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Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*

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Abstract. We show equivalence of several standard conditions for non-uniform hyperbolicity of complex rational functions, including the Topological Collet-Eckmann condition (TCE), Uniform Hyperbolicity on Periodic orbits, Exponential Shrinking of components of pre-images of small discs, backward Collet-Eckmann condition at one point, positivity of the infimum of Lyapunov exponents of finite invariant measures on the Julia set. The condition TCE is stated in purely topological terms, so we conclude that all these conditions are invariant under topological conjugacy.

For rational maps with one critical point in Julia set all the conditions above are equivalent to the usual Collet-Eckmann and backward Collet-Eckmann conditions. Thus the latter ones are invariant by topological conjugacy in the unicritical setting. We also prove that neither part of this stronger statement is valid in the multicritical case.

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Contents

Introduction

We establish the equivalence of several standard and widely used (and also some technical) conditions of non-uniform hyperbolicity of rational maps. Since one of them, the Topological Collet-Eckmann condition (TCE), is formulated in purely topological terms this shows their invariance under topological conjugacy. The paper provides a comprehensive insight into the exponential features for the maps satisfying TCE; surprisingly, there is no need to involve any Markov type structure.

The condition TCE is natural and important because it can be formulated in several terms: differential, topological, measure-theoretic, geometric. The class of maps satisfying TCE is likely to be generic in the complement of the class of uniformly hyperbolic maps in measure-theoretic sense (see [B] and compare with [S] and [GSw1]). This is true for quadratic maps of the interval (see [AM] and the earlier [Lyu]). Non-uniformly hyperbolic dynamics was introduced by J. Sinai in the sixties in the context of billiards and became a challenge in various areas of dynamical systems in the past 20 years. In this paper we deal with the complex 1-dimensional situation. Our methods also contribute to the development of the theory of unimodal and multimodal maps of the interval (see [GSw2] and G. Świątek's[1] address at the ICM, Berlin 1998).

1 The main results

Given a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere $\overline{\mathbb{C}}$ we denote by $J(f)$ the Julia set of f, which is the closure of the repelling periodic points of f. Denote by Crit the set of critical points of f, which are the points where the derivative of f is equal to zero. Derivatives are taken with respect to the spherical metric.

[1] See also recent *Collet-Eckmann condition in one-dimensional dynamics*, AMS Proc. Symposia Pure Math. 69 (2001), 489-498.

All conditions below imply there are no parabolic periodic points (that is no $f^n(p) = p$, $(f^n)'(p)$ being a root of unity). For elements of an analogous theory in the presence of parabolic periodic orbits (cycles) see [PU2].)

The conditions we discuss include the following ones.

- CE. *Collet-Eckmann condition.* There exist $\lambda_{CE} > 1$ and $C > 0$ such that for every critical point c in $J(f)$, whose forward orbit does not meet other critical points, and for every $n \geq 0$ we have

$$|(f^n)'(f(c))| \geq C\lambda_{CE}^n. \qquad (CE \text{ at } c)$$

  Moreover there are no parabolic cycles.
- CE2(z_0). *Backward* or *second Collet-Eckmann condition at* $z_0 \in \overline{\mathbb{C}}$. There exist $\lambda_{CE2} > 1$ and $C > 0$ such that for every $n \geq 1$ and every $w \in f^{-n}(z_0)$,

$$|(f^n)'(w)| \geq C\lambda_{CE2}^n.$$

  (In this case z_0 is necessarily not in the forward orbit of a critical point.)

  We write CE2(some z_0) if there exists $z_0 \in \overline{\mathbb{C}}$ such that CE2(z_0) holds.
- CE2. *Backward* or *second Collet-Eckmann condition.* The condition CE2(c) holds for all critical points $c \in J(f)$ which are not in the forward orbit of other critical points.
- UHP. *Uniform Hyperbolicity on Periodic orbits.* There exists $\lambda_{\mathrm{Per}} > 1$ such that every repelling periodic point $p \in J(f)$ of period $k \geq 1$ satisfies,

$$|(f^k)'(p)| \geq \lambda_{\mathrm{Per}}^k.$$

- ExpShrink. *Exponential shrinking of components.* There exist $\lambda_{\mathrm{Exp}} > 1$ and $r > 0$ such that for every $x \in J(f)$, every $n > 0$ and every connected component W of $f^{-n}(B(x, r))$ we have

$$\mathrm{diam}(W) \leq \lambda_{\mathrm{Exp}}^{-n}.$$

- Hölder Coding Tree. There are constants $\lambda_{\mathrm{Ho}} > 0$ and $C > 0$ and a point $w_0 \in \overline{\mathbb{C}}$ such that the following holds. For each preimage $w_1 \in f^{-1}(w_0)$ there exist a continuous path $\gamma_{w_1} \subset \overline{\mathbb{C}} \setminus \cup_{n\geq 1} f^n(\mathrm{Crit})$, without self-intersections, joining w_0 to w_1, such that for every $n \geq 0$ and every connected component γ of $f^{-n}(\gamma_{w_1})$ we have $\mathrm{diam}(\gamma) \leq C\lambda_{\mathrm{Ho}}^{-n}$.
- TCE. *Topological Collet-Eckmann condition.* There exist $M \geq 0$, $P \geq 1$ and $r > 0$ such that for every $x \in J(f)$ there exists a strictly increasing sequence of positive integers n_j, for $j = 1, 2, ...$ such that $n_j \leq P \cdot j$ and for each j

$$\#\big\{i : 0 \leq i < n_j, \mathrm{Comp}_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), r) \cap \mathrm{Crit} \neq \emptyset\big\} \leq M,$$

  where Comp_y means the connected component containing y (above $y = f^i(x)$).

  We call the condition in the display above: TCE at x for n_j with criticality bounded by M.

- Lyapunov. *Lyapunov exponents of invariant measures are bounded away from zero*. There is a constant $\lambda_{\mathrm{Lyap}} > 1$ such that the Lyapunov exponent of any invariant probability measure μ supported on Julia set satisfies $\Lambda(\mu) := \int \ln|f'|d\mu \geq \ln \lambda_{\mathrm{Lyap}}$.
- Negative Pressure. *Pressure for large t is negative*. For large values of t the pressure function $P(t) \equiv P_f(-t\ln|f'|) := \sup\{I(\mu) - t\Lambda(\mu) : \mu$ being f-invariant probability supported on $J(f)\}$ is negative. Here $I(\mu)$ is the entropy (information) for μ and f.

With the help of new technical ideas we complete the theory started in [GS1], [PR1], [PR2], and [P3] to obtain the following result.

Main Theorem. *For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the following conditions are equivalent.*

(a) *TCE.*
(b) *ExpShrink.*
(c) *CE2(z_0) holds for some $z_0 \in \overline{\mathbb{C}}$.*
(d) *Hölder Coding Tree.*
(e) *UHP.*
(f) *Lyapunov.*
(g) *Negative Pressure.*

Supremums over possible constants λ_{Exp}, λ_{CE2} (supremum over all z_0), λ_{Ho}, λ_{Per}, and λ_{Lyap} coincide.

Moreover each of conditions (h) CE and (i) CE2 implies all the previous ones, and there are counterexamples to any other implication not mentioned.

If all critical points in the Julia set have the same dynamical multiplicity, then conditions CE and CE2 are equivalent.

If there is only one critical point in the Julia set then all conditions (a)–(i) are equivalent.

The *dynamical* multiplicity takes into account the possibility that a critical point may be mapped to another critical point; see Preliminaries for a definition.

1.1 On topological invariance of the conditions. The condition TCE is stated in purely topological terms and hence is invariant under topological conjugacy. So we obtain the following immediate corollary.

Corollary. *All conditions (a)–(g) are invariant under topological conjugacy. So are conditions CE and CE2 in the case where there is only one critical point in the Julia set.*

If there are several critical points in the Julia set, conditions CE and CE2 need not be invariant under topological conjugacy. In Appendix C we provide examples of two real polynomials of degree 4 that are quasiconformally conjugated, one satisfying both conditions and the other satisfying neither.

1.2 Historical remarks. Conditions CE and CE2 were first introduced by P. Collet and J.-P. Eckmann in [CE], providing a large class of S-unimodal maps of an interval having a finite absolutely continuous invariant measure. Further investigations by T. Nowicki, S. van Strien, and D. Sands proved the equivalence of those and several similar properties in the real S-unimodal setting, in which our investigations have originated; see [NS] and also [N].

Rational Collet-Eckmann maps were first studied in [P1]. The condition TCE was first introduced in [PR1] in the complex setting and later in [NP] in the real setting.

The existence of a unique non-atomic conformal measure and of an absolutely continuous finite invariant measure for rational Collet-Eckmann maps was proved in [GS2], see also [P1]. For TCE maps with more than 1 critical point in $J(f)$ the answer is not known.

The situation in the complex setting is not completely analogous to the real one and requires some new techniques. Furthermore, we obtain results in the presence of several critical points, whereas some results in the real setting are restricted to the unimodal case (one critical point), see [NS]. A natural question that arises for the real multimodal setting is whether condition UHP is equivalent to TCE.

1.3 Further conditions and remarks. For every rational function f *the pressure function* $P(t)$ is convex (by definition) and monotone decreasing with t_0 the least zero being equal to the hyperbolic dimension of $J(f)$, [P2]. Therefore condition Negative Pressure is equivalent to $P(t)$ being strictly decreasing. In Sect. 4 we prove that t_0 is equal to Hausdorff and box dimensions of $J(f)$ and $t_0 < 2$ provided $J(f) \neq \overline{\mathbb{C}}$. Negative Pressure not satisfied is equivalent to $P(t) \equiv 0$ for $t \geq t_0$.

We consider also further variants of condition TCE, that are more technical in nature. Several of them appear along the proof of the Main Theorem, see Sect. 5.

In Appendix B we prove that if the upper density in the set of all natural numbers of the set of integers n_j, such that for a critical point $c \in J(f)$ condition TCE at c for n_j with criticality bounded by M holds, is positive, then it is also positive with $M = 0$ at a critical value $f(c')$ for a critical point c' in the closure of the forward orbit of c.
(Note however that TCE is related to the lower density.)

We remark that in the real unimodal setting condition CE2(some z_0) does not imply neither ExpShrink or TCE. For example, for the Feigenbaum quadratic polynomial condition CE2(z_0) holds (in the real sense) for all z_0 not accumulated by the critical orbit.

2 Structure of the proof of the Main Theorem

The most involved part is the equivalence between conditions (b), (c) and (d) (Sects. 1 and 2), which is explained in more detail in 2.1, below.

The equivalence between conditions (a) TCE and (b) ExpShrink was proven in [PR1], see also [P3, Sect. 4].

Condition (e) UHP is discussed in Sect. 3. The implication (b) ExpShrink $\Rightarrow$ (e) UHP is easy (Proposition 3.2.) The implication (e) UHP $\Rightarrow$ (c) CE2(some z_0) is more difficult. We provide two proofs of this implication: one in Sect. 3 and another in Appendix A. Both proofs involve a sort of Bowen specification property (shadowing by periodic orbits.)

The proofs that conditions (f) Lyapunov and (g) Negative Pressure are equivalent to the previous ones are contained in Sect. 4. The only deep one (b) ExpShrink $\Rightarrow$ (f) Lyapunov relies on Birkhoff Ergodic Theorem and $\Lambda(\mu) \geq 0$ proved in [P5], which in turn has been based on Pesin theory.

The assertions concerning CE and CE2 follow from known results and the equivalence of (a)–(g), except the examples presented in Sect. 6. Note that in CE2 $\Rightarrow$ (a)–(g) the implication CE2 $\Rightarrow$ (c) is obvious.

2.1 On the equivalence of conditions (b) ExpShrink, (c) CE2(some z_0) and (d) Hölder Coding Tree. The implication (b) ExpShrink$\Rightarrow$(c) CE2(some z_0), is rather simple (Sect. 2). The most unexpected and involved implication is (c) CE2(some z_0) $\Rightarrow$ (b) ExpShrink. A natural approach would be to join z_0 to $B(x,r)$ with a curve slowly approximated by the forward orbit of critical points and deduce that the diameters of the connected components of $f^{-n}(B(x,r))$ shrink exponentially fast using condition CE2(z_0) and controlling distortion along the curve. However it is hard to succeed this way.[2]

We proceed in a different way, *via* (d), which is analogous to what was done for polynomials in [GS1] and [P3]. Consider a polynomial P and denote by B its attracting basin of infinity. In [GS1] it was proved that condition CE2(z_0) for some $z_0 \in B$ is equivalent to B being a *Hölder domain*, see [GS1] or [Po] for a definition. Then in [P3] it was proved that if B is a Hölder domain, then condition ExpShrink holds. The idea was to consider a *geometric coding tree* in B, see [PS] for a general theory and also [PUZ]. Here we follow a similar reasoning for general rational maps, constructing a geometric coding tree (see the terminology below) which is Hölder, even in absence of B. The tree itself plays the role of a basin.

The proof of (d) Hölder Coding Tree $\Rightarrow$ (b) ExpShrink (in Sect. 2) is similar to what was done for polynomials in [P3]. We now explain the implication (c) CE2(z_0) for some $z_0 \in \overline{\mathbb{C}} \Rightarrow$ (d) Hölder Coding Tree.

In general a *geometric coding tree* is a graph composed of vertices $f^{-n}(w_0), n = 0, 1, ...,$ for an arbitrary $w_0 \in \overline{\mathbb{C}}$ called the *root* of the tree, and edges being connected components of $f^{-n}(\gamma_{w_1})$ for γ_{w_1} as in Hölder Coding Tree condition before.

[2] In an early version of the paper we proved that condition CE2 implies ExpShrink using this idea. Troubles with diamComp $f^{-n}(B(x,r))$ arising after meeting a critical point c can be overcome directly using the assumption CE2(c).

We say that such a geometric coding tree is *Hölder* or a *Hölder coding tree* if there are constants $\lambda_{\mathrm{Ho}} > 1$ and $C > 0$ such that for every edge γ in $f^{-n}(\gamma_{w_1})$ we have $\operatorname{diam}(\gamma) \le C\lambda_{\mathrm{Ho}}^{-n}$.

