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Harmonic Measure and SLE

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Abstract: In this paper we study the multifractal structure of Schramm's SLE curves. We derive the values of the (average) spectrum of harmonic measure and prove Duplantier's prediction for the multifractal spectrum of SLE curves. The spectrum can also be used to derive estimates of the dimension, Hölder exponent and other geometrical quantities. The SLE curves provide perhaps the only example of sets where the spectrum is non-trivial yet exactly computable.

1. Introduction

The motivation for this paper is twofold: to study multifractal spectrum of the harmonic measure and to better describe the geometry of Schramm's SLE curves (see Sects. 1.1 and 1.2 for brief introductions to the respective subjects). Our main result is the following theorem in which we rigorously compute the average spectrum of harmonic measure on domains bounded by SLE curves (see below for precise definitions).

Theorem 1. *The average integral means spectrum $\bar{\beta}(t)$ of SLE_κ is equal to*

$$\begin{array}{ll} -t+\kappa\dfrac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{4\kappa} & t\le -1-\dfrac{3\kappa}{8}, \\ -t+\dfrac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\,)}{4\kappa} & -1-\dfrac{3\kappa}{8}\le t\le \dfrac{3(4+\kappa)^2}{32\kappa}, \\ t-\dfrac{(4+\kappa)^2}{16\kappa} & t\ge \dfrac{3(4+\kappa)^2}{32\kappa}. \end{array}$$

The average integral means spectrum $\bar{\beta}(t)$ of the bulk of SLE (see definition below) is equal to

$$\begin{aligned} 5 - t + \frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\,)}{4\kappa} \qquad & t \le \frac{3(4+\kappa)^2}{32\kappa}, \\ t - \frac{(4+\kappa)^2}{16\kappa}, \qquad & t \ge \frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

Several results can be easily derived from this theorem: dimension estimates of the boundary of SLE_κ hulls, Hölder continuity of SLE_κ Riemann maps, Hölder continuity of SLE_κ trace, and more. We also would like to point out that SLE_κ seems to be *the only family of models* where the spectrum (even average) of harmonic measure is non-trivial and known explicitly.

1.1. Integral means spectrum. There are several equivalent definitions of harmonic measure that are useful in different contexts. For a domain Ω with a regular boundary we define the harmonic measure with a pole at $z \in \Omega$ as the exit distribution of the standard Brownian motion started at z. Namely, $\omega_z(A) = \mathbb{P}(B^z_\tau \in A)$, where $\tau = \inf\{t : B^z_t \notin \Omega\}$ is the first time the standard two-dimensional Brownian motion started at z leaves Ω.

Alternatively, for a simply connected planar domain the harmonic measure is the image of the normalized length on the unit circle under the Riemann mapping that sends the origin to z.

It is easy to see that harmonic measure depends on z in a smooth (actually harmonic) way, thus the geometric properties do not depend on the choice of the pole. So we fix the pole to be the origin or infinity and eliminate it from notation.

Over the last twenty years it became clear that many extremal problems in the geometric function theory are related to the geometrical properties of harmonic measure and the proper language for these problems is *the multifractal analysis*.

Multifractal analysis operates with different spectra of measures and relations between them. In this paper we study the harmonic measure on simply connected domains, so we give the rigorous definition for this case only.

Let $\Omega = \mathbb{C}\backslash K$, where K is a connected compact set and let ϕ be a Riemann mapping from $\mathbb{D}_-$ (i.e. the complement of the unit disc) onto Ω such that $\phi(\infty) = \infty$. *The integral means spectrum* of ϕ (or Ω) is defined as

$$\beta_\phi(t) = \beta_\Omega(t) = \limsup_{r\to 1+} \frac{\log \int |\phi'(re^{i\theta})|^t d\theta}{-\log(r-1)}.$$

The universal integral means spectrum is defined as

$$B(t) = \sup \beta_\Omega(t),$$

where supremum is taken over all simply connected domains with compact boundary.

On the basis of work of Brennan, Carleson, Clunie, Jones, Makarov, Pommerenke and computer experiments for quadratic Julia sets Kraetzer [17] in 1996 formulated the following universal conjecture:

$$\begin{aligned} B(t) &= t^2/4, \qquad |t| < 2, \\ B(t) &= |t| - 1, \quad |t| \ge 2. \end{aligned}$$

It is known that many other conjectures follow from Kraetzer's conjecture. In particular, Brennan's conjecture [5] about integrability of $|\psi'|$, where ψ is a conformal map from a domain to the unit disc is equivalent to $B(-2) = 1$, while Carleson-Jones conjecture [6] that if $\phi(z) = z + \sum a_n z^n$ is a bounded univalent function in the unit disc then $|a_n| \lesssim n^{-3/4}$ is equivalent to $B(1) = 1/4$.

There are many partial results in both directions: estimates of $B(t)$ from above and below (see surveys [3, 15]). Upper bounds are more difficult and they are still not that far from the trivial bounds like $B(1) \leq 1/2$. Currently the best upper bound is $B(1) \leq 0.46$ [14]. Until recently lower bounds were also quite far from the conjectured value.

The main problem in finding lower bounds is that it is almost impossible to compute the spectrum explicitly for any non-trivial domain. The origin of difficulties is easy to see: only fractal domains have interesting spectrum, but for them the boundary behavior of $|\phi'(re^{i\theta})|^t$ depends on θ in a very non smooth way, making it hard to find the average growth rate.

We claim that in order to overcome these problems one should work with regular random fractals instead of deterministic ones. For random fractals it is natural to study *the average integral means spectrum* which is defined as

$$\bar{\beta}(t) = \limsup_{r \to 1} \frac{\log \int \mathbb{E}\left[|\phi'(re^{i\theta})|^t\right] d\theta}{-\log |r-1|}.$$

The advantage of this approach it that for many random fractals the average boundary behavior of $|\phi'|$ is a very smooth function of θ. Therefore it is sufficient to study average behavior along any particular radius. Regular (random) fractals are invariant under some (random) transformation, making $\mathbb{E}|\phi'|^t$ a solution of a specific equation. Solving this equation one can find the average spectrum.

Note that $\bar{\beta}(t)$ and $\beta(t)$ do not necessarily coincide. It can even happen (and in this paper we consider exactly this case) that $\bar{\beta}(t)$ is not a spectrum of any particular domain. But $\bar{\beta}(t)$ is still bounded by the universal spectrum $B(t)$. If there is a random fractal with $\bar{\beta}(t) > B(t)$, then for each scale $r_n = 1 + 1/2^n$ there is a realization of the random fractal for which the integral mean on the scale r_n is at least $c2^{n(\bar{\beta}(t)-\epsilon)}$, where c is a universal constant. Then by Makarov's fractal approximation [25] we can glue together all these realizations and find a domain which has a large spectrum on all scales.

Another important notion is the *dimension* or *multifractal spectrum* of harmonic measure which can be non-rigorously defined as

$$f(\alpha) = dim\{z : \omega(B(z,r)) \approx r^{\alpha}\}, \quad \alpha \geq 1/2,$$

where $\omega(B(z,r))$ is the harmonic measure of the disc of radius r centered at z. The condition $\alpha \geq 1/2$ is equivalent to Beurling's estimate $\omega(B(z,r)) \leq cr^{1/2}$.

