

## ON LITTLEWOOD'S CONSTANTS

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### ABSTRACT

In two papers, Littlewood studied seemingly unrelated constants: (i) the best  $\alpha$  such that for any polynomial  $f$ , of degree  $n$ , the areal integral of its spherical derivative is at most  $\text{const} \cdot n^\alpha$ , and (ii) the extremal growth rate  $\beta$  of the length of Green's equipotentials for simply connected domains. These two constants are shown to coincide, thus greatly improving known estimates on  $\alpha$ .

### 1. Introduction

In this paper, we study the growth rate as  $n \rightarrow \infty$  of the quantity

$$A_n = \sup \int_{\mathbb{D}} \frac{|g'|}{1 + |g|^2} dm,$$

where the supremum is taken over all polynomials  $g$  of degree  $n$ ,  $\mathbb{D}$  is the unit disc  $\{|z| < 1\}$ , and  $m$  denotes the two-dimensional Lebesgue measure. We are interested in the best  $\alpha$  such that  $A_n \lesssim n^\alpha$  (which means that for every  $\varepsilon > 0$ , there is a constant  $C_\varepsilon$  with  $A_n \leq C_\varepsilon n^{\alpha+\varepsilon}$ ). In [16], Littlewood observed that  $0 \leq \alpha \leq 1/2$ , and he conjectured that  $\alpha < 1/2$ . The problem of determining the best possible  $\alpha$  appears under the number 4.18 in Hayman's problem list [11].

It is easy to show that there is a constant  $c$  such that for any rational function  $g$  of degree  $n$ ,

$$\int_{\mathbb{D}} \frac{|g'|}{1 + |g|^2} dm \leq c\sqrt{n}.$$

Note that the integrand is a modulus of the spherical derivative  $g'_\sigma$  of  $g$  (that is, the derivative with respect to the spherical metric), and that in  $\mathbb{D}$  the spherical measure  $dm_\sigma$  is comparable to the Lebesgue measure  $dm$ . So our integral can be estimated by

$$\begin{aligned} \int_{\mathbb{D}} |g'_\sigma| dm_\sigma &\leq \int_{\mathbb{C}} |g'_\sigma| dm_\sigma \\ &\leq \left( \int_{\mathbb{C}} |g'_\sigma|^2 dm_\sigma \right)^{1/2} \left( \int_{\mathbb{C}} dm_\sigma \right)^{1/2} \\ &= (2\pi n)^{1/2} (2\pi)^{1/2} \\ &= 2\pi\sqrt{n}. \end{aligned}$$

Here we use the fact that a rational function of degree  $n$  maps the complex sphere to itself  $n$ -to-1, so the area of the image is  $n$  times bigger than the area of the sphere. In particular, this argument shows that  $\alpha \leq 1/2$ .

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Littlewood’s conjecture was proved in [14] by Lewis and Wu: improving upon the work of Eremenko and Sodin [8], they obtained an explicit upper estimate  $\alpha < 1/2 - 2^{-264}$ . Later, Eremenko obtained in [7] a positive lower bound on  $\alpha$ . Following the work by Eremenko and Sodin [8] and Lewis and Wu [14], we exploit a connection between this problem and the extremal behavior of the harmonic measure.

Our main result is that  $\alpha$  is related to the growth rate of the length of Green’s lines. In the case of simply connected domains  $\Omega$ , we define  $\beta_\Omega$  as

$$\limsup_{\varepsilon \rightarrow 0} \frac{\log \text{length}\{z : G(z) = \varepsilon\}}{\log 1/\varepsilon},$$

where  $G$  is Green’s function with a pole at infinity, and we define

$$\beta = \sup \beta_\Omega,$$

where the supremum is taken over all simply connected domains  $\Omega$ . In the non-simply connected case, one needs a more elaborate definition, and we use the multi-fractal analysis technique.

