CRITICAL PERCOLATION AND CONFORMAL INVARIANCE

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Many 2D critical lattice models are believed to have conformally invariant scaling limits. This belief allowed physicists to predict (unrigorously) many of their properties, including exact values of various dimensions and scaling exponents. We describe some of the recent progress in the mathematical understanding of these models, using critical percolation as an example.

1. Introduction

For several 2D lattice models physicists were able to make a number of spectacular predictions (non-rigorous, but very convincing) about exact values of various scaling exponents and dimensions. Many methods were employed (Coulomb Gas, Conformal Field Theory, Quantum Gravity) with one underlying idea: that in some sense the model concerned has a continuum scaling limit (as mesh of the lattice goes to zero) and the latter is conformally invariant.

Recently mathematicians came up with some new (and rigorous) approaches. We will describe some of the progress made, much of it due to Lawler, Schramm, and Werner. We will center on the percolation which has a simple definition, and is now fairly well understood.

In *Bernoulli site percolation* with $p \in [0, 1]$ each vertex of some graph is declared open (or colored blue) with probability p and closed (or colored yellow) with probability $1 - p$, independently of each other. We are interested in graphs, which approximate geometry of the plane, especially square, triangular, and hexagonal lattices. One studies *clusters*, which are connected subgraphs of a given color. One can also color edges (bond percolation) or, in a planar graph, faces.

Since vertices are colored independently, the model has locality property: observables for disjoint areas are independent. However, the model exhibits a complicated behavior. It is now well-known that in these models (both in site and bond cases) a *phase transition* occurs: there is a critical value $p_c \in (0, 1)$ such that when $p \leq p_c$, there is no infinite cluster of open vertices, while if $p > p_c$, there is a unique such infinite cluster. The value of p_c is lattice-dependent. It was shown by Kesten that for bond percolation on the square lattice $p_c = 1/2$, whereas for site percolation on the square lattice $p_c \approx 0.59$ with exact value still unknown. See the textbooks [7, 5] for a good introduction to the subject.

The *percolation probability* $\theta(p)$ that the origin belongs to the infinite open cluster is positive for $p > p_c$ (but strictly smaller than 1 as the origin might be surrounded by closed vertices), see [5]. Arguments from theoretical physics predict the exact power law for θ:

$$\theta(p) \asymp (p-p_c)^{5/36}\ , \quad p \to p_c+\ . \tag{1}$$

The exponent $5/36$ is supposed to be independent (unlike p_c) of the planar lattice involved. Several other predictions were made, for example if C_0 denotes the open cluster containing the origin (which might be empty), the expected number $\chi(p)$ of vertices in it, provided it is finite, was predicted to behave like

$$\chi(p) := \mathbb{E}\left(\#C_0 \middle| \#C_0 < \infty\right) \asymp |p-p_c|^{-43/18}\ , \quad p \to p_c\ . \tag{2}$$

Another power law was predicted for the *correlation length*:

$$\xi(p) := \mathbb{E}\left(\sum_{y\in C_0} \frac{|y|^2}{\#C_0} \middle| \#C_0 < \infty\right) \asymp |p-p_c|^{-4/3}\ , \quad p \to p_c\ , \tag{3}$$

where $|y|$ denotes the distance from the site y to the origin.

All these results were first conjectured (on the basis of experiments and heuristical arguments) and then predicted (using Coulomb Gas methods, Conformal Field Theory, or Quantum Gravity) by physicists. Now they have been rigorously established for site percolation on triangular lattice by a combination of [8, 11, 12, 13, 18, 19]:

Theorem 1. *Scaling laws (1), (2), (3) hold for percolation on the triangular lattice.*

Here we restrict ourselves to the discussion of rigorous methods, see [19] for references to the physics literature. Below we sketch some ideas which contributed to the proof. We start with conformally invariant scaling limits for percolation, and then describe their connections to SLE, or Schramm-Loewner Evolution, a way to obtain conformally invariant random curves, introduced by Schramm. We conclude with some open questions.

2. Harmonic Conformal Invariants

For some time it appeared difficult to formulate rigorously that percolation has a conformally invariant scaling limit, cf. [1]. One simple interpretation, which attracted much attention, was suggested by Langlands, Pouliot, and Saint-Aubin in [9].

Given a topological rectangle (a simply connected domain Ω with boundary points a, b, c, d) one can superimpose a lattice with mesh δ onto Ω and study the probability $\Pi_\delta\left(\Omega, [a,b], [c,d]\right)$ that there is an open cluster joining the arc $[a,b]$ to the arc $[c,d]$ on the boundary of Ω. In [9] extensive computer experiments were performed to check that there is a limit $\Pi := \lim_{\delta\to 0} \Pi_\delta$, which is independent of the lattice, and depends only on the conformal modulus of the configuration Ω, a, b, c, d. The latter conjecture authors attributed to Aizenman. The results were conclusively positive, leading to the conjecture that *crossing probabilities have a conformally invariant scaling limit.*

Using Conformal Field Theory Cardy was able to derive in [4] (unrigorously) the exact value of the crossing probabilities. Assuming conformal invariance, he mapped the domain to the upper half plane $\mathbb{C}_+$, three boundary points to $0, 1, \infty$, and derived a differential equation for Π as a function of the fourth point. The solution turned out to be a hypergeometric function ${}_2F_1$:

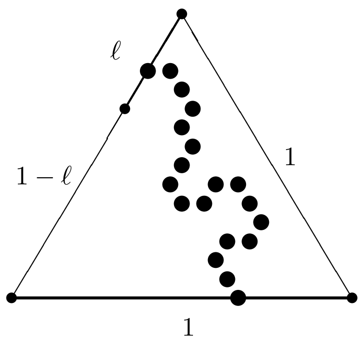
$$\Pi\left(\mathbb{C}_+, [1-u, 1], [\infty, 0]\right) = \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\, u^{1/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; u\right) =: F(u)\ . \tag{4}$$

There are several definitions for the function ${}_2F_1$. It can be written as an integral, leading to the following formula for F:

$$F(u) \;=\; \int_0^u (v(1-v))^{-2/3}\,dv \Big/ \int_0^1 (v(1-v))^{-2/3}\,dv\ .$$

Later Carleson observed that the very same hypergeometric function maps the half-plane to an equilateral triangle, and there Cardy's formula takes a particularly easy form. Namely, suppose that we map Ω conformally to an equilateral triangle with side length 1, with three points going to vertices a, b, c and the fourth point to some $z \in ab$. Then crossing probability $\Pi([a,z],[b,c])$ converges to the distance from a to z:

$\mathbb{P}$ (



) $\xrightarrow{\delta\to 0}$ ℓ .

Cardy's formula along with conformal invariance was proved in [18] for the site percolation on the triangular lattice:

Theorem 2. *For the critical site percolation on the triangular lattice, as mesh goes to zero crossing probabilities converge to a limit which is conformally invariant and satisfies Cardy's formula.*

We will start by sketching the proof. The method is different from that of Cardy, and the hypergeometric function arises in a different way. As we will see later, SLE gives this function yet another interpretation.

For this particular model it is known by results of Kesten and Wierman that $p_c = 1/2$, so each vertex of the triangular lattice with mesh δ (or equivalently each hexagon in the dual honeycomb lattice) is open or closed with equal probability $1/2$. The method utilizes self-duality of the model (there is a horizontal yellow crossing if and only if there is no vertical blue crossing) and the fact that $p_c = 1/2$ (which allows to change the colors to the opposite while preserving the probabilities).

Like Cardy, we fix a domain Ω and three boundary points a, b, c, and study the behavior of $\Pi(\Omega,[ab],[cz])$ as the forth point z moves. However we allow z to move inside domain Ω as well, considering a new function $H_a(z) := H(z,a,b,c,\Omega,\delta)$ which is the probability that a yellow path from the boundary arc ab to the boundary arc ca separates z from bc. On the arcs ab and ca this function becomes the crossing probability Π, whereas on the arc bc it vanishes. One defines functions H_b and H_c symmetrically. By Russo-Seymour-Welsh estimates (see [5, 18]) one can bound their Hölder norms, so as $\delta \to 0$ we can choose a uniformly converging subsequence.

It turns out that these three functions are (approximatively) discrete harmonic, and so their limits are harmonic. The discrete derivative of a function $\partial_\alpha H_a$ in the direction α is the difference $H_a(z') - H_a(z)$ for two neighboring sites z, z' with $z' - z = \alpha\delta$. Let Q denote the area above the lowest (i.e. closest to bc) yellow crossing from ab to ca. Then $H_a(z) = \mathbb{P}(z \in Q)$, and so

$$H_a(z') - H_a(z) \;=\; \mathbb{P}(z' \in Q) - \mathbb{P}(z \in Q) \;=\; \mathbb{P}(z' \in Q, z \notin Q) - \mathbb{P}(z \in Q, z' \notin Q)\,.$$

The two probabilities on the right hand side have an easy combinatorial interpretation: e.g. in the first case the lowest yellow crossing of hexagons passes though the edge zz', and by duality there is a blue crossing from z to the arc bc (which prevents existence of a lower yellow crossing). So the first term is the probability of three multicolored crossings from z to the three boundary arcs (whereas the second is the same for z').

A combinatorial argument, based on the fact that $p_c = 1/2$, and employed earlier by Aizenman, Duplantier, and Aharony [3], shows that such probabilities are independent of the colors of crossings, as long as both colors are present. If we change the colors from yellow–yellow–blue to blue–yellow–yellow, we will have a similar picture, but instead of crossing a yellow path between ab and ca in the direction α we will be crossing a yellow path between bc and ab in the rotated by $2\pi/3$ direction $\alpha\exp(2\pi i/3)$. So probabilities contributing to $\partial_\alpha H_a$ are identified with probabilities contributing to $\partial_{\alpha\exp(2\pi i/3)} H_b$.

In this way one arrives to Cauchy-Riemann equations (in the basis of cube roots of 1):

$$\partial_\alpha H_a \;\approx\; \partial_{\alpha\exp(2\pi i/3)} H_b \;\approx\; \partial_{\alpha\exp(4\pi i/3)} H_c\;. \tag{5}$$

This implies that the scaling limits of the functions H satisfy exactly such Cauchy-Riemann equations. Moreover, on every boundary arc we know the values of one of them: e.g. $H_a(a) = 1$, $H_a = 0$ on bc. Such boundary value problem has a unique solution, given in the half-plane by a complexification of Cardy's hypergeometric function. It is even easier to see this in Carleson's form: in an equilateral triangle these three functions are linear. Note also that this problem and hence its solution are conformally invariant.