There are several ways to make this definition rigorous, leading to slightly different notions of spectrum. But it is known [25] that the universal spectrum $F(\alpha) = \sup_{\Omega} f(\alpha)$ is the same for all definitions of $f(\alpha)$.

For regular (in some sense) fractals the integral means and dimension spectra are related by a Legendre type transform (for general domains there is only one-side inequality). It is also known [25] that the universal spectra are related by a Legendre type transform:

$$F(\alpha) = \inf_t (t + \alpha(B(t) + 1 - t)),$$

$$B(t) = \sup_{\alpha > 0} \frac{F(\alpha) - t}{\alpha} + t - 1.$$

1.2. Schramm-Loewner Evolution. It is a common belief (and it was proved in a few cases) that planar lattice models at criticality have conformally invariant scaling limits as the mesh of the lattice tends to zero. Schramm [32] introduced a one parametric family of random curves which are called SLE_κ (SLE stands for Stochastic Loewner Evolution or Schramm-Loewner Evolution) that are the only possible limits of cluster perimeters for critical lattice models. It turned out to be also a very useful tool in many related problems.

In this section we give the definition of SLE and the necessary background. The discussion of various versions of SLE and relations between them can be found in Lawler's book [19].

To define SLE we need a classical tool from complex analysis: the Loewner evolution. In general this is a method to describe by an ODE the evolution of the Riemann map from a growing (shrinking) domain to a uniformization domain. In this paper we use the radial Loewner evolution (where uniformization domain is the complement of the unit disc) and its modifications.

Definition 1. *The radial Loewner evolution in the complement of the unit disc with driving function $\xi(t) : \mathbb{R}_+ \to \mathbb{T}$ is the solution of the following ODE:*

$$\partial_t g_t(z) = g_t(z)\frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)}, \qquad g_0(z) = z. \tag{1}$$

It is a classical fact [19] that for any driving function ξ, g_t is a conformal map from $\Omega_t \to \mathbb{D}_-$, where $\mathbb{D}_-$ is the complement of the unit disc and $\Omega_t = \mathbb{D}_- \setminus K_t$ is the set of all points where solution of (1) exists up to the time t.

The *Schramm-Loewner Evolution SLE_κ* is defined as a Loewner evolution driven by the Brownian motion with speed $\sqrt{\kappa}$ on the unit circle, namely $\xi(t) = e^{i\sqrt{\kappa}B_t}$, where B_t is the standard Brownian motion and κ is a positive parameter. Since ξ is random, we obtain a family of random sets. The corresponding family of compacts K_t is also called SLE (or the *hull of SLE*).

A number of theorems was already established about SLE curves. Rohde and Schramm [29] proved that SLE for $\kappa \neq 8$ is a.s. generated by a curve. Namely, almost surely there is a random curve γ (called *trace*) such that Ω_t is the unbounded component of $\mathbb{D}_- \setminus \gamma_t$, where $\gamma_t = \gamma([0, t])$. The trace is almost surely a simple curve when $\kappa \le 4$. In this case the hull K_t is the same as the curve γ_t. For $\kappa \ge 8$ the trace γ_t is a space-filling curve. In the same paper they also proved that almost surely the Minkowski (and hence the Hausdorff) dimension of the SLE_κ trace is no more than $1 + \kappa/8$ for $\kappa \le 8$. Beffara [2] proved that the Hausdorff dimension is equal to $1+\kappa/8$ for $\kappa = 6$, later expanding the result to all $\kappa \le 8$. Recently Lawler presented in [20] a completely different proof of the Hausdorff dimension of SLE paths. Lind [24] proved that the trace is Hölder continuous.

Another natural object is the *boundary* of SLE hull, namely the boundary of K_t. For $\kappa \le 4$ the boundary of SLE is the same as SLE trace (since the trace is a simple curve). For $\kappa > 4$ the boundary is the subset of the trace. Rohde and Schramm [29] proved that for $\kappa > 4$ the dimension of the boundary is no more than $1 + 2/\kappa$.

In 1998 Lawler [18] proved that the a.s. multifractal spectrum of the Brownian frontier (which is the same as the boundary of SLE_6) can be expressed in terms of intersection exponents. He also showed that these exponents are non-trivial. They have been computed later by Lawler, Schramm, and Werner in [21–23]. In [9,10], physicist Duplantier using quantum gravity methods predicted the average multifractal spectrum of SLE. The same result was later derived using conformal field theory by Bettelheim, Rushkin, Gruzberg, and Wiegmann [4,30].

Another important property of SLE curves is the so-called duality property: the boundary of the SLE_κ hull for $\kappa > 4$ is in the same measure class as the trace of $SLE_{16/\kappa}$. This property was first discovered by Duplantier, and much later proved by Zhan [33] and Dubedat [8].

In this paper we rigorously compute the average integral means spectrum of SLE and show that it coincides with Duplantier's prediction. This gives new proofs that dimension of the boundary is no more than $1+2/\kappa$ for $\kappa > 4$ and SLE maps are Hölder continuous, and provides more evidence which supports the duality conjecture.

Since $\bar{\beta}$ is defined in terms of a Riemann mapping, it is more convenient to work with $f_t = g_t^{-1}$. From Eq. (1) one can derive an equation on f_t. Unfortunately this equation involves f_t' as well as $\partial_t f_t$, so we have a PDE instead of ODE.

There is another approach which leads to a nice equation. Changing the direction of the flow defined by Eq. (1) we get the equation for "inverse" function g_{-t}. For a given driving function ξ, maps g_t^{-1} and g_{-t} are different, but in the case of Brownian motion they have the same distribution.The precise meaning is given by the following lemma (which is an analog of the Lemma 3.1 from [29]):

Lemma 1. *Let g_t be a radial SLE, then for all $t \in \mathbb{R}$ the map $z \mapsto g_{-t}(z)$ has the same distribution as the map $z \mapsto \hat{f}_t(z)/\xi_t$, where $\hat{f}_t(z) = g_t^{-1}(z\xi_t)$.*

Proof. Fix $s \in \mathbb{R}$. Let $\hat{\xi}(t) = \xi(s+t)/\xi(s)$. Then $\hat{\xi}$ has the same distribution as ξ. Let

$$\hat{g}_t(z) = g_{s+t}(g_s^{-1}(z\xi(s)))/\xi(s).$$

It is easy to check that $\hat{g}_0(z) = z$ and

$$\hat{g}_{-s}(z) = g_0(g_s^{-1}(z\xi(s)))/\xi(s) = \hat{f}_s(z)/\xi(s).$$

Differentiating $\hat{g}_t(z)$ with respect to t we obtain

$$\partial_t \hat{g}_t(z) = \hat{g}_t(z)\frac{\hat{\xi}(t) + \hat{g}_t(z)}{\hat{\xi}(t) - \hat{g}_t(z)},$$

hence $\hat{g}_t$ has the same distribution as SLE. □

This lemma proves that the solution of the equation

$$\partial_t f_t(z) = f_t(z)\frac{f_t(z) + \xi(t)}{f_t(z) - \xi(t)}, \qquad f_0(z) = z, \tag{2}$$

where $\xi(t) = e^{i\sqrt{\kappa}B_t}$ has the same distribution as g_t^{-1}. Abusing notations we call it also SLE_κ.

