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Phase transition in subhyperbolic Julia sets

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Abstract. We study the pressure function for critically finite polynomials and analyze the case when this function fails to be real analytic.

1. *Introduction and results*

1.1. In this paper we study subhyperbolic polynomials

$$F(z) = z^d + \cdots \tag{1}$$

with connected Julia sets.

The *Julia set* $J = J_F$ of a polynomial F is defined as the boundary of the domain

$$\Omega = \Omega_F = \{z \in \mathbb{C} : F^n z \to \infty\},$$

the basin of attraction to ∞.

For $c \in \mathbb{C}$ let $k(c)$ denote the degree of F at c. Then

$$\operatorname{Crit} F = \{c : k(c) \geq 2\}$$

is the set of *critical points*. The Julia set is *connected* if and only if all critical points have bounded orbits. In this case the domain Ω is simply connected and we can consider the *conformal map*

$$\begin{aligned} \varphi : \mathbb{D}_- &\equiv \{|z| > 1\} \to \Omega, \\ \varphi(z) &= z + \cdots \quad \text{at } \infty. \end{aligned}$$

(We have $|\varphi'(\infty)| = 1$ because of the normalization (1).) This map conjugates $F : \Omega \circlearrowleft$ with the dynamics $T : z \mapsto z^d$ on $\mathbb{D}_-$:

$$\varphi \circ T = F \circ \varphi.$$

The main object of our study is the *spectrum* $\beta(t)$ which we define in terms of the derivative of the conformal map φ:

$$\beta(t) = \lim_{r \to 1+0} \frac{\log \int_{\partial\mathbb{D}} |\varphi'(r\zeta)|^t \, |d\zeta|}{\log \frac{1}{r-1}}, \quad t \in \mathbb{R}. \tag{2}$$

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It will be shown that this limit exists in the subhyperbolic case. The integral means in (2) have the sense of the partition functions corresponding to the measure of maximal entropy, or harmonic measure on J and therefore $\beta(t)$ can be seen as an analogue of the free-energy function in thermodynamics. Another thermodynamical interpretation of $\beta(t)$ comes from the following.

Definition. Let θ be an integrable function on $\mathbb{T} \equiv \partial\mathbb{D}$. Then the *pressure* $P(\theta)$ of θ with respect to the dynamics $T : z \mapsto z^d$ is defined as the limit

$$P(\theta) = \lim_{n\to\infty} \frac{1}{n} \log_d \sum_{\operatorname{rank} I = n} e^{(S_n\theta)_I}, \tag{3}$$

provided that this limit exists. In (3) the sum is taken over all d-adic intervals of rank n and

$$\theta_I \overset{\text{def}}{=} \frac{1}{|I|} \int_I \theta,$$

$$S_n\theta \overset{\text{def}}{=} \sum_{j=0}^{n-1} \theta \circ T^j.$$

1.2.

PROPOSITION. *Let $F(z) = z^d + \cdots$ be a polynomial with connected Julia set. Define*

$$\theta(\zeta) = \theta_F(\zeta) = \sum_{c \in \operatorname{Crit} F} (k(c) - 1) \log |\varphi(\zeta) - c|, \quad \zeta \in \partial\mathbb{D},$$

where $\varphi(\zeta)$ denotes the radial limit of φ at ζ. (It is well-known that $\varphi(\zeta)$ exists almost everywhere, and $\theta \in L^1(\partial\mathbb{D})$.) Then

$$\beta(t) = P(-t\theta) - 1, \quad t \in \mathbb{R},$$

in the sense that the existence of any of the limits (2) and (3) implies the existence of the other.

Proof. Differentiating the identity

$$\varphi(T^n z) = F^n(\varphi(z)), \quad z \in \mathbb{D}_-,$$

we obtain

$$\varphi'(T^n z) \prod_{j=0}^{n-1} T'(T^j z) = \prod_{j=0}^{n-1} F'(\varphi(T^j z))\varphi'(z).$$

Since $\varphi'(T^n z) \to 1$ for all $z \in \mathbb{D}_-$, we have

$$\varphi'(z) = \prod_{j=0}^{\infty} \left(\frac{T'}{F'\circ\varphi}\right)(T^j z), \quad z \in \mathbb{D}_-,$$
$$b(z) \equiv \log|\varphi'(z)| = -\sum_{j=0}^{\infty} \theta(T^j z),$$

where

$$\theta(z) = \log\left|\frac{F'(\varphi(z))}{T'(z)}\right|,$$

and $\theta(\zeta)$ is the radial limit function of $\theta(z)$.

The function $u(z) = \log \varphi'(z)$ has the Bloch norm

$$\|u\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right) |u(z)|$$

at most six (see [**Pom**, §4.2]). Therefore the function $b = \operatorname{Re} u$ has the following property (see [**Ma**, §I.1]): if $z \in \mathbb{D}_-$, $|z| < 2$, and $I(z)$ is the arc on $\mathbb{T}$ of length $|z| - 1$ centered at $|z|^{-1} z$, then

$$\left|b(z) - b_{I(z)}\right| \le \text{const}, \tag{4}$$

where the mean value b_I is defined as the limit

$$b_I = \lim_{r \to 1+0} (b(rz))_I \,.$$

For any d-adic interval I of rank n, we have

$$b_I = -\,(S_n \theta)_I \,.$$

By the distortion theorem,

$$\int_{r=1+d^{-n}} \left|\varphi'(r\zeta)\right|^t |d\zeta| \asymp d^{-n} \sum_{\operatorname{rank} I = n} e^{tb(z_I)},$$

(we write $z = z_I$ for $I = I(z)$), which is

$$\asymp\ d^{-n} \sum_{\operatorname{rank} I = n} e^{tb_I}$$

$$\asymp\ d^{-n} \sum_{\operatorname{rank} I = n} e^{-t(S_n\theta)_I}.$$

□

1.3. A metric $\sigma(z)\,|dz|$ defined in a neighbourhood of J is called *admissible* if σ is continuous as a map to $(0, +\infty]$, and satisfies the inequality

$$\sigma(z) \le \text{const} \sum_j \left|z - a_j\right|^{-\eta}$$

for some $\eta \in (0, 1)$ and some *finite* set $\{a_j\}$.

A polynomial F is called *subhyperbolic* if the map $F : J \circlearrowleft$ is *expanding* with respect to some admissible metric, i.e.

$$\sigma(F^n z)\left|(F^n)'(z)\right| \ge \text{const}\, q^n \sigma(z)$$

for some $q > 1$. It is known (see, e.g., [**M**, Theorem 14.4]) that F is subhyperbolic if and only if

(1) every critical point on J has a finite orbit, and
(2) every critical point outside of J has an orbit converging to some attracting periodic cycle.

A special case is when there is no critical point on J. In this case the dynamics is expanding for any non-singular metric (in particular, for $\sigma \equiv 1$) and F is called *hyperbolic*.

For hyperbolic polynomials F the function θ_F is continuous and therefore $P(\theta)$ is equal to the usual 'pressure' (see, e.g., [**B**], [**R1**]). Moreover, θ is Hölder continuous (this follows, for instance, from the fact that Ω_F is a John domain, see [**CJ**]) and by Ruelle's result [**R2**] we have the following:

PROPOSITION. *In the hyperbolic case, the spectrum $\beta(t)$ is a real analytic function on $\mathbb{R}$.*

For general subhyperbolic polynomials the situation is different. It may happen that the spectrum $\beta(t)$ has a point t_c at which the first derivative is discontinuous. In this case we will say that the polynomial has a *phase transition* and call t_c a *phase-transition point*. We will see that there is at most one phase-transition point and t_c is always negative. The simplest example is provided by the following.

1.4. *Chebyshev's polynomials.* The polynomials P_d are defined by the functional equation

$$P_d\left(z+z^{-1}\right)=z^d+z^{-d},$$

e.g., $P_2(z)=z^2-2$, $P_3(z)=z^3-3z$, etc. The critical points of P_d are the points

$$c_j=2\cos\frac{\pi j}{d},\qquad 1\le j\le d-1,$$

and their orbits are the following:

$$\begin{array}{ccc} c_1,c_3,\dots,c_{d-1} & \overset{F}{\twoheadrightarrow} & -2 \\ & & \downarrow \\ c_2,c_4,\dots,c_{d-2} & \twoheadrightarrow & 2\ \circlearrowleft \end{array}\qquad \text{if } d \text{ is even,}$$

and

$$\begin{array}{ccc} c_1,c_3,\dots,c_{d-2} & \overset{F}{\twoheadrightarrow} & -2\ \circlearrowleft \\ c_2,c_4,\dots,c_{d-1} & \twoheadrightarrow & 2\ \circlearrowleft \end{array}\qquad \text{if } d \text{ is odd.}$$

In the latter case the polynomials $-P_d$ are *not* conjugate to P_d:

$$\begin{array}{ccc} c_1,c_3,\dots,c_{d-2} & \overset{-F}{\twoheadrightarrow} & -2 \\ & & \updownarrow \\ c_2,c_4,\dots,c_{d-1} & \twoheadrightarrow & 2, \end{array}$$

(e.g., $-P_3\sim z^3+3z$). The Julia set of $\pm P_d$ is the segment $[-2,2]$, and the spectrum

$$\beta(t)=\max\{-t-1,0\}$$

has a phase-transition point $t_c=-1$ (see Figure 1).

The polynomials $\pm P_d$ and z^d are known to be the only polynomials with a smooth Julia set.

The following theorem is the main result of the paper.