Assuming condition (c) CE2(some z_0), we can construct a Hölder Coding Tree in the following way. First we choose an appropriate root point w_0 (it might be necessary to replace the point z_0 by a better point) and then for each preimage w_1 of w_0 we choose a path γ_{w_1} joining w_1 to w_0, in such a way that γ_{w_1} is slowly approximated by the forward orbit of critical points. This latter step is done with the help of a general geometric lemma (in Sect. 1), which is independent of dynamics. Then we deduce an exponential shrinking of $\operatorname{diam}(\gamma)$ from the exponential decay of $|(f^n)'(w)|^{-1}$ in the definition of CE2(z_0) and a distortion estimate.

Acknowledgements. The question we deal with in Appendix B was asked by Jacek Graczyk. The first author would like to thank Steffen Rohde for stimulating discussions concerning examples of CE and CE2.

Preliminaries

Distances, balls, diameters, and derivatives are considered with respect to the spherical metric on $\overline{\mathbb{C}}$. For $z \in \overline{\mathbb{C}}$ and $r > 0$, $B(z, r)$ denotes the ball centered at z with radius r. Sometimes we omit the origin and write $B(r)$. Then $B(ar)$ is the ball with the same origin and radius ar.

Const or C stand for various constants that can change even in one consideration. We write $a \sim b$ when numbers or functions a, b are comparable, i.e. there is a positive constant C such that $C^{-1} < a/b < C$.

Consider a rational map f. The measures we discuss are probabilities (probability measures) on $J(f)$, that is completed Borel measures μ supported on $J(f)$ with $\mu(J(f)) = 1$.

In the case that a critical point in $J(f)$ is eventually mapped to another critical point, our arguments work with whole blocks of critical points treated as just one critical point. Namely if the critical points $c_{i_1}, \dots, c_{i_k} \in J(f)$ are such that c_{i_s} is mapped to $c_{i_{s+1}}$ by an iterate of f (and not mapped to any critical point by a shorter iterate), then we treat this block of critical points as one critical point with multiplicity equal to the product of the multiplicities of the c_i. This is the *dynamical* multiplicity referred to in the introduction. So we can assume that no critical point in $J(f)$ is mapped to another critical point.

Fix two periodic orbits O_1 and O_2 of period at least 2 and let $r_{\mathrm{K}} > 0$ be such that for every $x \in \overline{\mathbb{C}}$ the disc $B(x, r_{\mathrm{K}})$ is disjoint from either O_1 or O_2. Then for every positive integer n and every component W of $f^{-n}(B(x, r_{\mathrm{K}}))$ we have

$$\operatorname{diam}(\overline{\mathbb{C}} \setminus W) \ge \min\{\operatorname{diam}(O_1), \operatorname{diam}(O_2)\} > 0.$$

Hence the following version of Koebe distortion Theorem for the spherical metric holds for inverse branches of f^n (see also [P1, Lemma 1.2[3]]).

Koebe distortion Theorem. *For every* $0 < \varepsilon < 1$ *there exists a constant* $C_{\mathrm{K}}(\varepsilon) > 1$ *such that for every disc* $B(x, b)$ *with* $b \le r_{\mathrm{K}}$*, every* n*, and every component* W *of* $f^{-n}(B(x, b))$ *such that* f^n *is univalent on* W*, we have*

$$|(f^n)'(z_1)|/|(f^n)'(z_2)| \le C_{\mathrm{K}}(\varepsilon),$$

for every $z_1, z_2 \in W$ *such that* $f^n(z_i) \in B(x, \varepsilon b)$ *for* $i = 1, 2$*. Moreover* $C_{\mathrm{K}}(\varepsilon) \to 1$ *as* $\varepsilon \to 0$.

We call the supremum of the left hand side ratios the *distortion* of f^n in $W(\varepsilon) = \mathrm{Comp}_W f^{-n}(B(x, \varepsilon b))$. In the sequel we shall discuss also the distortion $\sup_{z_1, z_2 \in A} |(f^n)'(z_1)|/|(f^n)'(z_2)|$ for an arbitrary set $A \subset \overline{\mathbb{C}}$.

1. A geometric lemma

The purpose of this section is to prove the following lemma.

Lemma 1.1. *Fix* $\alpha \in (\frac{1}{2}, 1)$ *and let* $f \in \mathbb{C}(z)$ *be a rational map. Then there is a set of full Lebesgue measure in* $\overline{\mathbb{C}}$ *of points* w*, so that we can join* w *to any preimage* $w_1 \in f^{-1}(w)$ *with a path* $\gamma_{w_1} \subset \overline{\mathbb{C}} \setminus \cup_{n \ge 1} f^n(\mathrm{Crit})$ *so that for* $n \ge n(w)$*, the distortion of* f^n *in any pull-back of* γ_{w_1} *by* f^{-n} *is bounded by* D^{n^α}*, where* $D > 1$ *is a constant independent of* f*.*

The proof of this lemma is based on the following observation. Fix a path $\gamma \subset \overline{\mathbb{C}} \setminus \cup_{n=1}^{\infty} f^n(\mathrm{Crit})$. For any $n \ge 1$ if we consider a topological disc $U \subset W_n = \overline{\mathbb{C}} \setminus \cup_{1 \le k \le n} f^n(\mathrm{Crit})$ containing a path γ, then there is an estimate of the distortion of f^n in any pull-back γ_n of γ by f^{-n}, given by Koebe distortion Theorem. It turns out that this distortion is bounded by D_0^h, where h is the hyperbolic diameter of γ in U and $D_0 > 1$ is a universal constant, see a version of Koebe distortion Theorem in [Po].

Hence to prove Lemma 1.1 it is enough to find paths $\gamma = \gamma_{w_1}$ with an estimate of the infimum of the hyperbolic diameter of γ in U, over all such discs $U \subset W_n$, $n \ge 1$. If γ is a rectifiable path, this latter quantity does not exceed (up to a constant factor C),

$$\int_\gamma \frac{|dw|}{\mathrm{dist}(w, \partial W_n)},$$

which is called the *quasihyperbolic* length of γ_{w_1} in W_n, or qh length for short, see [Po] and [P2, Sect. 3].

Thus Lemma 1.1 is an immediate consequence of the following lemma, with $D = D_0^{AC}$.

[3] The exponents -1 and -2 in the first and third estimates of distortion in [P1,(1.1)] are incorrect. One should replace them by the classical -3 and -4. Fortunately this error does not hurt any further results.

Geometric lemma. *There is a universal constant $A > 0$ such that for any $\alpha \in (\frac{1}{2}, 1)$ the following property holds. For any sequence of points $\{x_n\}_{n\geq 0} \subset \overline{\mathbb{C}}$ and every pair $z, w \in \overline{\mathbb{C}} \setminus \{x_0, x_1, ...\}$ outside an exceptional set of zero Lebesgue measure, there are a path γ joining z to w in $\overline{\mathbb{C}} \setminus \{x_0, x_1, ...\}$ and $n_0(z, w)$ such that for every $n \geq n_0(z, w)$, the qh length of γ in $\overline{\mathbb{C}} \setminus \{x_0, ..., x_n\}$ is less than $A \cdot n^{\alpha}$.*

Similar lemmas (however with γ depending on n), appeared in [P2] in a related context, see also Remark 1.4, and in [HH] by Hall and Hayman to study the distribution of zeros of Riemann ζ-function.

The proof of this lemma is based on Lemmas 1.2 and 1.3, below.

Lemma 1.2. *Fix $R > 0$ and let S be the strip $(-1, R+1) \times (-1, 1) \subset \mathbb{R}^2$ and $\ell = [0, R] \times \{0\}$. Consider $r \in (0, \frac{1}{3})$ and a finite set $Z \subset S$ so that $\mathrm{dist}(Z, \ell) > r$. Then the qh diameter of ℓ in $S \setminus Z$ is less than $R + 4\#Z \cdot \ln \frac{1}{r}$.*

Proof. Consider the vertical projection Z' of Z to $[-1, R+1] \times \{0\}$. Then,

$$\int_{\ell \cap B(Z', r)} \frac{1}{\mathrm{dist}(x, \partial(S \setminus Z))} dx \leq 2\#Z.$$

Note that for $x \in \ell \setminus B(Z', 1)$ we have $\mathrm{dist}(x, \partial(S \setminus Z)) \geq 1$ so,

$$\int_{\ell \setminus B(Z', 1)} \frac{1}{\mathrm{dist}(x, \partial(S \setminus Z))} dx \leq R.$$

Moreover, if $x \in \ell \cap (B(Z', 1) \setminus B(Z', r))$ then $\mathrm{dist}(x, \partial(S \setminus Z)) \geq \mathrm{dist}(x, Z')$. Thus,

$$\int_{\ell \cap (B(Z',1) \setminus B(Z',r))} \frac{1}{\mathrm{dist}(x, \partial(S \setminus Z))} dx \leq \#Z \int_{(-1,1) \setminus (-r,r)} \frac{1}{x} dx = 2\#Z \cdot \ln \frac{1}{r}.$$

This proves the desired estimate. $\square$

Fix $\alpha \in (\frac{1}{2}, 1)$ as in Lemma 1.1 and for $j \geq 0$ let n_j be the integer part of e^j.

Lemma 1.3. *Let $Q = (-2, 2) \times (-2, 2)$ and let $\{x_n\}_{n \geq 0}$ be a sequence of points in Q. There is a set of full Lebesgue measure of x in the interval $(-1, 1)$, for which there is $j(x)$ such that for all $j \geq j(x)$ the qh length of $\ell_x = (-1, 1) \times \{x\}$ in $Q \setminus \{x_0, ..., x_{n_j}\}$ is less than $5n_j^{\alpha}$.*

Proof. Since $\{\frac{n_j}{n_j^{2\alpha}}\}_{j \geq 0}$ is summable, it follows that for almost every $x \in (0, 1)$ there is $j_0(x)$ such that for all $j \geq j_0(x)$ we have that $\mathrm{dist}(\ell_x, x_n)$ is at least $n_j^{-2\alpha}$ if $0 \leq n \leq n_j$. The measure of those $x \in (-1, 1)$ for which the strip,

$$S_x^j = \left(-1 - \frac{1}{n_j^{\alpha}}, 1 + \frac{1}{n_j^{\alpha}}\right) \times \left(x - \frac{1}{n_j^{\alpha}}, x + \frac{1}{n_j^{\alpha}}\right)$$

contains at least $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ..., x_{n_j}\}$ is at most

$$2n_j^{-\alpha} \cdot \frac{n_j}{n_j^\alpha / \ln n_j} = 2(\ln n_j) n_j^{(1-2\alpha)} \sim je^{(1-2\alpha)j},$$

which is summable. Thus for almost every $x \in (-1, 1)$ there is $j_1(x)$ such that for all $j \geq j_1(x)$ the strip S_x^j contains at most $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ..., x_{n_j}\}$.

If $x \in (-1, 1)$ belongs to the two described sets, then for every $j \geq \max\{j_0(x), j_1(x)\}$ we have $\mathrm{dist}(\ell_x, \{x_0, ..., x_{n_j}\}) \geq \frac{1}{n_j^{2\alpha}}$ and the strip S_x^j contains at most $\frac{n_j^\alpha}{\ln n_j}$ points in $\{x_0, ..., x_{n_j}\}$. Applying Lemma 1.2 to a scaled copy of S_x^j by a factor of $n_j^\alpha = R$ and with $r = n_j^{-\alpha}$, it follows that the qh length of ℓ_x in $S_x^j \setminus \{x_0, ..., x_{n_j}\}$ is less than $n_j^\alpha + 4\frac{n_j^\alpha}{\ln n_j} \cdot \ln n_j^\alpha \leq 5n_j^\alpha$. □

Proof of the geometric lemma. Note that is enough to prove the analogous statement for the square Q instead of $\overline{\mathbb{C}}$.

Let $X, Y \subset (0, 1)$ be full measure sets from Lemma 1.3 found for the sets $\{x_0, ..., x_{n_j}\}$ and $\{I(x_0), ..., I(x_{n_j})\}$ respectively, where I is the map of $\mathbb{C}$ interchanging the real and imaginary parts.

Let $z = (z_Y, z_X)$, $w = (w_Y, w_X)$ be an arbitrary pair of points in the set $Y \times X \subset Q$. (Note that $Y \times X$ has full Lebesgue measure in Q.) Then we find a curve γ in $(-1, 1) \times \{z_X\} \cup \{w_Y\} \times (-1, 1)$ joining z to w, with qh length in $Q \setminus \{x_0, ..., x_{n_j}\}$, for large j, less than $10n_j^\alpha$.

It follows that for large j and any $n_j \leq n \leq n_{j+1}$, the qh length of γ in $Q \setminus \{x_0, ..., x_n\}$ is less than $10n_{j+1}^\alpha \leq 30n^\alpha$. □

Remark 1.4. With the same method it follows that, given any sequence $(x_n)_{n\geq 0} \subset \overline{\mathbb{C}}$ we can join any two points in $\overline{\mathbb{C}}$, outside an exceptional set of Lebesgue measure zero, by a path γ such that the qh length of γ in $\overline{\mathbb{C}} \setminus \{x_0, ..., x_n\}$ is $\mathcal{O}(n^{\frac{1}{2}} \ln n (\ln \ln n)^\beta)$ for any $\beta > \frac{1}{2}$. The mentioned lemma of Hall, Hayman and the first author yields a path *depending on* n, with qh length $\mathcal{O}(n^{\frac{1}{2}})$ in $\overline{\mathbb{C}} \setminus \{x_0, ..., x_n\}$. Furthermore, if we require the exceptional set in the Geometric Lemma of Hausdorff dimension at most $h > 0$, then we obtain the estimate $\mathcal{O}(n^{\frac{1}{2}} (\ln n)^{\frac{1}{2}+\frac{1}{h}})$ for the qh length.

2. Equivalence of conditions ExpShrink, CE2(some z_0) and Hölder

We begin with the easy implication.

Lemma 2.1. *Suppose that there are $\lambda_1 > 1$, $w \in J(f)$ and $r > 0$ such that for every $n \geq 1$ every connected component W of $f^{-n}(B(w, r))$ satisfies* $\mathrm{diam}(W) \leq \lambda_1^{-n}$. *Then condition CE2($z_0$) holds for a set of positive Lebesgue measure of $z_0 \in B(w, r)$.*

Proof. Note that we have

$$\mathrm{Area}\Big(\bigcup_{n\geq 1} B(f^n(\mathrm{Crit}), C_2/n)\Big) \leq C_2^2 \sum_{n\geq 1} (\#\mathrm{Crit})\pi/n^2 = C_2^2\, \#\mathrm{Crit}\, \pi^3/6,$$

which is smaller than the measure of $B(w, r)$, for C_2 small enough. So there is a set of positive measure of $z_0 \in B(w, r)$ not in $\bigcup_{n\geq 1} B(f^n(\mathrm{Crit}), C_2/n))$. Assume also that $C_2 \leq \mathrm{dist}(z_0, \partial B(w, r))$.

