NOTES ON RUELLE'S THEOREM

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ABSTRACT. This is an excerpt from the minicourse given in Stockholm in 1998. Following Ruelle's ideas, we give a self-contained proof of his theorem. By no means results or proofs are new, though some places are streamlined and some errors are corrected.

Let F be a rational map on the Riemann sphere, of degree $d \geq 2$. We write F^n for the n-th iterate of F. Distances and derivatives are measured in the spherical metric. The *Julia set* of F is denoted by J_F. We assume that F is hyperbolic. We refer to [6, 29] for definitions and basic facts of complex dynamics.

Theorem A. *For a hyperbolic rational map F, the Hausdorff dimension of the Julia set is real analytic in F.*

For a hyperbolic rational map F, one can introduce the pressure function $P_F(t)$ which turns out to be real analytic in $t \in \mathbb{R}$ and $F \in Hyp$. It has only one root, which is the Hausdorff dimension $\mathrm{HDim}\,(J_F)$. Thus application of the implicit function theorem reduces Theorem A to the following

Theorem B. *For a hyperbolic rational map F, the pressure function $P_F(t)$ is real analytic in $t \in \mathbb{R}$ and $F \in Hyp$. It is concave, strictly decreasing, and its only root is* $\mathrm{HDim}\,(J_F)$ *(Bowen's formula).*

We will derive the real analyticity of pressure by showing that it is equal to the logarithm of the spectral radius of an appropriate Ruelle transfer operator. In "good" functional spaces this spectral radius turns out to be a simple isolated eigenvalue; then one checks that Ruelle operator depends real-analytically on t and F and by perturbation theory derives the real-analyticity of pressure.

1. Ruelle transfer operator

In this Section, we consider Ruelle(-Perron-Frobenius) transfer operator with general smooth weight.

Let $L = L_g$ denote the *Ruelle transfer operator* with weight g which acts in appropriate function spaces according to the formula

$$L_g f(z) \ = \sum_{y \in F^{-1}(z)} g(y)\ f(y)\ . \tag{1}$$

The formal adjoint operator $L^* = L_g^*$ acts in the space of measures by

$$d\left(L_g^* \mu(z)\right) \ = \ g(z) d\mu(F(z))\ . \tag{2}$$

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It is easy to see that powers of these operators satisfy

$$\begin{aligned} L_g^n f(z) &= \sum_{y \in F^{-n}(z)} g_n(y)\ f(y)\ , \\ d\left(\left(L_g^*\right)^n \mu(z)\right) &= g_n(z) d\mu(F^n(z))\ , \end{aligned}$$

where $g_n(z) := \prod_{j=0}^{n-1} g(F^j z)$.

The hyperbolic dynamic is expanding, and looking at the definition one can see, that Ruelle operator makes "smooth" functions even "smoother." This makes the Ruelle operator "quasicompact" in various "smooth" spaces, which can be made rigorous in several ways:

- In very "smooth" spaces the operator is compact; e.g. this is easy to show when the weight is holomorphic: then transfer operator is a sum of compositions of multiplication operator (which is bounded), and "stretching" (which is compact by Cauchy formula). In our setting there seems to be no Banach space where the transfer operator would be compact; nevertheless similar to above arguments show that it is nuclear in the nuclear space C^∞. This is the original approach of Ruelle introduced in [35]. Unfortunately, though short and elegant, it requires invoking not so well-known and highly non-trivial (but beautiful) papers [13, 14] of Grothendieck on nuclear operators and their Fredholm determinants.
- We will follow another standard approach. In Hölder space the operator is quasicompact, i.e. it behaves like compact outside the essential spectral radius. The standard method of proof given by two-norm inequality of Ionescu-Tulcea and Marinescu from [19]. It will work in any space of "smooth" functions, the "smoother" is the space, the smaller will be essential radius. Among "smooth" spaces tried by different authors in various situations are Sobolev, BV, Zygmund, ...

  Space BV is especially good for one dimensional real dynamics, since composing with an isomorphism does not change the variation of a function, calculations thus are much simplified and can be done for very general dynamics. Unfortunately, there is only "partial" analog BV_2 of this space for complex situation, see [39].

