On Littlewood's constants

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Abstract

In 1952 Littlewood studied the following problem: what is the best constant α such that for any polynomial f of degree n the areal integral of its spherical derivative is at most $const \cdot n^{\alpha}$?

We show that α is equal to another constant β independently studied by Littlewood, namely the growth rate of the length of Green's lines. This greatly improves known estimates on α.

1 Introduction

In this paper we study the growth rate as $n \to \infty$ of the quantity

$$A_n = \sup \int_{\mathbb{D}} \frac{|g'|}{1+|g|^2} dm,$$

where supremum is taken over all polynomials g of degree n, $\mathbb{D}$ is a unit disc $\{|z| < 1\}$, and m denotes two-dimensional Lebesgue measure. We are interested in the best α such that $A_n \lesssim n^{\alpha}$ (which means that for every $\varepsilon > 0$ there is a constant C_ε with $A_n \le C_\varepsilon n^{\alpha+\varepsilon}$). In [L1] Littlewood observed that $0 \le \alpha \le 1/2$ and conjectured that $\alpha < 1/2$.

It is quite easy to show that for any rational function g of degree n

$$\int_{\mathbb{D}} \frac{|g'|}{1+|g|^2} dm \le \mathrm{const}\sqrt{n}.$$

Note that integrand is a modulus of spherical derivative g'_σ of g (i.e. derivative with respect to spherical metric) and that in $\mathbb{D}$ spherical measure dm_σ is comparable to Lebesgue measure dm. So our integral can be estimated by

$$\begin{aligned} \int_{\mathbb{D}} |g'_\sigma| dm_\sigma &\le \int_{\mathbb{C}} |g'_\sigma| dm_\sigma \le \left(\int_{\mathbb{C}} |g'_\sigma|^2 dm_\sigma\right)^{1/2} \left(\int_{\mathbb{C}} dm_\sigma\right)^{1/2} \\ &= (2\pi n)^{1/2} (2\pi)^{1/2} = 2\pi\sqrt{n}, \end{aligned}$$

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here we use that rational function of degree n maps complex sphere to itself n-to-1 so the area of the image is n times area of the sphere. In particular this argument shows that $\alpha \leq 1/2$.

Littlewood's conjecture was proved in [LW] by Lewis and Wu, who improving upon the work [ES] of Eremenko and Sodin obtained an explicit upper estimate $\alpha < 1/2 - 2^{-264}$. Later Eremenko in [E] obtained a positive lower bound on α.

Following works by Eremenko and Sodin [ES] and Lewis and Wu [LW] we exploit connection between this problem and extremal behavior of harmonic measure.

Our main result is that α is related to the growth rate of Green's lines length. In the case of simply connected domains Ω we define β_Ω as

$$\limsup_{\varepsilon \to 0} \frac{\log \operatorname{length}\{z : G(z) = \varepsilon\}}{\log 1/\varepsilon},$$

where G is the Green's function with pole at infinity and we define

$$\beta = \sup \beta_\Omega,$$

where supremum is taken over all simply connected domains Ω. In the non simply connected case one needs more elaborate definition and we use the multifractal analysis technique.

For a given domain Ω we define the *packing spectrum* $\pi_\Omega(t)$ as

$$\sup \left\{ q :\ \forall \delta > 0\ \exists\ \delta - \text{packing } \{B\} \ with \ \sum \operatorname{diam}(B)^t \omega(B)^q \ \geq\ 1 \right\},$$

where ω is the harmonic measure in Ω. Note that this definition is valid for any domain with compact boundary and for $t = 1$ it is analogue of β_Ω (see [M] for the proof of $\pi_\Omega(1) = \pi_\Omega(1)$ for simply connected Ω) .

We define the *universal spectrum* $\pi(t)$ as the supremum of $\pi_\Omega(t)$ over all planar domains Ω with compact boundary.

The main result of this paper is the following theorem:

Main Theorem. *For any positive ε there exists a constant $c = c(\varepsilon)$ such that*

$$A_n \leq c n^{\pi(1)+\varepsilon}.$$

Equivalently

$$\alpha \leq \pi(1).$$

Remark. Our proof actually implies that for $t \in [0, 2]$

$$\int\limits_{\mathbb{D}} \left(\frac{|g'|}{1 + |g|^2} \right)^t \leq c n^{\pi(2-t)+\varepsilon}.$$

Of interest to us are also $\pi_p(t)$ and $\pi_{p,sc}(t)$ which are suprema of $\pi_\Omega(t)$ over all domains of attraction to infinity for polynomial mappings and simply connected domains of attraction to infinity for polynomial mappings correspondingly (see [CG] for background on complex dynamics). It is clear that $\pi_{p,sc} \leq \pi_p \leq \pi$, but a priory they might differ.