Since the resulting limit is independent of subsequence chosen, we conclude that functions H have a scaling limit, which is conformally invariant and satisfies a complexification of Cardy's formula.

Once Cardy's formula is established, one can prove some statements about the scaling limit, [18].

For example, consider domain Ω with three boundary points a, b, c and the lowest (closest to bc) yellow path from the boundary arc ab to the boundary arc ca. By a priori estimates of Aizenman and Burchard [2] the family of the laws of the lowest crossings (for various values of the mesh) is weakly precompact, so every sequence has a weakly converging subsequence. To prove that there is a scaling limit, it is sufficient to show that this limit is independent of the chosen subsequence. But by Cardy's formula for any curve η we know the probability (in the limit) that the lowest crossing went completely below η, which is exactly the crossing probability for the domain obtained by cutting a part of Ω away along η. Such events generate (by disjoint unions and complements) the Borel σ-algebra of crossings with Hölder topology, and so the limiting law is determined uniquely. Moreover, it is conformally invariant since the events involved are.

Building upon this, one reasons similarly to show the existence and conformal invariance of the scaling limit for the perimeter of a cluster, i.e. an interface between two clusters of opposite color. It turns out that it is characterized by the same properties as Schramm's SLE(6), so they coincide. Alternatively one can aim from the beginning at proving that the interface converges to SLE(6) – an approach we discuss below. Once this connection is established, one can employ SLE in calculating percolation exponents.

3. Loewner Evolution

Loewner Evolution is a differential equation for a Riemann uniformization map for a domain with a growing slit. It was introduced by Loewner in [15] in his work on Bieberbach's conjecture. Loewner Evolution allows to write differentials of various functionals defined for planar domains when domain is perturbed by adding a slit, and it was successfully applied to many optimization problems.

There are two standard setups: *radial*, when the slit is growing towards a point inside the domain, and *chordal*, when the slit is growing towards a point on the boundary. In both cases we choose a particular Riemann map by fixing value and derivative at the target point. We will restrict our discussion to the chordal case, though radial is equally interesting in our context.

Chordal Loewner evolution describes uniformization for the upper half-plane $\mathbb{C}_+$ with a slit growing from 0 to ∞ (one deals with another domain Ω with boundary points a, b by mapping it to $\mathbb{C}_+$ so that $a, b \mapsto 0, \infty$).

Loewner dealt only with slits given by smooth simple curves, but more generally one allows any set which grows continuously in conformal metric when viewed from ∞. We will omit the precise definition of "allowed slits" (more extensive discussion in this context can be found in [10]), only noting that all simple curves are included. The random curves which arise from lattice models (e.g. cluster perimeters or interfaces) are simple curves. Their scaling limits are not necessarily simple, they have no "transversal" self-intersections. For such a curve to be an allowed slit it is sufficient to visit no point thrice.

Parameterizing the slit γ in some way by time t, we denote by $g_t(z)$ the conformal map sending $\mathbb{C}_+ \setminus \gamma_t$ to $\mathbb{C}_+$ normalized so that at infinity $g_t(z) = z + \alpha(t)/z + O(1/|z|^2)$, the so called *hydrodynamic normalization.* It turns out that $\alpha(t)$ is a continuous strictly increasing function (it is a sort of capacity-type parameter for γ_t), so one can change the time so that

$$g_t(z) \;=\; z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right) \tag{6}$$

Denote by $w(t)$ the image of the tip $\gamma(t)$. The family of maps g_t (also called a *Loewner chain*) is uniquely determined by the real-valued *"driving term"* $w(t)$. The general form of the Loewner's theorem can be stated as follows:

Loewner's theorem. *There is a bijection between allowed slits and continuous real valued functions $w(t)$ with $w(0) = 0$. This bijection is given by the differential equation*

$$\partial_t g_t(z) \;=\; \frac{2}{g_t(z) - w(t)} \;, \quad g_0(z) \;=\; z \;. \tag{7}$$

4. Schramm-Loewner Evolution

The Loewner's theorem states that a deterministic curve γ corresponds to a deterministic driving term $w(t)$. Similarly a random γ corresponds to a random $w(t)$. One obtains SLE(κ) by taking $w(t)$ to be a Brownian motion with speed κ:

Definition. *Schramm-Loewner Evolution*, or *SLE(κ)*, is the Loewner chain one obtains by taking $w(t) = \sqrt{\kappa} B_t$, $\kappa \in [0, \infty)$.

It is shown in [16] that the resulting slit will be almost surely a Hölder continuous curve. So we will also use the term SLE for the resulting random curve, i.e. a probability measure on the space of curves (to be rigorous one can think of a Borel measure on the space of curves with Hölder norm). Different speeds κ produce different curves, which become more "fractal" as κ increases: we grow the slit with constant speed (measured by capacity), while the driving term "wiggles" faster. For example, for $\kappa \leq 4$ the curve is a.s. simple, for $4 < \kappa < 8$ it a.s. touches itself, and for $\kappa \geq 8$ it is a.s. space-filling (i.e. visits every point in $\mathbb{C}_+$). See [16] for the discussion of basic properties of SLE's and references.