  If dynamics lacks expansion, one can still prove quasicompactness, provided additional conditions like "pressure is strictly bigger than topological pressure" are satisfied. Such conditions ensure that to "obtain" spectral radius of the transfer operator at least exponentially many inverse branches are needed, then one can use principle that out of exponentially many inverse branches most will be expanding.
- Another approach which works in the same situations, uses Banach space cones, and was developed by Liverani in [20].

We will consider transfer operator on the spaces $C(J)$ and $\mathcal{H}_\alpha(J)$, the latter with the norm

$$\|f\|_\alpha := \sup \frac{|f(x) - f(y)|}{|x-y|^\alpha} + \|f\|_\infty\ .$$

Proposition 1. *Assume that g is positive and Hölder-continuous with exponent α. Then operator L_g is bounded on the spaces $\mathcal{H}_\alpha$ and C, and its spectral radii satisfy*

$$r_{\mathrm{ess}}(L_g, \mathcal{H}_\alpha) < r(L_g, \mathcal{H}_\alpha) = r(L_g, C) =: \lambda_g \equiv \lambda\ .$$

Moreover, λ is a simple eigenvalue of L_g on $\mathcal{H}_\alpha$ and the only eigenvalue with modulus λ.

1.1. **Two-norm inequality.**

Lemma 1.1. *Operator L_g is bounded in the spaces C and $\mathcal{H}_\alpha$, moreover, the following* two-norm inequality *holds:*

$$\|L^n f\|_\alpha \le Cq^n \,\|L^n\|_\infty \cdot \|f\|_\alpha + M_n\|f\|_\infty \ . \tag{3}$$

Proof: It is trivial that L_g is bounded on the space C. To establish boundness and quasicompactness of L_g on the space $\mathcal{H}_\alpha$ we write

$$\begin{aligned}
&\frac{1}{|z-z'|^\alpha}\left|\left(L_g^n f\right)(z) - \left(L_g^n f\right) z')\right| \\
&\quad\le \frac{1}{|z-z'|^\alpha}\sum_{y\in F^{-n}z}\left|g_n(y)f(y)-g_n(y')f(y')\right| \\
&\quad\le \sup_{y\in F^{-n}z}\left|\frac{y-y'}{z-z'}\right|^\alpha \left(\sum_{y\in F^{-n}z}\frac{|g_n(y)-g_n(y')|}{|y-y'|^\alpha}|f(y')| \ + \sum_{y\in F^{-n}z} g_n(y)\frac{|f(y)-f(y')|}{|y-y'|^\alpha}\right) \\
&\quad\le Cq^n \left((\deg F)^n\,\|g_n\|_\alpha\,\|f\|_\infty + \sum_{y\in F^{-n}z} g_n(y)\,\|f\|_\alpha\right) \\
&\quad= \ C\ q^n\,\|f\|_\alpha\,\|L^n\|_\infty \ + \ M_n'\,\|f\|_\infty \ ,
\end{aligned}$$

with some positive constants C, M_n', and $q<1$. Above we used that for a hyperbolic Julia set preimages of two points z, z' can be assigned in pairs y, y' so that $|y-y'|<C_1q_1^n|z-z'|$ with some positive absolute (depending on J only) constants C_1 and $q_1<1$. This follows easily from the fact that preimages under F^{-n} of small balls centered on the Julia set are univalent and shrink exponentially as $n\to\infty$.

Thus the two-borm inequlaity (3) holds and it immedeately follows that L_g is bounded in $\mathcal{H}_\alpha$.

Since $\lambda := r(L,C) = \lim_{n\to\infty}\|L^n\|_\infty^{1/n}$, by the two-norm inequality (3), we can find an integer m such that

$$\|L^m f\|_\alpha \ \le \ \frac{1}{2}\,\lambda^m\|f\|_\alpha + M_m\|f\|_\infty \ . \tag{4}$$