For a given domain  $\Omega$ , we define the *packing spectrum*  $\pi_\Omega(t)$  as

$$\sup \left\{ q : \text{for all } \delta > 0, \text{ there exists a } \delta\text{-packing}\{B\} \text{ with } \sum \text{diam}(B)^t \omega(B)^q \geq 1 \right\},$$

where  $\omega$  is the harmonic measure in  $\Omega$ , and a  $\delta$ -packing is a collection of disjoint open sets whose diameters do not exceed  $\delta$ . Note that this definition is valid for any domain with compact boundary, and for  $t = 1$  it is analogous to  $\beta_\Omega$  (see [18] for the proof of  $\beta_\Omega = \pi_\Omega(1)$  for simply connected  $\Omega$ ).

We define the *universal spectrum*  $\pi(t)$  as the supremum of  $\pi_\Omega(t)$  over all planar domains  $\Omega$  with compact boundary.

The main result of this paper is the following theorem.

MAIN THEOREM. *For any positive  $\varepsilon$ , there exists a constant  $c = c(\varepsilon)$  such that*

$$A_n \leq cn^{\pi(1)+\varepsilon}.$$

Equivalently,

$$\alpha \leq \pi(1).$$

REMARK. Our proof actually implies that for  $t \in [0, 2]$ ,

$$\int_{\mathbb{D}} \left( \frac{|g'|}{1 + |g|^2} \right)^t \leq cn^{\pi(2-t)+\varepsilon}.$$

Also of interest to us are  $\pi_p(t)$  and  $\pi_{p,sc}(t)$ , which are, respectively, the suprema of  $\pi_\Omega(t)$  over all the domains of attraction to infinity for polynomial mappings, and over the simply connected domains of attraction to infinity for polynomial mappings (see [5] for background material on complex dynamics). It is clear that  $\pi_{p,sc} \leq \pi_p \leq \pi$ , but a priori they might differ.

In [7], Eremenko essentially proved that  $\pi_{p,sc}(1) \leq \alpha$ . (He works under the assumption that the polynomials are hyperbolic, but this can easily be avoided.) Binder, Makarov, and Smirnov show in [4] that  $\pi_{p,sc}(t) = \pi_p(t)$  for  $t > 0$ . Recently, Binder and Jones announced a proof of the identity  $\pi_p(t) = \pi(t)$ , which, together with our theorem, completes the circle:

$$\alpha \leq \pi(1) = \pi_p(1) = \pi_{p,sc}(1) \leq \alpha.$$

There is yet another growth (or, rather, decay) rate that is related to  $\alpha$ ; this was also studied by Littlewood. The growth rate  $\gamma$  of coefficients of univalent functions in  $\mathbb{D}^-$  is defined by

$$\gamma := \sup_{\phi} \limsup_{n \rightarrow \infty} \frac{\log |b_n|}{\log n} + 1,$$

where the first supremum is taken over all functions

$$\phi(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

that are univalent in  $\mathbb{D}^-$ . Littlewood proved (see [15, 17]) that  $\beta \geq \gamma$ . Much later, Carleson and Jones [6] showed that  $\gamma = \beta$ . Summing it all up, we arrive at the following corollary.

COROLLARY. *We have*

$$\alpha = \beta = \gamma = \pi(1) = \pi_{p,sc}(1). \tag{1.1}$$

The corollary uses the as yet unpublished result of Binder and Jones mentioned above. Note that a well-known conjecture (see [6, 18, 13, 20]) states that  $\pi(t) = (2 - t)^2/4$  for  $|t| \leq 2$ ; in particular,  $\alpha = \beta = \gamma = 1/4$ . The best published estimates to date for  $\beta$  are

$$0.17 \stackrel{[17]}{<} \beta \stackrel{[8]}{<} 0.4884,$$

so the same estimates hold for  $\alpha$ , which is a significant improvement over the previously known

$$1.11 \cdot 10^{-5} \stackrel{[2]}{\leq} \alpha \stackrel{[14]}{\leq} 1/2 - 2^{-264}.$$

Recently, Hedenmalm and Shimorin released a preprint [12] with the estimate  $\beta < 0.46$ . The authors have also recently obtained an estimate from below:  $\beta > 0.23$  (see [2, 3]).