We follow Schramm to show that this choice of $w(t)$ arises naturally. Schramm decided to describe the scaling limits of cluster perimeters, or interfaces for lattice models assuming their existence and conformal invariance. This naturally led to introduction of SLE in [17].

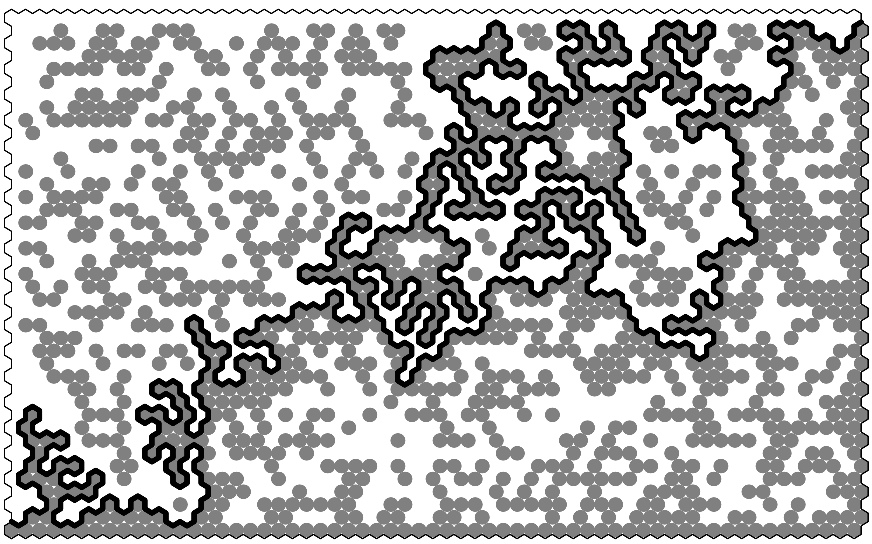


Figure 1. Critical site percolation on triangular lattice superimposed over a rectangle. Dobrushin boundary conditions produce the interface from the lower left to the upper right corner. The law of the interface converges to SLE(6) when mesh goes to zero.

The reasoning went similarly to the following: consider a simply connected domain Ω with two boundary points, a and b. Superimpose a lattice with mesh δ and consider some lattice model, say critical percolation with the Dobrushin boundary conditions, coloring the vertices blue on the boundary arc ab and yellow on the boundary arc ba.

This enforces existence (besides many loop interfaces) of an interface between yellow and blue clusters running from a to b, which is illustrated by Fig. 1 for a rectangle with two opposite corners as a and b. So we end up with a random simple curve (a broken line) connecting a to b inside Ω. The law of the curve depends of course on the lattice superimposed. If we believe the physicists' predictions, as mesh tends to zero, this law converges (in a weak-$*$ topology) to some law $\Lambda = \Lambda(\Omega, a, b)$ on curves from a to b inside Ω.

Furthermore, in this setup the conformal invariance prediction can be formulated as follows:

(A) Conformal Invariance: *The law is conformally invariant: for a conformal map ϕ of the domain Ω one has*

$$\phi\left(\Lambda(\Omega, a, b)\right) \;=\; \Lambda\left(\phi(\Omega), \phi(a), \phi(b)\right) \;.$$

Here a bijective map $\phi : \Omega \to \phi(\Omega)$ induces a map acting on the curves in Ω, which in turn induces a map on the probability measures on the space of such curves, which we denote by the same letter. By a conformal map we understand a bijection which locally preserves angles, i.e. is analytic or anti-analytic function (so it might change the orientation).

Moreover, if we start drawing the interface from the point a, we will be walking around the yellow cluster following the left-hand rule – see Fig. 1. If we stop at some point a' after drawing the part γ' of the interface, we cannot distinguish the boundary of Ω from the part of the interface we have drawn: they both are colored yellow on arc $a'b$ and blue on arc ba' of the domain $\Omega \setminus \gamma'$. So we can say that the conditional law of the interface (conditioned on it starting as γ') is the same as the law in a new domain with a slit. We expect the limit law Λ to have the same property:

(B) Markov-type property: *The law conditioned on the interface already drawn is the same as the law in the slit domain:*

$$\Lambda\left(\Omega, a, b\right)|\gamma' \;=\; \Lambda\left(\Omega \setminus \gamma', a', b\right)$$

If one wants to utilize these properties to characterize Λ, by (A) it is sufficient to study some reference domain (to which all others can be conformally mapped), say the upper half-plane $\mathbb{C}_+$ with a curve running from 0 to ∞.

Given (A), the second property (B) is easily seen to be equivalent to the following:

(B') Conformal Markov property: *The law conditioned on the interface is a conformal image of the original law. Namely, if $G = G_{\gamma'}$ is a conformal map from $\mathbb{C}_+ \setminus \gamma'$ to $\mathbb{C}_+$ preserving ∞ and sending the tip of γ' to 0, then*

$$\Lambda\left(\mathbb{C}_+, 0, \infty\right)|\gamma' \;=\; G^{-1}\left(\Lambda\left(\mathbb{C}_+, 0, \infty\right)\right) \;.$$