One of the most important properties of SLE is Markov property, roughly speaking it means that the composition of two independent copies of SLE is an SLE. The rigorous formulation is given by the following lemma.

Lemma 2. *Let $f_\tau^{(1)}$ be an SLE_κ driven by $\xi^{(1)}(\tau)$, $0 < \tau < t$ and $f_\tau^{(2)}$ be an SLE_κ driven by $\xi^{(2)}(\tau)$, $0 < \tau < s$, where $\xi^{(1)}$ and $\xi^{(2)}$ are two independent Brownian motions on the circle. Then $f_{s+t}(z) = f_s^{(2)}(f_t^{(1)}(z)/\xi^{(1)}(t))\xi^{(1)}(t)$ is SLE_κ at time $t+s$.*

Proof. This composition is the solution of Loewner Evolution driven by $\xi(\tau)$, where

$$\xi(\tau) = \begin{cases} \xi^{(1)}(\tau), & 0 < \tau \le t, \\ \xi^{(2)}(\tau - t)\xi^{(1)}(t), & t < \tau \le t + s. \end{cases}$$

It is easy to see that $\xi(\tau)$ is also a Brownian motion on the circle with the same speed $\sqrt{\kappa}$, hence f_{t+s} is also SLE_κ. □

We will need yet another modification of SLE which is in fact a manifestation of stationarity of radial SLE.

Definition 2. *Let $\xi(t) = \exp(i\sqrt{\kappa}B_t)$ be a two-sided Brownian motion on the unit circle. The whole plane SLE_κ is the family of conformal maps g_t satisfying*

$$\partial_t g_t(z) = g_t(z)\frac{\xi(t) + g_t(z)}{\xi(t) - g_t(z)},$$

with initial condition

$$\lim_{t\to-\infty} e^t g_t(z) = z, \qquad z \in \mathbb{C}\backslash\{0\}.$$

The whole-plane SLE satisfies the same differential equation as the radial SLE, the difference is in the initial conditions. One can think about the whole-plane SLE as about the radial SLE started at $t = -\infty$. And this is the way to construct the whole-plane SLE and prove the existence. Proposition 4.21 in [19] proves that the whole-plane Loewner Evolution g_t with the driving function $\xi(t)$ is the limit as $s \to -\infty$ of the following maps: $g_t^{(s)}(z) = e^{-t}z$ if $t \le s$, $g_t^{(s)}(z)$ is the solution to (1) with initial condition $g_s^{(s)}(x) = e^{-s}z$. The same is also true for inverse maps.

We use this argument to prove that there is a limit of $e^{-t}f_t$ as $t \to \infty$.

Lemma 3. *Let f_t be a radial SLE_κ then there is a limit in law of $e^{-t}f_t(z)$ as $t \to \infty$.*

Proof. The function $e^{-t}f_t$ is exactly the function which is used to define the whole-plane SLE. Multiplication by the exponent corresponds to the shift in time in the driving function. The function $e^{-t}f_t(z)$ has the same distribution as the inverse of $g_0^{(-t)}(z)$, hence it converges to F_0, where $F_\tau = g_\tau^{-1}$ and g_τ is a whole-plane SLE. □

1.3. Results, conjectures, and organization of the paper. It is easy to see that the geometry near "the tip" of SLE (the point of growth) is different from the geometry near "generic" points. This means that for some problems it is more convenient to work with the so-called *bulk* of SLE, i.e. the part of the SLE hull which is away from the tip. We repeat the statement of the main theorem in which we compute the average spectrum of SLE hull and SLE bulk.

Theorem 1. *The average integral means spectrum $\bar{\beta}(t)$ of SLE is equal to*

$$\begin{aligned} -t + \kappa\frac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{4\kappa} \qquad & t \le -1 - \frac{3\kappa}{8}, \\ -t + \frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\ )}{4\kappa} \qquad & -1 - \frac{3\kappa}{8} \le t \le \frac{3(4+\kappa)^2}{32\kappa}, \\ t - \frac{(4+\kappa)^2}{16\kappa} \qquad & t \ge \frac{3(4+\kappa)^2}{32\kappa}. \end{aligned}$$

The average integral means spectrum $\bar{\beta}(t)$ of the bulk of SLE is equal to

$$\begin{cases} 5 - t + \dfrac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2 - 8t\kappa}\,)}{4\kappa} & t \le \dfrac{3(4+\kappa)^2}{32\kappa}, \\ t - \dfrac{(4+\kappa)^2}{16\kappa}, & t \ge \dfrac{3(4+\kappa)^2}{32\kappa}. \end{cases}$$

Remark 1. The local structure of the SLE bulk is the same for all versions of SLE which means that they all have the same average spectrum.

Remark 2. To prove this theorem we show that

$$\mathbb{E}|f'(re^{i\theta})|^t \asymp (r-1)^\beta((r-1)^2+\theta^2)^\gamma,$$

where β and γ are given by (12) and (11). We would like to point out that β and γ are local exponents so they are the same for different versions of SLE.

There are several corollaries that one can derive from Theorem 1:

Corollary 1. *The SLE map f is Hölder continuous with any exponent less than*

$$\alpha_\kappa = 1 - \frac{1}{\mu} - \sqrt{\frac{1}{\mu^2} + \frac{2}{\mu}},$$

where $\mu = (4+\kappa)^2/4\kappa$.

Corollary 2. *The Hausdorff dimension of the boundary of the SLE hull for $\kappa \ge 4$ is at most $1 + 2/\kappa$.*

Corollary 3. *The SLE trace with time parametrization of SLE maps is Hölder continuous. The Hölder exponent is*

$$1 - \frac{\kappa}{24 - 2\kappa - 8\sqrt{8+\kappa}}.$$

The first two results are conjectured to be sharp. They both have been previously published in [16 and 29] correspondingly. Both results can be easily derived from the properties of the spectrum (see [25]) and Theorem 1.

The third corollary first appeared in a paper by Lind [24] where she uses derivatives estimated by Rohde and Schramm. One can use Theorem 1 to prove this result.

Theorem 1 gives the average spectrum of SLE. The question about spectra of individual realizations of SLE remains open. We believe that with probability one they all have the same spectrum $\beta(t)$ which we call the a.s. spectrum.

It is immediate that the tangent line at $t = 3(4+\kappa)^2/32\kappa$ intersects y-axis at $-(4+\kappa)^2/16\kappa < -1$. This contradicts Makarov's characterization of possible spectra [25] which in particular states that the tangent line to $\beta(t)$ should intersect y-axis between 0 and -1. Thus $\bar{\beta}$ can not be a spectrum of any given domain. In particular $\bar{\beta}$ is not the a.s. spectrum of SLE. On the other hand it suggests that the following conjecture is true.

