer $n$. Set $\Delta_n := \prod_{k \leq n} (1 - \delta_k)$. Let $B_r$ be a ball of radius $r$ around a point $z$ and $\{F^{-n} z\}$ be a sequence of preimages of $z$. We define $U_n$ and $U_n'$ as the connected components of $F^{-n} B_{r\Delta_n}$ and $F^{-n} B_{r\Delta_{n+1}}$, respectively, which contain $F^{-n} z$. Clearly,

$$F U_{n+1} \ = \ U_n' \ \subset \ U_n \ .$$

If $U_k$, for $1 \leq k \leq n$, do not contain critical points then distortion of $F^n \ : \ U_n' \to B_{r\Delta_{n+1}}$ is bounded (the Köbe distortion lemma) by a power of $\frac{1}{\delta_{n+1}}$, multiplied by an absolute constant.

Since $\sum_n \delta_n \ < \ \frac{1}{2}$, one also has $\prod_n (1 - \delta_n) \ > \ \frac{1}{2}$, and hence always $B_{r/2} \subset B_{r\Delta_n}$.

**Specification of orbits.** We call $F^{-n}(z), \ldots, z$ a sequence of preimages of $z$ by $F$ if for every $1 \leq j \leq n$,

$$F(F^{-j})(z) = F^{-j+1}(z) \ .$$

We will estimate expansion along the backward orbits by decomposing them into blocks of different types. We will introduce three types of orbits.

**Definition 2.1** *Let $R' < R < 1$.*

1. *A sequence $F^{-n}(z), \cdots, F^{-1}(z), z$ of preimages of $z$ is of the* first type *with respect to critical points $c_1$ and $c_2$ if*

(i) *Shrinking neighborhoods* $U_k$ *for* $B_r(z)$, $1 \leq k < n$, *avoid critical points and* $r \leq 2R'$.

(ii) *The critical point* $c_2 \in \partial U_n$,

(iii) *The critical point* $c_1 \in F^{-1}B_R(Fz)$.

2. *Let* $\operatorname{dist}(z, J_F) \leq R'/2$. *A sequence* $F^{-n}(z), \cdots, F^{-1}(z), z$ *of preimages of* $z$ *is of the* second type *if the ball* $B_{R'}(z)$ *can be pulled back univalently along it.*

3. *A sequence* $F^{-n}(z), \cdots, F^{-1}(z), z$ *of preimages of* $z$ *is of the* third type *with respect to the critical point* $c_2$ *if*

(i) *Shrinking neighborhoods* $U_k$ *for* $B_r(z)$, $1 \leq k < n$, *avoid critical points and* $r \leq 2R'$ ,

(ii) *The critical point* $c_2 \in \partial U_n$ .

We say that a backward orbit $y = F^{-n}(z), \ldots, z$ is decomposed into a sequence of *blocks* if there exists an increasing sequence of integers $0 = n_0 < \ldots < n_k = n$ so that for every $i = 0, \ldots, k-1$ the orbit $F^{-n_{i+1}}(z), \ldots, F^{-n_i}(z)$ is of type $1, 2$, or $3$. Given a pair of integers $0 \leq r < l \leq n$, we say that a subsequence $F^{-n_l}(y), \ldots, F^{-n_r}(y)$ yields expansion $M$ if

$$|(F^{n_l - n_{r-1}})'(y)| \geq M \ .$$

**Lemma 2.1** *Assume that a rational function* $F$ *satisfies the summability condition with an exponent* $\alpha \leq 1$ *and set* $\beta = \mu_{max}\alpha/(1-\alpha)$. *There exist* $\epsilon > 0$, *parameters* $R' < R < 1$, *and a sequence* $\gamma_n$, $\sum_{n=1}^{\infty} \gamma_n^{-\beta} < 1/(16 \deg F)^2$ , *with the following properties: if* $z$ *belongs to the* $\epsilon$*-neighborhood of the Julia set* $J$ *and a ball* $B_\Delta(z)$ *can be pulled back univalently by a branch of* $F^{-N}$ *then there exist positive constants* $L' > L, K$ *independent of* $z, \Delta$, *and* $\epsilon$ *such that the sequence* $F^{-N}(z), \ldots, z$ *can be decomposed into blocks of types type* 1, 2, *and* 3, *with the parameters* $R' < R$ *and*