Denote $C_n := \sup_{i+j<n}\{\lambda^i\left\|L^j\right\|_\infty\}$, clearly $\lim_{n\to\infty}(C_n)^{1/n}=\lambda$. By induction we have

$$\|L^{km} f\|_\alpha \ \le \ \frac{1}{2}\lambda^{km}\|f\|_\alpha + 2M_mC_{km}\|f\|_\infty, \quad (k=1,2,\dots)\ ,$$

therefore

$$r(L,\mathcal{H}_\alpha) = \lim_{n\to\infty}\|L^n\|_\alpha^{1/n} \le \lim_{n\to\infty}(\lambda^n + C_n)^{1/n} = \lambda = r(L,C)\ .$$

Inverse inequality follows from easy

$$\|L^n\|_\infty = \|L^n1\|_\infty \le \|L^n1\|_\alpha \le \|L^n\|_\alpha \ ,$$

and we deduce that $r(L,\mathcal{H}_\alpha) = r(L,C)$. □

1.2. **Finite rank approximation.**

Lemma 1.2. *There is a constant* C *such that for any* $\varepsilon > 0$ *there exists a finite rank operator* K *in* $\mathcal{H}_\alpha$ *with*

$$\begin{aligned} \|K\|_\alpha &\leq C, \\ \|f - Kf\|_\infty &\leq \varepsilon \, \|f\|_{1,p}. \end{aligned} \tag{5}$$

Proof: By the Whitney extension theorem there is a bounded in $\mathcal{H}_\alpha$ by some constant C linear operator, extending f to the whole plane. Consider a grid of equilateral triangles Δ of size $< \varepsilon/(2C)$. Define Kf to be a continuous function coinciding with f in the vertices and linear on each triangle. Clearly, $\|Kf\|_{\alpha,J} \leq \|f\|_{\alpha,\mathbb{C}} \leq C\|f\|_{\alpha,J}$. Moreover, on any triangle Δ one has

$$\|f - Kf\|_{L^\infty(\Delta)} \leq \frac{\varepsilon}{2} \cdot \|f - Kf\|_{\mathcal{H}_\alpha(\Delta)} \leq \varepsilon \|f\|_\alpha \, .$$

□

1.3. **Quasicompactness.** This method of establishing quasicompactness using two-norm inequality and finite rank approximation is due to Ionescu-Tulcea and Marinescu, [19] (compare also [30]).

Lemma 1.3. *Transfer operator* L_g *is quasicompact in the space* $\mathcal{H}_\alpha$*, that is its essential spectral radius is strictly smaller than its spectral radius.*

We remind that spectral radius of L in the space X is

$$r(L, X) := \lim_{n\to\infty} \|L^n\|_X^{\frac{1}{n}} \, ,$$

and the *essential spectral radius* is

$$r_{\text{ess}}(L, X) := \inf \, \{r(L - K, X) : \ K \text{ compact operator in } X\}.$$

The latter is the spectral radius of L in the *Calkin algebra* (= bounded operators modulo compact operators). The spectrum of L in X lying outside of the disk $\{|z| \leq r_{\text{ess}}\}$ consists of a finite number of eigenvalues that all have finite geometric multiplicity.

Proof: Chose n so large that in the two-norm inequality

$$Cq^n \, \|L^n\|_\infty < \frac{1}{4} r(L, C)^n \, ,$$

with n fixed take operator K from the Lemma 1.2 with such ϵ that $M_n \epsilon < \frac{1}{4} r(L, C)^n$. Then we have

$$\begin{aligned} \|L^n(f - Kf)\|_\alpha &\leq Cq^n \, \|L^n\|_\infty \cdot \|f - Kf\|_\alpha + M_n \|f - Kf\|_\infty \\ &\leq C' q^n \, \|L^n\|_\infty \cdot \|f\|_\alpha + M_n \epsilon \|f\|_\alpha \\ &< \frac{1}{2} r(L, C)^n \|f\|_\alpha \, . \end{aligned}$$

Since $L^n K$ is a finite rank operator, we deduce that

$$r_{\text{ess}}(L, \mathcal{H}_\alpha) \leq \|L^n (I - K)\|_\alpha^{1/n} < r(L, C) \, .$$