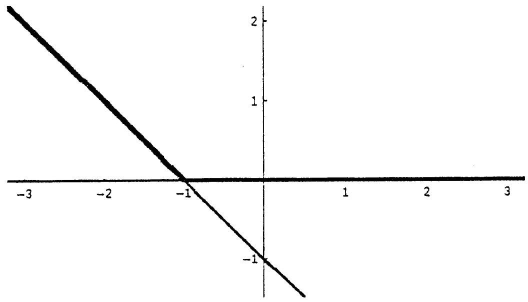

FIGURE 1.

1.5.

THEOREM.

(1) *Let F be a subhyperbolic polynomial with connected Julia set. Then either*
   (i) $\beta(t)$ *is real analytic on* $\mathbb{R}$, *or*
   (ii) *there is a (phase-transition) point* $t_c \leq -1$ *such that*

$$\beta'(t_c - 0) < \beta'(t_c + 0),$$

   $\beta(t)$ *is linear (i.e.* $= \beta'(t_c - 0)t - 1$*) on* $(-\infty, t_c]$, *and is real analytic on* $[t_c, +\infty)$ *(see Figure 2).*

(2) *If the degree of F is* 2 *or* 3, *then the only polynomials satisfying* (ii) *are* P_2 *and* $\pm P_3$ *(up to conjugation), but there are polynomials of degree* 4 *satisfying* (ii) *and not conjugate to* P_4.

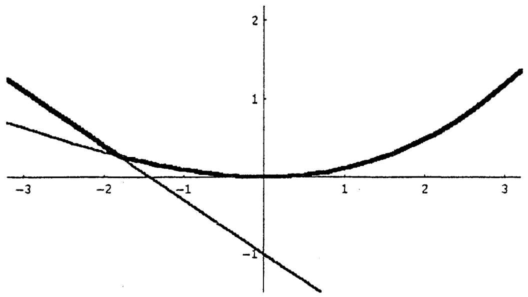

FIGURE 2.

We want to make some comments concerning the stated result.

1.6. It will be shown (in Corollary 6.2) that no phase transition occurs unless either $F \sim \pm P_d$, or F has a fixed point $a \in J$ and critical points $c_1, \ldots, c_m$ such that

$$\begin{cases} F(c_1) = \ldots = F(c_m) = a \\ \sum_{j=1}^{m} k(c_j) = d - 1. \end{cases} \tag{5}$$

It is easy to see that no quadratic polynomials satisfy (5). The situation is completely different in degree 3. There is a whole family $\{F_c\}_{c\in\mathbb{C}}$ of polynomials satisfying (5):

$$F_c(z) = z^3 - 3c^2z + 2(c^3 - c)$$

(in fact, every F satisfying (5) is conjugate to some F_c). To study this family we apply the following necessary condition (see §1.8. below):

$$\left|F'(a)\right| > \left|F'(b)\right| \text{ for any fixed point } b \neq a \tag{6}$$

which turns out (by pure chance!) to be inconsistent with the requirement that the Julia set $J_c = J_{F_c}$ must be connected. This can be seen from Figure 3 where we built the 'Mandelbrot set' $\{c : J_c \text{ is connected}\}$ and the region $\{c : F_c \in (6)\}$. The latter is the exterior domain bounded by the outer curve in the picture. In other words, there is no phase-transition phenomenon for cubic polynomials (except for $\pm P_3$) simply because we are considering only the polynomials with connected Julia set.

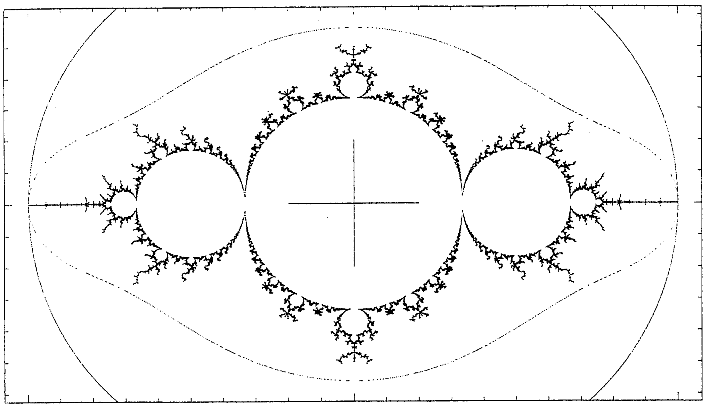

FIGURE 3.

1.7. It is natural to try to extend the notion of the $\beta(t)$-spectrum to disconnected Julia sets. It can be shown, for instance, that in the connected case,

$$\beta(t) = t - 1 + \lim_{\varepsilon\to 0} \frac{\log L_t(\varepsilon)}{|\log \varepsilon|}, \tag{7}$$

where

$$L_t(\varepsilon) = \inf \sum \delta_\nu^t,$$

the infimum being taken over all covers of the Julia set with discs of radius δ_ν and harmonic measure (evaluated at ∞) less than ε, cf **[CJ]**. This representation makes no reference to conformal maps and therefore (7) can be used as a definition of $\beta(t)$ in the general (subhyperbolic) case. Then the first part of Theorem 1.5 remains valid, as well as the statement concerning quadratic polynomials. But of course, there exist cubic polynomials (with disconnected Julia sets) such that $\beta(t)$ has a phase transition point.

1.8. There is a criterion (see §6.4) for the phase transition expressed in terms of the multipliers of the periodic points of F. For $b \in \text{Fix}\, F^p$ we denote

$$\mu(b) = \left|\left(F^p\right)'(b)\right|^{1/p}.$$

Then a subhyperbolic polynomial not conjugate to $\pm P_d$ has a phase transition if and only if there is a fixed point a and a positive number δ such that

$$b \in \text{Per}\, F, \quad b \neq a \Rightarrow \mu(b) \leq \mu(a) - \delta.$$

It is perhaps worth mentioning that no periodic point a of period ≥ 2 can have such a property (except for the case $F \sim -P_d$, d odd), and also that for every periodic point $a \in J$, and every $\delta > 0$ there is a periodic point $b \neq a$ such that

$$\mu(b) \leq \mu(a) + \delta.$$

1.9. The multipliers $\mu(b)$ characterize the local behaviour of the conformal map (or harmonic measure) at the periodic points, so that at points with larger multipliers, the domain Ω_F has 'a wider opening'. This provides the following geometric interpretation of the phase transition case: there is a point $a \in J$ (or two points, if $F \sim \pm P_d$) that is more 'exposed' than any other point of the Julia set. This point a provides a prevailing contribution to the integral means of the derivative of the conformal mapping. Every preimage of a must be a critical point (this is exactly our condition (5)) because otherwise the structure of the Julia set at the pre-image points would be the same as at a.

Figure 4 shows the Julia set of a critically finite polynomial

$$F(z) = (z - c)^3(z + 3c) - 3c, \quad c = \frac{1 + i\sqrt{2}}{3}.$$

The encircled point $a = -3c$ is responsible for the phase-transition phenomenon. It can be seen that this point is 'more exposed' than any other tip point in the picture. The Julia set has the same structure at all tip points except a. Figure 5 is the blow up of this Julia set near the point a.

It is instructive to compare Figure 4 with the Julia sets of the polynomials

$$z^2 + i \quad \text{(Figure 6)},$$

$$(z - c)^2(z + 2c) - 2c, \quad c = \sqrt{\frac{3 + i\sqrt{7}}{8}} \quad \text{(Figure 7)}.$$

These polynomials are also critically finite but they have a real analytic spectrum. The Julia set of $z^2 + i$ has tip points all similar to each other. The Julia set in Figure 7 does have a distinguished point ($a = -2c$, encircled) but the corresponding tip is 'less exposed' than the other tips.

2. *Some analytic properties*

In this section we study the analytic properties of the conformal map

$$\varphi : \mathbb{D}_- \to \Omega_F$$

(Ω_F is the basin of attraction to ∞ of a polynomial F with connected Julia set).

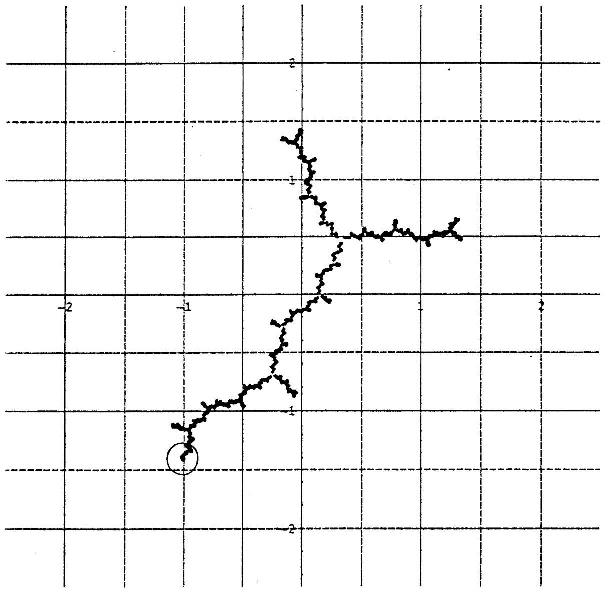

FIGURE 4.