- *every type* 2 *block, except possibly the leftmost one, has the length contained in* $[L, L')$ *and yields expansion* 6,
- *the leftmost type* 2 *block has the length contained in* $[0, L]$ *and yields expansion* $K > 0$,
- *all subsequences of the form* $1 \ldots 13$, *except possibly the rightmost one, yield expansion*

$$\gamma_{k_j} \cdots \gamma_{k_1} \gamma_{k_0} \ ,$$

  $k_i$ *being the lengths of the corresponding blocks,*

- *the rightmost sequence of the form* $1 \ldots 13$ *yields expansion*

$$\begin{array}{ll} \gamma_{k_j} \cdots \gamma_{k_1} \gamma_{k_0} \ \Delta^{(1-\mu(c)/\mu'_{max})} & \text{if a critical point } c \in B_\Delta(z) \ , \\ \gamma_{k_j} \cdots \gamma_{k_1} \gamma_{k_0} \ \Delta^{(1-1/\mu'_{max})} & \text{if otherwise} \ , \end{array}$$

  *where* $\mu'_{max}$ *is the largest multiplicity of critical points of* $F$ *met by shrinking neighborhoods involved in the decomposition procedure of the sequence* $F^{-N}(z), \ldots, z$ *into blocks of types* 1, 2, *and* 3, *with the parameters* $R' < R$.

**Proof:** If we replace in the last estimate $\mu'_{max}$ by $\mu_{max}$ then Lemma 2.1 is verbatim the Main Lemma of [4]. On the other hand it is clear that the critical points of $F$ which are not involved in the construction of blocks of types 1, 2, and 3 do not contribute to the estimates. Therefore, after this change in the formulation of the Main Lemma of [4], the estimates of expansion along the rightmost blocks are sill valid.

□

**Conditional estimates.** Fix a point $z$ and a positive number $\Delta$. Let $\mathcal{H}(z,\Delta)$ stand for a set of all preimages of $z$ such that a ball $B_\Delta(z)$ can be pulled back univalently along the corresponding branch. By Lemma 2.1, every backward orbit of $z$ which terminates at $y \in \mathcal{H}(z,\Delta)$ can be decomposed into blocks of type 1, 2, or 3.

In the decomposition of Lemma 2.1, let $x \in \mathcal{H}(z,\Delta)$ be a point which starts a type 3 block. Denote by $\mathbf{I}(x|z) = \mathbf{I}^\Delta(x|z)$ a set of all $y \in \mathcal{H}(z,\Delta)$ which are the endpoints of type 1 blocks preceded by exactly one type 3 block. For example, preimages of $x$ which are endpoints of blocks 13, 113,... belong to $\mathbf{I}(x|z)$. Note that the definition depends on the choice of $\Delta$.

We will drop $z$ from the notation of $\mathbf{I}(x|z)$ whenever no confusion can arise.

**Lemma 2.2** *Let $\beta = \mu_{max}\alpha/(1-\alpha)$. If a rational function $F$ satisfies the summability condition with an exponent $\alpha \le 1$ then there exist $\epsilon > 0$ and a sequence $\gamma_n$ summable with the power $-\beta = \mu_{max}\frac{\alpha}{1-\alpha}$ so that for every point $z$ from $\epsilon$-neighborhood of the Julia set $J$ and every set $\mathbf{I}(x|z) = \mathbf{I}^\Delta(x|z)$,*

$$\sum_{y\in\mathbf{I}(x|z)} \left|\left(F^{n(y)}\right)'(y)\right|^{-\beta} \;<\; \frac{1}{3}\sum_{k=1}^{\infty} \gamma_k^{-\beta}\Delta_k(x)^{-\beta(1-1/\mu_{max})}$$

*where*

$$\Delta_k(x) = \begin{cases} 1 & \text{if } x \neq z\ , \\ \operatorname{dist}\left(x, F^k(\mathrm{Crit}\cap J)\right) & \text{if } x = z\ . \end{cases}$$

**Proof:** Let $\gamma_n$ be the sequence from the assertion of Lemma 2.1.

Observe that any point $y \in F^{-k}(z)$ has at most $4\deg F$ preimages of a given length which are of the first or the third type. In fact, since pull-backs to the critical values are univalent, there is only one way to hit a specific critical value after particular number of steps, and thus only $\mu(c)$ ways to hit a critical point $c$, but

$$\sum_c \mu(c) \;=\; \#\{c\} + \sum_c(\mu(c)-1) \;\le\; 2(\deg F - 1) + 2(\deg F - 1) \;<\; 4\deg F\ . \tag{4}$$