It follows from Koebe distortion Theorem applied to the branch g of f^{-k} on $B(z_0, C_2/k)$ mapping z_0 to some $y \in f^{-k}(z_0)$, that

$$C_3 \cdot (C_2/k) \cdot |(f^k)'(y)|^{-1} \leq \mathrm{diamComp}_y f^{-k}(B(z_0, C_2/k)) \leq \\ \mathrm{diamComp}_y f^{-k}(B(w, r)) \leq \lambda_1^{-k}.$$

The constant C_3 plays the role of $1/4$; it takes into account that we consider the spherical metric, rather than the Euclidean one in $\mathbb{C}$ in which the Koebe distortion Theorem is usually considered. Hence for all $\lambda_2 \in (1, \lambda_1)$ there is $C_4 = C_4(\lambda_2)$ such that for all $y \in f^{-n}(z_0)$,

$$|(f^k)'(y)| \geq C_4 \lambda_2^k. \qquad \square$$

Remark 2.2. The same proof yields that there is an exceptional set $E \subset B(w, r)$ of Hausdorff dimension zero such that CE2(z_0) holds for every $z_0 \in B(w, r) \setminus E$ with any constant $\lambda \in (1, \lambda_{\mathrm{CE2}})$ and with a varying constant $C = C_4(z_0, \lambda)$. In fact E can be taken of zero Hausdorff measure with any gauge function ϕ (appearing in $\sum_j \phi(\mathrm{diam} B_j)$ in the definition of Hausdorff measure) satisfying $\sum_n \phi(\mu^n) < \infty$ for all $\mu < 1$.

The proofs of the implications CE2(some z_0) $\Rightarrow$ Hölder Coding Tree $\Rightarrow$ ExpShrink depend on the following lemma. This lemma (stated for individual backward trajectories) was proved in [P4, Lemma 5, Remark 1], where it was called "telescope lemma." For completeness we provide a simplified proof.

Lemma 2.3. *Suppose that* CE2(z_0) *holds for a rational map f and some point z_0 with some constant $\lambda_{\mathrm{CE2}} > 1$. Then for any $\lambda \in (1, \lambda_{\mathrm{CE2}})$ there is $r > 0$ such that for all $n \geq 0$ and every connected component W of $f^{-n}(B(z_0, r))$, we have* $\mathrm{diam}(W) \leq \lambda^{-n}$.

The assumption CE2(z_0) can be replaced by the existence of a continuum K containing z_0 and some other point such that for every connected component K_0 of $f^{-n}(K)$, we have $\mathrm{diam}(K_0) \leq \lambda_0^{-n}$, *where λ_0 replaces λ_{CE2}.*

Before proving Lemma 2.3 let us make two observations about distortion. First, it follows from Koebe distortion Theorem that for any $D > 1$ there is $\varepsilon \in (0, 1)$ such that if the disc W satisfies $\mathrm{diam}(W) \leq \varepsilon \mathrm{dist}(W, f(\mathrm{Crit}))$ (in particular $W \cap f(\mathrm{Crit}) = \emptyset$), then the distortion of f in any connected component W_1 of $f^{-1}(W)$ is bounded by D. Secondly, once a small $r_0 > 0$ is fixed, there is $C_2 \geq 1$ such that for every disc $W \subset \overline{\mathbb{C}}$ and for every component W_1 of $f^{-1}(W)$ the inequality $\operatorname{diam}(W_1) \leq C_2 \operatorname{diam}(W)|f'(z)|^{-1}$ holds for every $z \in W_1$. We shall use this estimate for W close to $f(\mathrm{Crit})$.

Proof of Lemma 2.3. Let $C_1 > 0$ be the constant C that stands in the property CE2(z_0). Fix $D \in (1, \lambda_{\mathrm{CE2}}/\lambda)$ and consider $\varepsilon > 0$ as above.

Let $C_2 > 0$ be as above. Let $l \geq 1$ be such that $C_2^{1+j/l} C_1^{-1} D^j \lambda_{\mathrm{CE2}}^{-j} \leq \frac{1}{2}\lambda^{-j}$ for $j \geq l$. Also let $r_1 > 0$ be so that for all $c \in \mathrm{Crit}$, $0 \leq k \leq l$, and every connected component W of $f^{-k}(B(c, 2r_1))$ we have $\operatorname{diam}(W) \leq \varepsilon \operatorname{dist}(W, \mathrm{Crit})$. Recalling the assumption that critical points are not hit by trajectories of other critical points allows to find such r_1.

Finally let $r > 0$ be so that for any connected component W of $f^{-k}(B(z_0, r))$ for $0 \leq k \leq l$ we have $\operatorname{diam}(W) \leq \varepsilon r_1$ (in particular $r \leq \varepsilon r_1$).

Choose any sequence of pull-backs: $W_0 = B(z_0, r)$, $W_1 = \operatorname{Comp} f^{-1}(W_0)$, ..., $W_{k+1} = \operatorname{Comp} f^{-1}(W_k)$, ... and choose $z_k \in W_k$ so that $f(z_{k+1}) = z_k$, for $k \geq 0$.

Let $k_0 < k_1 < \ldots$ be all the integers k such that $\operatorname{diam}(W_k) > \varepsilon \operatorname{dist}(W_k, \mathrm{Crit})$.

So, when $k_0 > 0$, we have for all $0 \leq j < k_0$,

$$\operatorname{diam}(W_j) \leq D^j |(f^j)'(z_j)|^{-1} \operatorname{diam}(W_0). \tag{2.1}$$

Setting $j = k_0 - 1$ we obtain

$$\begin{aligned} \operatorname{diam}(W_{k_0}) &\leq C_2 \operatorname{diam}(W_{k_0-1})|f'(z_{k_0})|^{-1} \\ &\leq C_2 D^{k_0-1} |(f^{k_0})'(z_{k_0})|^{-1} \operatorname{diam}(W_0) \\ &\leq C_2 C_1^{-1} D^{k_0} \lambda_{\mathrm{CE2}}^{-k_0} \operatorname{diam}(W_0) \leq \frac{1}{2}\lambda^{-k_0} \operatorname{diam}(W_0) \leq \varepsilon r_1, \end{aligned} \tag{2.2}$$

the last line being true when $k_0 > l$. If $k_0 \leq l$ then directly $\operatorname{diam}(W_{k_0}) \leq \varepsilon r_1$.

We conclude that in both cases $\operatorname{dist}(W_{k_0}, \mathrm{Crit}) < \frac{1}{\varepsilon}\operatorname{diam}(W_{k_0}) \leq r_1$, and $W_{k_0} \subset B(\mathrm{Crit}, 2r_1)$.

Now by induction we prove that

$$\operatorname{diam}(W_{k_i}) \leq C_2^{1+k_i/l} D^{k_i} |(f^{k_i})'(z_{k_i})|^{-1} \operatorname{diam}(W_0), \tag{2.3}$$

and the estimate can be continued by writing

$$\cdots \leq C_2^{1+k_i/l} C_1^{-1} D^{k_i} \lambda_{\mathrm{CE2}}^{-k_i} \operatorname{diam}(W_0) \leq \frac{1}{2}\lambda^{-k_i} \operatorname{diam}(W_0) \leq \varepsilon r_1. \tag{2.4}$$

In fact, we already established (2.3) and (2.4) for $i = 0$. Assuming that (2.3) and (2.4) hold for $i < \ell$, we see that $\operatorname{dist}(W_{k_i}, \mathrm{Crit}) < \frac{1}{\varepsilon}\operatorname{diam}(W_{k_i}) \leq r_1$ for $i < \ell$. Hence W_{k_i} is contained in a ball of radius $2r_1$ around some critical point and thus $k_{i+1} - k_i > l$ by the definition of r_1. Therefore for $i = \ell$

estimate (2.4) holds (since $k_\ell > l$) and to prove (2.3) we can write similarly to (2.2)

$$\begin{aligned}\operatorname{diam}(W_{k_\ell}) &\leq C_2 D^{k_\ell - k_{\ell-1}} |(f^{k_\ell - k_{\ell-1}})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_{k_{\ell-1}})\\ &\leq C_2^{2+k_{\ell-1}/l} D^{k_\ell} |(f^{k_\ell})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_0)\\ &\leq C_2^{1+k_\ell/l} D^{k_\ell} |(f^{k_\ell})'(z_{k_\ell})|^{-1} \operatorname{diam}(W_0).\end{aligned}$$

We have proved exponential decay of diameters of preimages of orders k_i. To deal with arbitrary order j, we deduce for $j \in [k_i, k_{i+1})$, similarly to (2.1), the desired estimate

$$\begin{aligned}\operatorname{diam}(W_j) &\leq D^{j-k_i} |(f^{j-k_i})'(z_j)|^{-1} \operatorname{diam}(W_{k_i})\\ &\leq C_2^{1+j/l} D^j |(f^j)'(z_j)|^{-1} \operatorname{diam}(W_0) \leq \lambda^{-j},\end{aligned}$$

when $j \geq l$, and by decreasing r it follows for all j.

This proves the first part of the Lemma. The same proof works for the second part, where z_0 is replaced by a continuum K in the assumptions, with the following alterations in the estimates. We replace $|(f^j)'(z_j)|^{-1}$ for $j < k_0$, by the ratio

$$\operatorname{diam}\big(\operatorname{Comp}_{z_j} f^{-j}(K \cap B(z_0, r))\big)/r;$$

we replace $|f'(z_{k_\ell})|^{-1}$ by

$$\operatorname{diam}\big(\operatorname{Comp}_{z_{k_\ell}} f^{-(k_\ell)}(K \cap B(z_0, r))\big)\Big/ \operatorname{diam}\big(\operatorname{Comp}_{z_{k_\ell - 1}} f^{-(k_\ell - 1)}(K \cap B(z_0, r))\big);$$

and finally we replace $|(f^{j-k_i})'(z_j)|^{-1}$ for $j \in [k_i, k_{i+1})$ by

$$\operatorname{diam}\big(\operatorname{Comp}_{z_j} f^{-j}(K \cap B(z_0, r))\big)/\operatorname{diam}\big(\operatorname{Comp}_{z_{k_i}} f^{-k_i}(K \cap B(z_0, r))\big).$$

The radius r has been chosen so that $r \leq \operatorname{diam} K$. □

Now we are ready to prove our key implication.

Lemma 2.4 (CE2(z_0) at some $z_0 \in \overline{\mathbb{C}}$ implies Hölder Coding Tree). *Suppose that* CE2(z_0) *holds for some* $z_0 \in \overline{\mathbb{C}}$ *with constant* $\lambda_{\mathrm{CE2}} > 1$ *and fix* $\lambda_{\mathrm{Ho}} \in (1, \lambda_{\mathrm{CE2}})$. *Then there is a set of positive Lebesgue measure of* $w_0 \in \overline{\mathbb{C}}$ *for which we can construct a Hölder coding tree with root* w_0 *and constant* λ_{Ho}.

Proof. Fix $\lambda \in (\lambda_{\mathrm{Ho}}, \lambda_{\mathrm{CE2}})$, then by Lemmas 2.1 and 2.3 there is a set of positive Lebesgue measure of points w_0 for which CE2(w_0) holds with the constants $\lambda > 1$ and $C > 0$. Thus one may choose such a point w_0 also satisfying the conclusions of Lemma 1.1 in Sect. 1. That is, for a fixed $\alpha \in (\frac{1}{2}, 1)$, we can join w_0 to any $w_1 \in f^{-1}(w_0)$ with a non-constant path γ_{w_1} so that the distortion of f^n, $n \gg 1$ in any curve being a pull-back γ_n of γ_{w_1} by f^{-n} is bounded by D^{n^α} for some definite constant $D > 1$.

Then condition CE2(w_0) implies $\operatorname{diam}(\gamma_n) = \mathcal{O}(D^{n^\alpha}\lambda^{-n}) = \mathcal{O}(\lambda_{\mathrm{Ho}}^{-n})$, for $n \geq 1$. □

Hölder coding tree implies ExpShrink. We proceed like in the proof that Hölder implies ExpShrink in [P3].

Let w_0 be a root of a Hölder coding tree with constant $\lambda_{\mathrm{Ho}} = \lambda_0 > 1$. We define a map σ on the vertices of the tree which sends $w_n \in f^{-n}(w_0)$ to its neighbor closest to the root, i.e. to the unique vertex $\sigma(w_n) \in f^{-(n-1)}(w_0)$ such that they are joined by the pull-back of some path γ_{w_1} along f^{n-1}. The latter path, which is the unique edge of the tree going from w_n in the direction of the root, we denote by γ_{w_n}.

Fix an arbitrary $z \in J(f)$ and let w_{n_j} be a sequence of n_j-preimages of w converging to z as $j \to \infty$. Since $f^{-n}(w_0)$ has finite cardinality, we can choose a subsequence so that for all $n \geq 1$, $z_n = \sigma^{n_j - n}(w_{n_j}) \in f^{-n}(w)$ does not depend on j for big $j > \mathrm{Const}(n)$. By Hölder it follows that,

$$\mathrm{dist}(z, z_n) = \lim_{j\to\infty} \mathrm{dist}\big(z, \sigma^{n_j - n}(w_{n_j})\big) \leq C \sum_{k=n}^{\infty} \lambda_0^{-k} \leq C_1 \lambda_0^{-n},$$

where $C_1 = C_1(\lambda_0, w) = \frac{C(\lambda_0, w)}{\lambda_0 - 1}$ does not depend on n. Moreover $\gamma_{z_n} \subset B(z, C_1\lambda_0^{1-n})$. Thus the paths γ_{z_n} form a *ray* δ from w landing at z. Since we choose γ_{w_1} to be non-constant, $a := \min_{w_1}\{\mathrm{diam}(\gamma_{w_1})\} > 0$.