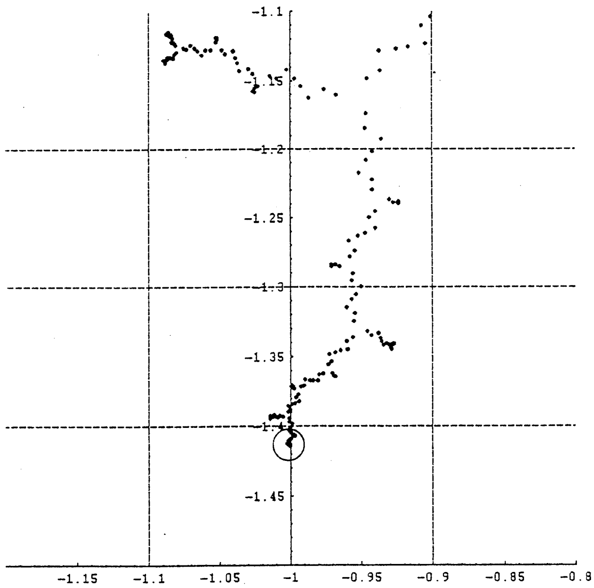

FIGURE 5.

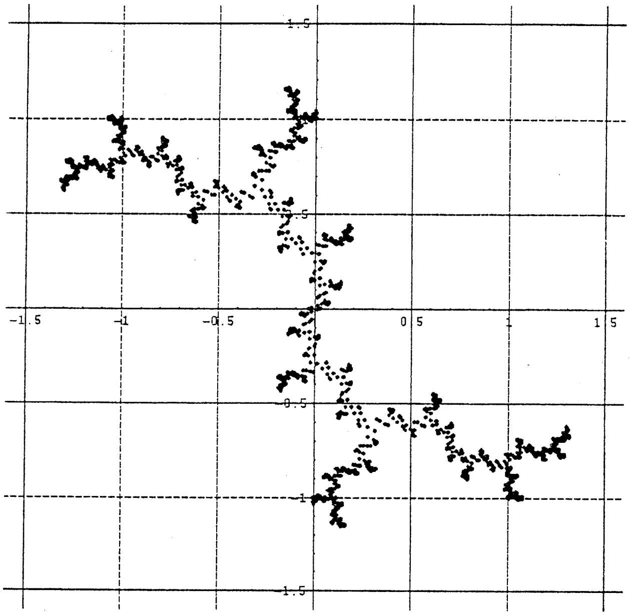

FIGURE 6.

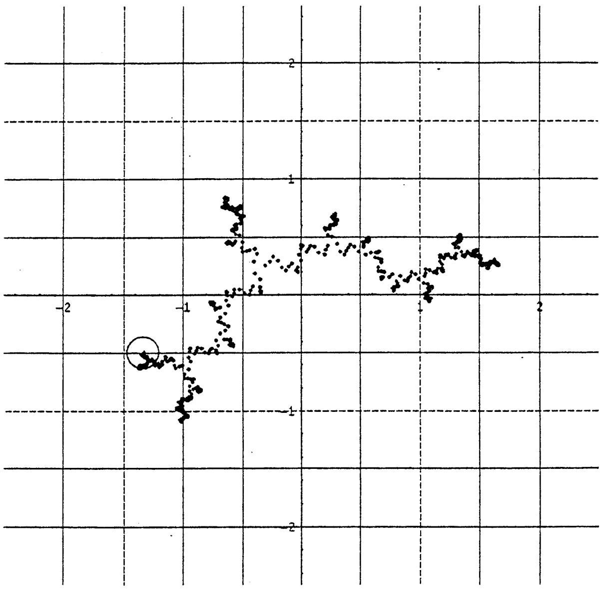

FIGURE 7.

2.1.

LEMMA. *Suppose that the Julia set J_F is locally connected. Let a be a repelling periodic point of F. Then for any point*

$$\zeta_0 \in \varphi^{-1} \bigcup_{m \geq 0} F^{-m} a$$

there is $\delta > 0$ such that if $\zeta \in \mathbb{T}$, $|\zeta - \zeta_0| < \delta$, then

$$|\varphi(\zeta) - \varphi(\zeta_0)| \asymp \operatorname{diam} \varphi \left(\widehat{\zeta \zeta_0}\right).$$

In fact both quantities in the latter relation are

$$\asymp |\zeta - \zeta_0|^{\log \mu(a)/D \log d}$$

where D is the degree of F^m at $\varphi(\zeta_0)$ for (all) sufficiently large m.

Proof. If k is the degree of F at the point $\varphi(\zeta_0)$, then for ζ close to ζ_0 we have

$$|F\varphi(\zeta) - F\varphi(\zeta_0)| = |\varphi(T\zeta) - \varphi(T\zeta_0)| \asymp |\varphi(\zeta) - \varphi(\zeta_0)|^k,$$
$$\operatorname{diam} F\varphi \left(\widehat{\zeta \zeta_0}\right) = \operatorname{diam} \varphi \left(\widehat{T\zeta T\zeta_0}\right) \asymp \left(\operatorname{diam} \varphi \left(\widehat{\zeta \zeta_0}\right)\right)^k.$$

Thus it is sufficient to consider only the case where ζ_0 is a fixed point of the map $T : \zeta \mapsto \zeta^d$.

Since the point a is repelling, there is a neighbourhood $\mathcal{U}$ such that $F|_{\mathcal{U}}$ is univalent and $\mathcal{U} \subset F\mathcal{U}$, and there is a conformal map $\tau : F\mathcal{U} \to \mathbb{D}$ such that

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad F \quad} & F\mathcal{U} \\ \Big\downarrow {\scriptstyle \tau|_{\mathcal{U}}} & & \Big\downarrow {\scriptstyle \tau} \\ \mu^{-1}\mathbb{D} & \xrightarrow{\; z \mapsto F'(a)z \;} & \mathbb{D} \end{array}$$

Moreover, we can take $\mathcal{U}$ such that τ is bi-Lipschitz.

By [**M**, Theorem 18.3] we have $\#\varphi^{-1}a < \infty$, and there is an arc $I \subset \mathbb{T}$ with center at ζ_0 such that

$$\varphi\left(I \backslash \{\zeta_0\}\right) \subset \mathcal{U} \backslash \{a\}.$$

Let $I^{\pm}$ denote the two components of $I \backslash T^{-1} I$ (T^{-1} denotes the branch that takes ζ_0 to ζ_0). We have $\operatorname{dist}\left(\varphi(I^{\pm}), a\right) > 0$, and

$$\left|(\tau\varphi)\left(T^{-n}\zeta\right)\right| = \mu^{-n} \left|(\tau\varphi)(\zeta)\right|, \quad \zeta \in I.$$

Therefore if $\zeta \in T^{-n} I^{\pm}$, then

$$|(\tau\varphi)(\zeta)| \asymp \operatorname{diam} \tau\varphi \left(\widehat{\zeta \zeta_0}\right) \asymp \mu^{-n}.$$

This implies the statement because the map τ has bounded distortion. □

2.2.

LEMMA. *Suppose F is a subhyperbolic polynomial and $a \in J_F$ is a preperiodic point. Then for any $\eta > 0$ there is $\alpha > 0$ such that*

$$\int |\varphi(\zeta) - a|^{-\alpha} \, d\mu(\zeta) < \infty$$

for every positive measure μ on $\mathbb{T}$ satisfying

$$\mu(I) \lesssim |I|^{\eta}, \quad (I \text{ is an arbitrary open arc on } \mathbb{T}).$$

Proof. This is a consequence of the last statement of Lemma 2.1. □

2.3. *John's property.* Let F be a subhyperbolic polynomial with connected Julia set. Then $\Omega = \Omega_F$ is a *John domain* (see [**CJ**]), which means that for some $c > 0$ and all $z_0 \in \Omega$ there is an arc $\gamma \subset \Omega$, $\infty \in \gamma$, such that

$$\operatorname{dist}(z, J) \geq c\,|z - z_0|, \quad z \in \gamma.$$

In this case the conformal map φ is *Hölder continuous* and

$$\operatorname{diam} \varphi(I(z)) \asymp (|z| - 1)\left|\varphi'(z)\right| \asymp \operatorname{dist}(\varphi(z), \partial\Omega) \tag{8}$$

for any $z \in \mathbb{D}_-$. In fact the Hölder condition holds on every scale, which implies the following diameter version of the (A_∞)-*condition*: there is $\eta < 1$ such that if I and $\tilde{I}$ are two arcs on $\mathbb{T}$, and $I \subset \tilde{I}$, then

$$\frac{\operatorname{diam} \varphi(I)}{\operatorname{diam} \varphi(\tilde{I})} \leq \operatorname{const} \left(\frac{|I|}{|\tilde{I}|}\right)^{\eta}. \tag{9}$$

One more property of John's domains is stated in the following lemma.

LEMMA. *Let $\varphi : \mathbb{D}_- \to \Omega$ be a conformal map onto a John domain Ω. Then there is $C > 0$ such that for any arc $I \subset \partial\mathbb{D}$ and any point $a \in \partial\Omega$,*

$$\max_I \log|\varphi - a| \leq \frac{1}{I}\int_I \log|\varphi - a| + C. \tag{10}$$

Proof. Because of the Bloch property (4) of the function $b = \log|\varphi - a|$, it is sufficient to show that

$$\max_I |\varphi - a| \lesssim |\varphi(z_I) - a|.$$

Let $\zeta \in I$ be the maximum point. If $\operatorname{diam}\varphi(I) \gtrsim |\varphi(\zeta) - a|$, we have

$$\begin{aligned} |\varphi(z_I) - a| &\geq \operatorname{dist}(\varphi(z_I), \partial\Omega) \\ &\overset{(8)}{\asymp} \operatorname{diam}\varphi(I) \gtrsim |\varphi(\zeta) - a|. \end{aligned}$$

On the other hand, if $\operatorname{diam}\varphi I \ll |\varphi(\zeta) - a|$, we have

$$\begin{aligned} |\varphi(z_I) - a| &\geq |\varphi(\zeta) - a| - |\varphi(z_I) - \varphi(\zeta)| \\ &\overset{(8)}{\geq} |\varphi(\zeta) - a| - \operatorname{const}\operatorname{diam}\varphi(I) \geq \frac{1}{2}|\varphi(\zeta) - a|. \end{aligned}$$

□

2.4.