Therefore, for every sequence $k_0, k_1, \ldots, k_m$ of positive integers there are at most $(2\deg F)^{m+1}$ sequences $1\ldots13$ with the corresponding lengths of the pieces of type 1 and 3. By Lemma 2.1,

$$\left|\left(F^{n(y)}\right)'(y)\right| \ge \gamma_{k_m}\cdots\gamma_{k_1}\gamma_{k_0}\Delta_{k_0}(x)^{(1-1/\mu_{max})}\ .$$

We obtain that

$$\begin{aligned}
\sum_{y\in \mathbf{I}(x)} \left|\left(F^{n(y)}\right)'(y)\right|^{-\beta} &< \sum_{m,k_0,k_1,\ldots,k_m} (4\deg F)^{m+1} \ (\gamma_{k_m}\cdots\gamma_{k_1}\gamma_{k_0})^{-\beta}\ \Delta_{k_0}(x)^{-\beta(1-1/\mu_{max})} \\
&< \sum_{m=1}^{\infty} \left(4\deg F\sum_{k_m}\gamma_{k_m}^{-\beta}\right)\cdot\ldots\cdot\left(4\deg F\sum_{k_1}\gamma_{k_1}^{-\beta}\right) \\
&\quad \cdot \left(4\deg F\sum_{k_0}\gamma_{k_0}^{-\beta}\ \Delta_{k_0}(x)^{-\beta(1-1/\mu_{max})}\right) \\
&< \sum_{m=1}^{\infty}\left(\frac{1}{16\deg F}\right)^m \left(4\deg F\sum_{k_0}\gamma_{k_0}^{-\beta}\ \Delta_{k_0}(x)^{-\beta(1-1/\mu_{max})}\right) \\
&< \frac{1}{3}\sum_{k_0}\gamma_{k_0}^{-\beta}\ \Delta_{k_0}(x)^{-\beta(1-1/\mu_{max})}\ .
\end{aligned}$$

This completes the proof of Lemma 2.2.

□

Let $L' > L$ be the constants supplied by Lemma 2.1. In the decomposition of Lemma 2.1, let $x \in \mathcal{H}(z,\Delta)$ be a point which starts a type 2 block. Denote by $\mathbf{II}_l(x|z)$ and $\mathbf{II}_s(x|z)$ correspondingly the sets of all "long" (of order $L' > n(y) \geq L$) and "short" (of order $n(y) < L$) type 2 preimages $y$ of $x$ obtained in the decomposition of Lemma 2.1. This definition also depends on the choice of $\Delta$, but as Lemma 2.3 shows we will use only estimates independent of $\Delta$, so we simplify the notation by omitting $\Delta$. Lemma 2.3 is Lemma 3.5 of [4].

**Lemma 2.3** *Assume that the Poincaré series with exponent $q$ is summable for some point $v \in \hat{\mathbb{C}}$. Then there exists $\epsilon > 0$ so that for every point $z$ from $\epsilon$-neighborhood of the Julia set $J$ and every set $\mathbf{II}_l(x|z)$ and $\mathbf{II}_s(x|z)$,*

$$\begin{aligned}
\textstyle\sum_{y\in \mathbf{II}_l(x|z)} \left|\left(F^{n(y)}\right)'(y)\right|^{-q} &< \tfrac{1}{36}\quad , \\
\textstyle\sum_{y\in \mathbf{II}_s(x|z)} \left|\left(F^{n(y)}\right)'(y)\right|^{-p} &< C(p) \quad \text{for any } p\ .
\end{aligned}$$

**Conditional Poincaré series.** Recall that $\mathcal{H}(z,\Delta)$ stands for a set of all preimages of $z$ such that a ball $B_\Delta(z)$ can be pulled back univalently along the corresponding branch.