Conjecture 1. Let t_{min} and t_{max} be the two points such that the tangents to $\bar{\beta}(t)$ at t_{min} and t_{max} intersect the y-axis at -1. The almost sure value of the spectrum is equal to $\bar{\beta}(t)$ for $t_{min} \le t \le t_{max}$ and continues as the tangents for $t < t_{min}$ and $t > t_{max}$. Explicit formulas for t_{min}, t_{max}, and tangent lines are given in (4) and (5). See Fig. 1 for plots of β and $\bar{\beta}$.

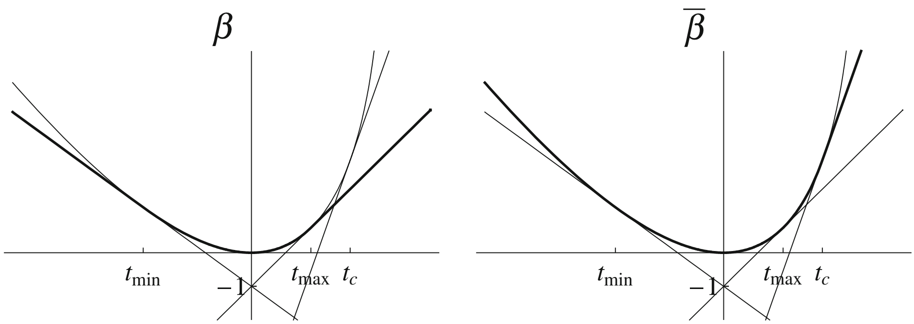

Fig. 1. Plots of β and $\bar{\beta}$ spectra. We also show the graph of $\tilde{\beta}$ (the analytical part of the spectra) as well as tangent lines at t_{min}, t_{max}, and $t_c = 3(4+\kappa)^2/32\kappa$. The almost sure spectrum is equal to $\tilde{\beta}$ as long as it does not violate Makarov's condition that tangent lines should intersect the y-axis above -1. This happens for $t_{min} < t < t_{max}$. Outside of this interval β continues as tangent lines. The average spectrum is given by $\tilde{\beta}$ as long as the derivative is less than 1. At $t = t_c$ the derivative is equal to 1 and $\bar{\beta}$ continues as a straight line for $t > t_c$

The rest of the paper is organized in the following way. In the first part of the Sect. 2 we discuss Duplantier's prediction and Conjecture 1. In second part we compute the moments of $|f'|$ and prove Theorem 1. In Sect. 3.1 me make some remarks about possible generalizations of SLE. In the last Sect. 3.2 we explain a possible approach to Conjecture 1.

2. Integral Means Spectrum of SLE

2.1. Duplantier's prediction for the spectrum of the bulk. In 2000, by means of quantum gravity, Duplantier predicted that the Hausdorff dimension spectrum of the bulk of SLE is

$$f(\alpha) = \alpha - \frac{(25-c)(\alpha-1)^2}{12(2\alpha-1)}, \quad \alpha \geq 1/2,$$

where c is the central charge which is related to κ by

$$c = \frac{(6-\kappa)(6-16/\kappa)}{4}.$$

The negative values of f do not have a simple geometric interpretation, they correspond to negative dimensions (see papers by Mandelbrot [26,27]) which appear only in the random setting. They correspond to the events that have zero probability in the limit, but appear on finite scales as exceptional events. There is another interpretation in terms of beta spectrum which we explain below.

Since negative values of f correspond to zero probability events, it makes sense to introduce the positive part of the spectrum: $f^+ = \max\{f, 0\}$. We believe that f^+ is the almost sure value of the dimension spectrum. This is the dimension spectrum counterpart of Conjecture 1. The function f^+ is equal to f for $\alpha \in [\alpha_{min}, \alpha_{max}]$, where

$$\alpha_{min} = \frac{16 + 4\kappa + \kappa^2 - 2\sqrt{2}\sqrt{16\kappa + 10\kappa^2 + \kappa^3}}{(4-\kappa)^2}, \quad \kappa \neq 4,$$
$$\alpha_{max} = \frac{16 + 4\kappa + \kappa^2 + 2\sqrt{2}\sqrt{16\kappa + 10\kappa^2 + \kappa^3}}{(4-\kappa)^2}, \quad \kappa \neq 4,$$
$$\alpha_{min} = \frac{2}{3}, \quad \kappa = 4,$$
$$\alpha_{max} = \infty, \quad \kappa = 4.$$

It is known (see [25]) that for regular fractals the $\beta(t)$ spectrum is related to the $f(\alpha)$ spectrum by the Legendre transform. We believe those relations to hold for SLE as well:

$$\beta(t) - t + 1 = \sup_{\alpha > 0} (f(\alpha) - t)/\alpha,$$
$$f(\alpha) = \inf_t (t + \alpha(\beta(t) - t + 1)).$$

The Legendre transform of f^+ is supposed to be equal to the almost sure value of the integral means spectrum $\beta(t)$, while the Legendre transform of f is believed to be equal to the average integral means spectrum $\bar{\beta}(t)$.

The Legendre transform of f^+ has two phase transitions: one for negative t and one for positive. The Legendre transform of f^+ is equal to

$$\beta(t) = t\left(1 - \frac{1}{\alpha_{min}}\right) - 1, \quad t \leq t_{min},$$
$$\beta(t) = -t + \frac{(4+\kappa)\left(4 + \kappa - \sqrt{(4+\kappa)^2 - 8t\kappa}\right)}{4\kappa}, \quad t_{min} < t < t_{max}, \tag{3}$$
$$\beta(t) = t\left(1 - \frac{1}{\alpha_{max}}\right) - 1, \quad t \geq t_{max},$$

where

$$t_{min} = -f'(\alpha_{min})\alpha_{min}, \quad \kappa > 0,$$
$$t_{max} = -f'(\alpha_{max})\alpha_{max}, \quad \kappa \neq 4,$$
$$t_{max} = 3/2, \quad \kappa = 4.$$

We can also express t_{min} and t_{max} in terms of $\mu = 4/\kappa + 2 + \kappa/4 = (4+\kappa)^2/4\kappa$:

$$t_{min} = \frac{-1 - 2\mu - (1+\mu)\sqrt{1+2\mu}}{\mu},$$
$$t_{max} = \frac{-1 - 2\mu + (1+\mu)\sqrt{1+2\mu}}{\mu}. \tag{4}$$

And the linear functions in (3) can be written as

$$t\left(\frac{1}{\sqrt{1 - 2t_{min}/\mu}} - 1\right) - 1,$$
$$t\left(\frac{1}{\sqrt{1 - 2t_{max}/\mu}} - 1\right) - 1. \tag{5}$$

For convenience we introduce

$$\tilde{\beta}(t) = -t + \frac{(4+\kappa)\left(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\right)}{4\kappa},$$

which is the analytic part of the spectrum and defined for all $t < (4+\kappa)^2/8\kappa$. This function is the analytic part of the Legendre transform of f. The critical points t_{max} and t_{min} are the points where the tangent line to the graph of $\tilde{\beta}(t)$ intersects the y-axis at -1. The Legendre transform of f^+ is equal to $\tilde{\beta}(t)$ between these two critical points and then continues as a linear function.