In [E] Eremenko essentially proved that $\pi_{p,sc}(1) \leq \alpha$ (he works under assumption that polynomials are hyperbolic, but it can be easily avoided). Binder, Makarov, and Smirnov in [BMS] showed that $\pi_{p,sc}(t) = \pi_p(t)$ for $t > 0$. Recently Binder and Jones [BJ] announced the proof of the identity $\pi_p = \pi$, which together with our theorem completes the circle:

$$\alpha \overset{\text{MainThm.}}{\leq} \pi(1) \overset{[BJ]}{=} \pi_p(1) \overset{[BMS]}{=} \pi_{p,sc}(1) \overset{[E]}{\leq} \alpha.$$

There is yet another growth rate that is related to α and was also studied by Littlewood. The growth rate γ of coefficients of univalent functions in $\mathbb{D}^-$ is defined

$$\sup_{\phi} \limsup_{n\to\infty} \frac{\log |b_n|}{\log n} + 1,$$

where the first supremum is taken over all functions $\phi(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$ that are univalent in $\mathbb{D}^-$. Littlewood proved [L2, LP] that $\beta \geq \gamma$. Much later Carleson and Jones [CJ] showed that $\gamma = \beta$. Summing it all up, we arrive at:

Corollary:

$$\alpha = \beta = \gamma = \pi(1) = \pi_{p,sc}(1). \tag{1}$$

Note that a well-known conjecture (see [CJ, M, K, P1]) states that $\pi(t) = (2-t)^2/4$ for $|t| \leq 2$, and particularly $\alpha = \beta = \gamma = 1/4$. The best known estimates for β are

$$0.17 \overset{[P]}{<} \beta \overset{[GP]}{<} 0.4884,$$

so the same estimates hold for α, which is a significant improvement over previously known $1.11 \cdot 10^{-5} \leq \alpha \leq 1/2 - 2^{-264}$ (see [BS] and [LW]).

Connection to the value distribution of entire functions

Before giving the proof we would like to remark that the reason for Littlewood's interest in this problem was the following striking corollary of his conjecture (see [L1]):

Littlewood's conditional theorem *Assume that $\alpha < 1/2$. If f is an entire function of order $0 < \rho < \infty$ then for any $0 < \theta < 1/2 - \alpha$ there is a "small" set S such that "almost all" roots of any equation $f(z) = w$ lie in S. Namely*

$$\frac{Area(S \cap B(R))}{Area(B(R))} = O\left(\frac{1}{R^{2\theta\rho}}\right), \qquad R \to \infty,$$

and for all w

$$\frac{\#\{z \in B(R) \setminus S : f(z) = w\}}{\#\{z \in B(R) : f(z) = w\}} = O\left(\frac{1}{R^{\rho(1/2-\alpha-\theta)}}\right), \qquad R \to \infty,$$

where $B(R)$ is a disc of radius R centered at the origin.

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2 Proof of The Main Theorem

It is a standard fact (for details see [M] and [F]) that $\pi(t)$ is finite, convex, and strictly decreasing on $[0,2]$, and hence for any small δ we can choose ε so small that

$$\pi(1-2\varepsilon)-\pi(1)<\delta.$$

We will also use a more elaborate fact (which follows from multifractal formalism and fractal approximation – see [M]), that there is a constant $\mathrm{const}(t,\varepsilon)$, such that for any disjoint collection of cubes $\{Q\}$ of size ≤ 1 one has

$$\sum_Q \omega(Q)^{\pi(t)} l(Q)^{t+\varepsilon} < \mathrm{const}(t,\varepsilon).$$

Let g be a polynomial of degree at most n and a_i its zeros. Consider a set where $|g|$ is big, i.e. $|g|\geq n$. We can easily estimate integral over this set by

$$\begin{aligned}\int\limits_{\mathbb{D}\cap\{|g|\geq n\}} \frac{|g'|}{1+|g|^2} &\leq n^{-1}\int\limits_{\mathbb{D}}\frac{|g'|}{|g|} = n^{-1}\int\limits_{\mathbb{D}}|(\log g)'| \\ &= n^{-1}\int\limits_{\mathbb{D}}\left|\sum_{i=1}^{n}\frac{1}{z-a_i}\right| \leq n^{-1}\cdot 2\pi n = 2\pi.\end{aligned} \tag{2}$$