**Proposition 1** *Suppose that a rational function $F$ satisfies the summability condition with an exponent*

$$\alpha < \frac{\delta_{Poin}(J)}{\mu_{max} + \delta_{Poin}(J)}\ .$$

*Then there exists $\epsilon > 0$ so that for every $p > \delta_{Poin}(J)$ and every point $z$ in the $\epsilon$-neighborhood of the Julia set, there exists $C(\epsilon, p)$ so that*

$$\sum_{y \in \mathcal{H}(z,\Delta)} \left|(F^n)'(y)\right|^{-p} \;<\; C \sum_{k=1}^{\infty} \gamma^{-p}\, \Delta_k(z)^{p(\frac{1}{\mu_{max}}-1)} \;.$$

**Proof:** The proof follows closely the proof of the self-improving property of the Poincaré series from [4]. Let $p \in (\delta_{Poin}(J), 2]$ and $c$ be a point of maximal multiplicity. Then, by (2), the Poincaré series $\Sigma_p(c)$ is summable.

We take $v := c$ in the hypotheses of Lemma 2.3. By Lemma 2.3, there exists $\epsilon > 0$ so that for every $z$ from the $\epsilon$-neighborhood of the Julia set of $F$,

$$\sum_{y \in \mathbf{II}_l(x)} \left|\left(F^{n(y)}\right)'(y)\right|^{-p} \;<\; \frac{1}{36} \;.$$

Also by Lemma 2.3

$$\sum_{y \in \mathbf{II}_s(x)} \left|\left(F^{n(y)}\right)'(y)\right|^{-p} \;<\; C \;=\; C(p) \;.$$

We expand $\sum_{y \in \mathcal{H}(z,\Delta)} \left|(F^n)'(y)\right|^{-p}$ by grouping preimages of the same kind into clusters. We begin with $z$ obtaining preimages of three kinds: $\mathbf{I}(z) = \mathbf{I}^{\Delta}(z)$, $\mathbf{II}_l(z)$ and $\mathbf{II}_s(z)$. Points in $\mathbf{II}_s(z)$ are terminal while preimages $y$ of the points in $\mathbf{I}(z)$ and $\mathbf{II}_l(z)$ are divided further. We proceed in this fashion down the tree of preimages of $z$. Using Lemmas 2.2 and 2.3, we obtain that

$$\begin{aligned}
\sum_{y \in \mathcal{H}(z,\Delta)} \left|(F^n)'(y)\right|^{-p} \;&=\; \sum_{z' \in \mathbf{II}_s(z)} \left|\left(F^{n(z')}\right)'(z')\right|^{-p} \;+\; \sum_{z' \in \mathbf{I},\mathbf{II}_l(z)} \left|\left(F^{n(z')}\right)'(z')\right|^{-p} \\
&\cdot\; \left( \sum_{z'' \in \mathbf{II}_s(z')} \left|\left(F^{n(z'')}\right)'(z'')\right|^{-p} \;+\; \sum_{z'' \in \mathbf{I},\mathbf{II}_l(z')} \left|\left(F^{n(z'')}\right)'(z'')\right|^{-p} \right. \\
&\cdot\; \left.\left( \sum_{z''' \in \mathbf{II}_s(z'')} \left|\left(F^{n(z''')}\right)'(z''')\right|^{-p} \;+\; \ldots \;\right)\right) \\
&\leq\; C + \left(\frac{1}{3} + \frac{1}{3}\sum_{k=1}^{\infty} \gamma_k^{-p} \Delta_k(z)^{-p(1-1/\mu_{max})}\right) \left(C + \frac{2}{3}(C + \ldots)\right) \\
&=\; C + \frac{1}{3}\left(1 + \sum_{k=1}^{\infty} \gamma_k^{-p}\, \Delta_k(z)^{-p(1-1/\mu_{max})}\right) C \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right) \\
&<\; 3C \sum_{k=1}^{\infty} \gamma_k^{-p}\, \Delta_k(z)^{-p(1-1/\mu_{max})} \;.
\end{aligned}$$

This proves Proposition 1.

□

**Proof of Theorem 1** Let $\sigma$ be the normalized Lebesgue measure on $\hat{\mathbb{C}}$. Observe that for every integer $k \geq 0$ and every $p \leq 2$,

$$\int \Delta_k(z)^{-p(1-1/\mu_{max})}\, d\sigma \leq \sum_{c \in \mathrm{Crit} \cap J} \int |z - F^k(c)|^{-p(1-1/\mu_{max})}\, d\sigma < \infty .$$

Hence for every $p \in (\delta_{Poin}(J), 2)$,

$$\int \sum_{k=1}^{\infty} \gamma_k^{-p}\, \Delta_k(z)^{p(1/\mu_{max}-1)}\, d\sigma \ \leq\ C \sum_{k=1}^{\infty} \gamma_k^{-p} < C' < \infty$$

and consequently, for almost all $z$ in the sense of Lebesgue measure $\sigma$,

$$\sum_{y \in \mathcal{H}(z,\Delta)} \left| (F^n)'(y) \right|^{-p} < C'$$

is bounded independently from $\Delta$. Passing with $\Delta$ to zero we obtain Theorem 1.