For $r : 0 < r < a$ and $W_0 = B(z, r)$ let δ_0 be a piece of the ray inside W_0 that joins z to ∂W_0. Choose an integer m so that $C_1\lambda_0^{-m} \leq 1$ and fix r satisfying $a(\sup|f'|)^{-(m-1)} > 2r$. So, due to $\mathrm{diam}(\gamma_{w_n}) \geq a(\sup|f'|)^{-(n-1)}$, we get $\delta_0 \subset \cup_{n\geq m}\gamma_{z_n}$.

So, for every component δ_n of $f^{-n}(\delta_0)$ we get $\mathrm{diam}(\delta_n) \leq C_1\lambda_0^{-(n+m)} \leq \lambda_0^{-n}$. So $K = \delta_0$ satisfies the assumptions of Lemma 2.3. This proves ExpShrink with λ_{Exp} arbitrarily close to λ_{Ho}. □

3. Uniform hyperbolicity at periodic orbits (UHP)

In this section we prove that UHP implies CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$ and that ExpShrink implies UHP plus the non-existence of indifferent cycles. The first implication is an easy consequence of Lemma 3.1 and the second implication is stated as Proposition 3.2. Together with Sect. 2, these implications show that these conditions are equivalent.

We will work with a hyperbolic set $K \subset J(f)$ of positive Hausdorff dimension $\mathrm{HD}(K)$ and with points $z_0 \in K$ which are away from the forward iterates of the critical points in the following sense:

$$(\forall n > 0)\ \mathrm{dist}\big(z_0, f^n(\mathrm{Crit})\big) \ \geq\ \max\{n, n_0\}^{-\alpha}, \qquad (*)$$

with positive constants α, $n_0 = n_0(z_0)$. Recall that a compact set $K \subset \overline{\mathbb{C}}$ is called *hyperbolic*, or *expanding*, if there exist $\lambda_K > 1$ and $n = n_K > 0$ such that $|(f^n)'(x)| \geq \lambda_K$ for every $x \in K$. The existence of such a set follows from general theory (cf. construction in the proof of Proposition 4.4.)

First observe that for any rational map $f \in \mathbb{C}(z)$ there is an abundance of such points. Namely, when $\alpha > HD(K)^{-1}$ all z_0 in K, except for a set of Hausdorff dimension $\alpha^{-1} < HD(K)$ satisfy $(*)$. Indeed, for a fixed positive n_0 consider the union A_{n_0} over all positive integers n of balls centered at n-th iterates of critical points with radii $\max\{n_0, n\}^{-\alpha}$. Then the intersection $A := \cap_{n_0} A_{n_0}$ is exactly the set of all points z failing $(*)$. On the other hand, for every $h > \alpha^{-1}$,

$$\frac{n_0}{n_0^{\alpha h}} + \sum_{n \geq n_0} \frac{1}{n^{\alpha h}} \to 0 \text{ as } n_0 \to \infty.$$

Up to a multiplicative constant (the number of critical points) the sum above contains h's powers of the radii of all balls constituting A_{n_0}. So we conclude by the definition of Hausdorff dimension that $HD(A) \leq h$. Letting $h \searrow \alpha^{-1}$ we obtain $HD(A) \leq \alpha^{-1} < HD(K)$.

Note that for z_0 satisfying $(*)$ all pull-backs of $B(z_0, n^{-\alpha})$ by f^n, for $n \geq n_0$, are univalent (we mean by this that f^n is univalent on all the components $\mathrm{Comp}\, f^{-n}(B(z_0, n^{-\alpha}))$).

Lemma 3.1. *Let $f \in \mathbb{C}(z)$ be a rational map such that every repelling cycle of period n has eigenvalue of modulus at least $\lambda_n > 1$. Then there are positive constants c, C such that the following holds if $z_0 \in K$ satisfies $(*)$: For any n-th preimage z_n of z_0, with $n \geq n_0(z_0)$, we have $|(f^n)'(z_n)| > c\lambda_{n+k} n^{-\alpha}$ for some $0 \leq k \leq C \ln n$.*

Proof. Since f is expanding on K, there exists $\delta > 0$ and $C(\delta) > 0$ such that for all $z \in K$ and for all $n \geq 0$ the map f^n is univalent on $\mathrm{Comp}_z f^{-n}(B(f^n(z), 2\delta))$, and its distortion is bounded by $C(\delta)$. This follows either from Koebe distortion Theorem or directly from the expanding property and Hölder continuity of $\ln |f'|$ in a neighborhood of K. One can even show that $C(\delta) \to 1$ as $\delta \to 0$.

For a ball of radius $r \leq \delta$ around z_0 let $l = l(r)$ be the smallest integer such that $|(f^l)'(z_0)| \geq C(\delta)2\delta/r$. Then for $U := f^{-l}(B(f^l(z_0), 2\delta)) \subset B(z_0, r)$. Also we can write $\lambda_K^{l/n_K - 2} \leq |(f^{l-1})'(z_0)| < C(\delta)2\delta/r$ and so l satisfies $l \leq C'' + C' \ln \frac{1}{r}$ with constants $C'' := n_K(2 + \ln(C(\delta)2\delta)/\ln \lambda_K)$ and $C' := n_K/\ln \lambda_K$.

By the eventually onto property of Julia sets there is m such that m-th image of any ball of radius δ centered on $J(f)$ covers the entire Julia set.

Now consider some point $z_0 \in K$ which satisfies $(*)$.

Take $n \geq n_0(z_0)$ and fix $z_n \in f^{-n}(z_0)$. By $(*)$ any pull-back V' of $B(z_0, (n+m)^{-\alpha})$ by f^{n+m} is univalent. Relying on Koebe distortion Theorem, we may choose $\varepsilon > 0$ independent of n such that the pull-back $V \subset V'$ of $B(z_0, \varepsilon(n+m)^{-\alpha})$ by f^{n+m} satisfies $\mathrm{diam}(V) \leq \delta$.

Now take $r := \varepsilon(n+m)^{-\alpha}$ and set $w := f^l(z_0)$. By the discussion above, one of the preimages $f^{-m} z_n$, say z_{n+m}, lies inside $B(w, \delta)$. Let V and V' be the corresponding pull-backs.

Then $V \subset B(w, 2\delta)$. There is a univalent pull-back W' of $B(w, 2\delta)$ by f^l such that $W' \subset B(z_0, r)$. Denote by W the pull back of V along the same branch and note that $l \leq C'' + C' \ln n$ for corrected constants C'' and C'.

Since $f^{n+m+l} : W \longrightarrow B(z_0, r)$ is a bijection, there is a repelling fixed point p of f^{n+m+l} in W. Now take $k := m + l$, then $k \leq m + C'' + C' \ln n \leq 2C' \ln n$ for n large enough, so it satisfies the last assertion of the lemma with $C = 2C'$. To check the first assertion write

$$\lambda_{n+k} \leq |(f^{n+k})'(p)| \leq \mathrm{Const}\, n^{\alpha} |(f^n)'(f^k(p))| \leq \mathrm{Const}\, n^{\alpha} |(f^n)'(z_n)|,$$

where the last inequality follows by Koebe distortion Theorem. Therefore $|(f^n)'(z_n)| \geq c\lambda_{n+k} n^{-\alpha}$ as claimed. □

UHP implies CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$. Suppose that UHP holds with constant $\lambda_{\mathrm{Per}} > 1$. Letting $\lambda_n = \lambda_{\mathrm{Per}}^n$ in Lemma 3.1 we obtain that for every $\lambda \in (1, \lambda_{\mathrm{Per}})$ there is $C = C(\lambda) > 0$ such that CE2(z_0) holds with constants $C > 0$ and $\lambda > 1$. □

Proposition 3.2. *If a rational map $f \in \mathbb{C}(z)$ satisfies* ExpShrink *with a constant $\lambda_{\mathrm{Exp}} > 1$, then for every non-attracting periodic point p of period k we have $|(f^k)'(p)| \geq \lambda_{\mathrm{Exp}}^k$. In particular* ExpShrink *implies* UHP *with $\lambda_{\mathrm{Per}} \geq \lambda_{\mathrm{Exp}}$.*

Proof. Let $r > 0$ be the radius constant involved in ExpShrink. Suppose that S is a Siegel disc for f. Take a ball B with center in the boundary of S and radius r. Let g be the restriction of f to S, and U be a connectivity component of $B \cap S$. Then because of the linearization, preimages $g^{-n}(U)$ have diameters bounded away from zero and so do preimages $f^{-n}(B)$ along the corresponding branches, which contradicts ExpShrink. Therefore f has no Siegel discs, so every non-attracting periodic point belongs to $J(f)$.

Let $p \in J(f)$ be a periodic point of period $k \geq 1$, then ExpShrink implies that for a sufficiently small ball $B(p, r)$ its preimages W_{kn} along the branches of f^{kn} preserving p are univalent. Applying Schwarz lemma to the inverse of $f^{kn} : W_{kn} \to B(p, r)$ we obtain that,

$$|(f^k)'(p)|^n = |(f^{kn})'(p)| \geq \mathrm{Const}\, \mathrm{diam}(W_{kn})^{-1} \geq \mathrm{Const}\, \lambda_{\mathrm{Exp}}^{kn}.$$

Hence $|(f^k)'(p)| \geq \lambda_{\mathrm{Exp}}^k$, as desired. □

Remark 3.3. Note that the definition of UHP imposes a condition only for repelling cycles. However as a consequence of the implications UHP $\Rightarrow$ CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$ $\Rightarrow$ExpShrink and by Proposition 3.2 we obtain that UHP implies the non existence of indifferent cycles, even outside the Julia set.

4. Lyapunov exponents, pressure and dimension

In this section we prove that conditions Lyapunov and Negative Pressure are equivalent to the conditions (a)–(e), as stated in the Main Theorem. We also consider some variants of condition Lyapunov (Sect. 4.2).

The implication Lyapunov⇒UHP is immediate. In fact suppose that condition Lyapunov holds with constant $\lambda_{\mathrm{Lyap}} > 1$. Given a repelling periodic orbit $p, f(p), \ldots, f^n(p) = p$ we consider the probability measure μ equidistributed on it; we have $\frac{1}{n}\ln|(f^n)'(p)| = \Lambda(\mu) \geq \ln\lambda_{\mathrm{Lyap}}$. So condition UHP holds with constant λ_{Lyap}.

In the opposite direction we prove

Proposition 4.1. *The condition* ExpShrink *implies* Lyapunov *with* $\lambda_{\mathrm{Lyap}} \geq \lambda_{\mathrm{Exp}}$.

Proof. Take any f-invariant probability measure μ on $J(f)$. By [P5, Theorem A] we have $\Lambda(\mu) \geq 0$. In particular the function $\ln|f'|$ is μ-integrable. Hence by Birkhoff Ergodic Theorem for μ-a.e. x there exists a limit $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ln|f'(f^j(x))|$. Hence for $a_n := \ln|f'(f^n(x))|$ we get $\lim_{n\to\infty} a_n/n = 0$. There exist constants $C_1, C_2 > 0$ depending only on f such that for every $z \in \overline{\mathbb{C}}$ we have $\operatorname{dist}(z, \mathrm{Crit}) \geq C_1|f'(z)|^{C_2}$. Therefore for every $\delta > 0$ x as above and n large, setting $z = f^n(x)$ we get $\operatorname{dist}(f^n(x), \mathrm{Crit}(f)) \geq \exp(-\delta n)$.

Write $M := \sup|f'|$. Fix an arbitrary $\delta_1 > 0$. Fix r as in the definition of ExpShrink. Set $\epsilon := \frac{\delta_1}{2\ln M}$. Note that for all n large $rM^{-\epsilon n} \geq \exp -\delta_1 n$ and therefore $\mathrm{Comp}_{f^n(x)} f^{-\epsilon n}(B(f^{n+\epsilon n}(x), r)) \supset B(f^n(x), \exp -\delta_1 n)$

Hence for every $0 \leq s \leq n$

$$\begin{aligned}
&\operatorname{diam}\mathrm{Comp}_{f^s(x)} f^{s-n}\big(B(f^n(x), \exp(-\delta_1 n))\big)\\
&\qquad\leq \operatorname{diam}\mathrm{Comp}_{f^s(x)} f^{s-(n+\epsilon n)}(B(f^{n+\epsilon n}(x), r))\\
&\qquad\leq \lambda_{\mathrm{Exp}}^{s-(n+\epsilon n)} \leq \lambda_{\mathrm{Exp}}^{-\epsilon n} = \exp(-\epsilon n \ln\lambda_{\mathrm{Exp}}) = \exp\frac{-\delta_1\ln\lambda_{\mathrm{Exp}}}{2\ln M} n.
\end{aligned}$$

Setting $\delta := \frac{\delta_1\ln\lambda_{\mathrm{Exp}}}{2\ln M}$ we obtain for n large and all $0 \leq s \leq n$

$$\mathrm{Comp}_{f^s(x)} f^{s-n}\Big(B\big(f^n(x), \exp(-\delta_1 n)\big)\Big) \cap \mathrm{Crit} = \emptyset.$$

Hence distortion of f^n on $\mathrm{Comp}_x f^{-n}(B(f^n(x), \frac{1}{2}\exp(-\delta_1 n)))$ is bounded by a constant independent of n. Hence $|(f^n)'(x)| \geq C\exp(-\delta_1 n)/\lambda_{\mathrm{Exp}}^{-n}$. Letting $\delta_1 \to 0$ we get $\Lambda(\mu) \geq \ln\lambda_{\mathrm{Exp}}$. □

4.1. On pressure and dimension

Proposition 4.2. *The conditions* Lyapunov *and* Negative Pressure *are equivalent.*

Proof. By the definition of $P(t)$, since entropy is by definition non-negative, for every f-invariant probability measure μ on $J(f)$ we have $-t\Lambda(\mu) \leq P(t)$. Hence the condition $P(t_1) < 0$ implies $\Lambda(\mu) \geq -P(t_1)/t_1 > 0$. Thus Negative Pressure⇒Lyapunov.

On the other hand by the definition of $P(t)$ there exists the limit $\lim_{t\to\infty} P(t)/t = -\inf\{\Lambda(\mu) : \mu\}$. Hence Lyapunov⇒Negative Pressure. □

In general $P(t)$ is decreasing since $\Lambda(\mu) \geq 0$, see [P5]. Under condition Lyapunov, where $\Lambda(\mu) \geq \ln \lambda_{\mathrm{Lyap}} > 0$, the function $P(t)$ is strictly decreasing so its zero is unique.