Now consider a complementary set where $|g|$ is small, which is contained inside the disc of radius $3/2$

$$\Omega := \{z : |g(z)|<n, |z|<\frac{3}{2}\}\ ,$$

and let $W=\{Q_j\}$ be a Whitney decomposition of Ω. We note that

$$d\mu(z) = 4n^{-1}\frac{|g'(z)|^2}{(1+|g(z)|^2)^2}dxdy,$$

is the Riesz measure associated with the nonnegative subharmonic function

$$u = \frac{\log(1+|g|^2)}{n}\ .$$

Then by Riesz representation formula

$$\mu(B(z,r)) \leq \frac{c}{2\pi}\int\limits_0^{2\pi}(u(z+2re^{i\theta})-u(z))d\theta. \tag{3}$$

Hence for every cube Q_j we have

$$\mu(Q_j)\leq c\frac{\log(1+n)}{n}.$$

Fix a cube Q_j such that $Q_j\cap\mathbb{D}\neq\emptyset$ and denote by ξ_j a point at $\partial\Omega$ such that $d(\xi_j,Q_j)\leq 2l(Q_j)$. Then from (3)

$$\begin{aligned}\mu(Q_j) &\leq \tfrac{c}{2\pi}\int\limits_0^{2\pi}(u(\xi_j+8l(Q_j)e^{i\theta})-u(\xi_j))d\theta \\ &\leq \max_{z\in B(\xi_j,8l(Q_j))} c[u(z)-u(\xi_j)]^+\ .\end{aligned} \tag{4}$$

Denote by $G(z)$ the Green's function for $\mathbb{C}\backslash\overline{\Omega}$ with pole at infinity. Extend G to a continuous subharmonic function in $\mathbb{C}$ by setting $G = 0$ on Ω.

By the maximum principle for domain $\mathbb{C}\backslash\overline{\Omega}$ we obtain $G(z) \geq \log|2z/3|$ for any $z \in \mathbb{C} \setminus \Omega$, hence

$$G(z) \geq \log\frac{4}{3} \qquad \text{for } |z| = 2.$$

By the maximum principle for domain $2\mathbb{D} \setminus \overline{\Omega}$, we have

$$u(z) - u(\xi_j) \leq M_2 G(z) \left(\log\frac{4}{3}\right)^{-1} \qquad \text{for } |z| \leq 2,$$

where $M_2 = \max_{|z|=2} u(z)$.

If we let z_j be a center of Q_j and ω be a harmonic measure on $\mathbb{C} \setminus \overline{\Omega}$ with pole at infinity then by previous inequality and (4) we have

$$\begin{aligned}
\mu(Q_j) &\leq \max_{z\in B(\xi_j, 8l(Q_j))} c[u(z) - u(\xi_j)]^+ \\
&\leq c\left(\log\frac{4}{3}\right)^{-1} M_2 \max_{z\in B(\xi_j,8l(Q_j))} G(z) \leq cM_2 \max_{z\in B(z_j,16l(Q_j))} G(z).
\end{aligned}$$

By Harnack's inequality right hand side is less than

$$cM_2 3 \int\limits_{\partial B(z_j, 32l(Q_j))} G(z)\frac{|dz|}{2\pi 32 l(Q_j)},$$

which by Riesz representation formula equals to

$$cM_2 \int\limits_0^{32l(Q_j)} \frac{\omega(B(z_j,t))}{t} dt \leq cM_2\omega(B(z_j, 32l(Q_j))).$$

So finally we have

$$\mu(Q_j) \leq \text{const } M_2\ \omega(B(z_j, 32l(Q_j))). \tag{5}$$

By Schwarz's inequality we have

$$\begin{aligned}
&\int\limits_\Omega \frac{|g'|}{1+|g|^2} dxdy = n^{1/2} \sum_{Q_j\in W} n^{-1/2} \int\limits_{Q_j} \frac{|g'|}{1+|g|^2} dxdy \\
&\leq \frac{1}{2} n^{1/2} \sum_{Q_j\in W} \left(\int\limits_{Q_j} \frac{4|g'|^2}{n(1+|g|^2)^2} dxdy\right)^{1/2} \left(\int\limits_{Q_j} dxdy\right)^{1/2} \\
&\leq \frac{1}{2} n^{1/2} \sum_{Q_j\in W} \mu(Q_j)^{1/2} l(Q_j) = \frac{1}{2} n^{1/2} \sum_{Q_j\in W} \mu(Q_j)^{1/2-\pi(1-2\varepsilon)} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\
&\leq C n^{1/2} \left(\frac{\log(1+n)}{n}\right)^{1/2-\pi(1-2\varepsilon)} \sum_{Q_j\in W} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\
&\leq C n^{\pi(1)+\delta} \sum_{Q_j\in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j),
\end{aligned}$$