LEMMA. *Let F be a subhyperbolic polynomial with connected Julia set and let $a \in J$ be a* strictly *preperiodic point of F. Denote*

$$\theta(\zeta) = \log|\varphi(\zeta) - a|, \quad \zeta \in \mathbb{T}.$$

Then for any integer n and any d-adic interval I of rank n,

$$\max_I S_n\theta - (S_n\theta)_I \leq C \tag{11}$$

Proof. For an arc $I \subset \mathbb{T}$ let $\tilde{I}$ denote the minimal arc on $\mathbb{T}$ containing I and the point of $\varphi^{-1}a$ which is the nearest to I. We claim that

$$\max_I \theta - \theta_I \lesssim \frac{\operatorname{diam}\varphi(I)}{\operatorname{diam}\varphi(\tilde{I})}. \tag{12}$$

Indeed, if $\operatorname{diam}\varphi(I) \asymp \operatorname{diam}\varphi(\tilde{I})$, then (12) follows from Lemma 2.3. Suppose now that $\operatorname{diam}\varphi(I) \ll \operatorname{diam}\varphi(\tilde{I})$. Let $\zeta_{\min}$, $\zeta_{\max}$ be the points of I at which θ attains the minimal and the maximal values. We have

$$|\varphi(\zeta_{\min}) - a| \geq |\varphi(\zeta_{\max}) - a| - \operatorname{diam}\varphi(I)$$

and, by Lemma 2.1,

$$\begin{aligned} |\varphi(\zeta_{\max}) - a| &\asymp \operatorname{diam}\varphi\left(\widehat{\zeta_0\zeta_{\max}}\right) \\ &\asymp \operatorname{diam}\varphi(\tilde{I}). \end{aligned}$$

(We can assume that I is sufficiently close to $\varphi^{-1}a$, otherwise the relation is still true because both sides are $\asymp 1$.) Therefore,

$$\begin{aligned} \max_I \theta - \theta_I &\leq \log\left|\frac{\varphi(\zeta_{\max}) - a}{\varphi(\zeta_{\min}) - a}\right| \\ &\leq \log\frac{|\varphi(\zeta_{\max}) - a|}{|\varphi(\zeta_{\max}) - a| - \operatorname{diam}\varphi(I)} \\ &\lesssim \frac{\operatorname{diam}\varphi(I)}{|\varphi(\zeta_{\max}) - a| - \operatorname{diam}\varphi(I)} \\ &\lesssim \frac{\operatorname{diam}\varphi(I)}{\operatorname{diam}\varphi(\tilde{I})}. \end{aligned}$$

To prove the lemma we must show that if I is an interval of rank n, then

$$\sum_j \frac{\operatorname{diam}\varphi(T^jI)}{\operatorname{diam}\varphi(\widetilde{T^jI})} \leq \text{const}.$$

By (9), it suffices to prove that if $\eta > 0$, then

$$\sum_j \left(\frac{|T^jI|}{|\widetilde{T^jI}|}\right)^\eta \leq C. \tag{13}$$

Let integers l_j be such that

$$|\widetilde{T^j I}| \asymp d^{-l_j}.$$

Then the left-hand side of (13) is

$$\asymp \sum \left(\frac{d^{j-n}}{d^{-l_j}}\right)^\eta = \sum Q^{-n+j+l_j},$$

where $Q = d^\eta > 1$. We now fix some number M and consider the sequence $1 \leq j_1 < \cdots < j_m \leq n$ of the indices satisfying $l_j \geq M$. Then we have

$$j_{k+1} - j_k \geq l_{j_k} - \text{const}, \quad (k < m), \tag{14}$$

and

$$\sum_k Q^{-n+j_k+l_{j_k}} \lesssim \sum_k Q^{-n+j_{k+1}} \leq \text{const}.$$

This implies (13) because

$$\sum_{\{j:l_j \leq M\}} Q^{-n+j+l_j} \leq \text{const}.$$

To prove (14) observe that $\text{dist}(T^j I, \varphi^{-1}a) \lesssim d^{-l_j}$ and, consequently, the intervals $T^{j+1}I, T^{j+2}I, \ldots$ are at a distance $\lesssim d^{-l_j+1}, d^{-l_j+2}, \ldots$ from the finite set $\varphi^{-1}\{Fa, F^2a, \ldots\}$. This set is separated from $\varphi^{-1}a$ because the point a is strictly preperiodic, and it will take at least $(l_j - \text{const})$ iterations to get anywhere close to $\varphi^{-1}a$. □

2.5.

Remark. The same argument shows that if I is an interval of rank n and

$$\zeta_I = I \cap \text{Fix}\, T^n,$$

then

$$\left[\max_I S_n\theta\right] - S_n\theta(\zeta_I) \leq C.$$

The only change in the proof is that instead of Lemma 2.3 we use the inequality

$$\max_I \theta - \theta(\zeta_I) \leq C. \tag{15}$$

If ζ_0 is the point of $\varphi^{-1}a$ nearest to I, then (15) is equivalent, by Lemma 2.1, to the inequality

$$\left|\frac{\zeta_I - \zeta_0}{\zeta_{\max} - \zeta_0}\right| \geq \text{const}.$$

The latter follows from the fact that

$$\text{dist}\left(\zeta_0, \text{Fix}\, T^n\right) \asymp d^{-n}$$

i.e., if $|\zeta_0 - \eta| \ll d^{-n}$ for some $\eta \in \text{Fix}\, F^n$, then $|T^n\zeta_0 - \eta| \ll 1$ which is impossible since ζ_0 is strictly preperiodic.

COROLLARY. *If F is a subhyperbolic polynomial of degree d with connected Julia set, then*

$$\beta(t) - t + 1 = \lim_{n\to\infty} \frac{1}{n} \log_d \sum_{\zeta \in \mathrm{Fix}\, T^n} \left| \left(F^n\right)' (\varphi(\zeta)) \right|^{-t}.$$

Proof. This follows from Proposition 1.2, Lemma 2.3 and the above remark. □

3. *Perron–Frobenius operator*

3.1. Let $C(\mathbb{T})$ denote the space of continuous functions, and $C_+(\mathbb{T}) = \{f \in C(\mathbb{T}) : f \geq 0\}$. Suppose $g \in C_+(\mathbb{T})$. The *Perron–Frobenius* operator $L = L_g$ corresponding to g and the dynamics

$$T : z \mapsto z^d, \quad z \in \mathbb{T},$$

is defined on $C(\mathbb{T})$ by the equation

$$(Lf)(x) = \sum_{y \in T^{-1}x} g(y) f(y), \quad (x \in \mathbb{T},\ f \in C(\mathbb{T})).$$

For the iterates of L, we have the formula

$$\left(L^n f\right)(x) = \sum_{y \in T^{-n}x} g_n(y) f(y), \tag{16}$$

where

$$g_n \equiv \prod_{j=0}^{n-1} g \circ T^j.$$

The conjugate operator L^* acts on the space $M(\mathbb{T})$ of Borel complex measures. For example, if δ_x is the δ-measure at $x \in \mathbb{T}$, then

$$L^* \delta_x = \sum_{y \in T^{-1}x} g(y) \delta_y. \tag{17}$$

The best known case is when the function g satisfies the Hölder condition and *does not vanish* on $\mathbb{T}$ (see, e.g., [**B**], [**R1**]). For $\alpha > 0$, we denote by $\mathcal{H}_\alpha$ the space of α-Hölder continuous functions on $\mathbb{T}$:

$$\mathcal{H}_\alpha = \{f \in C(\mathbb{T}) : \|f\|_\alpha < \infty\},$$
$$\|f\|_\alpha = \|f\|_\infty + \sup_{x, x' \in \mathbb{T}} \frac{\left|f(x) - f(x')\right|}{|x - x'|^\alpha}.$$

It is clear that if $g \in \mathcal{H}_\alpha$, then L_g also acts on $\mathcal{H}_\alpha$.

THEOREM. (Ruelle's Theorem.) *If g is a strictly positive function in $\mathcal{H}_\alpha$, then the number*

$$\lambda = d^{P(\log g)}$$

is the spectral radius and an isolated eigenvalue of multiplicity one of the operator $L_g : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$.

It is immediate from Ruelle's theorem that the *pressure function*

$$t \in \mathbb{R} \mapsto P(t \log g)$$

is real analytic, and this implies the corresponding fact for the spectrum $\beta(t)$ in the *hyperbolic* case (see §1). What we would like to do is to extend Ruelle's theorem to certain Hölder continuous functions g *vanishing* at some points of $\mathbb{T}$. We introduce the following.