□

1.4. **Existence of eigenmeasures.**

Lemma 1.4. *There exists a probability measure ν_g supported on the whole Julia set such that*

$$L_g^* \nu_g \; = \; \lambda_g \, \nu_g.$$

Proof: Clearly, operator L^* is bounded on $M(J)$, and maps positive measures to positive measures. Thus we can consider the following operator on the (convex and weak-$*$ compact) space of probability measures $\mathrm{Prob}(J)$:

$$P: \; \mu \; \mapsto \; \frac{L^*\mu}{\mathrm{Var} L^*\mu} \; .$$

By the fixed point theorem, P fixes some measure ν, which is the eigenmeasure of L^*: $L^*\nu = \lambda'\nu$. To show that the eigenvalue λ' is equal to λ, we write

$$\lambda^n \asymp \|L^n\|_\infty \asymp (L^n 1, \nu) = (1, (L^*)^n \nu) = (\lambda')^n \; ,$$

where $\asymp$ means up to a multiplicative $e^{o(n)}$.

It is clear from (2) that $F^{-1}\mathrm{spt}\,\nu \subset \mathrm{spt}\,\nu$, and since preimages of any point are dense in J_F we deduce that $\mathrm{spt}\,\nu = J_F$. □

1.5. **Multiplicity of λ.**

Lemma 1.5. *$\lambda \equiv \lambda_g$ is a simple eigenvalue of the operator L_t in $\mathcal{H}_\alpha$, and the only eigenvalue with modulus λ:*

$$\dim \; \ker \, (L_g - \lambda_g)^2 \; = \; 1 \; .$$

Proof: Consider some eigenvalue λ', $|\lambda'| = \lambda$ with a non-zero eigenfunction p, such should exist since $r_{\mathrm{ess}}(L, \mathcal{H}_\alpha) < r(L, \mathcal{H}_\alpha) = \lambda$. Then we can write

$$|\lambda'|(|p|, \nu) \; = \; (|Lp|, \nu) \; \le \; (L|p|, \nu) \; = \; (|p|, L^*\nu) \; = \; \lambda(|p|, \nu) \; .$$

The integral $(|p|, \nu)$ is positive, the first and the last terms above are equal, and using that $\mathrm{spt}\,\nu = J$ we can write $|Lp| \equiv L|p| \equiv \lambda|p|$.

By the definition of L the latter implies that for any z values of p at all preimages of z have the same argument (i.e. there is no cancelation, which would lead to $|Lp| < L|p|$). Since preimages are dense in the Julia set, we conclude, that p has constant argument and without loss of generality is positive.

It immediately follows that $\lambda' = \lambda$ and that $\dim \; \ker \, (L_g - \lambda_g) \; = \; 1$ (otherwise a difference of two distinct eigenfunctions would have non-constant argument)

Suppose now that

$$(L_g - \lambda)^2 h = 0$$

for some $h \in \mathcal{H}_\alpha$. We need to show that $p := (L_g - \lambda)h$ is trivial. Since there is a positive eigenfunction and $\dim \; \ker \, (L_t - \lambda) = 1$, non-trivial p can be assumed to be strictly positive. Then we have

$$0 \; < \; (p, \nu) \; = \; (L_g h, \nu) \; - \; (\lambda h, \nu) \; = \; (h, L_g^* \nu) \; - \; \lambda \, (h, \nu) \; = \; 0 \; ,$$

leading to contradiction, which proves the Lemma. □

Remark 1. *One really needs to check that geometric multiplicity of λ is one, and not just that eigenfunction is unique, as is shown by the following example of a Jordan cell. Operator*

$$Q_t \ := \ \begin{pmatrix} o & 1 \\ x^2 & 0 \end{pmatrix} \ ,$$

acting on $\mathbb{R}^2$ depends real analytically on t. But $\sigma(Q_t) = \{\pm t\}$ and therefore $r(Q_t) = |t|$ is not a real analytic function.

1.6. **Alternative approaches.** There are alternative ways to do lemmas 1.4 and 1.5 which work in more general situation than methods discussed above, and we outline them.
Alternative construction of eigenfunctions: The following argument, showing that there is a positive eigenfunction with eigenvalue λ (before existence of an eigenmeasure is known) is taken from [39].