To use the property (B'), we describe the random curve by the Loewner evolution with a certain random driving force $w(t)$ (we assume that the curve is a.s. an allowed slit). If we fix the time s, the property (B') with the slit $\gamma[0, s]$ and the map$G(z) = g_s(z) - w(s)$ can be rewritten for random conformal map G_{t+s} conditioned on G_s (which is the same as conditioning on γ') as

$$G_{t+s}|G_s \;=\; G_s\left(G_t\right) \;.$$

Expanding G's near infinity we obtain

$$z - w(t+s) + \dots | G_s \;=\; (z - w(s) + \dots) \circ (z - w(t) + \dots) \;=\; z - (w(s) + w(t)) + \dots ,$$

concluding that

$$w(t+s) - w(s) | G_s \;=\; w(t) \;.$$

This means that $w(t)$ is a continuous (by Loewner's theorem) stochastic process with independent stationary symmetric (apply (A) with anti-conformal reflection $\phi(u+iv) = -u+iv$) increments. Thus $w(t)$ has to be a Brownian motion with certain speed $\kappa \in [0, \infty)$: $w(t) = \sqrt{\kappa} B_t$.

So one logically arrives at the definition of SLE, and what we call

Schramm's principle. *A random curve satisfies (A) and (B) if and only if it is given by SLE(κ) with certain $\kappa \in [0, \infty)$.*

The discussion above is essentially contained in Schramm's [17] for the radial version, when slit is growing towards a point inside and the Loewner differential equation takes a slightly different form. To make this principle a rigorous statement, one has to ask the curve to be almost surely an allowed slit.

In order to use the above principle one still has to show the *existence* and *conformal invariance* of the scaling limit, and then calculate some observable to pin down the value of κ. For percolation one can employ locality property or Cardy's formula to show that $\kappa = 6$ (we will discuss below why SLE(6) is the only SLE matching Cardy's formula). So Schramm concluded in [17] that *if* percolation interface has a conformally invariant scaling limit, it must be SLE(6).

5. SLE as a scaling limit

One way to show that certain random curve coincides with SLE is to determine infinitely many observables, which could prove difficult (for percolation, locality helps to create many observables from just one).

Fortunately, the situation turns out to be much nicer: if one can show that just one (non-trivial) observable has a limit satisfying analogues of (A) and (B), convergence to SLE(κ) (with κ determined by the values of the observable) follows.

This was demonstrated by Lawler, Schramm and Werner in [14] in establishing the convergence of two related models: of Loop Erased Random Walk to SLE(2) and of Uniform Spanning Tree to SLE(8). We describe how a modification of their ideas gives another proof that percolation perimeter converges to SLE(6).

For percolation on a fixed lattice in a domain Ω with boundary points a, b and Dobrushin boundary conditions consider interface running from a to b, see Fig. 1. Already mentioned estimates [2, 1] imply that collection of interface laws on lattices with different mesh is precompact (in the weak-$*$ topology on the space of Borel measures on Hölder continuous curves). So to show that as mesh goes to zero the interface law converge to the law of SLE(6), it is sufficient to show that the limit of any converging subsequence is in fact SLE(6).

Take some converging subsequence, whose limit is a random curve γ with law Λ. Though γ is a scaling limit of a simple curve, γ itself will a.s. be non-simple. But known a priori estimates [1] show that γ a.s. visits no point thrice and so is an allowed slit. So we can (by mapping Ω to $\mathbb{C}_+$ and applying Loewner's theorem) describe γ by a Loewner evolution with a (random) driving force $w(t)$. It remains to show that $w(t) = \sqrt{6}B_t$.

Add two more points on the boundary, making Ω a topological rectangle $axby$ and consider the crossing probability $\Pi_\delta\left(\Omega, [a,x], [b,y]\right)$ (from the arc ax to the arc by on a lattice with mesh δ).

Parameterize the curve γ in some way by time and assume t to be small enough so that $\gamma[0,t]$ does not reach x, y. The crossing probability conditioned on $\gamma[0,t]$ coincides with crossing probability in the slit domain $\Omega \setminus \gamma[0,t]$, (an analogue of the property (A)) we can write by the total probability theorem

$$\Pi_\delta\left(\Omega, [a,x], [b,y]\right) = \mathbb{E}\left(\Pi_\delta\left(\Omega \setminus \gamma[0,t], [\gamma(t), x], [b,y]\right)\right) \ . \tag{8}$$

If enough a priori estimates are available, identity (8) also holds for the scaling limit $\Pi := \lim_{\delta\to 0}\Pi_\delta$ of the crossing probabilities, which we know to exist, be conformally invariant, and satisfy Cardy's formula. Mapping to half-plane and applying conformal invariance for the map $g_t(z) - w(t)$ we write

$$\begin{aligned}\Pi\left(\mathbb{C}_+, [0,x], [\infty,y]\right) &= \mathbb{E}\left(\Pi\left(\mathbb{C}_+ \setminus \gamma[0,t], [\gamma(t), x], [\infty, y]\right)\right)\\ &= \mathbb{E}\left(\Pi\left(\mathbb{C}_+, [0, g_t(x) - w(t)], [\infty, g_t(y) - w(t)]\right)\right) \ . \end{aligned} \tag{9}$$