Definition. (cf Lemmas 2.3 and 2.4). We say that a function $g \in C_+(\mathbb{T})$ satisfies the *condition* $(\mathcal{B}_w)$, or $(\mathcal{B})$, if

$$\#g^{-1}(0) < \infty, \quad \log g \in L^1(\mathbb{T}),$$

and for any d-adic interval $I \in \mathbb{T}$ of rank n we have

$$\max_I g \asymp \exp\{(\log g)_I\}, \tag{$\mathcal{B}_w$}$$

or

$$\max_I g_n \asymp \exp\{(\log g_n)_I\}. \tag{$\mathcal{B}$}$$

3.2.

THEOREM. *Suppose a function $g \in \mathcal{H}_\alpha$ satisfies the condition $(\mathcal{B})$, and suppose there are continuous functions $\tilde{g} \in (\mathcal{B})$ and $u \in (\mathcal{B}_w)$ such that*

$$g = \frac{1}{\tilde{g}} \frac{u \circ T}{u}. \tag{H}$$

I. *If there exist a probability measure ν, and a positive number λ such that*

$$L^*\nu = \lambda\nu, \quad \int u\, d\nu \neq 0, \tag{18}$$

then the conclusion of Ruelle's theorem holds.

II. *The following condition is* sufficient *for the existence of ν and λ satisfying (18):*

$$\begin{cases} S \subset u^{-1}(0) \\ S = T^{-1}S \backslash g^{-1}(0) \end{cases} \Rightarrow S = \emptyset. \tag{$\mathcal{X}$}$$

The proof of the theorem is obtained by adjusting a known method to the case under consideration. For the sake of completeness, we repeat some arguments of the papers [**HK**], [**Ry**] and [**P**]. The first part of the theorem is proved in §3.3–3.6 below, and the second in §3.7 and 3.8.

3.3.

LEMMA. *Suppose $g \in C_+(\mathbb{T})$ satisfies $(\mathcal{B})$ and (H) with some $\tilde{g} \in (\mathcal{B})$, $u \in (\mathcal{B}_w)$, and suppose there are ν and λ satisfying (18). Then*

$$\nu(I) \asymp \lambda^{-n} \max_I g_n \asymp \lambda^{-n} \exp\{(\log g_n)_I\} \tag{19}$$

for any n and any interval I of rank *n. Futhermore,*

$$\|L_g^n\|_{C(\mathbb{T})} \leq \lambda^n,$$

and hence λ is the spectral radius of L_g. The pressure $P(\log g)$ exists and is equal to $\log_d \lambda$.

Proof. Let $h_\varepsilon \downarrow \chi_I$ as $\varepsilon \to 0$, $h_\varepsilon \in C_+(T)$. Then

$$\begin{aligned}\nu(I) &= \lim_{e\to 0} \langle h_\varepsilon, \nu\rangle \\ &= \lambda^{-n} \lim_{e\to 0} \langle L^n h_\varepsilon, \nu\rangle \\ &\overset{(16)}{=} \lambda^{-n} \int_{\partial\mathbb{D}\setminus\{1\}} g_n(Ix)d\nu(x) + \nu(1)\left[g_n(\partial_+) + g_n(\partial_-)\right],\end{aligned}$$

where (Ix) is the symbolic notation for the point $I \cap T^{-n}x$, and $\partial_\pm$ are the endpoints of I. Therefore,

$$\nu(I) \lesssim \lambda^{-n} \max_I g_n.$$

On the other hand, (H) implies

$$g_n = \frac{1}{\tilde{g}_n} \frac{u \circ T^n}{u},$$

and

$$\nu(I) \gtrsim \frac{\lambda^{-n}}{\max_I \tilde{g}_n \max_I u} \int_{\partial\mathbb{D}} u d\nu.$$

Since the latter integral is positive, and

$$\begin{aligned}\max_I \tilde{g}_n \max_I u &\asymp \exp\{(\log \tilde{g}_n u)_I\} \\ &\asymp \exp\{-(\log g_n)_I\},\end{aligned}$$

we have (19). Since L is a positive operator,

$$\begin{aligned}\|L^n\| &= \|L^n 1\|_\infty = \left\| \sum_{y\in T^{-n}x} g_n(y) \right\|_\infty \\ &\leq \sum_{\operatorname{rank} I=n} \max_I g_n \asymp \lambda^n \sum \nu(I) \asymp \lambda^n.\end{aligned}$$

□

3.4.

LEMMA. *Suppose* $g \in \mathcal{H}_\alpha$, $g \geq 0$, *and*

$$\|L_g^n\|_{C(\mathbb{T})} \lesssim 1.$$

Then $\forall N\ \exists M = M(N)$ *such that*

$$\|L^N f\|_\alpha \lesssim d^{-N\alpha}\|f\|_\alpha + M\|f\|_\infty. \tag{20}$$

In particular we have

$$\|L_g^n f\|_{\mathcal{H}_\alpha} \lesssim 1. \tag{21}$$

Proof. We have

$$\begin{aligned}&\frac{\left|\left(L^N f\right)(x) - \left(L^N f\right)(x')\right|}{|x-x'|^\alpha} \\ &\leq \sum_{\operatorname{rank} I=N} \frac{\left|g_N(Ix) - g_N(Ix')\right|}{|x-x'|^\alpha} |f(Ix)| + \sum_{\operatorname{rank} I=N} \frac{\left|f(Ix) - f(Ix')\right|}{|x-x'|^\alpha} g_N(Ix') \\ &\leq d^N d^{-\alpha N}\|g_N\|_\alpha \|f\|_\infty + d^{-\alpha N}\|f\|_\alpha \left\|L^N 1\right\|_\infty,\end{aligned}$$

and we see that (20) holds with

$$M = 1 + d^{(1-\alpha)N} \|g_N\|_\alpha.$$

To prove (21) we fix N such that

$$\left\|L^N f\right\|_\alpha \le q\|f\|_\alpha + M\|f\|_\infty$$

for some $q < 1$. Then, by induction, we have

$$\left\|L^{Nk} f\right\|_\alpha \le q^k \|f\|_\alpha + \frac{CM}{1-q}\|f\|_\infty$$

with $C = \max \|L^n\|_{C(\mathbb{T})}$. □

3.5. *Quasicompactness.* A Banach space linear operator L is called *quasicompact* if

$$\left\|L^N - K\right\| < 1$$

for some compact operator K and some integer N. An equivalent statement is that the image $[L^N]$ of L^N in the *Calkin algebra* (i.e. bounded operators modulo compact operators) has norm strictly less than one. This implies that the spectral radius of $[L]$ in the Calkin algebra is strictly less than one, and therefore there is $r < 1$ such that the part of the spectrum of L lying outside of the disk $\{|z| \le r\}$ consists of a finite number of eigenvalues that all have finite geometric multiplicity.

LEMMA. *Any operator L on $\mathcal{H}_\alpha$ satisfying (20) is quasicompact.*

Proof. Fix N such that

$$\left\|L^N f\right\|_\alpha \le \frac{1}{100}\|f\|_\alpha + M\|f\|_\infty, \quad f \in \mathcal{H}_\alpha.$$

Also fix $m \gg 1$ ($Md^{-\alpha m} \ll 1$ is sufficient) and define a projection operator P on $\mathcal{H}_\alpha$ by the requirement

$$\begin{cases} Pf = f & \text{on } T^{-m}1, \\ Pf(e^{it}) \text{ is a linear function in } t \text{ for } e^{it} \in I, & \operatorname{rank} I = m. \end{cases}$$

It is clear that $\dim P\mathcal{H}_\alpha = d^m < \infty$, and it remains to check that

$$\left\|L^N - L^N P\right\| < 1.$$

Since $\|P\| \le 4$, and

$$\|f - Pf\|_\infty \le d^{-\alpha m}\|f - Pf\|_\alpha,$$

we have

$$\begin{aligned} \left\|L^N (f - Pf)\right\|_\alpha &\le \tfrac{1}{100}\|f - Pf\|_\alpha + M\|f - Pf\|_\infty \\ &\le \tfrac{1}{10}\|f - Pf\|_\alpha \le \tfrac{1}{2}\|f\|_\alpha \end{aligned}$$

□

Returning to the proof of the first part of Theorem 3.2, we see that the last three lemmas imply that λ is the spectral radius and an isolated eigenvalue of $L = L_g : \mathcal{H}_\alpha \circlearrowleft$. It remains to show that

$$\dim \bigcap_{n \ge 1} \ker(L - \lambda)^n = 1.$$

3.6. *Multiplicity of* λ. We first prove

$$\dim\ker(L-\lambda)=1. \tag{22}$$

Observe that if f satisfies $Lf=\lambda f$ with $\lambda>0$, then $L\,|f|=\lambda\,|f|$. Indeed,

$$\begin{aligned}\langle\lambda\,|f|,\nu\rangle &= \langle|Lf|,\nu\rangle\le\langle L\,|f|,\nu\rangle\\ &= \left\langle|f|,L^*\nu\right\rangle=\langle|f|,\lambda\nu\rangle\end{aligned}$$

and $L\,|f|=\lambda\,|f|$ ν-a.e. and hence everywhere because $\operatorname{supp}\nu=\mathbb{T}$ (by Lemma 3.3). Next observe that if f is a real-valued eigenfunction, $Lf=\lambda f$, then either $f\ge 0$ or $f\le 0$. Otherwise, we can find a non-trivial eigenfunction $\tilde f\ge 0$ (e.g. $\tilde f=|f|-f$) which is zero on some interval. The equation

$$\lambda^n\tilde f(x)=\sum_{y\in T^{-n}x} g_n(y)\tilde f(y)$$

then implies that $\tilde f$ must vanish on a dense set, hence everywhere.