Note that Makarov's theorem [25] states that all possible integral means spectra satisfy the following conditions: they are non-negative convex functions bounded by the universal spectrum such that the tangent line at any point intersects the y-axis between 0 and -1. So there is another way to describe the Legendre transform of f^+: it coincides with $\tilde{\beta}$ as long as this does not contradict Makarov's criteria and then continues in the only possible way.

If we do not cut off the negative part of f, then the picture is a bit different. There is no phase transition for negative t. For positive t, phase transition occurs later, and it happens because the derivative of $f(\alpha)$ is bounded at infinity. For large α,

$$f(\alpha) = \alpha\left(1-\frac{(4+\kappa)^2}{16\kappa}\right) + \frac{3(4+\kappa)^2}{32\kappa} + O\left(\frac{1}{\alpha}\right),$$

hence

$$\bar{\beta}(t) = -t + \frac{(4+\kappa)\left(4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}\right)}{4\kappa}, \qquad t \le \frac{3(4+\kappa)^2}{32\kappa},$$
$$\bar{\beta}(t) = 1-\frac{(4+\kappa)^2}{16\kappa}+t-1 = t-\frac{(4+\kappa)^2}{16\kappa}, \qquad t > \frac{3(4+\kappa)^2}{32\kappa}.$$

The explanation of this phase transition is rather simple. It is obvious that $\bar{\beta}(t)$ is a convex function, and it follows from Makarov's fractal approximation that the average spectrum is bounded by the universal spectrum. It is known that for the large values of $|t|$ the universal spectrum is equal to $|t|-1$. Altogether it implies that $|\bar{\beta}'(t)| \le 1$ and if it is equal to 1 at some point then $\bar{\beta}$ should be linear after this point. And $\bar{\beta}' = 1$ exactly at $t = 3(4+\kappa)^2/32\kappa$.

2.2. Rigorous computation of the spectrum. In this section we compute the average integral means spectrum of SLE (and its bulk) and show that it coincides with the Legendre transform of the dimension spectrum predicted by Duplantier.

The average integral means spectrum is the growth rate of $\tilde{F}(z,\tau) = \mathbb{E}\left[|f_\tau'(z)|^t\right]$, where f_τ is a radial SLE_κ. Actually, this function depends also on t and κ, but they are fixed throughout the proof and we will not mention this dependence to simplify the notation.

Lemma 4. *The function $\tilde{F}(z,\tau)$ is a solution of*

$$t\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}\tilde{F} + \frac{r(r^2-1)}{r^2-2r\cos\theta+1}\tilde{F}_r$$
$$-\frac{2r\sin\theta}{r^2-2r\cos\theta+1}\tilde{F}_\theta + \frac{\kappa}{2}\tilde{F}_{\theta,\theta} - \tilde{F}_\tau = 0, \tag{6}$$

where $z = re^{i\theta}$.

Proof. The idea of the proof is to construct a martingale $\mathcal{M}_s$ (w.r.t filtration defining SLE) which involves $\tilde{F}$. The ds term in its Itô derivative should vanish. This will give us a partial differential equation on $\tilde{F}$. We set

$$\mathcal{M}_s = \mathbb{E}\left[|f_\tau'(z)|^t \mid \mathcal{F}_s\right].$$

By Lemma 2,

$$\begin{aligned}\mathbb{E}\left[|f_\tau'(z)|^t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[|f_s'(z)|^t |f_{\tau-s}'(f_s(z)/\xi_s)|^t \mid \mathcal{F}_s\right]\\ &= |f_s'(z)|^t \tilde{F}(z_s, \tau - s),\end{aligned}$$

where $z_s = f_s(z)/\xi_s$.

We will need derivatives of z_s and $|f_s'|^t$,

$$\begin{aligned}\partial_s \log |f_s'(z)| &= \operatorname{Re} \frac{\partial_z f_s \frac{f_s+\xi_s}{f_s-\xi_s}}{f_s'} = \operatorname{Re}\left[\frac{f_s+\xi_s}{f_s-\xi_s} - \frac{2\xi_s f_s}{(f_s-\xi_s)^2}\right]\\ &= \operatorname{Re} \frac{z_s^2 - 1 - 2z_s}{(z_s-1)^2} = \frac{r^4 + 4r^2(1 - r\cos\theta) - 1}{(r^2 - 2r\cos\theta + 1)^2},\end{aligned}$$

where $z_s = r\exp(i\theta)$. Next we have to find the derivative of z_s,

$$d\log z_s = d\log r + i d\theta = d\log f_s - i\sqrt{\kappa}\, dB_s,$$

where

$$d\log f_s = \frac{df_s}{f_s} = \frac{z_s+1}{z_s-1} ds.$$

Writing everything in terms of r and θ we get

$$\begin{aligned}d\log r + i d\theta &= \frac{z_s+1}{z_s-1} ds - i\sqrt{\kappa}\, dB_s\\ &= \frac{r^2-1}{r^2 - 2r\cos\theta + 1} ds + i\left(-\frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1} ds - \sqrt{\kappa}\, dB_s\right).\end{aligned}$$

Summing it all up we obtain

$$\partial_s \log |f_s'(z)| = \frac{r^4 + 4r^2(1 - r\cos\theta) - 1}{(r^2 - 2r\cos\theta + 1)^2}, \tag{7}$$

$$d\theta = -\frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1} ds - \sqrt{\kappa}\, dB_s, \tag{8}$$

$$dr = r\, d\log r = \frac{r(r^2-1)}{r^2 - 2r\cos\theta + 1} ds. \tag{9}$$

Let us write $F(z, \tau)$ as $F(r, \theta, \tau)$. The ds term in the Itô derivative of $\mathcal{M}$ is equal to

$$\begin{aligned}|f_s'(z)|^t \Bigg(t \frac{r^4 + 4r^2(1 - r\cos\theta) - 1}{(r^2 - 2r\cos\theta + 1)^2} \tilde{F} + \frac{r(r^2-1)}{r^2 - 2r\cos\theta + 1} \tilde{F}_r\\ - \frac{2r\sin\theta}{r^2 - 2r\cos\theta + 1} \tilde{F}_\theta + \frac{\kappa}{2} \tilde{F}_{\theta,\theta} - \tilde{F}_\tau\Bigg).\end{aligned}$$

This derivative should be 0 and, since f_s is a univalent function and its derivative never vanishes, $\tilde{F}$ is a solution of (6). □

By Lemma 3 there is a limit of $e^{-\tau} f_\tau$ as $\tau \to \infty$. Hence we can introduce

$$F(z) = \mathbb{E}[|F_0'(z)|^t] = \lim_{\tau\to\infty} e^{-\tau t} \tilde{F}(z,\tau),$$

where F_0 is a whole-plane SLE map at time zero. Passing to the limit in (6) we can see that $F(z)$ is a solution of

$$t\left(\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}-1\right)F+\frac{r(r^2-1)}{r^2-2r\cos\theta+1}F_r$$
$$-\frac{2r\sin\theta}{r^2-2r\cos\theta+1}F_\theta+\frac{\kappa}{2}F_{\theta,\theta}=0. \tag{10}$$

Notation 1. We define two constants β and γ:

$$\gamma = \gamma(t,\kappa) = \frac{4+\kappa-\sqrt{(4+\kappa)^2-8t\kappa}}{2\kappa}, \tag{11}$$

$$\beta = \beta(t,\kappa) = t - \frac{(4+\kappa)\gamma}{2}. \tag{12}$$

It is easy to see that the second constant β is equal to $-\tilde{\beta}$.