By conformal invariance (under Möbius transformation $z \mapsto (z-y)/(x-y)$), and Cardy's formula

$$\Pi\left(\mathbb{C}_+, [0,x], [\infty,y]\right) \ = \ \Pi\left(\mathbb{C}_+, \left[-\frac{y}{x-y}, 1\right], [\infty, 0]\right) \ = \ F\left(\frac{x}{x-y}\right) \ ,$$

for Cardy's hypergeometric function F. Rewriting in this way both sides of the equation (9) we arrive at

$$F\left(\frac{x}{x-y}\right) \ = \ \mathbb{E}\, F\left(\frac{g_t(x) - w(t)}{g_t(x) - g_t(y)}\right) \ .$$

This provides some information about $w(t)$, but it is difficult to use before we get rid of g_t. To that purpose we fix the ratio $x/(x-y) =: 1/3$ (anything not equal to $1/2$ would do) and let x tend to infinity: $y := -2x$, $x \to +\infty$. Using the normalization $g_t(z) = z + 2t/z + O(1/z^2)$ at infinity, writing Taylor expansion for F, and plugging in values of derivatives of F at $1/3$, we obtain

$$\begin{aligned}
F\left(\frac{1}{3}\right) &= \mathbb{E}\, F\left(\frac{g_t(x)-w(t)}{g_t(x)-g_t(-2x)}\right) \\
&= \mathbb{E}\, F\left(\frac{x-w(t)+2t/x+O(1/x^2)}{(x+2t/x+O(1/x^2))-(-2x+2t/(-2x)+O(1/x^2))}\right) \\
&= \mathbb{E}\left(F\left(\frac{1}{3}-\frac{w(t)}{3}\frac{1}{x}+\frac{t}{3}\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right)\right) \\
&= \mathbb{E}\left(F\left(\frac{1}{3}\right)-\frac{w(t)}{3}F'\left(\frac{1}{3}\right)\frac{1}{x}+\left(\frac{t}{3}F'\left(\frac{1}{3}\right)+\frac{w(t)^2}{3^2\cdot 2}F''\left(\frac{1}{3}\right)\right)\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right) \\
&= F\left(\frac{1}{3}\right)-\frac{1}{x}\frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\frac{3^{1/3}}{2^{2/3}}\mathbb{E}\, w(t) \\
&\quad -\frac{1}{x^2}\frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\frac{1}{3^{2/3}2^{5/3}}\mathbb{E}\left(w(t)^2-6t\right)+O\left(\frac{1}{x^3}\right) .
\end{aligned}$$

Observing that coefficients by $1/x$ and $1/x^2$ on the right hand side should vanish, we conclude that

$$\mathbb{E}\, w(t) \;=\; 0\ , \quad \mathbb{E}\, w(t)^2-6t \;=\; 0\ . \tag{10}$$

Applying the same reasoning to domain $\Omega\setminus\gamma[0,t]$ relative to $\Omega\setminus\gamma[0,s]$ gives identities (10) for the increments of $w(t)$. Thus $w(t)$ is a continuous (by Loewner's theorem) process such that both $w(t)$ and $w(t)^2-6t$ are local martingales so by Lévy's theorem $w(t)=\sqrt{6}B_t$, and therefore SLE(6) is the scaling limit of the critical percolation interface. The argument will work wherever Cardy's formula and a priori estimates are available, particularly for triangular lattice, giving the following [18]:

Theorem 3. *Consider the site percolation on the triangular lattice in a simply connected domain Ω with boundary points a and b*

and Dobrushin boundary conditions. Then the interface running from a to b has a conformally invariant scaling limit, which coincides with SLE(6).

6. Calculations for SLE

The value $\kappa=6$ appears in the argument above because of the specific values of the conformal martingale considered. The above reasoning also indicates that very few functions can arise as conformal invariant martingales for SLE's. Indeed, this was used by Lawler, Schramm, and Werner to determine the values of various probabilities and expectations which are conformally invariant or covariant.

We will sketch one of their calculations, showing that assuming convergence of the percolation interface to SLE(6) one can derive Cardy's formula from the properties of SLE. Of course, Cardy's formula was used to establish the convergence in the first place. But the argument below provides new insight into it and can also be applied in many other situations.

The scaling limit of the crossing probability $\Pi\left(\mathbb{C}_+,[0,x],[\infty,y]\right)$ is equal to the probability that SLE(6) (which is the scaling limit of the percolation interface) touches the interval

$[x,+\infty]$ before the interval $[-\infty,y]$. This can be seen in the Fig. 1: the vertical gray crossing forces the interface to touch the upper side of the rectangle before it touches the right side.

We will calculate similar probability for SLE(κ), which is driven by $w(t) = \sqrt{\kappa}B_t$. Denote this probability by Π_κ. By conformal invariance (under the Möbius transformation $z \mapsto (z-y)/(x-y)$), it depends only on $x/(x-y)$:

$$\Pi_\kappa\left(\mathbb{C}_+,[0,x],[\infty,y]\right) \;=\; \Pi_\kappa\left(\mathbb{C}_+,\left[1-\frac{x}{x-y},1\right],[\infty,0]\right) \;=:\; F_\kappa\left(\frac{x}{x-y}\right) \;. \qquad (11)$$