Since $r_{\mathrm{ess}}(L_g, \mathcal{H}_\alpha) < r(L_g, \mathcal{H}_\alpha)\lambda$, there should be eigenvalues λ_j satisfying $|\lambda_j| = \lambda$, but there are only finitely many of them and the corresponding spectral projections have finite ranks. Denote

$$g_j := P_j\ 1; \quad g_0 := 1 - \sum g_j.$$

Applying L_g^n, we have

$$L_g^n\ g_0\ +\ \sum L_g^n\ g_j\ =\ L_g^n\ 1,$$

and since

$$\|L_g^n\ g_0\|_\infty\ \lesssim\ \|L_g^n\ g_0\|_\alpha\ =\ o(\|L_g^n\ 1\|_\infty) \quad \text{as} \quad n \to \infty,$$

at least one of g_j's is not zero.

We also have

$$\|L_g^n\ g_j\|_\alpha\ \asymp\ n^{k_j}\ \lambda^n \quad \text{as} \quad n \to \infty,$$

where $k_j \geq 0$ is the maximal integer number such that

$$\varphi_j := (L - \lambda_j)^{k_j}\ g_j \neq 0,$$

(i.e. k_j is the size of the corresponding Jordan cell). Let $k := \max\{k_j\}$. Then

$$p_n\ :=\ n^{-k}\ (L_g^n\ 1)\ =\ \sum_{j:\ k_j = k} \lambda_j^n\ \varphi_j + o(\lambda^n) \tag{6}$$

in $\mathcal{H}_\alpha$ and also in $C(\bar{\Omega})$. Since the functions φ_j are linearly independent, we have

$$\|p_n\|_\infty\ \asymp\ \|p_n\|_\alpha\ \asymp\ \lambda^n,$$

and we also have $p_n(z_0) \asymp \lambda^n$ for some fixed $z_0 \in \partial\Omega$. Since $p_n \geq 0$, it follows that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} \frac{p_n}{\lambda^n} \right\|_\infty\ \gtrsim\ \frac{1}{N} \sum_{n=1}^{N} \frac{p_n(z_0)}{\lambda^n}\ \asymp\ 1\ .$$

By (6), this is possible only if one of the eigenvalues λ_j is positive. □
Alternative construction of eigenmeasures: What follows below is a version of construction of conformal measures due to Patterson [32] and Sullivan [42].

Fix a point z and consider the sequence of positive measures

$$\mu_n\ :=\ \lambda^{-n}\, (L_g^*)^n\, \delta_z\ =\ \lambda^{-n} \sum_{y \in F^{-n}(z)} g_n(y)\, \delta_y\ .$$

Clearly, $L_g^*\mu_n \,=\, \lambda\, \mu_{n+1}$, and by the proof above we have

$$\|\mu_n\| \;=\; \lambda^{-n}\, L_g^n 1(z) \;\asymp\; n^k$$

for some integer $k \geq 0$. Next we define

$$\nu_n \;:=\; \sum_{j=0}^{n} \mu_n \;,$$

and take some (weak-$*$) limit point ν of the sequence $\nu_n \,/\, \|\nu_n\|$. Then ν is a probability measure supported on J_F, and since

$$\frac{\|L_g^*\nu_n \,-\, \lambda\nu_n\|}{\|\nu_n\|} \;=\; \frac{\|\lambda(\mu_{n+1} - \mu_0)\|}{\|\nu_n\|} \;\asymp\; \frac{n^k}{n^{k+1}} \;=\; \frac{1}{n} \;\to 0,$$

we have $L_g^*\nu \,=\, \lambda\nu$. □

2. Analyticity of pressure

Now we turn to the proof of Theorem B.

Consider the transfer operator $L_{(t,F)}$ with (positive Hölder) weight $g_{(t,F)} \,=\, |F'|^{-t}$. It has the main eigenvalue $\lambda(t,F)$, and we define *pressure* by $P_F(t) \;:=\; \log\lambda(t,F)$.

For a general weight g the pressure is usually defined by $P(\log g) := \log\lambda_g$, so our particular case $P_F(t)$ corresponds to $P(-t\log|F'|)$ in standard notation.