where D is the family of all dyadic squares with the side length less then 32 that intersect Ω, ε is a small positive number, and C is a constant. By (5) we can estimate the last sum

$$
\begin{aligned}
& Cn^{\pi(1)+\delta} \sum_{Q_j \in D} \mu(Q_j)^{\pi(1-2\varepsilon)} l(Q_j) \\
& \leq C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} l(Q_j)^{\varepsilon} \\
& = C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \tfrac{1}{2^{k\varepsilon}} \sum_{l(Q_j)=1/2^k} \omega(Q_j)^{\pi(1-2\varepsilon)} l(Q_j)^{(1-2\varepsilon)+\varepsilon} \\
& \leq C\, M_2 n^{\pi(1)+\delta} \sum_{k=1}^{\infty} \tfrac{1}{2^{k\varepsilon}} \operatorname{const}(\varepsilon) \leq n^{\pi(1)+\delta} \operatorname{const}(\varepsilon) M_2.
\end{aligned}
\tag{6}
$$

Now assume the following dichotomy: *For any* $\varepsilon > 0$ *there exists a constant* $\operatorname{const}(\varepsilon)$ *such that for any polynomial* g *of degree* n

$$M_2 \leq \operatorname{const}(\varepsilon) n^{\varepsilon}, \tag{7}$$

or

$$|\Omega| \leq 1/n. \tag{8}$$

Then if (7) holds then the desired estimate follows from (6):

$$\int_{\Omega} \frac{|g'|}{1+|g|^2} \leq \operatorname{const} n^{\pi(1)+\varepsilon+\delta}.$$

But both ε and δ can be made arbitrary small we have the desired estimate.

If (8) holds then even better estimate follows from Schwartz' inequality:

$$
\begin{aligned}
\int_{\Omega} \frac{|g'|}{1+|g|^2} & \leq \left(\int_{\Omega} \frac{|g'|^2}{(1+|g|^2)^2} \right)^{1/2} \left(\int_{\Omega} 1 \right)^{1/2} \\
& \leq \left(\int_{\mathbb{C}} |g'_{\sigma}| \right)^{1/2} \sqrt{|\Omega|} \leq \sqrt{2\pi n |\Omega|} \leq \sqrt{2\pi}.
\end{aligned}
$$

Therefore it remains to prove the dichotomy.

Proof of the dichotomy. Assume that $M_2 > n^{\varepsilon}$. Recalling the definition $M_2 = \sup_{|z|=2} \log(1+|g|^2)/n$, we deduce that $\sup_{|z|=2} |g| > \exp(n^{1+\varepsilon})$ so the set Ω where $|g| < n$ cannot have big measure.

We can write g as $g = PQ$, where

$$P(z) = \lambda \prod_{|a_i|>4} (z-a_i), \quad Q(z) = \lambda \prod_{|a_i|\leq 4} (z-a_i).$$

Let m be the degree of P, then

$$\log\left(|\lambda|\prod_{|a_i|>4}|a_i|\right)-m\log 2\leq \log|P(z)|\leq \log\left(|\lambda|\prod_{|a_i|>4}|a_i|\right)+m\log 2,$$

when $|z|\leq 2$. Since $|Q(z)|<6^n$ for $|z|\leq 2$, it follows that

$$\inf_{z\in\Omega}\log|P(z)|\geq \sup_{|z|=2}\log|P(z)|3^{-n}\geq \log\left(\exp(n^{1+\varepsilon})3^{-n}6^{-n}\right)\geq \frac{1}{2}n^{1+\varepsilon},$$

if n is sufficiently large. Since $\log|PQ|=\log|g|<n$ in Ω we can write

$$\log|Q(z)|\leq \log|PQ|-\log|P|\leq n-\frac{1}{2}n^{1+\varepsilon}\leq -\frac{1}{4}n^{1+\varepsilon},\ z\in\Omega$$

if n is large enough. Therefore, Ω is contained in the union of disks $\{z:|z-a_i|\leq \exp(-n^{\varepsilon}/4)\}$. Hence

$$|\Omega|\leq n\pi\exp(-\frac{1}{2}n^{\varepsilon})\leq 1/n \qquad \text{when } n \text{ is sufficiently large.}$$

This proves the dichotomy for $n>N(\varepsilon)$, for degree n bounded from above by $N(\varepsilon)$ the dichotomy is easy by compactness argument (and anyway, it suffices to prove the estimate for polynomials of sufficiently large degree). This completes the proof of the main theorem. ☐

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