Suppose now that we have two real functions f_1, f_2 satisfying

$$Lf_j=\lambda f_j,\qquad f_j\ge 0\ (j=1,2).$$

Normalizing them by the condition

$$\int f_j d\nu=1,$$

we have

$$\int|f_1-f_2|\,d\nu=\left|\int(f_1-f_2)d\nu\right|=0,$$

and $f_1=f_2$. This proves (22).

It remains to show that

$$\ker(L-\lambda)^2=\ker(L-\lambda).$$

Suppose this is not true. Then there are $f\ne 0$ and h such that

$$Lf=\lambda f,\ \ Lh=\lambda h+cf,\qquad (c\ne 0).$$

By induction, we have

$$L^n h=\lambda^n\left(h+nc\lambda^{-1}f\right)$$

which contradicts the relation $\|L^n\|\asymp\lambda^n$. □

We turn now to the proof of the second part of Theorem 3.2.

3.7.

LEMMA. *Let g be a Hölder continuous function satisfying $\#g^{-1}(0)<\infty$. Then there exist $\lambda>0$ and $\nu\in M(\mathbb{T})$ such that*

$$L^*\nu=\lambda\nu.$$

Proof. For $\varepsilon > 0$ let L_ε denote the Perron–Frobenius operator correspomding to the function $g + \varepsilon$. By Ruelle's theorem we have

$$L_\varepsilon^* \nu_\varepsilon = \lambda_\varepsilon \nu_\varepsilon \tag{23}$$

for some probability measure ν_ε and $\lambda_\varepsilon > 0$. Since the numbers λ_ε are bounded (for $\varepsilon < 1$), we can find a sequence $\varepsilon_j \to 0$ such that

$$\lambda_{\varepsilon_j} \to \lambda, \quad \nu_{\varepsilon_j} \xrightarrow{w^*} \nu,$$

and take a w^*-limit in (23). To prove that $\lambda > 0$, it remains to show that the numbers λ_ε are bounded away from zero. But it follows from the condition $\#g^{-1}(0) < \infty$, that there is a T-periodic point $\zeta \in \mathbb{T}$ such that $g \neq 0$ on the orbit of ζ. Therefore,

$$g_n(\zeta) \geq c^n \text{ for some } c > 0$$

and

$$\begin{aligned} \lambda_\varepsilon^n &\asymp \sum_{\operatorname{rank} I = n} e^{(\log(g+\varepsilon)_n)_I} \\ &\asymp \sum_{\operatorname{rank} I = n} \max_I (g+\varepsilon)_n \\ &\geq g_n(\zeta) \geq c^n. \end{aligned}$$

Thus $\lambda_\varepsilon \geq c > 0$. □

3.8. *Condition $(\mathcal{X})$.* We finally show that $(\mathcal{X})$ implies

$$\int u d\nu > 0$$

for any positive measure ν satisfying $L^*\nu = \lambda\nu$, $\lambda \neq 0$.

Suppose $\int u d\nu = 0$. Then ν is concentrated on a finite set $S \subset u^{-1}(0)$. From (17) it follows that

$$\operatorname{supp} L^* \delta_x = T^{-1}\{x\} \backslash g^{-1}(0).$$

Consequently,

$$\operatorname{supp} L^* \nu = T^{-1} S \backslash g^{-1}(0),$$

and $L^*\nu = \lambda\nu$ implies

$$S = T^{-1} S \backslash g^{-1}(0).$$

□

4. *The pressure function*

4.1. Let g be as in Theorem 3.2 and assume that the condition $(\mathcal{X})$ is satisfied. For every $t \geq 0$ we have

$$P(t \log g) = \log_d \lambda(t),$$

where $\lambda(t)$ is the spectral radius of L_{g^t}.

COROLLARY. *The function $t \mapsto P(t \log g)$ is real analytic on $\{t > 0\}$.*

Proof. This function is convex on $\{t \geq 0\}$ and therefore $\lambda(t)$ is continuous for $t > 0$. Fix some point $t_0 > 0$. Then there is $\beta > 0$ such that $g^t \in \mathcal{H}_\beta$ for all ts in some neighbourhood of t_0. Let $L_{(t)}$ denote the Perron–Frobenius operator on $\mathcal{H}_\beta$ corresponding to g^t. It is clear that the map $t \mapsto L_{(t)}$ is real analytic.

Chose a single closed curve γ separating $\lambda(t_0)$ from the rest of the spectrum of $L_{(t_0)}$. If t is sufficiently close to t_0, the point $\lambda(t)$ lies inside γ and the operators $\left(L_{(t)} - z\right)$ are invertible for all $z \in \gamma$. Consider the spectral projection

$$P_t = \frac{1}{2\pi i} \int_\gamma (L_t - z)^{-1}\, dz.$$

Then $t \mapsto P_t$ is an analytic map and hence $\operatorname{rank} P_t = \operatorname{rank} P_{t_0} = 1$. It follows that P_t is a projection onto the eigenspace of $L_{(t)}$ corresponding to $\lambda(t)$, and

$$\lambda(t) = \frac{L_{(t)} P_t f}{P_t f}, \quad (f \not\equiv 0,\ P_{t_0} f = f),$$

is an analytic function. □

4.2.

THEOREM. *Let g and $\tilde{g}$ be Hölder continuous functions satisfying ($\mathcal{B}$) and*

$$g = \frac{1}{\tilde{g}} \frac{u \circ T}{u} \qquad (H)$$

for some continuous function u satisfying ($\mathcal{B}_w$) and the following condition:

> *there is $\alpha > 0$ such that $\int u^{-\alpha} d\mu < \infty$ for every positive measure μ satisfying $\mu(I) \lesssim |I|^{\frac{1}{2}}$.*

Suppose also that both g and $\tilde{g}$ satisfy the ($\mathcal{X}$)-condition:

$$\left\{ \begin{array}{l} S \subset u^{-1}(0) \\ S = T^{-1}S \backslash g^{-1}(0) \end{array} \right. \quad or \quad \left\{ \begin{array}{l} S \subset u^{-1}(0) \\ S = T^{-1}S \backslash \tilde{g}^{-1}(0) \end{array} \right. \Rightarrow S = \emptyset.$$

Then the pressure $P(t \log g)$ exists for all $t \in \mathbb{R}$ and is real analytic as a function of t.

Proof. For $t > 0$ the statement is proved in Corollary 4.1. It is also true for $t < 0$ because

$$P(t \log g) = P(|t| \log \tilde{g}), \quad t < 0. \qquad (24)$$

In fact, we always can (by multiplying by a constant) normalize u so that $u_{\mathbb{T}} = 0$. Then for an interval I of rank n we have $(u \circ T^n)_I = u_{\mathbb{T}} = 0$ and

$$\begin{aligned} \left(\log g_n^t\right)_I &= t\,(\log g_n)_I \\ &= t\left(\log\left(\frac{1}{\tilde{g}}\frac{u \circ T}{u}\right)_n\right)_I \\ &= -t\,(\log \tilde{g}_n)_I + t\left(\log u \circ T^n\right)_I - t\,(\log u)_I \\ &= |t|\,(\log \tilde{g}_n)_I + |t|\,(\log u)_I, \quad (\operatorname{rank} I = n). \end{aligned}$$

Therefore

$$\sum_{\operatorname{rank} I=n} e^{(\log g_n^t)_I} \asymp \tilde{\lambda}^n \sum_{\operatorname{rank} I=n} e^{|t|(\log u)_I} \tilde{\nu}(I), \tag{25}$$

where

$$\tilde{\lambda} = d^{P(|t|\log \tilde{g})}$$

and $\tilde{\nu}$ is the probability measure satisfying

$$L_{\tilde{g}^t} \tilde{\nu} = \tilde{\lambda}\tilde{\nu}.$$

The last sum in (25) is $\asymp 1$ because u is bounded and $\operatorname{supp}\tilde{\nu} = \mathbb{T}$.

It remains to show that the pressure function is analytic at $t = 0$. This function is convex and therefore continuous. Let g_t denote the function

$$\begin{cases} g^t & t \geq 0 \\ \tilde{g}^{-t} & t \leq 0, \end{cases}$$

and L_t the corresponding Perron–Frobenius operator. Define λ_t and ν_t with respect to L_t:

$$L_t^* \nu_t = \lambda_t \nu_t,$$
$$P(t \log g) = \log_d \lambda_t, \qquad \text{(see (24))}.$$

Since $\lambda(0) = d$, for small values of t we have $\lambda(t) \geq d^{2/3}$ and

$$\begin{aligned} \nu_t(I) &\asymp \lambda_t^{-n} \max_I (g_t)_n, \quad (n = \operatorname{rank} I), \\ &\leq d^{-n/2}\left[d^{-1/6}\|g_t\|_\infty\right]^n \leq |I|^{1/2}; \end{aligned}$$

in particular, the measures ν_t have no atoms. From the condition on u, it follows that there is $\alpha > 0$ such that

$$|t| < \alpha \Rightarrow \int u^{-\alpha} d\nu_t < \infty. \tag{26}$$

From now on we will be considering only the values $t \in (-\alpha/2, \alpha/2)$. Define

$$h_t = g^{t+\alpha}\tilde{g}^\alpha = g_t \frac{v_t \circ T}{v_t}$$

where

$$v_t = \begin{cases} u^\alpha & t \geq 0 \\ u^{\alpha+t} & t \leq 0. \end{cases}$$

These functions all belong to some Hölder space $\mathcal{H}_\beta$, and satisfy the relations

$$h_t = \frac{1}{\tilde{h}_t}\frac{u_t \circ T}{u_t},$$

with $\tilde{h}_t = h_{-t}$ and $u_t = u^{|t|}v_t$. We claim that

$$L_{h_t}^* \mu_t = \lambda_t \mu_t \tag{27}$$

for some finite positive measures μ_t, namely for

$$d\mu_t = \frac{1}{v_t} d\nu_t, \qquad \text{(see (26))},$$

and

$$\int u_t d\mu_t = \int u^{|t|} d\nu_t \neq 0.$$

Then by Theorem 3.2, λ_t is an isolated eigenvalue of $L_{h_t} : \mathcal{H}_\beta \to \mathcal{H}_\beta$. The correspondence $t \mapsto L_{h_t}$ is an analytic map from $(-\alpha/2, \alpha/2)$ to the space of operators on $\mathcal{H}_\beta$, and the argument in the previous section shows that the function $t \mapsto \lambda_t$ is analytic at 0.