Recall that a compact and forward invariant set $X \subset \overline{\mathbb{C}}$ is called *hyperbolic*, or *expanding*, if there exists $n > 0$ such that $|(f^n)'(x)| > 1$ for every $x \in X$. Then the *hyperbolic dimension* of $J(f)$, is the supremum of Hausdorff dimensions of f-invariant hyperbolic subsets of $J(f)$.

Theorem 4.3. *If f satisfies* Topological Collet-Eckmann (TCE)*, then the pressure function $P(t)$ has a unique zero that is equal to the hyperbolic dimension, Hausdorff dimension and box dimension of $J(f)$. All these dimensions of $J(f)$ coincide and are less than 2 provided $J(f) \neq \overline{\mathbb{C}}$.*

This equality of dimensions was proved under the Collet-Eckmann condition in [P1] and under a summability condition in [GS2].

Proof of Theorem 4.3. The equality of the hyperbolic dimension of $J(f)$ to the first zero of $P(t)$ holds for every rational function f, see [P2, (2.2) and Appendix 2]. As in the proof of mean porosity of $J(f)$ in [PR1], we see that there exist $p_1 > 1, \xi > 0, \delta > 0, M \geq 0$ such that for every $x \in J(f)$ there exist increasing sequences of integers n_j, k_j such that $n_j \leq p_1 j$, $W_j := \mathrm{Comp}_x f^{-k_j}(B(f^{k_j}(x), \delta))$ is contained in $B(x, 2^{-n_j})$ and contains $B(x, \xi 2^{-n_j})$ and for each j there is only finite, bounded by M, number of $i : 0 \leq i \leq k_j$ such that $f^i(W_j)$ contains a critical point (δ replaced by 2δ in the latter W_j). In fact in the definition of TCE the constant P, hence p_1 here, can be arbitrarily close to 1, see [PU2, Remark 4.3.c]. The trick to lessen P (on the cost of M) is to use secondary, etc., telescopes as in Remark 5.2.

Let m be any probability α-conformal measure on $J(f)$. Then there exists a constant C depending on M such that $m(W_j) \geq C(\mathrm{diam} W_j)^\alpha$, compare [P1]. Hence for every $r > 0$ and every $x \in J(f)$ we obtain $m(B(x, r)) \geq C_1 r^{\alpha+\epsilon}$ where ϵ depends on p_1 and tends to 0 for $p_1 \searrow 1$ and $C_1 = C_1(\varepsilon) > 0$ is a constant not depending on x and r.

For every r one can find a bounded multiplicity cover of $J(f)$ by discs B_i of diameter r and $m(B_i) \geq C_1 r^{\alpha+\epsilon}$ (Besicovitch) hence $\sum_i (\mathrm{diam} B_i)^{\alpha+\epsilon} \leq C_1^{-1} \sum_i m(B_i) \leq \mathrm{Const}$. Hence upper box dimension $\mathrm{BD}(J(f)) \leq \alpha + \epsilon$. Taking α equal to hyperbolic dimension of $J(f)$ (possible by [DU] and [P5, Corollary A and comments in Introduction]) proves the existence of box dimension of $J(f)$ and its equality to Hausdorff dimension which is always between hyperbolic and box ones. They are strictly less than 2 by mean porosity of $J(f)$, [PR1]. □

4.2. On variants of condition Lyapunov

Note that condition Lyapunov trivially implies the following condition.

L1. Lyapunov exponents of ergodic invariant measures with positive Lyapunov exponent are bounded away from zero.

Moreover by Ruelle's inequality: $I(\mu) \le 2\Lambda(\mu)$ for every ergodic f-invariant measure μ on $J(f)$, this condition implies in turn the following one.

L2. Lyapunov exponents of ergodic invariant measures with positive entropy are bounded away from zero.

Denote possible bounds in L1 and L2 by λ_{L1} and λ_{L2} respectively. We obtain immediately $\sup \lambda_{L2} \ge \sup \lambda_{L1} \ge \sup \lambda_{\text{Lyap}}$. Thus the following proposition together with the Main Theorem imply that conditions L1 and L2 above are equivalent to condition Lyapunov, with the same suprema over all possible λ's.

Proposition 4.4. *Condition L2 implies condition UHP with* $\lambda_{\text{Per}} \ge \lambda_{L2}$.

Proof. For every f-periodic p write $\Lambda(p) := \frac{1}{n(p)} \ln |(f^{n(p)})'(p)|$, where $n(p)$ is a period of p. Denote the periodic orbit of p by $O(p)$.

Given a repelling periodic point p of period $n(p)$ we want, for an arbitrary positive ε, to find a measure μ of positive entropy such that $\Lambda(\mu) \le \Lambda(p) + \varepsilon$. We shall construct an invariant hyperbolic set of positive entropy and demanded exponent using an idea of "building bridges" from $O(p)$ to itself, by E. Prado [Prado]. (This corresponds in higher dimension to finding a "horseshoe" close to a homoclinic point.) Let $x_0 = p, x_1, \ldots$ be a backward trajectory such that $x_1, x_2, \ldots \notin O(p)$ and $x_n \to O(p)$ as $n \to \infty$. There exists $\delta > 0, m > 0$ such that all $f^t(B(p,\delta))$ for $0 < t \le n(p)$ are pairwise disjoint with f univalent on them, $f^{n(p)}(B(p,\delta)) \supset \mathrm{cl} B(p,\delta)$, $x_m \in f^{n(p)}(B(p,\delta)) \setminus \mathrm{cl} B(p,\delta)$, $x_n \notin f^{n(p)}(B(p,\delta))$ for all $0 < n < m$ and $x_n \in \bigcup_{0 \le t < n(p)} f^t(B(p,\delta))$ for $n > m$.

Fix k large enough so that for $V_0 = \mathrm{Comp}_p f^{-kn(p)}(B(p,\delta))$ and $V_n = \mathrm{Comp}_{x_n} f^{-n}(V_0)$ for $n = 1, 2, \ldots$, the following properties hold:

1. $V_m \subset f^{n(p)}(B(p,\delta)) \setminus \mathrm{cl} B(p,\delta)$;
2. All the sets $V_n, n = 0, \ldots, m-1$ are disjoint from $f^{n(p)}(B(p,\delta))$.

Considering $f^t(V_0), 0 \le t < n(p)$ and $V_n, 0 < n \le kn(p) + m$, we obtain a family of topological discs U_j on which f is "generalized polynomial-like", that is satisfies the Markov property $f(\bigcup_j \partial U_j) \cap \bigcup_j \partial U_j = \emptyset$. More precisely, we can guarantee this by making small changes of U_j to destroy $f(\partial V_{n+1}) = \partial V_n$ and $f^t(\partial V_0) = \partial f^{t+1}(V_0)$.

If x_n do not meet Crit then choosing δ small enough, denoting $\bigcup_j U_j$ by U, we guarantee $U \cap \mathrm{Crit} = \emptyset$, hence f on the set of non-escaping points $X = \{z : (\forall n \geq 0) f^n(z) \in U\}$ is expanding (that is $(\forall x \in X)|(f^n)'(x)| > 1$).

If all backward trajectories from $O(p)$ meet Crit, we replace first $O(p)$ by $O(p')$ of minimal period $kn(p) + m$, with $p' \in V_0$ defined above, with large k. $O(p')$ follows $O(p)$ for time $kn(p)$. For every $\varepsilon > 0$ we find k large enough that $\Lambda(p') \leq \Lambda(p) + \varepsilon$ and backward trajectories from $O(p')$ do not meet Crit (use $\sharp\mathrm{Crit} < \infty$) so we can repeat the previous construction for p replaced by p'. (Note that $\Lambda(p')$ can be much smaller than $\Lambda(p)$ for every $p' \neq p$, consider for example $z \mapsto z^2 - 2$ and $p = 2$.)

We conclude with expanding X on which f is a topological Markov chain, topologically transitive and not just one periodic orbit. Hence the topological entropy of f on X is positive.

Note finally that for every $\varepsilon > 0$ one can find δ so that Lyapunov exponents of all points in X are smaller than $\Lambda(p) + \varepsilon$ because the smaller V_0 the longer time each trajectory follows $O(p)$ before it passes a bridge $x_m, x_{m-1}, \ldots, x_0$. So every μ on X has Lyapunov exponent less than $\Lambda(p)+\varepsilon$, in particular the balanced measure, that is ergodic and has maximal, hence positive, entropy. $\square$

Remark 4.5. The proof of the implication ExpShrink$\Rightarrow$Lyapunov uses [P5], so it is not easy. To prove L1 is easier. One can prove directly UHP$\Rightarrow$L1 with $\lambda_{L1} \geq \lambda_{\mathrm{Per}}$ using Pesin-Katok Theory. The idea is that given an ergodic probability f-invariant measure μ supported on $J(f)$ with positive Lyapunov exponent, we can shadow a typical backward trajectory by a periodic repelling orbit, cf. the proofs of UHP$\Rightarrow$CE2(z_0) in Sect. 3, and in Appendix A (using Lemma A.4).

5. Further conditions equivalent to TCE

In this section we consider further conditions that are equivalent to TCE. We are mainly interested in the following condition.

- TCE at x. The same as TCE but only for a fixed x.

  Furthermore we consider the following auxiliary property.

- BLS. *Backward Lyapunov stability.* For every $\delta > 0$ there exists $r > 0$ such that for every $x \in J(f)$ and any positive integer n, $\mathrm{diam}\,\mathrm{Comp}_x f^{-n} B(f^n(x), r)) \leq \delta$.

We will prove that the condition (TCE at critical points & BLS) is equivalent to TCE, which gives a new topological condition equivalent to TCE. This leads us to state the following natural problem.

Problem. Suppose that a rational map satisfies TCE at critical points. Does it satisfy condition BLS?

Moreover we consider the further variants of condition ExpShrink.

- *Exponential shrinking of components at critical points.* There is $\lambda > 1$ and $r > 0$ such that for every critical point $c \in J(f)$ and for every n, the connected component W of $f^{-n}(B(f^n(c), r))$ containing c, satisfies $\operatorname{diam}(W) \le \lambda^{-n}$.
- *Exponential shrinking of components at x.* The same as above but only for a fixed x in place of c.
- *Backward exponential shrinking of components at $z_0 \in \overline{\mathbb{C}}$.* There is $\lambda > 1$ and $r > 0$ such that for every $n \ge 0$ and every connected component W of $f^{-n}(B(z_0, r))$ we have $\operatorname{diam}(W) \le \lambda^{-n}$.

Complement to the Main Theorem. *For a rational map f of degree at least two, condition TCE is equivalent to the following ones.*

1. *TCE at all critical points in $J(f)$ & BLS.*
2. *Exponential shrinking of components at critical points and no parabolic periodic orbits.*
3. *Backward exponential shrinking of components at some $z_0 \in \overline{\mathbb{C}}$.*
4. *$CE2(z_0)$ holds for all $z_0 \in \overline{\mathbb{C}}$, except for an exceptional set E of Hausdorff dimension zero, with the same $\lambda_{CE2} > 1$ (and the same supremums of λ_{CE2}'s at each z_0) and possibly with different constant $C > 0$.*
5. *Lyapunov exponents of invariant measures with positive entropy (resp. positive Lyapunov exponent) are bounded away from zero.*

Lemma 5.1. *For an arbitrary $x \in J(f)$ the condition (TCE at x & BLS) implies the condition: Exponential shriking of components at x.*

Proof. TCE at $x \in J(f)$ implies *mean exponential shrinking of components at x* (compare with [NP]); that is there is $P > 0$, $\delta > 0$, $\lambda > 1$ and a strictly increasing sequence of positive integers n_j, $j = 1, 2, ...$ such that $n_j \le Pj$ and for each j and $i : 0 \le i \le n_j$

$$\operatorname{diamComp}_{f^i(x)} f^{-(n_j - i)} B(f^{n_j}(x), \delta) \le \lambda^{-(n_j - i)}.$$

The proof is a part of the proof of Proposition 3.1 in [PR] (rewritten in [NP, Sect. 2] for unimodal maps of the interval.) The idea is the *telescope* construction. We can assume that all $n_{j+1} - n_j$ are larger than a constant so that that

$$\operatorname{diamComp}_{f^{n_j}(x)} f^{-(n_{j+1} - n_j)} B(f^{n_{j+1}}(x), \delta) \le \delta/2.$$

Hence

$$\operatorname{diamComp}_x f^{-n_j} B(f^{n_j}(x), r) \le \xi^j \le (\xi^{1/P})^{n_j}.$$

for a constant $\xi < 1$. See [M] and also [P1, Lemma 1.1] or [GSw2, pp. 279–280].

Next notice that for $n : n_j < n < n_{j+1}$, by BLS, we have

$$\mathrm{diamComp}_{f^{n_j}(x)} f^{-(n-n_j)} B(f^n(x), r) < \delta.$$

Hence $\mathrm{Comp}_{f^{n_j}(x)} f^{-(n-n_j)} B(f^n(x), r) \subset B(f^{n_j}(x), \delta)$ and finally $\mathrm{diam}_x f^{-n} B(f^n(x), r) \le \xi^j \le \xi^{-1}\xi^{j+1} \le \xi^{-1}(\xi^{1/P})^n$. □

Remark 5.2. In [PR1, Proposition 3.1] we proved Exponential shrinking of components at critical points assuming CE. We also went via TCE, but coped with $n : n_j < n < n_{j+1}$ using again CE, referring to [P1, Sect. 3] where the proof of CE⇒BLS was sketched (for a detailed proof see [PU2, Appendix B]. Here we just assume condition BLS instead.

For TCE assumed at all points, we coped in [PR1] with $n : n_j < n < n_{j+1}$ proving ExpShrink, by using secondary telescopes at $f^{n_j}(x)$. Namely for each $y = f^{n_{j_0}}(x)$, due to TCE at y, there exists a sequence n'_j with the similar properties as n_j with c replaced by y; we are interested only in $n'_j < n_{j_0+1} - n_{j_0}$. The secondary gaps (n'_j, n'_{j+1}) can be filled in with the next generation telescopes, etc, yielding finally ExpShrink. See [P3, Lemma 4.3] for details.

Proof of the complement to the Main Theorem.
1 and 2. For the equivalence between TCE and condition 2, see [PR1] and [P3]. Lemma 5.1 applied to x being critical points in $J(f)$ and the observation that BLS implies the absence of parabolic periodic orbits yields the implication 1 ⇒ 2. So it remains to prove that TCE implies condition 1. Obviously TCE implies TCE at critical points. Moreover by [PR1] and [P3] condition TCE implies condition ExpShrink, which clearly implies BLS.