## 2.1 Proof of Theorem 2.

**Construction.** Let $z \in J$ be a periodic point with period $n$ and a sequence

$$F^{-n}(z), \dots F^{-1}(z), z$$

form a chain of preimages, that is $F(F^{-i}(z)) = F^{-i+1}(z)$ for $i = 1, \dots n$. We will decompose the chain into blocks of preimages of the types 2 and $1 \dots 1$. Consider shrinking neighborhoods $\{U_k\}$ for $B_{2R'}(z)$. If they do not contain the critical points we form one block of type 2 of the length $n$. Otherwise, we set $r = 0$ and increase it continuously until certain shrinking neighborhood $U_k$ hits a critical point $c$, $c \in \partial U_k$. It must happen for some $0 < r < 2R'$. We set $n_1 := k$ and $z_1 := F^{-n_1}(z)$. Then $z_1$ is a third type preimage of $z$ and the ball $B_r(z)$ can be pulled back univalently by $F^{n_1}$ along the chain.

**Inductive procedure.** Suppose we have already constructed $z_j = F^{-n_j}(z)$ which is of type 1 or 3. We enlarge the ball $B_r(z_j)$ continuously increasing the radius $r$ from 0 until one of the following conditions is met:

1) for some $k \leq n - n_j$ the shrinking neighborhood $U_k$ for $B_r(z_j)$ hits a critical point $c \in \mathrm{Crit} \cap J$, $c \in \partial U_k$,

2) radius $r$ reaches the value of $2R'$.

In the case 1) we put $n_{j+1} := n_j + k$. Clearly, $z_{j+1} := F^{-n_{j+1}}(z)$ is a type 1 preimage of $z_j$. If 2) holds, we set $z_{j+1} := F^{-n}(z_1)$ which is a type 2 preimage of $z_j$. This terminates the construction in this case.

**Coding.** As a result of the inductive procedure, we can decompose the backward orbit of every point $z$ from a given cycle $\mathcal{C}$ into pieces of type 1, 2 and 3. We ascribe to the cycle a code 2 if there is a point $z \in \mathcal{C}$ so that its backward orbit $F^{-n}(z), \ldots, z$ consists of one block of type 2. Otherwise, for every $z \in \mathcal{C}$ its backward orbit $F^{-n}(z), \ldots, zx$ must contain a least one block of type 3 or 1. Consider now all critical points which end or begin the blocks of type 3 and 1. Denote $\mu_{max}(\mathcal{C})$ a maximal multiplicity of these critical points. Let $y \in \mathcal{C}$ be a point which begins or ends a type 3 or a type 1 block with the corresponding critical point of multiplicity $\mu_{max}(\mathcal{C})$. We attach to $\mathcal{C}$ a code of the backward orbit $F^{-n}(y), \ldots, y$.

This gives a coding of cycles by sequences of 1's, 2's. By the construction, only the following three types of codings are allowable: $2, 1 \ldots 1, 21 \ldots 1$. We recall that according to our convention, during the inductive procedure we put symbols in the coding from the right to the left.

We attach to every chain of preimages of $z$ the sequence $k_l, \ldots k_0$ of the lengths of the blocks of preimages of a given type in its coding. Again our convention requires that $k_0$ always stands for the length of the rightmost block in the code. Clearly, $k_0 + \cdots + k_l = n$.

**Estimates.** We recall that the last estimate of Lemma 2.1 ($\mu'_{max} =: \mu_{max}(\mathcal{C})$) implies that every sequence of the form $11 \ldots 1$ with the length of the corresponding pieces $k_l, \ldots k_0$ yields expansion

$$\gamma_{k_l} \cdot \cdots \cdot \gamma_{k_0}$$

Let $\nu$ be a $\delta_{conf}$-conformal measure and $\delta \geq \delta_{conf}$. Next, observe that for singleton sequences $\{2\}$ of the length $k \leq n$ we have the following estimate,

$$\sum_{y \in F^{-k}(x)} \frac{1}{|(F^k)'(y)|^\delta} \lesssim \sum_{F^{-k}} \frac{\nu(F^{-k}(B_{R'}(x)))}{\nu(B_{R'}(x))} \leq \frac{1}{\nu(B_{R'}(x))} \leq K_{R'} \; , \tag{5}$$

which is independent from $k$.