Denote $X_t := g_t(x)-\sqrt{\kappa}B_t$, $Y_t := g_t(y)-\sqrt{\kappa}B_t$, $U_t := X_t/(X_t-Y_t)$. The function $\Pi_\kappa\left(\mathbb{C}_+,[0,x],[\infty,y]|\gamma[0,t]\right)$ is a local martingale (with respect to the usual filtration for B_t). Applying conformal invariance for the map $g_t(z)-\sqrt{\kappa}B_t$ as before, we write

$$\begin{aligned}\Pi_\kappa\left(\mathbb{C}_+,[0,x],[\infty,y]|\gamma[0,t]\right) &= \Pi_\kappa\left(\mathbb{C}_+\setminus\gamma[0,t],[\gamma(t),x],[\infty,y]\right)\\ &= \Pi_\kappa\left(\mathbb{C}_+,[0,g_t(x)-\sqrt{\kappa}B_t],[\infty,g_t(y)-\sqrt{\kappa}B_t]\right)\\ &= \Pi_\kappa\left(\mathbb{C}_+,[0,X_t],[\infty,Y_t]\right) \;=\; F_\kappa\left(U_t\right)\;.\end{aligned}$$

Thus $F_\kappa(U_t)$ is a local martingale. Let us calculate its differential. By Loewner equation,

$$dX_t \;=\; \frac{2}{X_t}dt-\sqrt{\kappa}dB_t\;,\quad dY_t \;=\; \frac{2}{Y_t}dt-\sqrt{\kappa}dB_t\;.$$

So by Itô's formula

$$\begin{aligned}dU_t = d\frac{X_t}{X_t-Y_t} \;&=\; 2\frac{X_t+Y_t}{X_tY_t(X_t-Y_t)}dt-\frac{\sqrt{\kappa}}{X_t-Y_t}dB_t\\ &= 2\frac{(X_t+Y_t)(X_t-Y_t)}{X_tY_t}ds-\sqrt{\kappa}dB_s \;=\; 2\frac{1-2U_t}{U_t(1-U_t)}ds-\sqrt{\kappa}dB_s\,, \end{aligned}\qquad (12)$$

where we changed the time so that $ds = dt/(X_t-Y_t)^2$.

Applying Itô's formula to F_κ (one also has to prove that it is smooth) we get

$$\begin{aligned}dF_\kappa(U_t) &= F'(U_t)\left(2\frac{1-2U_t}{U_t(1-U_t)}ds-\sqrt{\kappa}dB_s\right)+\frac{F''(U_t)}{2}\kappa ds\\ &= \left(2\frac{1-2U_t}{U_t(1-U_t)}F'(U_t)+\frac{\kappa}{2}F''(U_t)\right)ds-\sqrt{\kappa}F'(U_t)dB_s\,.\end{aligned}$$

Since $F_\kappa(U_t)$ is a martingale, the drift term has to vanish, leading to the following differential equation for F_κ (which is similar to the Cardy's equation):

$$2\frac{1-2u}{u(1-u)}F'_\kappa(u)+\frac{\kappa}{2}F''_\kappa(u) \;=\; 0\;.$$

This equation can be integrated by writing

$$\left(\log F'_\kappa(u)\right)' = \frac{F''_\kappa(u)}{F'_\kappa(u)} = -\frac{4}{\kappa}\frac{1-2u}{u(1-u)} = \left(-\frac{4}{\kappa}\log\left(u(1-u)\right)\right)'\;,$$

which leads to

$$F_\kappa(u) \;=\; C_1+C_2\int_0^u \left(v(1-v)\right)^{-4/\kappa}dv\;.$$

From obvious boundary conditions $F_\kappa(0) = 0$, $F_\kappa(1) = 1$ we derive the values of constants, arriving at

$$F_\kappa(u) \;=\; \int_0^u (v(1-v))^{-4/\kappa}\, dv \Big/ \int_0^1 (v(1-v))^{-4/\kappa}\, dv \;\; .$$

For $\kappa = 6$ (and only for this value of κ) solution coincides with Cardy's hypergeometric function. For $\kappa \le 4$ there is no solution, reflecting the fact that SLE is then a.s. a simple curve which a.s. doesn't touch the boundary.

7. Scaling exponents

Calculations for SLE like the one above and knowledge about percolation can be combined to establish the values of mentioned scaling exponents. The method is explained in [19], and proceeds as follows.

For a variety of percolation models Kesten has reduced in [8] most of the predictions about behavior near p_c to establishing the so called one- and four-arm exponents for critical percolation. Namely, one has to prove that on the lattice with mesh δ the probability of a yellow path extending distance ≥ 1 from the δ-neighborhood of the origin is $\asymp \delta^{5/48}$, while similar probability for two yellow and two blue paths simultaneously is $\asymp \delta^{4/3}$, $\delta \to 0$. Kesten's proof is quite involved and based on differential inequalities.

These probabilities can be expressed in terms of the percolation interfaces, and with some technical work, it can be shown that they have the same asymptotics as similar probabilities for the interface scaling limit, which is SLE(6) by [18]. The required 5/48 and 4/3 exponents for SLE(6) are established by Lawler, Schramm, and Werner in [11, 12, 13].

8. Conclusions

We described advances related to percolation on triangular lattice in the plane. It is interesting that the mathematical progress did not come along the lines of theoretical physics arguments, but from the different new directions. But as the physicists' approaches before, all of the described techniques use conformal invariance in an essential way and therefore are two-dimensional in nature. So unfortunately the progress made does not seem to have implications in higher dimensions.