3 and 4. Clearly condition ExpShrink, which is equivalent to TCE, implies condition 3. The condition ExpShrink also implies condition 4 by Remark 2.2. Lemma 2.1 states that 3 implies condition CE2(z_0) for some $z_0 \in \overline{\mathbb{C}}$, which is equivalent to TCE by the Main Theorem. That condition 4 implies TCE follows trivially from the Main Theorem.

5. That the conditions in 5 are equivalent to TCE was proven in Sect. 4.2. □

6. On conditions CE and CE2. Counterexamples

We now explain the assertions of the Main Theorem related to the conditions CE and CE2. Remind that we have already proved that conditions (a)–(g) in the Main Theorem are equivalent.

The implication CE2 ⇒ (c) CE2(some z_0), is obvious and the implication CE ⇒ (a) TCE was proven in [PR1]. So each of the conditions CE and CE2 implies the equivalent conditions (a)–(g).

Below we give examples of real polynomials of degree 5 that satisfy CE but not CE2, and vice-versa. S. van Strien considered such an example (though not polynomial) for maps of the interval [vS]. We also provide a real polynomial of degree 3 satisfying TCE but neither CE nor CE2. A similar example (in degree 4) was considered in [PR2], but the measurable Riemann mapping theorem was used to justify it.

The troubles with CE or CE2 arise when the forward trajectory of one critical point closely approaches another critical point.

Rational maps with only one critical point in its Julia set. In [GS1] it is proved that condition CE implies CE2(c), for all critical points c of f in $J(f)$ with maximal (dynamical) multiplicity. On the other hand it is not difficult to prove that condition CE2 together with condition ExpShrink imply CE at c for all critical points $c \in J(f)$ with maximal (dynamical) multiplicity, see [GS1] and also [R-L] for a simpler case.

Hence, due to our new implication (c)CE2(some z_0) $\Rightarrow$(b)ExpShrink, we obtain the equivalence of conditions CE and CE2 in the class of all rational functions for which all critical points in the Julia set have the same (dynamical) multiplicity. This class includes rational maps with only one critical point in the Julia set.

It remains to note that for rational functions with only one critical point in the Julia set the implication TCE $\Rightarrow$ CE was proved in [P3]. Let us mention that in [GP] J. Graczyk and I. Popovici proved directly the implication UHP $\Rightarrow$ CE for quadratic non infinitely renormalizable polynomials.

6.1. Counterexamples

A class of rational maps of a large interest are the semi-hyperbolic ones, see [CJY].

Definition. *We call a rational function f* semi-hyperbolic *if it does not have parabolic periodic points and if all its critical points in $J(f)$ are non-recurrent.*

Semi-hyperbolicity does not exclude that one critical point belongs to the forward limit set of another. It was proved in [CJY], with the use of [M], that semi-hyperbolic maps satisfy condition ExpShrink. In [CJY] it was also proved that semi-hyperbolicity is equivalent to TCE with $P = 1$ (in this case $\{n_j\}$ is the sequence of all non-negative numbers numbers.)

All counterexamples constructed in this section and in Appendix C are semi-hyperbolic.

Consider a holomorphic one-parameter family of polynomials f_σ with the following properties: f_σ is real for σ real, 0 is a repelling fixed point with multiplier $f'(0) = \lambda_\sigma$ and such that there are critical points c_1 and c_2 with the criticalities μ_1 and μ_2 respectively such that $f_\sigma^2(c_2) \equiv 0$. Assume furthermore that for some real σ_0, $f_{\sigma_0}^k(c_1) = 0$ for a natural number k, but $f_\sigma^k(c_1) \not\equiv 0$, or stronger: $\partial_\sigma f_\sigma^k(c_1) \neq 0$ at $\sigma = \sigma_0$.

Moreover suppose that there is an open set $W \subset \mathbb{C}$ such that the restriction of f_σ to W is polynomial-like with c_2 as unique critical point. Then there is an invariant interval $I \subset W \cap \mathbb{R}$ and an holomorphic motion i of I compatible with dynamics such that i_{σ_0} is the identity. Note that $|i_\sigma(x)| \sim |x|^{\alpha_\sigma}$ for x close to 0, where α_σ is defined by $|\lambda_\sigma| = |\lambda_{\sigma_0}|^{\alpha_\sigma}$.

Fix $\eta > 0$ close to 1 and consider a point $x \in I$ with lower Lyapunov exponent $\liminf_{n\to\infty} \frac{1}{n} \ln |(f_{\sigma_0}^n)'(x)| = \ln \eta$ for f_{σ_0} with the following properties. The orbit of x accumulates at c_2 and the closest return times of x to c_2 are $n_1 < n_2 < \ldots$ which are also the only times that $f_{\sigma_0}^n(x)$ is close to c_2.

Such x can be found with the use of induction. Suppose x^i satisfies $|(f_{\sigma_0}^{n_j+1})'(x^i)| \sim \eta^{n_j+1}$ for all $j \le i$ and the closest return times to c_2 are are $n_1, \ldots, n_i$. Put $\rho_i = |f_{\sigma_0}^{n_i+1}(x^i) - f_{\sigma_0}(c_2)|$. Consider the first time $m > n_i + 1$ so that $|f_{\sigma_0}^m(x^i)| \sim 1$. By bounded distortion $|(f_{\sigma_0}^{m-n_i-1})'(f_{\sigma_0}^{n_i+1}(x^i))| \sim \rho_i^{-1}$ and therefore $m - n_i \sim \ln(\rho_i^{-1})$.

Choose y close to c_2, such that $|(f_{\sigma_0}^m)'(x^i)| \cdot |y - c_2|^{\mu_2 - 1} \sim \eta^{m+1}$. Set $n_{i+1} := m$ and define x^{i+1} as the preimage of y under the branch of $f_{\sigma_0}^{-m}$ mapping $f_{\sigma_0}^m(x^i)$ to x^i defined on an *a priori* fixed disc D disjoint from the set $\{f_{\sigma_0}(c_2), 0\}$ and such that $f_{\sigma_0}^m(x^i)$, $y \in D'$ for another *a priori* fixed disc D' with closure in D. Hence by bounded distortion $|(f_{\sigma_0}^{n_j+1})'(x^{i+1})| \sim |(f_{\sigma_0}^{n_j+1})'(x^j)|$ for all $j = 1, \ldots, i$ and $|(f_{\sigma_0}^{n_{i+1}})'(x^{i+1})| \cdot |y - c_2|^{\mu_2 - 1} \sim |(f_{\sigma_0}^{n_{i+1}+1})'(x^{i+1})| \sim \eta^{n_{i+1}+1}$. For $x := \lim_{i\to\infty} x^i$ for each j we obtain $|(f_{\sigma_0}^{n_j+1})'(x)| \sim \eta^{n_j+1}$

Note that

$$\left| f_{\sigma_0}^{n_{i+1}}(x) - c_2 \right|^{\mu_2 - 1} \sim \rho_i \eta^{n_{i+1} - n_i} \ll \rho_i^{\frac{\mu_2 - 1}{\mu_2}} \sim \left| f_{\sigma_0}^{n_i}(x) - c_2 \right|^{\mu_2 - 1},$$

if η is close enough to one, since $n_{i+1} - n_i \sim \ln(\rho_i^{-1})$. Thus n_{i+1} is a closer return than n_i, as we wanted, provided ρ_1 is small enough (to define it, let us set $n_0 = 0$), that is x close to 0. Moreover $n_{i+1} - n_i \to \infty$ exponentially fast; this assures the convergence of x^i.

Hence

$$\begin{aligned}
&\left| \left(f_\sigma^{n_i+1} \right)' (i_\sigma(x)) \right| \\
&\quad \sim \prod_{j=1}^{i-1} \left| \left(f_\sigma^{n_{j+1} - n_j - 1} \right)' \left(f_\sigma^{n_j+1}(i_\sigma(x)) \right) \right| \cdot \left| f_{\sigma_0}^{n_{j+1}}(i_\sigma(x)) - c_2 \right|^{\mu_2 - 1} \\
&\quad \sim \prod_{j=1}^{i-1} \rho_j^{-\alpha_\sigma} \left| f_{\sigma_0}^{n_{j+1}}(i_\sigma(x)) - c_2 \right|^{\alpha_\sigma(\mu_2 - 1)} \sim \eta^{\alpha_\sigma n_i}.
\end{aligned}$$

We conclude that the lower Lyapunov exponent of f_σ at the point $i_\sigma(x)$ is $\alpha_\sigma \ln \eta$.

6.1.1. Semi-hyperbolicity does not imply neither CE nor CE2. An example, a degree 4 polynomial, was constructed already in [P4], with the use of Douady-Hubbard straightening of polynomial-like maps. Here we find directly degree 3 polynomials.

Just consider a family f_σ as above with an arbitrary $\eta < 1$. $f_{a,b}(z) = az(1+z)(1-z) - b$ in appropriate affine coordinates on $\mathbb{C}$ satisfies our assumptions. We have critical points $c_1 = -\sqrt{\frac{1}{3}}$ and $c_2 = \sqrt{\frac{1}{3}}$.

Parameterize the family by σ, with $a(\sigma)$ and $b(\sigma)$, such that $a(\sigma_0) = \frac{3\sqrt{3}}{2}$ and $b(\sigma_0) = 0$; hence $f_{\sigma_0}(z) = \frac{3\sqrt{3}}{2} z(1+z)(1-z)$. We get $f^2_{\sigma_0}(c_i) = 0$ for $i = 1, 2$.

There exists a curve $(a(\sigma), b(\sigma))$ with $\partial_\sigma a, \partial_\sigma b > 0$ at $\sigma = \sigma_0$ and $f_\sigma(p) = p = f^2_\sigma(c_2)$ for $p = p(\sigma)$. One can see this by computing the gradients of the functions $f_{a,b}(p) - p$ and $f^2_{a,b}(c_2) - p$ in the coordinates $(a, b, p) \in \mathbb{R}^3$ at $a = \frac{3\sqrt{3}}{2}$, $b = p = 0$. One can see that $f^2_\sigma(c_1) - p(\sigma) > 0$ for $\sigma > \sigma_0$ by computing for example that $\partial_a f^2_{a,b}(c_1)$ and $\partial_b f^2_{a,b}(c_1)$ are large at $a = \frac{3\sqrt{3}}{2}$, $b = 0$.

Therefore there exists $\sigma > \sigma_0$ (arbitrarily close to σ_0) and $k \geq 2$ such that $f^k_\sigma(c_1) = i_\sigma(x)$. Hence the lower Lyapunov exponent at the critical value $f_\sigma(c_1)$ is $\ln(\eta^{\alpha_\sigma}) < 0$. Thus f_σ is not CE. Since $\mu_1 = \mu_2 (= 2)$ here, f_σ does not satisfy CE2 neither, see introduction and [GS1].

6.1.2. CE2 does not imply CE. Consider now a family f_σ with $\mu_2 > \mu_1$. Start for example with $f_{a,b}(z) = az^4(1-z) + b$, write $c_1 = \frac{3}{4}$ and $c_2 = 0$, find an appropriate curve $a(\sigma), b(\sigma)$ and change coordinates so that the repelling fixed point is always at 0.

Put $\eta = 1$ and let σ close to σ_0 such that $f^k_\sigma(c_1) = i_\sigma(x)$ for some $k \geq 2$, there are such σ since $f_\sigma(c_1) \not\equiv 0$. By the above the lower Lyapunov exponent of f_σ is 0 so f_σ is not CE. Let us prove that f_σ is CE2.

Since c_1 is not an accumulation point of forward trajectories neither of c_1 nor of c_2, there is a neighborhood U_1 of c_1 such that all f^j on all components of $f^{-j}_\sigma(U_1)$ are univalent and, since f_σ as semi-hyperbolic is TCE (see [CJY]), the diameters shrink uniformly exponentially as $j \to \infty$. So c_1 satisfies CE2.

Similarly there is a neighborhood U_2 of c_2 such that the diameters of all pull-backs shrink uniformly exponentially. Note that a pull-back can encounter now the critical point c_1, but for any (x_j), a backward trajectory of c_2, this can happen at most for one component $\mathrm{Comp}_{x_j} f^{-j}(U_2)$ (since the diameters of the components shrink uniformly to 0 and c_1 is non-recurrent).

We can find a smaller neighborhood $\hat{U}_2 \subset U_2$ so that in all the components of $f^{-n}(\hat{U}_2)$, as long as f^n are univalent on the respective components of $f^{-n}(U_2)$, the derivatives of f^n are exponential at the respective preimages of c_2.

Consider now a pull-back $\hat{V}$ of $\hat{U}_2$ by f_σ^{n+1} so that $c_1 \in \hat{V}$. So f_σ^n is univalent in $f_\sigma(V)$, the respective pull-back of U_2 so the distortion of f_σ^n in $f_\sigma(\hat{V})$ is bounded by some definite constant, see Lemma 1.4 for, say, $U_2 = B(c_2, r)$ and $\hat{U}_2 = B(c_2, r/2)$, with $r \le r_K$.

(If it happens that $c_1 \in V \setminus \hat{V}$ then we replace for all backward trajectories of c_2 through V the set U_2 by $\hat{U}_2$. As we cannot encounter c_1 again, only univalent cases happen.)