To prove (27), let $M = M_t$ denote the operator of multiplication by v_t in $C(\mathbb{T})$. Then

$$\begin{aligned}\left(L_{h_t} M f\right)(x) &= \sum_{y \in T^{-1}x} h_t(y) v_t(y) f(y) \\ &= \sum_{y \in T^{-1}x} g_t(y) \frac{v_t(x)}{v_t(y)} v_t(y) f(y) \\ &= (M L_t f)(x),\end{aligned}$$

and

$$M^* L_{h_t}^* = L_t^* M^*.$$

Since $M^* \mu_t = \nu_t$, we have

$$\begin{aligned} M^*(\lambda_t \mu_t) &= \lambda_t \nu_t = L_t^* \nu_t \\ &= L_t^* M^* \mu_t = M^* L_{h_t}^* \mu_t, \end{aligned}$$

which implies (27) because the measures μ_t and $L_{h_t}^* \mu_t$ have no atoms. □

5. *Homology*

5.1. Let F be a polynomial, and let C be a non-empty subset of $\operatorname{Crit} F \cap J$. Denote

$$\mathcal{O}^+ = \mathcal{O}^+(C) = \bigcup_{n \geq 1} F^n C,$$
$$\mathcal{O} = \mathcal{O}(C) = \mathcal{O}^+(C) \cup C.$$

LEMMA. *Suppose that the set $\mathcal{O}$ is finite. Let*

$$G(z) = \prod_{c \in C} |z - c|^{\nu(c)}$$

for some positive numbers $\nu(c) \leq k(c) - 1$. Then G satisfies the homological equation

$$G = \frac{1}{Q} \frac{H \circ F}{H}$$

where H and Q have the form

$$H(z) = \prod_{a \in \mathcal{O}^+} |z - a|^{\alpha(a)}, \quad \alpha(a) > 0, \tag{28}$$

$$Q(z) = \prod_{b \in F^{-1}\mathcal{O} \setminus \operatorname{Per} F} |z - b|^{\alpha(b)}, \quad \alpha(b) \geq 0. \tag{29}$$

Futhermore, the zero set $Q^{-1}(0)$ has the property

$$\begin{cases} \Sigma \subset \mathcal{O}^+ \ (= H^{-1}(0)) \\ \Sigma = F^{-1}\Sigma \setminus Q^{-1}(0). \end{cases} \Rightarrow \Sigma = \emptyset. \tag{30}$$

Proof. The map F provides an oriented graph structure on $\mathcal{O}$: $a \to F(a)$. Without loss of generality we can assume that this graph is connected. Deleting from the graph all bonds originating from the critical points, we obtain some family $\Gamma = \{\gamma\}$ of the components. One of the components, γ_{final}, contains a periodic cycle and has no critical points. All other γs contain exactly one critical point $c_\gamma \in \gamma$. Thus Γ can be identified with the set

$$(\text{Crit}\, F \cap \mathcal{O}) \cup \{\gamma_{\text{final}}\}$$

and has the natural structure of an oriented graph.

Define the *rank of* γ as the maximal number $\nu = 0, 1, \ldots$ such that

$$\gamma_1 \to \gamma_2 \to \cdots \to \gamma_\nu \to \gamma$$

for some components $\gamma_1, \ldots, \gamma_\nu \in \Gamma$. In particular, γ has rank zero iff $\gamma \neq \gamma_{\text{final}}$, and $c_\gamma \in C \backslash \mathcal{O}^+$.

We define $\alpha(\gamma) = 0$ for γs of rank zero, and then define inductively the numbers $\alpha(\gamma)$ for γ's of rank $1, 2, \ldots$ by the formula

$$\alpha(\gamma) = \max_{\gamma' \to \gamma} \check{\alpha}(\gamma'),$$

where

$$\check{\alpha}(\gamma') = \frac{\alpha(\gamma') + \nu(c')}{k(c')}, \tag{31}$$

($c' \stackrel{\text{def}}{=} c_{\gamma'}$ and $\nu(c') = 0$ if $c' \notin C$).

Let H_γ denote the polynomial

$$H_\gamma = \prod_{a \in \gamma} (z - a).$$

If $\gamma \neq \gamma_{\text{final}}$, the divisor $F^{-1}\gamma$ consists of the set

$$\{a \in \gamma : a \neq c_\gamma\},$$

of the critical points $c' \in \mathcal{O}$ such that $c' \to \gamma$ (they have multiplicity $k(c')$ in $F^{-1}\gamma$), and of some divisor B_γ in $F^{-1}\mathcal{O} \backslash \mathcal{O}$. Therefore,

$$\begin{aligned} H_\gamma(F(z)) &= \prod_{a \in F^{-1}\gamma} (z - a) \\ &= \frac{H_\gamma(z)}{z - c_\gamma} \prod_{c' \to \gamma} (z - c')^{k(c')} \prod_{b \in B_\gamma} (z - b). \end{aligned}$$

If $\gamma = \gamma_{\text{final}}$, we have the same formula but without the term $(z - c_\gamma)$.

Define

$$H = \prod_{\gamma \in \Gamma} |H_\gamma|^{\alpha(\gamma)}.$$

Then H has the the form (28) and

$$\begin{aligned} Q(z) &\equiv \frac{H(F(z))}{G(z) H(z)} \\ &= \prod_{\gamma \in \Gamma} \prod_{\gamma' \to \gamma} |z - c'|^{k(c')[\alpha(\gamma) - \check{\alpha}(\gamma')]} \prod_{\gamma \in \Gamma} \prod_{b \in B_\gamma} |z - b|^{\alpha(\gamma)} \end{aligned}$$

has the form (29). Moreover, the choice (31) implies that if rank $\gamma > 0$, then there is γ' such that $\gamma' \to \gamma$ and $c' \in Q^{-1}(0)$. Thus we can find a critical point $c^* \in C\backslash\mathcal{O}^+$, i.e. γ of rank zero, such that the orbit $\mathcal{O}^*$ of c^* does not intersect $Q^{-1}(0)$. This implies (30): if $\Sigma \neq 0$, then $\Sigma \cap \mathcal{O}^* \neq \emptyset$ (because $F\Sigma \subset \Sigma$), and since

$$\left(F^{-1}\Sigma\right) \cap \mathcal{O}^* \subset F^{-1}\Sigma\backslash Q^{-1}(0) \subset \Sigma,$$

we have $c^* \in \Sigma$, which contradicts the assumption $\Sigma \subset \mathcal{O}^+$. □

The condition (30) is a version of the condition $(\mathcal{X})$ in Theorem 3.2. In fact, we have the following.

5.2.

LEMMA. *Let F be a polynomial with connected, locally connected Julia set, and let $\varphi : \mathbb{D}_- \to \Omega_F$ be the corresponding conformal map. Then for any $A \subset J$ and $S \subset \mathbb{T}$, we have*

$$S = T^{-1}S\backslash\varphi^{-1}A \Rightarrow \Sigma = F^{-1}\Sigma\backslash A, \quad (\Sigma \equiv \varphi S).$$

Proof. By the monodromy theorem applied to the inverse branches of F on the outer ray $\{\varphi(r\eta) : r > 1\}$, we have

$$b \in J,\ \eta \in \mathbb{T}, \quad Fb = \varphi(\eta) \Rightarrow b \in \varphi\left(T^{-1}\eta\right).$$

Consequently,

$$F^{-1}\varphi(S) = \varphi\left(T^{-1}S\right),$$

and if $S = T^{-1}S\backslash\varphi^{-1}A$, then

$$\begin{aligned}\Sigma = \varphi(S) &= \varphi\left(T^{-1}S\backslash\varphi^{-1}A\right)\\ &= \varphi\left(T^{-1}S\right)\backslash A\\ &= F^{-1}\varphi S\backslash A = F^{-1}\Sigma\backslash A.\end{aligned}$$

□

Now we are ready to apply our results to the $\beta(t)$-spectrum of Julia sets.

5.3.

THEOREM. *Let F be a subhyperbolic polynomial with connected Julia set and $\beta(t)$ be the spectrum defined in §1. Then*

(1) *$\beta(t)$ is real analytic on $(0, +\infty)$;*

(2) *$\beta(t)$ is real analytic on $\mathbb{R}$ if the following condition is satisfied:*

$$\begin{cases}\Sigma \subset CV \cap J\\ \Sigma = F^{-1}\Sigma\backslash C\end{cases} \Rightarrow \Sigma = \emptyset, \qquad (\Sigma)$$

where $C = \operatorname{Crit} F$, $CV = \cap_{n\geq 1} F^n C$.