As for the other planar models, the Schramm-Loewner Evolution describes all possible conformally invariant scaling limits for interfaces. Calculation for other values of κ is not much different from the case $\kappa = 6$: most problems can be reduced via stochastic analysis to solving ordinary or partial differential equations. Of course, some of the resulting PDE's might be non-integrable. But it seems that more problems can be solved in this way than by theoretical physics' methods. For example, the so called "backbone exponent" for percolation which was out of reach for physicists was proven by Lawler, Schramm, and Werner [13] to coincide with the main eigenvalue of a certain PDE. Unfortunately, this eigenvalue does not seem to admit a nice formula.

However some difficulties might arise with technical points in calculations: SLE(6) shares the locality property of percolation, so writing estimates is easier. Moreover, the relation between various scaling exponents, which was rigorously established for percolation by Kesten is mostly missing for other models, where it might be technically more difficult.

The remaining part, establishing that other models have conformally invariant scaling limits, seems even more challenging. Lawler, Schramm, and Werner [14] have shown that the Loop Erased Random Walk converges to SLE(2), while a related model, perimeter of the Uniform Spanning Tree, converges to SLE(8). Recently Schramm and Sheffield defined a new random curve, Harmonic Explorer, which converges to SLE(4).

In all these cases, like in percolation, some observables were shown to be discrete harmonic functions with various boundary conditions implying existence of a conformally invariant harmonic limit. There are two differences though: for these models discrete observables turn out to be exactly harmonic (for percolation only approximatively), and the proofs (similar to one sketched above for percolation) rely heavily on Loewner Evolution, whereas for percolation one can construct the scaling limit without appealing to SLE.

Despite extensive predictions, similar problems are open for other critical lattice models in the plane. Probably there one should also look for discrete harmonic or analytic observables. A few cases which stand out are:

- Other critical percolation models: interface should also converge to SLE(6). Bond percolation on square lattice and Voronoi percolation share some duality properties of the site percolation on triangular lattice, so one might start with those.
- Ising model at critical temperature: interface should converge to SLE(3). Here Kenyon's approach to dimer models provides some insight, but technical estimates are still missing.
- Self Avoiding Random Walk: it should converge to SLE(8/3), which also describes Brownian Frontier or percolation lowest crossing. Detailed study of SLE(8/3) by Lawler, Schramm, and Werner is discussed in Lawler's paper in this volume.

There are also conjectures for the full spectrum of Q-state Potts models and Fortuin-Kasteleyn models. We refer the reader to expository works [6, 10, 20] for more details.

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References

1. M. Aizenman, The geometry of critical percolation and conformal invariance, In *STATPHYS 19 (Xiamen, 1995)*, pages 104–120, World Sci. Publishing, River Edge, NJ, 1996.
2. M. Aizenman, A. Burchard, Hölder regularity and dimension bounds for random curves. *Duke Math. J.* **99**, 419–453 (1999).
3. M. Aizenman, B. Duplantier, A. Aharony, Path crossing exponents and the external perimeter in 2D percolation. *Phys. Rev. Let.* **83**, 1359–1362 (1999).
4. J. L. Cardy, Critical percolation in finite geometries, *J. Phys. A* **25**, L201–L206 (1992).
5. G. Grimmett, *Percolation*, Springer-Verlag, Berlin, second edition, 1999.
6. W. Kager, B. Nienhuis, A Guide to Stochastic Löwner Evolution and its Applications, *J. of Statistical Physics*, to appear, arXiv:math-ph/0312056.
7. H. Kesten, *Percolation theory for mathematicians*, Birkhaüser, Boston, 1982.

8. H. Kesten, Scaling relations for 2D-percolation, *Comm. Math. Phys.* **109**, 109–156 (1987).
9. R. Langlands, Ph. Pouliot, Y. Saint-Aubin, Conformal invariance in two-dimensional percolation, *Bull. Amer. Math. Soc. (N.S.)* **30**, 1–61 (1994).
10. G. F. Lawler, Conformally invariant processes in the plane, book in progress, available from http://www.math.cornell.edu/~lawler/books.html
11. G. F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents I: Half-plane exponents, *Acta Mathematica* **187** 237–273 (2001).
12. G. F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents II: Plane exponents, *Acta Mathematica* **187**, 275–308 (2001).
13. G. F. Lawler, O. Schramm, W. Werner, One-arm exponent for $2D$ critical percolation, *Electron. J. Probab.* **7**, 13 pages (2001).
14. G. F. Lawler, O. Schramm, W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Ann. Prob.*, to appear, arXiv: math.PR/0112234.
15. K. Löwner, Untersuchungen über schlichte konforme Abbildung des Einheitskreises, I. *Math. Ann.* **89**, 103–121 (1923).
16. S. Rohde, O. Schramm, Basic properties of SLE, *Ann. Math.*, to appear, arXiv: math.PR/0106036 (2001).
17. O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* **118**, 221–288 (2000).
18. S. Smirnov, Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits, *C. R. Acad. Sci. Paris* **333**, 239–244 (2001).
19. S. Smirnov, W. Werner, Critical exponents for two-dimensional percolation, *Math. Res. Lett.* **8**, 729-744 (2001).
20. W. Werner, Random planar curves and Schramm-Löwner Evolutions, *Lecture notes from the 2002 Saint-Flour summer school*, LNM, Springer, to appear, arXiv: math.PR/0303354.