Note that $n = n_i + k - 1$ for some i, since the n_i are the only times that the orbit of $i_\sigma(x)$ approaches c_2. By the previous considerations $|(f_\sigma^{n_i+k})'(f_\sigma(c_1))| > \mathrm{Const}\theta^{i\alpha_\sigma}$, hence,

$$\left|f_\sigma^{n_i+k}(c_1) - c_2\right|^{\mu_2-1} \ge \mathrm{Const} \cdot \theta^{i\alpha_\sigma} \left|\left(f_\sigma^{n_i+k-1}\right)'(f_\sigma(c_1))\right|^{-1}.$$

Let ζ close to c_1 be such that $f_\sigma(\zeta)$ is the unique (n_i+k)-th preimage of c_2 in $\hat{V}$. So, by bounded distortion,

$$|f_\sigma(\zeta) - f_\sigma(c_1)| \sim \left|f_\sigma^{n_i+k}(c_1) - c_2\right| \left|\left(f_\sigma^{n_i+k-1}\right)'(f_\sigma(c_1))\right|^{-1}$$

$$\ge \mathrm{Const}\theta^{i\frac{\alpha_\sigma}{\mu_2-1}} \left|\left(f_\sigma^{n_i+k-1}\right)'(f_\sigma(c_1))\right|^{-\frac{\mu_2}{\mu_2-1}},$$

and considering that $|f_\sigma'(\zeta)| \sim |f_\sigma(\zeta) - f_\sigma(c_1)|^{\frac{\mu_1-1}{\mu_1}}$,

$$\left|\left(f_\sigma^{n_i+k}\right)'(\zeta)\right| \ge \mathrm{Const}\theta^{i\frac{\alpha_\sigma}{\mu_2-1}\frac{\mu_1-1}{\mu_1}} \left|\left(f_\sigma^{n_i+k-1}\right)'(f_\sigma(\zeta))\right|^{1-\frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1}}.$$

Since $\mu_2 > \mu_1$ it follows that $1 - \frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1} > 0$ so the above estimate is exponentially large in n_i. Finally note that, by taking U_2 small we may assume that $\zeta \in U_1$ so it follows that the derivative at preimages of this ζ is exponential so c_2 also satisfies CE2.

6.1.3. CE does not imply CE2. Such an example was already considered by S. van Strien [vS]. Choose $\mu_1 > \mu_2$ this time and $\eta > 1$ and let f_σ as above.

Use for example again the family $f_{a,b}(z) = az^4(1-z) + b$ but change the roles of the critical points, $c_1 = 0,\ c_2 = \frac{3}{4}$, translate the repelling fixed point to 0 and find an appropriate curve $a(\sigma), b(\sigma)$.

Thus f_σ is CE and just as before we have an estimate,

$$\left|\left(f_\sigma^{n_i+k+1}\right)'(\zeta)\right| \le \mathrm{Const}(\theta^{-i}\eta^{n_i})^{\frac{\mu_1-1}{\mu_1}\frac{\alpha_\sigma}{\mu_2-1}} \left|\left(f_\sigma^{n_i+k}\right)'(f_\sigma(\zeta))\right|^{1-\frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1}},$$

now $1 - \frac{\mu_1-1}{\mu_1}\frac{\mu_2}{\mu_2-1} < 0$ so the above estimate is exponentially small in n_i, if η is chosen close enough to 1. So c_2 does not satisfies CE2. Note that f_σ is not CE2 even in the real sense (for backward trajectories in $\mathbb{R}$).

Appendix A. An alternative proof of UHP $\Rightarrow$ CE2(some z_0)

In this Appendix we present an alternative proof that UHP implies CE2(some z_0).

Lemma A.1. *For any integer $L \geq 2$ there exists $R(L) > 0$ such that if for a ball $B(x, R)$, intersecting $J(f)$ and of radius $R \leq R(L)$, a component of $f^{-k}B(x, R)$ contains a critical point, and also $f^{-l}B(x_k, R)$ contains a critical point, then $k + l \geq L$. Here $x_k \in f^{-k}x$ is a preimage of x belonging to the component of $f^{-k}B(x, R)$ under consideration.*

Proof. Take δ to be so small that the return time of the set of critical points to its δ-neighborhood is greater than L (recall our convention that no critical point in $J(f)$ is in the forward orbit of another critical point). By compactness there is such R that for $k \leq L$ any component of connectivity of the pull-back $f^{-k}B$ of any ball of radius R has diameter less than δ. This R is the desired one. □

Lemma A.2. *Fix ε and let R be sufficiently small. Then for sufficiently large n any $a \in J(f)$ has a "good" preimage $b \in f^{-[\varepsilon n]}a$, meaning that for some $k < \varepsilon n$ the ball $B(f^k b, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through a.*

Proof. Denote by d the degree of f. Take $R < R(L)$ with L large enough to satisfy $4\ln(4dL)/L < \epsilon \ln(d)$ with $\epsilon < \varepsilon/2$, and assume $n > L/\epsilon$. For every branch of $f^{-(n+[\epsilon n])}$ at a do the following procedure: take a ball $B(a, R)$ and pull it back until it hits some critical point after k_1 iterations, take a ball $B(f^{-k_1}a, R)$ and pull it back until it hits some critical point after k_2 iterations. We continue by induction until we obtain a ball $B(f^{-m}a, R)$, with $m = k_1 + \cdots + k_\ell$, which can be pulled back univalently till $f^{-(n+[\epsilon n])}a$.

For any ball B and positive integer i for at most $4d$ (more precisely: at most $2(2d - 2)$) (counting multiplicities) branches of f^{-i} preimage of B hits critical points first time exactly after i iterates. Thus one specification $(k_1, \ldots, k_\ell)$ can correspond to at most $(4d)^\ell$ distinct branches of length m (depending on the branch). Also, as was shown above for any specification we have $k_i + k_{i+1} > L$, thus in any interval $[j, j + L)$, with $0 \leq j \leq [(n + [\epsilon n])/L]$, there are at most two numbers of the form $k_1 + \cdots + k_i$. Considering every second i each block $[tL, (t + 1)L)$ contains no $\sum_{s=1}^{i} k_s$ or at most one with L possible positions. Therefore there are at most $(L + 1)^{2+2(n+[\epsilon n])/L} < L^{4n/L}$ ways to choose a specification, and necessarily $\ell \leq 2 + 2(n + [\epsilon n])/L < 4n/L$. Hence the number of such "critical" branches is at most

$$(4d)^\ell L^{4n/L} < (4dL)^{\frac{4n}{L}} = \exp\left(4n \ln(4dL)/L\right) < d^{\epsilon n} .$$

But counting multiplicities there are $d^{[\varepsilon n]} > d^{\epsilon n}$ preimages $f^{-[\varepsilon n]}a$, thus there is a point $b \in f^{-[\varepsilon n]}a$, such that for all branches of $f^{-(n+[\varepsilon n])}a$ passing through it $m < [\varepsilon n]$. Clearly b is the desired point, and $k = [\varepsilon n] - m$ is the desired iterate. □

Lemma A.3. *Let μ be the invariant measure of maximal entropy. Then for almost all z with respect to μ there is a positive integer $n(z)$ such that for $n > n(z)$*

1. *The ball $B(f^k z, R)$ can be pulled back univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.*
2. *The ball $B(f^{n+k} z, R)$ can be pulled back univalently along branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.*

Proof. Choosing arbitrarily small ϵ in the Lemma A.2, we deduce that when n is large, except for a set of exponentially small measure μ, for all z in the Julia set the ball $B(f^k z, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$. Here we use the fact that Jacobian of f with respect to μ is equal to d, hence exponentially small proportion of preimages supports only exponentially small proportion of μ. Letting $n \to \infty$ and using Borel-Cantelli lemma, we deduce the first claim.

Similarly, passing to preimages (and using invariance of the measure μ), we obtain the second claim. It also follows immediately from Pesin's theory. □

In the sequel we shall need the following general fact.

Lemma A.4. *There is a constant $j = \mathrm{Const}(R)$ such that if a component U of $f^{-n}B(R)$ is a univalent pull back of some ball $B(R)$ of radius R for sufficiently large n, then any component U' of $f^{-n}B(R/2)$ inside U contains a repelling periodic point of period $n + j$.*

Proof. Choose j to be large enough so that preimages of any point in $J(f)$ by f^j are $R/(9\deg(f))$-dense in $J(f)$. By compactness there is $\delta > 0$ such that any component of the pull-back $f^{-j}B(\delta)$ of any ball $B(\delta)$ of radius δ has diameter less than $R/(9\deg(f))$. By a standard argument (dating back to Fatou) for sufficiently large n component U' would have diameter less than δ.

Since $B(R/3)$ contains (for $R \ll 1$) at least $\frac{2}{3}\pi(R/3)^2/(\sqrt{3}R/9\deg(f))^2 > 4\deg(f)$ points that are $4R/(9\deg(f))$-separated, we obtain at least $4\deg(f)$ different components of $f^{-j}(U')$ inside $B(R/2)$. By Fatou's estimate on the number of non-repelling cycles, at least one of them, say V, would intersect only repelling cycles. Using an index argument for the map $f^{n+j} : V \to B(R/2)$, we find the desired periodic point, which is forced to be repelling. □

Now assume UHP. Take any $\lambda < \lambda_{\mathrm{Per}}$. Choose ε small enough, so that $\lambda_{\mathrm{Per}}/\lambda > M^{\varepsilon}$. Denote $M := \sup_{J(f)} |f'|$. Parameters R, j are given by Lemmas above. Then for n larger than some n' we have $(\lambda_{\mathrm{Per}}/\lambda)^n > CM^{\varepsilon n + j}$.

Take as above the measure μ of maximal entropy (balanced measure). Then by Lemma A.3 for μ-almost all points z_0 for $n > n(z_0)$ the ball $B(f^k z_0, R)$ can be pulled univalently along any branch of $f^{-(n+k)}$ passing through z, where $k = k(n) < \epsilon n$.

Take one of such points z_0, some $n > n(z_0)$, and some $w \in f^{-n}(z_0)$. Let W be the component of $f^{-(n+k)}B(f^k z_0, R/2)$ containing w. By the Lemma A.4 there is a periodic point $p \in W$ with a period $n+k+j$. Using Koebe distortion Theorem and UHP, we can write

$$|(f^n)'(w)| \geq |(f^n)'(p)|/C \geq |(f^{n+k+j})'(p)|/(CM^{k+j}) \geq \lambda_{\mathrm{Per}}^n/(CM^{k+j}),$$

which is greater than λ^n whenever n is large enough.

So we have shown that UHP implies CE2(z_0) for almost all z_0 with respect to μ.

Using the second statement of Lemma A.3 we similarly note that for μ-almost all points $|(f^n)'(z)| > \lambda^n$ whenever n is large enough. Thus the lower Lyapunov exponent $\liminf_{n\to\infty} \frac{1}{n} \ln |(f^n)'(z)|$ is at least $\ln \lambda > 0$ for μ-almost all points z, and we infer that $\Lambda(\mu) \geq \ln \lambda > 0$.

Remark A.5. Lemma A.3, hence the assertions above, hold for any invariant measure μ of Jacobian bounded away from 1 on a set of full measure μ. These are however not the best results. In Remark 4.5 it is noted that with the use of Pesin-Katok method one can prove $\Lambda(\mu) \geq \ln \lambda > 0$ for every μ of positive Lyapunov exponent (in fact even this assumption can be skipped) and every $\lambda < \lambda_{\mathrm{Per}}$. Furthermore CE($z_0$) for just one z_0 implies CE(z_0) for almost all z_0 for all invariant ergodic measures of positive entropy (even for a larger class of invariant measures) by Remark 1.1; any set of positive measure has positive Hausdorff dimension by the formula: dimension of measure = entropy / Lyapunov exponent. Nevertheless the direct method *via* elementary Lemma A.2 seems to be of an independent interest.

Appendix B. Does bounded criticality imply zero criticality?

We shall discuss here the set of integers for which TCE holds at a critical point with a given criticality.

Notation. For any A, a subset of the set of natural numbers, we write

$$\underline{d}(A) := \liminf_{n\to\infty} \#([1, ..., n] \cap A)/n$$

$$\overline{d}(A) := \limsup_{n\to\infty} \#([1, ..., n] \cap A)/n.$$

For every $x \in \overline{\mathbb{C}}, \delta > 0$ and $n \geq k \geq 0$ denote the set $\mathrm{Comp}_{f^k(x)} f^{-(n-k)}(B(f^n(x), \delta))$ by $W_{n,k,x,\delta}$.

Given $M \geq 0, \delta > 0, x \in \overline{\mathbb{C}}$ we say $n \geq 0$ is an (x, M, δ) good time, and write $n \in G(x, M, \delta)$ if

$$\sharp\{0 \leq k < n : W_{n,k,x,\delta} \cap \mathrm{Crit} \neq \emptyset\} \leq M.$$

We assume that δ is small enough that for all $n \in G(x, M, \delta)$ and $0 \leq k \leq n$, the sets $W_{n,k,x,\delta}$ are topological discs of diameters smaller than $\min \mathrm{dist}(c, c')$, the minimum taken over all pairs of distinct critical points, see [M]; therefore each $W_{n,k,x,\delta}$ contains at most one critical point.

Theorem B.1. $\overline{d}(G(c, M, \delta)) > 0$ *at a critical point* $c \in J(f)$ *implies* $\overline{d}(G(f(c'), 0, \delta')) > 0$ *for some* $\delta' > 0$ *and* $c' \in \mathrm{Crit} \cap \mathrm{cl} \bigcup_{n \geq 0} f^n(c)$.

Remark B.2. We do not know whether $\overline{d}$ can be replaced by $\underline{d}$ in this this Theorem, which would be in accordance to condition TCE at c and c'. We expect a negative answer, even in unicritical case.

Proof. The proof goes by induction with respect to M. Fix c, M, δ and assume $\overline{d}(G(c, M, \delta)) > 0$, $M > 1$. For every $n \in G(c, M, \delta)$ let $\delta \geq \delta_n \geq 2^{-M}\delta$ be chosen so that for every $0 < k < n$, $(W_{n,k,x,\delta_n} \setminus W_{n,k,x,\delta_n/2}) \cap \mathrm{Crit} = \emptyset$. Denote $W_{n,k} := W_{n,k,c,\delta_n}$.

Shrinking δ and taking only every n_0-th n for n_0 large enough, we can assume the following *telescope property*: For every $n_1 \leq n_2 \leq n_3 \in G(c, M, \delta)$ it holds $W_{n_3,n_1} \subset W_{n_2,n_1}$.

For all $s = 0, ..., n$, $n \in G(c, M, \delta)$, for $\delta'(s) := \frac{\delta}{2^M \sup |f'|^s}$, we have

$$B(f^{n-s}(c), \delta'(s)) \subset W_{n,n-s}$$

hence, for all $k : 0 < k \leq n - s$,

$$W_{n-s,k,c,\delta'(s)} \subset W_{n,k}.$$

Let $t(n) : 0 \leq t(n) < n$ be the largest integer such that $W_{n,t(n)}$ contains a critical point.

Since $W_{n,t(n)}$ captures a critical point, there are at most $M - 1$ integers $k : 0 \leq k < t(n)$ such that $W_{n,k}$ captures a critical point. Therefore for $s \geq n - t(n)$, $n - s \in G(c, M - 1, \delta'(s))$.