It follows from the definition of transfer operator and Koebe distortion theorem that for a fixed point z

$$P_F(t) \;=\; \frac{1}{n}\,\|L^n\|_\infty + o\left(\frac{1}{n}\right) \;=\; \frac{1}{n}\log\left(\sum_{y\in F^{-n}z} \left|(F^n)'(y)\right|^{-t}\right) \;+\; o\left(\frac{1}{n}\right) \;, \quad n\to\infty \;.$$

And since $|(F^n)'| > CQ^n$ with $Q > 1$, it easily follows that $P_F(t)$ is strictly decreasing and convex in t, therefore continuous. Exponential decay of correlation, discussed below, implies that the error term $o(1/n)$ actually decays like τ^n with $\tau < 1$.

2.1. **Bowen's formula.** Sketch:

Pressure is strictly decreasing, $P(0) > 0$, $P(2) \leq 0$, thus there is unique δ such that $P(\delta) = 0$ and hence $\lambda_\delta = 1$. Thus the measure ν_δ has Jacobian $|F'|^\delta$ (like δ-dimensional Haudorff measure would), it is called *δ-conformal measure.*

Alternatively one can arrive at it via Patterson-Sullivan construction.

Pulling back a fixed cover by small balls and applying Koebe distortion one shows that for every balll B of radius R, centered on the Julia set, $\nu_g(B) \asymp R^\delta$. Then with Besikovitch covering theorem we deduce that δ-dimensional Hausdorff measure of J is positive and finite, hence Hausdorff dimension of J is δ.

2.2. **Perturbation theory.** Fix hyperbolic rational function $\tilde{F}$ and number $\tilde{t}$. Then for hyperbolic rational function $F \approx \tilde{F}$ by Mañé-Sad-Sullivan [28] there is a quasiconformal conjugation H_F : $J_{\tilde{F}} \to J_F$ between dynamical systems (J_F, F) and $(J_{\tilde{F}}, \tilde{F})$, which depends holomorphically on F.

Consider the transfer operator $\tilde{L} = \tilde{L}_{(t,F)}$ corresponding to the (fixed!) dynamics $\tilde{F}$ and weight $|F'(H_F)|^{-t}$.

First note that conjugation H_F transforms operator $\tilde{L}_{(t,F)}$ into operator $L_{(t,F)}$ (corresponding to dynamics F and weight $|F'|^{-t}$), preserving the space of continuous functions; therefore they have the same main eigenvalue. Thus $P_F(t)$ is the logarithm of the main eigenvalue $\lambda(t,F)$ of $\tilde{L}_{(t,F)}$.

Take $\alpha > 0$ such that weight $|F'(H_F)|^{-t} \in \mathcal{H}_\alpha$ for all t and F's in some neighborhoods of $\tilde{t}$ and $\tilde{F}$. The dynamics for operator $\tilde{L}_{(t,F)}$ is fixed (i.e. independent of F and t). Direct checking shows that the weight changes with real analytically (as a function in $\mathcal{H}_\alpha$) with t and F. From the definition of the transfer operator, the map $(t,F) \mapsto L_{(t,F)}$ is real analytic as a map to the Banach space of bounded operators on $\mathcal{H}_\alpha$. Then standard perturbation theory implies that $P_F(t)$ is real analytic in F and t.

In fact, chose a single closed curve γ separating $\lambda(\tilde{t},\tilde{F})$ from the rest of the spectrum of $\tilde{L}_{(\tilde{t},\tilde{F})}$. If t, F are sufficiently close to $\tilde{t}$, $\tilde{F}$, the point $\lambda(t,F)$ lies inside γ and the operators $\left(\tilde{L}_{(t,F)} - z\right)$ are invertible for all $z \in \gamma$. Consider the spectral projection

$$\Pi_{(t,F)} \;=\; \frac{1}{2\pi i}\int_\gamma \left(\tilde{L}_{(t,F)} - z\right)^{-1} \partial z.$$

Then $(t,F) \mapsto \Pi_{(t,F)}$ is a real analytic map and hence $\operatorname{rank}\Pi_{(t,F)} \equiv \operatorname{rank}\Pi_{(\tilde{t},\tilde{F})} = 1$. It follows that $\Pi_{(t,F)}$ is a projection onto the eigenspace of $\tilde{L}_{(t,F)}$ corresponding to $\lambda(t,F)$, and

$$\lambda(t,F) \;=\; \frac{\tilde{L}_{(t,F)}\Pi_{(t,F)}f}{\Pi_{(t,F)}f},\ (f \not\equiv 0,\ P_{(\tilde{t},\tilde{F})}f = f),$$

is an analytic function.