Let $\mathcal{A}_n$ be a set of all periodic points of period $n$ in the Julia set with codes of the form $1 \ldots 1$ and $21 \ldots 1$. If $x$ starts a block of type 2 in the code of $F^{-n}(z), \ldots z$ then we define $n_x$ by $F^{n_x}(x) = z$. Hence,

$$\begin{aligned} \sum_{y \in \mathcal{A}_n} \frac{1}{|(F^n)'(y)|^\delta} &\leq \sum_{n_x=1}^{n} \left( \sum_{11\ldots13} \frac{1}{|(F^{n_x})'(x)|^\delta} \right) \left( \sum_{y \in F^{-n+n_x}} \frac{1}{|(F^n)'(y)|^\delta} \right) \\ &\leq K_{R'} \sum_{11\ldots13} (\gamma_{k_l} \cdot \cdots \cdot \gamma_{k_0})^{-\delta} \end{aligned}$$

For every sequence $k_l, \ldots k_0$ of positive integers, there is at most $(2 \deg F)^{l+1}$ different branches involved in the construction of the codings $11 \ldots 1$ with the corresponding lengths of the pieces of type 1 (see the estimate (4)). Every periodic point $z$ defines a unique inverse branch of $F^{-n}$ by the condition that $F^{-n}(z) = z$. Therefore, we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathcal{A}_n} \frac{1}{|(F^n)'(y)|^{\delta}} &\lesssim \sum_{n=1}^{\infty} \sum_{l,k_l+\dots+k_0=n} (4 \deg F)^{l+1} (\gamma_{k_l} \cdot \ldots \cdot \gamma_{k_0})^{-\delta} \\
&< \sum_{l=1}^{\infty} \left( 4 \deg F \sum_{k_l} \gamma_{k_l}^{-\delta} \right) \cdot \ldots \cdot \left( 4 \deg F \sum_{k_0} \gamma_{k_0}^{-\delta} \right) \\
&< \sum_{l=1}^{\infty} \left( \frac{1}{4} \right)^l < \infty \,.
\end{aligned}$$

Let $\mathcal{B}_n$ be a set of all repelling points of period $n$ coded by a sequence 2. By eventually 'onto' property, there exists $m > 0$ so that every disk with center in $J$ and radius $R'$ contains a preimage $F^{-k}(c)$, $k \leq m$, of a critical point $c$ of the maximal multiplicity $\mu_{max}$. Since the derivative of $F$ is bounded on $J$ (if $J = \hat{\mathbb{C}}$ then we work with the spherical metric), we arrive at the following estimate,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathcal{B}_n} |(F^y)'(z)|^{-\delta} &\lesssim \sum_{n=1}^{\infty} \sum_{k=0}^{m} \sum_{y \in F^{-n-k}(c)} |(F^{n+k})'(y)|^{-\delta} \\
&\lesssim m \sum_{n=1}^{\infty} \sum_{y \in F^{-n}(c)} |(F^n)'(y)|^{-\delta} < \infty
\end{aligned}$$

provided $\delta \geq \delta_{Poin}(J) = \delta_{conf}$, see (2) and (3). Since $\mathcal{A}_n \cup \mathcal{B}_n$ is the set of all periodic points of period $n$, we obtain that $\delta_{Per}(J) \leq \delta_{Poin}(J)$.

The reversed inequality follows from the general theory. Indeed, let $X$ be a Cantor repeller contained in $J_c$. Denote by $\delta_{Per}(X)$ the Poincaré exponent of the restriction of $F$ to $X$. For Cantor repellers we have that

$$\delta_{Per}(X) = \delta_{conf}(X) = \mathrm{HDim}(X) \,.$$

Since $\delta_{conf}(J) = \sup_X \delta_{conf}(X)$, for any $\delta < \delta_{conf}$ there exist a Cantor repeller $X \subset J$ and a vicinity $W$ of $X$ so that for any point $z \in W$, the Poincaré series $\mathbf{\Sigma}_{\delta,X}(z) = \infty$. Consequently, for any non-exceptional $z \in \mathbb{C}$ there exists $y \in W$ so that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{y \in \mathcal{C}_n} |(F^y)'(z)|^{-\delta} \gtrsim \mathbf{\Sigma}_{\delta,X}(z) = \infty \,,$$

which completes the proof of Theorem 2.