For every integer $s > 0$ denote $G_s := \{n \in G(c, M, \delta) : n - t(n) \leq s\}$. Suppose there exists $s > 0$ such that

$$\overline{d}(G(c, M - 1, \delta 2^{-M})) \cup G_s) > 0.$$

Suppose $n \in G_s$. Then $n - s \in G(c, M - 1, \delta'(s))$. Of course if $n \in G(c, M - 1, \delta 2^{-M})$ then $n - s \in G(c, M - 1, \delta'(s))$. Therefore

$$\overline{d}(G(c, M - 1, \delta'(s)) \geq \overline{d}(G(c, M - 1, \delta 2^{-M}) \cup G_s) > 0.$$

Suppose now that s as above does not exist. Then there are sequences $n_j, s_j \to \infty$ such that for

$$G^j := \left\{n \in G(c, M, \delta) \setminus G(c, M - 1, \delta 2^{-M}) : 0 < n \leq n_j, n - t(n) \geq s_j\right\}$$

it holds

$$\#G^j/n_j \geq \varepsilon_0 := \frac{1}{2}\overline{d}(G(c, M, \delta)) > 0.$$

Note that for an arbitrary j and all $n \in G^j$ and $t(n) < m < n, m \in G(c, M, \delta)$, we have $t(m) = t(n)$. Suppose to the contrary that $t(m) > t(n)$. The set $W_{m,t(m)}$ contains a critical point by definition. If $W_{n,k}$ for $k \leq t(n)$

contains a critical point then also $W_{m,k}$ contains it by the *telescope property*. Since $n \notin G(c, M-1, \delta 2^{-M})$ there are M of such k's. So $M+1$ number of $W_{m,k}$'s, for $k < m$ capture critical points, i.e. $m \notin G(c, M, \delta)$, a contradiction.

We obtain $\#\{t(n) : n \in G^j\}/n_j \le 1/s_j \to 0$ as $j \to \infty$, since for each $t(m)$, the closest $n > t(m), n \in G^j$, satisfies $n \ge t(n) + s_j = t(m) + s_j$. Hence there exists a pair $n < n' \in G^j$ such that there is no integer of the form $t(m)$ between $t(n)$ and $t(n')$ and

$$\#(G^j \cap [t(n)+1, t(n')])/s_j \ge \varepsilon_0.$$

(We include to the set of $t(n')$'s the last integer n_j in G^j.)

Note that when $m \in G^j \cap [t(n)+1, t(n')]$ and $k \in [t(n)+1, m-1]$ the set $W_{m,k}$ cannot contain a critical point. Recall finally that a critical point $c' \in W_{m,k} = \operatorname{Comp} f^{-(m-k)}(B(f^m(c), \delta_m))$ belongs in fact to $\operatorname{Comp} f^{-(m-k)}(B(f^m(c), \delta_m/2))$ by the definition of δ_m. Hence $m \in G(f(c'), 0, \delta_m/2) \subset G(f(c'), 0, \delta 2^{-(M+1)})$.

Thus, by $s_j \to \infty$, choosing c' appearing infinitely many times for j's, we get

$$\overline{d}(G(f(c'), 0, \delta 2^{-(M+1)})) \ge \varepsilon_0 > 0.$$

By the construction, $c' \in B(\bigcup_{n\ge 0} f^n(c), \delta)$, so $c' \in \operatorname{cl} \bigcup_{n\ge 0} f^n(c)$ if we assumed that $\delta \le \operatorname{dist}(\bigcup_{n\ge 0} f^n(c), \operatorname{Crit} \setminus \operatorname{cl} \bigcup_{n\ge 0} f^n(c))$. □

Appendix C. Neither CE nor CE2 are preserved by quasiconformal conjugacy

It is proved in [P3] that, in the presence of only one critical point in the Julia set, CE is equivalent to TCE and therefore CE is invariant by topological conjugacy, see also [PR2].

In this section we prove the existence of two quasiconformally conjugated real polynomials of degree 4, so that one of them satisfies conditions CE and CE2 and the other one does not satisfies neither one of these two properties. So it follows that neither CE nor CE2 are invariant by quasiconformal conjugacy.

1.– Consider the family of polynomials

$$f_{\lambda,c_1}(z) = 2 + \lambda(z^2 - 4)\left(1 - \frac{z}{c_1}\right)^2, \text{ for } c_1 \neq 0.$$

For every such f_{λ,c_1}, 2 is a fixed point, c_1 is a critical point mapped to 2 and $f_{\lambda,c_1}(-2) = 2$. We are mainly interested in $c_1 \in [2, +\infty)$ big. If c_1 is big there is a unique critical point $c_2 = c_2(c_1)$ of f_{λ,c_1} close to 0. In fact such c_2 satisfies $c_2^2 = \frac{c_1}{2}c_2 + 2$ so $c_2 = -4c_1^{-1} + \mathcal{O}(c_1^{-2})$. Then we will fix $\lambda = \lambda(c_1)$ close to one so that $f_{\lambda,c_1}(c_2) = -2$ and we denote f_{λ,c_1}

just by f_{c_1}. Note that $\lambda(c_1) = 1 + \mathcal{O}(c_1^{-2})$. Note also that f_{c_1} converges uniformly in compact sets to $z^2 - 2$ as $c_1 \to \infty$ and moreover the restriction of f_{c_1} to some definite neighborhood W of $[-2, 2]$ is polynomial like with c_2 as unique critical point and it is quasiconformally conjugated to $z^2 - 2$. Then we consider the fixed point $\alpha(c_1)$ of f_{c_1} in W, different from 2, so that $\alpha(c_1) = -1 - 2c_1^{-1} + \mathcal{O}(c_1^{-2})$. Moreover let $\lambda_1(c_1)$ and $\lambda_2(c_1)$ be the eigenvalues of 2 and $\alpha(c_1)$ respectively so that,

$$\lambda_1(c_1) = f_{c_1}'(2) = 4\big(1 - 4c_1^{-1}\big) + \mathcal{O}\big(c_1^{-2}\big) \text{ and}$$
$$\lambda_2(c_1) = f_{c_1}'(\alpha(c_1)) = -2\big(1 + c_1^{-1}\big) + \mathcal{O}\big(c_1^{-2}\big).$$

Note that the function

$$s(c_1) := \frac{\ln|\lambda_1(c_2)|}{\ln|\lambda_2(c_1)|} = 2\left(1 - \frac{3}{\ln 2}c_1^{-1}\right) + \mathcal{O}\big(c_1^{-2}\big),$$

is not constant. Moreover, for different values of c_1 the f_{c_1} are quasiconformally conjugated. We will find small perturbations $f_{c_1,\sigma(c_1)}$ of this family so that all elements are quasiconformally conjugated and so that for some values of c_1, $f_{c_1,\sigma(c_1)}$ is CE and CE2 but for other values of c_1 the polynomial $f_{c_1,\sigma(c_1)}$ does not satisfy neither CE nor CE2.

2.– Given η close to one we can choose $x_{c_1} \in [-2, 2]$ with lower Lyapunov exponent $\ln\eta$ as x in Sect. 6 so that the orbit $x_n = f^n(x_{c_1})$ for $n \geq 0$ of x_{c_1}, is in W and accumulates c_2. Let $n_1 < n_2 < \ldots$ be the for which x_{n_i} is closer to c_2 than the previous times. Then we may choose x_{c_1} so that these times are also the only times that the x_n are close to c_2 and moreover $|(f_{c_1}^{n_i+1})'(x_0)| \sim \eta^{n_i}$.

So if m_i is such that $|x_{m_i} - 2| \sim 1$, then there is m_i' such that $|m_i' - m_i| <$ Const so $|x_{m_i'} - \alpha(c_1)| \sim |\lambda_2(c_1)|^{-2^{m_i}}$. Therefore, for some definite $k \geq 0$, we have that $|x_{m_i'+2^{m_i}+k} - \alpha(c_1)| \sim 1$. We further assume that n_{i+1} is such that $|n_{i+1} - (m_i' + 2^{m_i})| <$ Const. Note that $m_i - n_i \sim m_i' - n_i \sim \ln(n_i) \ll 2^{m_i}$,

$$\left|\left(f_{c_1}^{m_i'-n_i-1}\right)'(x_{n_i+1})\right| \sim |\lambda_1(c_1)|^{m_i'-n_i} \text{ and } \left|\left(f_{c_1}^{2^{m_i}}\right)'(x_{m_i'})\right| \sim |\lambda_2(c_1)|^{2^{m_i}}, \text{ so,}$$
$$|f_{c_1}'(x_{n_{i+1}})|^{-1} \sim \eta^{-(n_{i+1}-n_i)}|\lambda_1(c_1)|^{m_i'-n_i}|\lambda_2(c_1)|^{2^{m_i}}.$$

3.– Consider the family,

$$f_{c_1,\sigma}(z) = f_{c_1}(z) - \sigma(z^2 - 4)(z - c_2(c_1))^2 \text{ for } \sigma \in \mathbb{C},$$

note that $f_{c_1,\sigma}$ is real when c_1 and σ are real. Moreover 2 is a fixed point for $f_{c_1,\sigma}$, $c_2 = c_2(c_1)$ is a critical point mapped to 2 and $f_{c_1,\sigma}(-2) = 2$. Fix c_1 so there is a continuation $\tilde{c}_1 = \tilde{c}_1(\hat{c}_1, \hat{\sigma})$ of the critical point c_1 for $(\hat{c}_1, \hat{\sigma})$ close to $(c_1, 0)$ and note that $\tilde{c}_1$ is real when $\hat{c}_1$ and $\hat{\sigma}$ are real and $\tilde{c}_1(\hat{c}_1, 0) \equiv \hat{c}_1$.

Furthermore there is canonic holomorphic motion i of $[-2, 2]$ defined in a neighborhood of $(c_1, 0)$ which is compatible with dynamics and so that

$i_{c_1,0}$ is the inclusion. Note that $i_{\hat{c}_1,\hat{\sigma}}(\alpha(c_1))$ is a fixed point of $f_{\hat{c}_1,\hat{\sigma}}$. Denote by $\lambda_i(\hat{c}_1,\hat{\sigma})$, $i = 1,2$ the eigenvalues of 2 and $i_{\hat{c}_1,\hat{\sigma}}(\alpha(c_1))$ respectively. Thus,

$$\left|\left(f^{2^{m_i}}_{\hat{c}_1,\hat{\sigma}}\right)'(i_{\hat{c}_1,\hat{\sigma}}(x_{m'_i}))\right| \sim |\lambda_2(\hat{c}_1,\hat{\sigma})|^{2^{m_i}},$$

and as in Sect. 6,

$$|(f_{\hat{c}_1,\hat{\sigma}})'(i_{\hat{c},\hat{\sigma}}(x_{n_i+1}))| \sim |f'_{c_1}(x_{n_i})|^{\frac{\ln|\lambda_1(\hat{c}_1,\hat{\sigma})|}{\ln|\lambda_1(c_1)|}}.$$

Considering that $m'_i - n_i$, $n_{i+1} - n_i - 2^{m_i} \ll 2^{m_i}$ it follows that the lower Lyapunov exponent of $f_{\hat{c}_1,\hat{\sigma}_n}$ at $i_{\hat{c}_1,\hat{\sigma}}(x_{c_1})$ is equal to

$$\ln|\lambda_2(\hat{c}_1,\hat{\sigma})| + \frac{\ln|\lambda_1(\hat{c}_1,\hat{\sigma})|}{\ln|\lambda_1(c_1)|}(\ln\eta - \ln|\lambda_2(c_1)|)$$

$$= \left(\frac{\ln|\lambda_2(\hat{c}_1,\hat{\sigma})|}{\ln|\lambda_1(\hat{c}_1,\hat{\sigma})|} - \frac{\ln|\lambda_2(c_1)|}{\ln|\lambda_1(c_1)|}\right)\cdot\ln|\lambda_1(\hat{c}_1,\hat{\sigma})| + \frac{\ln|\lambda_1(\hat{c}_1,\hat{\sigma})|}{\ln|\lambda_1(c_1)|}\ln\eta.$$

4.– For $\sigma > 0$ and c_1 real, $f_{c_1,\sigma}(\tilde{c}_1) < 2$ so we may choose a sequence $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \to 0$ and such that $\tilde{c}_1$ is eventually mapped to $i_{c_1,\sigma_n}(x_{c_1})$ by f_{c_1,σ_n}. Hence, for c'_1 in some definite neighborhood of c_1 there is $\sigma_n(c'_1)$ so that

$$f_{c'_1,\sigma_n(c'_1)}(\tilde{c}_1) = i_{c'_1,\sigma_n(c'_1)} \circ i^{-1}_{c_1,\sigma_n}(f_{c_1,\sigma_n}(\tilde{c}_1)).$$

Therefore $f_{c'_1,\sigma_n(c'_1)}$ is quasiconformally conjugated to f_{c_1,σ_n} and moreover the lower Lyapunov exponent of $f_{c'_1,\sigma_n(c'_1)}$ at $f_{c'_1,\sigma_n(c'_1)}(\tilde{c_1})$ is equal to the lower Lyapunov exponent of $i_{c'_1,\sigma_n(c'_1)}(x_{c_1})$. By 3 these lower Lyapunov exponent converges to,

$$(s(c'_1) - s(c_1))\cdot\ln|\lambda_1(c'_1)| + \frac{\ln|\lambda_1(c'_1)|}{\ln|\lambda_1(c_1)|}\ln\eta$$

as $n \to \infty$, where the function s is as in 1. By 1 the function s is not constant and since it is analytic one may choose c'_1 arbitrarily close to c_1 so that $s(c'_1) \neq s(c_1)$. Once fixed such c'_1 one may choose η so that,

$$\ln\eta \quad\text{and}\quad (s(c'_1) - s(c_1))\cdot\ln|\lambda_1(c'_1)| + \frac{\ln|\lambda_1(c'_1)|}{\ln|\lambda_1(c_1)|}\ln\eta,$$

have different signs. Hence, for n big one of the f_{c_1,σ_n} or $f_{c'_1,\sigma_n(c'_1)}$ satisfies CE and the other one does not. Since the polynomials considered have all critical points with the same multiplicity, CEis equivalent to CE2. Hence, neither CE nor CE2 are preserved by quasiconformal conjugacy.

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