### 1.1. Connection to the value distribution of entire functions

Before giving the proof of our main theorem, we would like to note that the reason for Littlewood's interest in this problem was the following striking corollary to his conjecture (see [16]).

LITTLEWOOD'S CONDITIONAL THEOREM. *Assume that  $\alpha < 1/2$ . If  $f$  is an entire function of order  $0 < \rho < \infty$ , then for any  $0 < \theta < 1/2 - \alpha$  there is a 'small' set  $S$  such that 'almost all' roots of any equation  $f(z) = w$  lie in  $S$ . In other words,*

$$\frac{\text{Area}(S \cap B(R))}{\text{Area}(B(R))} = O\left(\frac{1}{R^{2\theta\rho}}\right), \quad R \rightarrow \infty,$$

and for all  $w$ ,

$$\frac{\#\{z \in B(R) \setminus S : f(z) = w\}}{\#\{z \in B(R) : f(z) = w\}} = O\left(\frac{1}{R^{\rho(1/2 - \alpha - \theta)}}\right), \quad R \rightarrow \infty,$$

where  $B(R)$  is a disc of radius  $R$ , centered at the origin.

2. Proof of the main theorem

It is a standard fact (for details, see [18] and [9]) that  $\pi(t)$  is finite, convex, and strictly decreasing on  $[0, 2]$ . Hence, for any small  $\delta$ , we can choose  $\varepsilon$  so small that

$$\pi(1 - 2\varepsilon) - \pi(1) < \delta.$$

We will also use a more elaborate fact (which follows from multifractal formalism and fractal approximation – see [18]), that there is a constant  $\text{const}(t, \varepsilon)$ , such that for any disjoint collection of cubes  $\{Q\}$  of size at most 1, one has

$$\sum_Q \omega(Q)^{\pi(t)} l(Q)^{t+\varepsilon} < \text{const}(t, \varepsilon).$$

Let  $g$  be a polynomial of degree at most  $n$ , and  $a_i$  its zeros. Consider a set where  $|g|$  is big; that is,  $|g| \geq n$ . We can easily estimate the integral over this set by

$$\begin{aligned} \int_{\mathbb{D} \cap \{|g| \geq n\}} \frac{|g'|}{1 + |g|^2} &\leq n^{-1} \int_{\mathbb{D}} \frac{|g'|}{|g|} = n^{-1} \int_{\mathbb{D}} |(\log g)'| \\ &= n^{-1} \int_{\mathbb{D}} \left| \sum_{i=1}^n \frac{1}{z - a_i} \right| \leq n^{-1} \cdot 2\pi n = 2\pi. \end{aligned} \tag{2.1}$$

Now consider a complementary set, where  $|g|$  is small, which is contained inside the disc of radius  $3/2$ :

$$\Omega := \left\{ z : |g(z)| < n, |z| < \frac{3}{2} \right\},$$

and let  $W = \{Q_j\}$  be a Whitney decomposition of  $\Omega$ . We note that

$$d\mu(z) = 4n^{-1} \frac{|g'(z)|^2}{(1 + |g(z)|^2)^2} dx dy$$

is the Riesz measure associated with the nonnegative subharmonic function

$$u = \frac{\log(1 + |g|^2)}{n}.$$

Then, by the Riesz representation theorem,

$$\mu(B(z, r)) \leq \frac{c}{2\pi} \int_0^{2\pi} (u(z + 2re^{i\theta}) - u(z)) d\theta. \tag{2.2}$$

Hence, for every cube  $Q_j$ , we have

$$\mu(Q_j) \leq c \frac{\log(1 + n)}{n}.$$

Fix a cube  $Q_j$  such that  $Q_j \cap \mathbb{D} \neq \emptyset$ , and denote by  $\xi_j$  a point at  $\partial\Omega$  such that  $d(\xi_j, Q_j) \leq 2l(Q_j)$ . Then, from (2.2),

$$\begin{aligned} \mu(Q_j) &\leq \frac{c}{2\pi} \int_0^{2\pi} (u(\xi_j + 8l(Q_j)e^{i\theta}) - u(\xi_j)) d\theta \\ &\leq \max_{z \in B(\xi_j, 8l(Q_j))} c[u(z) - u(\xi_j)]^+. \end{aligned} \tag{2.3}$$

Denote by  $G(z)$  the Green function for  $\mathbb{C} \setminus \bar{\Omega}$  with a pole at infinity. Extend  $G$  to a continuous subharmonic function in  $\mathbb{C}$  by setting  $G = 0$  on  $\Omega$ .