Let us explain where these constants come from. Roughly speaking spectrum $\beta(t)$ is the growth rate of F as $r \to 1$. F is a solution of Eq. (10) which is parabolic as $r \to 1$. It has a singularity when $|z| = 1$ which corresponds to the large time singularity in the usual parabolic equation. Coefficients of (10) have singularities at $z = 1$ which means that solutions could have an additional singularity at $z = 1$. Let us assume that F has a power series expansion near 1. Then we can write the power series expansion of coefficients of (10) and assuming that the leading term is $(r-1)^\beta((r-1)^2+\theta^2)^\gamma$ we get an equation on β and γ. Constants γ and β are solutions of these equations. Now let us explain why it makes sense to consider this expansion.

There is another (and more popular) version of SLE: the chordal SLE in the upper half-plane, which is defined as the solution of

$$\partial_\tau f_\tau(z) = -\frac{2}{f_\tau(z)-\sqrt{\kappa}B_\tau}.$$

If we define $F(x,y,\tau) = \mathbb{E}|f_\tau'(x+iy)|^t$, then the argument similar to the one presented above proves that F satisfies a certain PDE. If we remove the F_τ term (which should be irrelevant for large τ) then the equation will be

$$2t\frac{x^2-y^2}{(x^2+y^2)^2}F-\frac{2x}{x^2+y^2}F_x+\frac{2y}{x^2+y^2}F_y+\frac{\kappa}{2}F_{xx}=0. \tag{13}$$

This equation is "tangent" to (10) at $r = 1$ and $\theta = 0$.

This equation has a solution of the form $y^\beta(x^2+y^2)^\gamma$, where β and γ as above. Actually, this is the way we found these exponents. This approach seems to be easier, but there are two major problems. First it is not easy to argue that we can neglect the derivative with respect to τ. Another problem is that $y^\beta(x^2+y^2)^\gamma$ can not be equal to F since it blows up at infinity and we have to show that the local behavior does not depend on the boundary conditions at infinity.

When this work was finished we learned from Gruzberg that several years ago Hastings in [13] derived Eq. (13) by completely different methods (and for completely different purposes).

Theorem 2. *Let*

$$t \leq \frac{3(4+\kappa)^2}{32\kappa}.$$

Then we have

$$\mathbb{E}\left[\int_{|z|=r} |F_0'(re^{i\theta})|^t d\theta\right] \asymp \left(\frac{1}{r-1}\right)^{\bar{\beta}(t)},$$

where the expectation is taken for a whole-plane SLE map $F_0 = \lim e^{-\tau} f_\tau$ and $\bar{\beta}(t)$ is equal to

$$\begin{aligned} -\beta(t,\kappa), \qquad & t > -1 - \frac{3\kappa}{8}, \\ -\beta(t,\kappa) - 2\gamma(t,\kappa) - 1, \qquad & t \leq -1 - \frac{3\kappa}{8}. \end{aligned} \tag{14}$$

Proof. Let Λ be the differential operator which corresponds to Eq. (10). This is a parabolic operator where θ corresponds to the spatial variable and $r \to 1$ corresponds to the time variable. It is clear that $F(z)$ is bounded on any circle of radius $r_0 > 1$.

Suppose that we can find positive functions ϕ_+ and ϕ_- which are bounded on the circle of radius r_0 and such that $\Lambda\phi_- < 0$ and $\Lambda\phi_+ > 0$. Then there are positive constants c_+ and c_- such that F is between $c_+\phi_+$ and $c_-\phi_-$ on the circle of radius r_0. By the maximum principle it will be between $c_+\phi_+$ and $c_-\phi_-$ for all $1 < r < r_0$.

In Lemma 6 we will construct such functions ϕ_- and ϕ_+. They are of the form

$$\phi_\pm = (r-1)^\beta (r^2 - 2r\cos\theta + 1)^\gamma (-\log(r-1))^{\mp 1} g(r^2 - 2r\cos\theta + 1),$$

where $g > 0$ for $r = 1$. Both functions have the same polynomial growth rate as $r \to 1$, thus F has also the same growth rate. By the Tonelli theorem

$$\mathbb{E}\left[\int |F_0'|^t\right] = \int \mathbb{E}\left[|F_0'(r,\theta)|^t\right] d\theta \approx \int (r-1)^\beta (r^2 - 2r\cos\theta + 1)^\gamma d\theta,$$

where $\approx$ means that functions have the same polynomial growth rate. For $\gamma > -1/2$ the weight $(r^2 - 2r\cos\theta + 1)^\gamma$ is integrable up to the boundary and we immediately get

$$\mathbb{E}\left[\int_{|z|=r} |F_0'|^t\right] \approx \left(\frac{1}{r-1}\right)^{-\beta}.$$

For $\gamma \leq -1/2$ the situation is a bit different. In this case the integral of the weight blows up as $(r-1)^{2\gamma+1}$, which gives us $\mathbb{E}\left[\int |F_0'|^t d\theta\right] \approx (r-1)^{\beta+2\gamma+1}$. It is easy to check that $\gamma \leq -1/2$ if and only if $t \leq -1 - 3\kappa/8$. $\square$

Remark 3. The growth rate of $\mathbb{E}\left[\int |F_0'|^t\right]$ is similar to $\bar{\beta}(t)$ predicted by Duplantier. The phase transition at $t = -1 - 3\kappa/8$ is due to the exceptional behavior of SLE at the tip. If we integrate over values of θ bounded away from 0 then the weight $|z-1|^{2\gamma}$ does not blow up and we have no phase transition at $t = -1 - 3\kappa/8$ any more. This gives us the spectrum of the bulk of SLE.

Now we can prove Theorem 1 which is actually Theorem 2 stated in terms of integral means spectrum. This theorem proves that Duplantier's prediction for $\bar{\beta}(t)$ is correct.

Proof (Theorem 1). Theorem 2 gives us the value of $\bar\beta(t)$ for $t \le 3(4+\kappa)^2/32\kappa$. Direct computations show that the derivative of $-\beta(t,\kappa)$ at $t = 3(4+\kappa)^2/32\kappa$ is equal to one. As we mentioned before, the $\bar\beta$ spectrum is a convex function bounded by the universal spectrum, and the universal spectrum is equal to $|t|-1$ for the large values of $|t|$ (see [7]). This means that if $\bar\beta' = 1$ at some point then it should continue as a linear function with slope one. Hence $\bar\beta$ should continue as $t-(4+\kappa)^2/16\kappa$ for $t > 3(4+\kappa)^2/32\kappa$. Plugging in the values of β and γ we finish the proof of the theorem.

To complete the proof of Theorem 2 we have to construct functions ϕ_- and ϕ_+. We do it in three steps, first we write the restriction of Eq. (10) to the unit circle, then we find a positive solution g of the resulting equation. Finally we construct ϕ_- and ϕ_+ out of g.