## 2.2 Proof of Theorem 3.

We start with a remark that $P(t) \leq H(t)$ by Katok's theory, [5, 6]. Indeed, let $X$ be a Cantor repeller contained in $J_c$. Denote by $P_X(t)$ the pressure of the restriction of $F$ to $X$. Then

$$P(t) = \sup_{X \subset J} P_X(t) = \sup_{X \subset J} H_X(t) \leq H(t) \,.$$

**Different types of periodic orbits.** As in the proof of Thereom 3, we divide all periodic points of $F$ of period $n$ into two parts, $\mathcal{A}_n$ and $\mathcal{B}_n$. Every cycle in $\mathcal{A}_n$ has a code of the form $21\ldots1$ or $1\ldots1$ with the corresponding lengths of the blocks of 1's and 2 adding up to $n$. If the code of a cycle $\mathcal{C}$ of period $n$ contains a terminating block of type 2 starting at $y \in \mathcal{C}$ then we will replace it by a block of type 1 preimages starting at $y$ in the decomposition of the chain $F^{-n}(y),\ldots,y$. A new code of $\mathcal{C}$ is of the form $1\ldots1$ with the corresponding lengths of the blocks of 1's adding up to a number in $[n,2n)$. As in the proof of Theorem 2, we observe that for every sequence $k_l,\ldots,k_0$ of positive integers there is at most $(2\deg F)^{l+1}$ different branches involved in the construction of the codes $11\ldots1$ with the corresponding lengths of the pieces associated to a given choice of the critical points. Therefore, denoting $H_n(t) = \sum_{y\in\mathcal{A}_n} |(F^n)'(y)|^{-t}$, we obtain that

$$
\begin{aligned}
H_n(t) &\lesssim \sum_{l,n\le k_l+\cdots+k_0<2n} (4\deg F)^{l+1}(\gamma_{k_l}\cdot\ldots\cdot\gamma_{k_0})^{-t} \\
&< \sum_{l=1}^{2n}\left(4\deg F\sum_{k_l}\gamma_{k_l}^{-t}\right)\cdot\ldots\cdot\left(4\deg F\sum_{k_0}\gamma_{k_0})^{-t}\right) \\
&< \sum_{l=1}^{2n}\left(4\deg F\sum_{i=1}^{n}\gamma_i^{-t}\right)^{l} \\
&\le \sum_{l=1}^{2n}(2n)^{1-\frac{t}{\beta}}\left(\frac{1}{4}\right)^{\frac{lt}{\beta}} \lesssim n^2\,,
\end{aligned}
$$

where the estimate in the last line follows from the Hölder inequality. By eventually 'onto' property, for every non-exceptional $w \in \hat{\mathbb{C}}$, there exists $m > 0$ so that every disk with center in $J$ and radius $R'$ contains a preimage $F^{-k}(w)$, $k \le m$. Since the derivative of $F$ is bounded on $J$, say by $M$, (if $J = \hat{\mathbb{C}}$ then we work with the spherical metric), we have that

$$
\begin{aligned}
\sum_{y\in\mathcal{B}_n}|(F^y)'(y)|^{-t} &\lesssim \sum_{y\in F^{-n-k}(w)}|(F^{n+k})'(y)|^{-t} \\
&\lesssim M^m\sum_{y\in F^{-n-m}(w)}|(F^{n+m})'(y)|^{-t}\,.
\end{aligned}
$$

For every $w \in \hat{\mathbb{C}}$ and $0 \le t < \delta_{conf}$ the pressure $P(t,w)$ is positive, hence

$$
H(t) \le \lim_{n\to\infty}\frac{1}{n}\log\left(n^2 + M^m\sum_{y\in F^{-n-m}(w)}|(F^{n+m})'(y)|^{-t}\right) = P(t,w)
$$

and $H_{R'}(t) = H(t)$. Consequently, $H(t) \le \inf_{w\in\hat{\mathbb{C}}} P(t,w)$ for $t \in [0,\mathrm{HypDim}(J)]$ which completes the proof of Theorem 3.

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