By the maximum principle, for the domain  $\mathbb{C} \setminus \overline{\Omega}$  we obtain  $G(z) \geq \log|2z/3|$  for any  $z \in \mathbb{C} \setminus \Omega$ ; hence

$$G(z) \geq \log \frac{4}{3} \quad \text{for } |z| = 2.$$

By the maximum principle, for the domain  $2\mathbb{D} \setminus \overline{\Omega}$  we have

$$u(z) - u(\xi_j) \leq M_2 G(z) \left(\log \frac{4}{3}\right)^{-1} \quad \text{for } |z| \leq 2,$$

where  $M_2 = \max_{|z|=2} u(z)$ .

If we let  $z_j$  be a center of  $Q_j$ , and  $\omega$  a harmonic measure on  $\mathbb{C} \setminus \overline{\Omega}$  with a pole at infinity, then by the previous inequality and (2.3) we have

$$\begin{aligned} \mu(Q_j) &\leq \max_{z \in B(\xi_j, 8l(Q_j))} c[u(z) - u(\xi_j)]^+ \\ &\leq c \left(\log \frac{4}{3}\right)^{-1} M_2 \max_{z \in B(\xi_j, 8l(Q_j))} G(z) \\ &\leq cM_2 \max_{z \in B(z_j, 16l(Q_j))} G(z). \end{aligned}$$

By Harnack's inequality, the right-hand side is less than

$$cM_2 3 \int_{\partial B(z_j, 32l(Q_j))} G(z) \frac{|dz|}{2\pi 32l(Q_j)},$$

which, by the Riesz representation formula, is equal to

$$cM_2 \int_0^{32l(Q_j)} \frac{\omega(B(z_j, t))}{t} dt \leq cM_2 \omega(B(z_j, 32l(Q_j))).$$

So, finally, we have

$$\mu(Q_j) \leq \text{const } M_2 \omega(B(z_j, 32l(Q_j))). \tag{2.4}$$

By Schwarz's inequality we have

$$\begin{aligned} \int_{\Omega} \frac{|g'|}{1+|g|^2} dx dy &= n^{1/2} \sum_{Q_j \in W} n^{-1/2} \int_{Q_j} \frac{|g'|}{1+|g|^2} dx dy \\ &\leq \frac{1}{2} n^{1/2} \sum_{Q_j \in W} \left( \int_{Q_j} \frac{4|g'|^2}{n(1+|g|^2)^2} dx dy \right)^{1/2} \left( \int_{Q_j} dx dy \right)^{1/2} \\ &\leq \frac{1}{2} n^{1/2} \sum_{Q_j \in W} \mu(Q_j)^{1/2} l(Q_j) \\ &= \frac{1}{2} n^{1/2} \sum_{Q_j \in W} \mu(Q_j)^{1/2-\pi(1-2\varepsilon)} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\ &\leq C n^{1/2} \left( \frac{\log(1+n)}{n} \right)^{1/2-\pi(1-2\varepsilon)} \sum_{Q_j \in W} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\ &\leq C n^{\pi(1)+\delta} \sum_{Q_j \in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j), \end{aligned}$$

where  $\varepsilon$  is a small positive number,  $C$  is a constant, and  $D$  is the family of all dyadic squares with side length less than 32, that intersect  $\Omega$ .