We look for a solution in the following form:

$$f(r,\theta) = (r-1)^\beta (r^2 - 2r\cos\theta + 1)^\gamma g(r^2 - 2r\cos\theta + 1).$$

Plugging f into (10), factoring $(r-1)^\beta (r^2 - 2r\cos\theta + 1)^{\gamma-2}$ out, and taking $r=1$, we obtain a differential equation on $g(2-2\cos\theta)$. Using relations between β, γ, t, and κ we can simplify coefficients and write the equation in the following form:

$$\begin{aligned}&-2(2+\kappa)\gamma(1-\cos\theta)^2 g(2-2\cos\theta)\\&\quad+(2-2\cos\theta)\left[-2-\kappa+2\gamma\kappa+2\kappa\cos\theta-(\kappa-2+2\gamma\kappa)\cos(2\theta)\right]g'(2-2\cos\theta)\\&\quad+2\kappa(2-2\cos\theta)^2\sin\theta^2 g''(2-2\cos\theta)=0.\end{aligned}\tag{15}$$

Lemma 5. *Equation (15) has a smooth (with possible exception at $\theta = 0$) positive bounded solution on the circle if and only if*

$$t \le \frac{3(4+\kappa)^2}{32\kappa}.\tag{16}$$

Proof. Changing the variable to $x = 2-2\cos\theta$ we rewrite (15) as a hypergeometric equation

$$\gamma(2+\kappa)g(x) + (8-2x+\kappa(x-2)+2\gamma\kappa(x-4))g'(x) + \kappa(x-4)xg''(x) = 0,\tag{17}$$

which has two independent solutions

$$g_1(x) = {}_2F_1(a, b, \frac{1}{2}+a+b, \frac{x}{4})$$

and

$$g_2(x) = x^{1/2-a-b}{}_2F_1\left(\frac{1}{2}-a, \frac{1}{2}-b, \frac{3}{2}-a-b, \frac{x}{4}\right),$$

where

$$\begin{aligned}a &= \gamma - \frac{1}{\kappa} - \frac{\sqrt{1-2t\kappa}}{\kappa},\\ b &= \gamma - \frac{1}{\kappa} + \frac{\sqrt{1-2t\kappa}}{\kappa}.\end{aligned}$$

Function $g(2-2\cos\theta)$ is a non-singular part of F and should have a second derivative everywhere on the unit circle except at the point $\theta=0$ (the equation on F has a singularity at this point). Note that $2-2\cos\theta=4$ corresponds to the point -1 on the unit circle: this is not a singular point, hence $g(x)$ should have expansion $c+O(4-x)$ at the endpoint 4.

Any solution of (15) is a linear combination of g_1 and g_2: $g=c_1g_1+c_2g_2$. We want to find coefficients c_1 and c_2 such that this sum is bounded and has a correct expansion at $x=4$.

Expansions of g_1 and g_2 at 4 are

$$g_1(x)=\frac{\sqrt{\pi}\Gamma(1/2+a+b)}{\Gamma(1/2+a)\Gamma(1/2+b)}-\frac{\sqrt{\pi}\Gamma(1/2+a+b)}{\Gamma(a)\Gamma(b)}\sqrt{4-x}+O(4-x),$$

and

$$\begin{aligned}g_2(x)=&\frac{2^{1-2a-2b}\sqrt{\pi}\Gamma(3/2-a-b)}{\Gamma(1-a)\Gamma(1-b)}\\&-\frac{2^{1-2a-2b}\sqrt{\pi}\Gamma(3/2-a-b)}{\Gamma(1/2-a)\Gamma(1/2-b)}\sqrt{4-x}+O(4-x).\end{aligned}$$

If $c_2\neq 0$ then $1/2-a-b$ should be nonnegative, otherwise g is not bounded at 0. Note that

$$\frac{1}{2}-a-b=\frac{4+\kappa-4\gamma\kappa}{2\kappa}$$

which is nonnegative if and only if

$$t\le\frac{3(4+\kappa)^2}{32\kappa}$$

which is exactly the restriction from the statement of the lemma. If $t>3(4+\kappa)^2/32\kappa$, then $c_2=0$. In this case g has a correct expansion at 4 if and only if $\Gamma(a)=0$ or $\Gamma(b)=0$, but $1-2t\kappa<0$, so both a and b are not a real number and the gamma function has only real roots.

We can introduce

$$C=\frac{\Gamma(1/2+a+b)\Gamma(1/2-a)\Gamma(1/2-b)}{2^{1-2a-2b}\Gamma(a)\Gamma(b)\Gamma(3/2-a-b)},$$

and

$$g_3(x)=g_1(x)-Cg_2(x).$$

By construction $g_3(x)=\text{const}+O(4-x)$ near 4. Finally we have to prove that g_3 is a positive function. Note that in (17) g and g'' have coefficients of different signs. Obviously, $g_3(0)=1$. Suppose that g_3 has a local minimum inside the interval $(0,4)$, then $g_3' = 0$ and $g_3'' \geq 0$ at this point, hence g_3 is also positive. Thus it is sufficient to check that $g_3(4) > 0$. The value of $g_3(4)$ is easy to evaluate:

$$\begin{aligned} g_3(4) &= \sqrt{\pi}\Gamma(1/2+a+b) \\ &\quad\times\left(\frac{1}{\Gamma(1/2+a)\Gamma(1/2+b)} - \frac{\Gamma(1/2-a)\Gamma(1/2-b)}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)}\right) \\ &= \frac{\sqrt{\pi}\Gamma(1/2+a+b)\cos(\pi(a+b))}{\Gamma(1/2+a)\Gamma(1/2+b)\cos(\pi a)\cos(\pi b)} \\ &= \pi^{-3/2}\Gamma(1/2+a+b)\cos(\pi(a+b))\Gamma(1/2-a)\Gamma(1/2-b). \end{aligned}$$

By (16), $a+b < 1/2$, hence $\Gamma(1/2+a+b)\cos(\pi(a+b)) > 0$. Finally we have to show that $\Gamma(1/2-a)\Gamma(1/2-b) > 0$. We consider two different cases: when $t \leq 1/2\kappa$ and $t > 1/2\kappa$. In the second case a and b are conjugated and $\Gamma(1/2-a)\Gamma(1/2-b) = |\Gamma(1/2-a)|^2 > 0$. In the first case, we will prove that $1/2-a > 0$ and $1/2-b > 0$. It is easy to see that $1/2-b < 1/2-a$, hence it is sufficient to prove that $1/2-b > 0$. Recall that

$$\frac{1}{2} - b = \frac{1}{2} - \gamma + \frac{1}{\kappa} - \frac{\sqrt{1-2t\kappa}}{\kappa},$$

hence

$$\partial_t(1/2-b) = \frac{1}{\sqrt{1-2\kappa t}} - \frac{2}{\sqrt{(4+\kappa)^2 - 8t\kappa}} > 0.$$

This means that $1/2-b$ has a minimum when $t=0$, this minimum is

$$\frac{1}{2} - b(0) = \frac{1}{2} - \gamma(0) = \frac{1}{2} > 0.$$

This proves that $g_3(x) > 0$ on $[0, 4]$. □

Lemma 6. *Let g be a positive bounded solution of (15) and*

$$\begin{aligned} F =& f(r,\theta)(-\log(r-1))^\delta \\ =&(r-1)^\beta(r^2-2r\cos\theta+1)^\gamma g(r^2-2r\cos\theta+1)(-\log(r-1))^\delta. \end{aligned}$$

Then

$$\begin{aligned} \Lambda F > 0, &\quad \delta < 0, \\ \Lambda F < 0, &\quad \delta > 0, \end{aligned}$$

for r sufficiently close to 1.