3. Bonuses

(i) *Perron-Frobenius Theorem: exponential decay of correlations.*

The probability eigenmeasure $\nu \equiv \nu_g$ in Lemma 1.4 is unique, and if $f_g \in \mathcal{H}_\alpha$ denotes the non-negative eigenfunction of L_g satisfying

$$\nu_g(f_g) = 1,$$

then the rank one operator

$$\mathcal{P} := (\cdot, \nu_g) f_g$$

is the spectral projection of $L_g : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ corresponding to the isolated eigenvalue $\lambda \equiv \lambda_g$. We have shown that λ is the only eigenvalue with modulus λ (or bigger) and it is simple, hence

$$r((I - \mathcal{P})L_g, \mathcal{H}_\alpha) < \lambda, \tag{7}$$

which implies that

$$\lambda^{-n} L_g^n \;\to\; \mathcal{P}\ ,$$

with exponential rate of convergence in the uniform operator topology. The latter can be rewritten as exponential decay of correlations with respect to (invariant) measure $d\mu_g := f_g d\nu_g$:

$$\left| \int \phi d\mu_g \int \psi d\mu_g - \int \phi(F^n)\psi d\mu_g \right| \;\le\; C\tau^n$$

for some $\tau < 1$.

(ii) *Equilibrium states.*

Let μ_g denote the probability measure $f_g\nu_g$. It is immediate that μ_g is an ergodic, F-invariant measure. We claim that μ_g is a unique *equilibrium state*, i.e. it is a unique invariant measure μ, for which the *free energy*

$$h(\mu) - \chi_g(\mu) \ ,$$

attains its maximum, and the latter appears to be equal to the pressure $P(\log g) := \log \lambda_g$. Here we write $h(\mu)$ for the entropy of μ and $\chi_g(\mu) := \int \log g \, d\mu$. Partcularly,

$$P(\log g) = h(\mu_g) - \chi_g(\mu_g) =: h_g - \chi_g \ . \tag{8}$$

The equality (8) follows from the Rokhlin-type formula

$$h_g = \int \log J_g \ d\mu_g, \tag{9}$$

where

$$J_g := \frac{d\mu_g \circ F}{d\mu_g} \equiv \lambda_g \frac{f_g \circ F}{f_g} |F'|^t \ \in L^1(\mu_g)$$

is the *Jacobian* of μ_g. The formula (9) follows from the well-known estimate

$$h_g \geq \int \log J_g \ d\mu_g$$

and from the variational principle.

To prove the uniqueness result, it is sufficient to show that if μ is an equilibrium state, then

$$\mu(\Psi) = \mu_g(\Psi) \qquad \text{for all} \qquad \Psi \in C^\infty.$$

For the latter see [34].

(iii) *Derivatives of the pressure function.*

One can establish formulas for the derivatives of $P(t)$ (see [36, 37, 38]). Denote $\chi(\mu) := \int \log |F'| d\mu$, so that $\chi_{|F'|^{-t}} \equiv -t\chi(\mu_t)$.

Then for the first derivative we have

$$P'(t) = -\chi(\mu_t) \ ,$$

and also

$$P'(+\infty) = - \inf_{\text{invariant } \mu} \chi(\mu) \ = \ \lim_{n\to\infty} \frac{1}{n} \inf_J \log |(F^n)'| \ ,$$

$$P'(-\infty) = - \sup_{\text{invariant } \mu} \chi(\mu) \ = \ \lim_{n\to\infty} \frac{1}{n} \sup_J \log |(F^n)'| \ .$$

These statements follow easily from the variational principle applied to particular case $g = |F'|^{-t}$.

(iii) *Ruelle expansion for the Hausdorff dimension of* $J_{z^2+\epsilon}$.

(iv) *Zdunik's theorem for hyperbolic polynomials.*

(v) *Multifractal formalism for measure of maximal entropy.*

(vi) *Generalizations to non-hyperbolic polynomials.*

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