By (2.4), we can estimate the last sum as follows:

$$\begin{aligned}
 & Cn^{\pi(1)+\delta} \sum_{Q_j \in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\
 & \leq CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} l(Q_j)^\varepsilon \\
 & = CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} \tag{2.5} \\
 & \leq CM_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \text{const}(\varepsilon) \\
 & \leq n^{\pi(1)+\delta} \text{const}(\varepsilon) M_2.
 \end{aligned}$$

Now assume that the following dichotomy holds: *For any  $\varepsilon > 0$ , there exists a constant  $\text{const}(\varepsilon)$  such that for any polynomial  $g$  of degree  $n$ , we have*

$$M_2 \leq \text{const}(\varepsilon) n^\varepsilon, \tag{2.6}$$

or

$$|\Omega| \leq 1/n. \tag{2.7}$$

Thus, if (2.6) holds, then the desired estimate follows from (2.5):

$$\int_{\Omega} \frac{|g'|}{1+|g|^2} \leq \text{const} n^{\pi(1)+\varepsilon+\delta}.$$

But both  $\varepsilon$  and  $\delta$  can be made arbitrarily small, so we have the desired estimate.

If (2.7) holds, then an even better estimate follows from Schwartz's inequality:

$$\begin{aligned}
 \int_{\Omega} \frac{|g'|}{1+|g|^2} & \leq \left( \int_{\Omega} \frac{|g'|^2}{(1+|g|^2)^2} \right)^{1/2} \left( \int_{\Omega} 1 \right)^{1/2} \\
 & \leq \left( \int_{\mathbb{C}} |g'_\sigma| \right)^{1/2} \sqrt{|\Omega|} \\
 & \leq \sqrt{2\pi n |\Omega|} \\
 & \leq \sqrt{2\pi}.
 \end{aligned}$$

Therefore it remains only to prove the dichotomy.

*Proof of the dichotomy.* Assume that  $M_2 > n^\varepsilon$ . Recalling the definition that

$$M_2 = \sup_{|z|=2} \frac{\log(1+|g|^2)}{n},$$

we deduce that  $\sup_{|z|=2} |g| > \exp(n^{1+\varepsilon})$ , so the set  $\Omega$  where  $|g| < n$  cannot have big measure.

We can write  $g$  as  $g = PQ$ , where

$$P(z) = \lambda \prod_{|a_i|>4} (z - a_i), \quad Q(z) = \lambda \prod_{|a_i|\leq 4} (z - a_i).$$

Let  $m$  be the degree of  $P$ ; then

$$\begin{aligned} \log \left( |\lambda| \prod_{|a_i| > 4} |a_i| \right) - m \log 2 &\leq \log |P(z)| \\ &\leq \log \left( |\lambda| \prod_{|a_i| > 4} |a_i| \right) + m \log 2, \end{aligned}$$

when  $|z| \leq 2$ . Since  $|Q(z)| < 6^n$  for  $|z| \leq 2$ , it follows that

$$\begin{aligned} \inf_{z \in \Omega} \log |P(z)| &\geq \sup_{|z|=2} \log |P(z)| 3^{-n} \\ &\geq \log (\exp(n^{1+\varepsilon}) 3^{-n} 6^{-n}) \\ &\geq \frac{1}{2} n^{1+\varepsilon}, \end{aligned}$$

if  $n$  is sufficiently large. Since  $\log |PQ| = \log |g| < n$  in  $\Omega$ , we can write

$$\begin{aligned} \log |Q(z)| &\leq \log |PQ| - \log |P| \\ &\leq n - \frac{1}{2} n^{1+\varepsilon} \\ &\leq -\frac{1}{4} n^{1+\varepsilon}, \quad z \in \Omega, \end{aligned}$$

if  $n$  is large enough. Therefore,  $\Omega$  is contained in the union of disks  $\{z : |z - a_i| \leq \exp(-n^\varepsilon/4)\}$ . Hence

$$|\Omega| \leq n\pi \exp\left(-\frac{1}{2}n^\varepsilon\right) \leq 1/n \quad \text{when } n \text{ is sufficiently large.}$$

This proves the dichotomy for  $n > N(\varepsilon)$ . For degree  $n$  bounded from above by  $N(\varepsilon)$ , the dichotomy is easy to prove by a compactness argument (and, anyway, it suffices to prove that the estimate holds for polynomials of sufficiently large degree). This completes the proof of the main theorem.  $\square$

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