Proof. Applying Λ we find

$$\Lambda F = (-\log(r-1))^\delta\left(\Lambda f - f\frac{r(r+1)\delta}{(r^2-2r\cos\theta+1)(-\log(r-1))}\right).$$

By Lemma 5 $\Lambda f = (r-1)^\beta(r^2-2r\cos\theta+1)^\gamma O(r-1)$, hence

$$\begin{aligned} \Lambda F =&(-\log(r-1))^\delta(r-1)^\beta(r^2-2r\cos\theta+1)^\gamma \\ &\times\left(O(r-1) - \frac{r(r+1)\delta(g(2-2\cos\theta)+O(r-1))}{w(-\log(r-1))}\right). \end{aligned}$$

The sign of the main term is opposite to the sign of δ. This proves the claim. □

Remark 4. Note that we proved a stronger result than announced in Theorem 2: $\mathbb{E}\int|F'|^t$ has growth rate $(r-1)^\beta$ up to a factor $\log^\delta(r-1)$ for *arbitrary small* $|\delta|$.

3. Concluding Remarks

3.1. Loewner Evolution driven by other processes. It is known that Loewner Evolution can be defined for a very large class of driving functions. In particular, they do not have to be continuous. In [3], we proposed to study Lévy-Loewner Evolution (LLE), which is the Loewner Evolution driven by a Lévy process (i.e. process with independent stationary increments). This defines a very rich class of random fractals. It seems that it is still possible to find the spectrum of harmonic measure for this class explicitly.

In the fundamental Lemma 4 we only use the fact that the Brownian motion is a Lévy process. So the same argument can be applied for LLE. As a result we get that $F=\mathbb{E}\left[|e^{-\tau}f'_\tau(z)|^t\right]$ is the solution of

$$t\left(\frac{r^4+4r^2(1-r\cos\theta)-1}{(r^2-2r\cos\theta+1)^2}-1\right)F+\frac{r(r^2-1)}{r^2-2r\cos\theta+1}F_r$$
$$-\frac{2r\sin\theta}{r^2-2r\cos\theta+1}F_\theta+\Lambda F=0,$$

where Λ is the generator of the driving Lévy process. Thus again finding the spectrum boils down to the analysis of a parabolic type integro-differential equation. We have freedom to choose the driving process (and the generator Λ), so it seems possible to find a driving process such that this equation could be solved and gives large spectrum.

Unpublished computer experiments by Meyer [28] suggested that the spectrum for 1-stable process could be large (and possibly equal to the conjectured universal spectrum). Unfortunately later work by Gruzberg, Guan, Kadanoff, Oikonomou, Rohde, Rushkin, Winkel, and others [11,12,31] showed that this is wrong. But there is still a possibility that computer experiments exposed an existing phenomenon. It could be that the integral means grow fast for a few (relatively) large scales and when we approach the boundary their growth slows down. If this is true, one can use LLE as a building block in a snowflake (or any other construction which allows to replicate scales). In this way one can hope to construct a domain with large integral means on *all* scales.

3.2. Almost sure value of the spectrum. In this section, we speculate about what should be done to prove that the almost sure value of the spectrum is given by (3).

Let us introduce random variables $X_k(n)=|f'((1+2^{-n})e^{2\pi ik/2^n})|^t$. The spectrum is the growth rate of $2^{-n}\sum_k X_k$. We know that

$$2^{-n}\sum_{k=1}^{2^n}\mathbb{E}X_k\asymp 2^{n\bar\beta(t)}.$$

We want to show that the probability

$$\mathbb{P}\left\{2^{-n}|\sum X_k-\mathbb{E}X_k|>2^{n(\bar\beta(t)-\delta)}\right\} \tag{18}$$

is summable for some positive δ. This will clearly imply that the spectrum of SLE is equal to $\beta(t)$ with probability one.

Conformal field theory considerations suggest that X_k and X_l are essentially independent if $|k-l| \gg 1$ (in other words the distance between points should be much larger than their distance to the boundary). In fact it is believed that derivatives are essentially independent if the distance between points is greater than any power (less than one) of the distance to the boundary. Let us exaggerate it a little bit more and assume that X_k and X_l are independent for any $k \neq l$.

Let us denote $X_k - \mathbb{E}X_k$ by Y_k. By the Chebyshev inequality the probability (18) is less than

$$\frac{\mathbb{E}|\sum Y_k|^{1+\epsilon}}{2^{n(1+\epsilon)(\bar{\beta}(t)+1-\delta)}}.$$

It is known (see [1]) that for independent random variables with zero mean $\mathbb{E}|\sum Y_k|^{1+\epsilon} \le c \sum \mathbb{E}|Y_k|^{1+\epsilon}$, where c is an absolute constant which does not depend on the number of terms. Using this we can estimate the fraction above by

$$\frac{\sum \mathbb{E}|Y_k|^{1+\epsilon}}{2^{n(1+\epsilon)(\bar{\beta}(t)+1-\delta)}} \le c\frac{2^n 2^{n\bar{\beta}(t+t\epsilon)}}{2^{n(1+\epsilon)(\bar{\beta}(t)+1-\delta)}} = c2^{n(1+\bar{\beta}(t+t\epsilon)-\bar{\beta}(t)-1+\delta-\epsilon\bar{\beta}(t)-\epsilon+\epsilon\delta)}. \tag{19}$$

For small $\epsilon < \epsilon_0(t)$ the exponent in the last formula is bounded by

$$n(\bar{\beta}'(t)t\epsilon + \epsilon^{3/2} + \delta - \epsilon\bar{\beta}(t) - \epsilon + \epsilon\delta) = n(\epsilon(\bar{\beta}'(t)t - \bar{\beta}(t) - 1) + \epsilon^{3/2} + \delta + \epsilon\delta).$$

If $\bar{\beta}'(t)t - \bar{\beta}(t) - 1 = c(t) < 0$, then we can find a small ϵ_t (depending on t only) such that $\epsilon_t(\bar{\beta}'(t)t - \bar{\beta}(t) - 1) + \epsilon_t^{3/2} < c(t)\epsilon_t/2$. Fix $\delta = -\epsilon_t c(t)/4$, then the exponent in (19) is negative. This implies that the probability in (18) is summable if $-1 < \bar{\beta}(t) - t\bar{\beta}'(t)$. The last inequality means that the tangent line to β at point t intersects the y axis above -1. This is exactly the condition which appeared in (3).

Thus, assuming the independence of derivatives, we can prove that the almost sure value of the spectrum is equal to $\bar{\beta}(t)$ for $t_{min} < t < t_{max}$. For other values of t Makarov's theorem implies that the spectrum should continue as a straight line tangent to $\bar{\beta}(t)$ at t_{min} and t_{max} correspondingly.

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