Proof. By Proposition 1.2,

$$\beta(t) = P\left(-t \log_d g\right)$$

where

$$g = \prod_{c \in C} |\varphi - c|^{k(c)-1}.$$

Represent g in the form

$$g = g_J g_0,$$

where

$$g_J = G \circ \varphi, \quad G(z) = \prod_{c \in J} |z - c|^{k(c)-1}.$$

Applying Lemma 5.1 to G (and to the set $C \cap J$), we obtain functions Q and H satisfying

$$G = \frac{1}{Q} \frac{H \circ F}{H}.$$

Define

$$\tilde{g} = g_0^{-1} Q \circ \varphi, \quad u = H \circ \varphi.$$

Then the functions g, $\tilde{g}$, u satisfy the homological relation (H) in Theorem 3.2 and have the following properties.

(i) They are Hölder continuous (see §2.3).

(ii) g and $\tilde{g} \in (\mathcal{B})$. (By Lemma 5.1 the zeros of G and Q are strictly preperiodic, by Lemma 2.4 this implies $(\mathcal{B})$.)

(iii) $u \in (\mathcal{B}_w)$ (by Lemma 2.3).

(iv) The zero sets of u and $\tilde{g}$ satisfy $(\mathcal{X})$:

$$\begin{cases} S \subset u^{-1}(0) \\ S = T^{-1} S \backslash \tilde{g}^{-1}(0) \end{cases} \Rightarrow S = \emptyset.$$

(We have $u^{-1}(0) = \varphi^{-1}(CV \cap J)$ and $\tilde{g}^{-1}(0) = \varphi^{-1} Q^{-1}(0)$. If $S \subset u^{-1}(0)$, $S = T^{-1} S \backslash \tilde{g}^{-1}(0)$, then

$$\Sigma \stackrel{\text{def}}{=} \varphi S \subset CV \cap J \quad \text{and} \quad \Sigma = F^{-1} \Sigma \backslash Q^{-1}(0),$$

by Lemma 5.2. Then Lemma 5.1 implies $\Sigma = \emptyset$ and $S = \emptyset$.)

(v) There is $\alpha > 0$ such that $\int u^{-\alpha} d\mu < \infty$ for every positive measure μ satisfying $\mu(I) < |I|^{1/2}$ (see Lemma 2.2).

If $t > 0$, then

$$\beta(t) = P\left(t \log_d \tilde{g}\right)$$

by (24), and the analyticity on $\{t > 0\}$ follows from Corollary 4.1.

If, in addition, we assume (Σ), then the above argument shows that the condition $(\mathcal{X})$ is satisfied also for the zero set of g, and by Theorem 4.2 we have the analyticity of $\beta(t)$ on the whole real line. □

6. *Phase transition*

6.1.

LEMMA. *Let F be a subhyperbolic polynomial of degree d with connected, non-smooth Julia set J and let C be a subset of* $\operatorname{Crit} F \cap J$. *Then a non-empty set $\Sigma \subset \mathcal{O}^+(C)$ satisfying*

$$\Sigma = F^{-1}\Sigma \setminus C$$

exists if and only if there are points $c_1, \dots, c_m \in C$ such that

$$\sum_{j=1}^{m} k(c_j) = d - 1$$

and

$$F(c_1) = \cdots = Fc_m = a$$

for some fixed point $a = Fa$.

Proof. Denote $n = \#\Sigma$. Since

$$\Sigma \supset F^{-1}\Sigma \setminus \operatorname{Crit} F,$$

we have

$$n \geq dn - \sum_{Fc\in\Sigma} k(c) \geq dn - 2(d-1), \tag{32}$$

which is possible only if $n = 1$ or 2.

Suppose $n = 2$. Then we must have an equality in (32), which means that all critical points $c_1, \dots, c_{d-1}$ of F are simple and $Fc_j \in \Sigma$. Let $\Sigma = \{a, b\}$, and $c_1, \dots, c_{\nu_1} \to a$, $c_{\nu+1}, \dots, c_{d-1} \to b$, $\nu_1 \leq \nu_2$. Observe that ν_1 and $\nu_2 = d - 1 - \nu_1$ are $\leq d$, and that $F\Sigma \subset \Sigma$. If d is even, $d = 2k$, we then have $\nu_1 = k - 1$, $\nu_2 = k$, $F(a) = a$, $F(b) = a$; that is

$$\begin{array}{ccc} c_1, \dots, c_{k-1} & \twoheadrightarrow & a \;\circlearrowleft \\ & & \uparrow \\ c_k, \dots, c_{2k-1} & \twoheadrightarrow & b \end{array} \qquad (\text{if } k > 1).$$

It is easy to see that any such polynomial is conjugate to P_d.

If d is even, a similar argument shows that F is conjugate to either P_d or $-P_d$. Since we assume that the Julia set is not smooth, the case $n = 2$ is ruled out. The statement is obvious for the remaining case $n = 1$. □

6.2.

COROLLARY. *Let F be a subhyperbolic polynomial with connected, non-smooth Julia set. Then the spectrum $\beta(t)$ is real analytic unless F has a fixed point $a \in J$ and critical points $c_1, \dots, c_m$ such that*

$$\begin{cases} F(c_1) = \cdots = F(c_n) = a \\ \sum k(c_j) = d - 1. \end{cases}$$

This result has the following consequences for polynomials of degrees 2, 3 and 4.

If $d = 2$, then a polynomial has a phase transition only if it is conjugate to $z^2 - 2$.
If $d = 3$, then the only (non-smooth) case where we can have a phase transition is

$$c \twoheadrightarrow a \circlearrowleft,$$

i.e.

$$F(z) = (z - c)^2(z - a) + a. \tag{33}$$

Every polynomial of degree 3 is conjugate to a polynomial with coefficient zero at z^2, in which case we have $a = -2c$ in (33). It follows that F has a real analytic spectrum unless it is conjugate to either $z^3 \pm 3z$ or some polynomial in the one-parameter family

$$F_c(z) = (z + 2c)(z - c)^2 - 2c = z^3 - 3c^2 z + 2(c^3 - c).$$

If $d = 4$, then we can have the phase transition only if

$$c \twoheadrightarrow\!\!\!\!\rightarrow a \circlearrowleft$$

(except for Chebychev's case). Reasoning as above, we see that any such polynomial is conjugate to an element of the family

$$F_c(z) = (z + 3c)(z - c)^3 - 3z.$$

For polynomials of degree $d \geq 5$ there are more possibilities. For example, if $d = 5$, then there are two families:

$$c \twoheadrightarrow\!\!\!\!\twoheadrightarrow a \circlearrowleft,$$

and

$$c_1, c_2 \twoheadrightarrow a \circlearrowleft \qquad \text{(a two-parameter family)},$$

that we can suspect of a phase transition.

We are now going to analyse the phase-transition case.

6.3.

THEOREM. *Let F be a subhyperbolic polynomial with connected, non-smooth Julia set. Suppose that there are critical points $c_1, \ldots, c_m$ and a fixed point $a \in J$ and such that*

$$\begin{cases} Fc_1 = \cdots = Fc_m = a, \\ \sum_{j=1}^{m} k(c_j) = d - 1. \end{cases}$$

Denote

$$g = \prod_{c \in \operatorname{Crit} F} |\varphi - c|^{\nu(c)},$$

where

$$\nu(c) = \begin{cases} k(c) - 1 & c \in \operatorname{Crit} F \setminus \{c_1, \ldots, c_m\} \\ \dfrac{k(c)}{\kappa} - 1 & c \in \{c_1, \ldots, c_m\}, \end{cases}$$

and

$$\kappa = \min_{1 \leq j \leq m} k(c_j).$$

Then the function $t \to P(t \log g)$ *is real analytic, and*

$$1 + \beta(-t) = \max\{P(t \log g), t \log_d(\mu/d)\}, \tag{34}$$

where $\mu = |F'(a)|$.

Proof. At least one of the points $c_1, \ldots, c_m$ does not belong to the set

$$C = \{c \in J \cap \operatorname{Crit} F : \nu(c) \neq 0\},$$

and therefore

$$\begin{cases} \Sigma \subset \mathcal{O}^+(C) \\ \Sigma = F^{-1}\Sigma \setminus C \end{cases} \Rightarrow \Sigma = \emptyset. \tag{35}$$

Otherwise, by Lemma 6.1 we would have another set $\{c_{m+1}, \ldots, c_{m+l}\}$ of critical points such that

$$\sum_{k+1}^{m+l} k(c_j) = d - 1,$$
$$F(c_{m+1}) = \cdots = F(c_{m+l}) = b = F(b).$$

Then $\sum_{k=1}^{m+l} k(c_j) = 2d - 2$, which means that all critical points are simple, and $m = l = (d-1)/2$. It is easy to see that F is then conjugate to a Chebychev polynomial.

As in Theorem 5.3, this proves that the pressure function $t \to P(t \log g)$ is real analytic and it remains to verify the formula (34). The case $t \leq 0$ is easier, so we will concentrate on positive ts. Since

$$\begin{aligned} \prod_{c \in \operatorname{Crit} F} |z - c|^{k(c)-1} &= g \left[\prod_{j=1}^{m} |z - c_j|^{k(c)} \right]^{1-\frac{1}{\kappa}} \\ &= g \left[\frac{|F(z) - a|}{|z - a|} \right]^{1-\frac{1}{\kappa}}, \end{aligned}$$

we have

$$1 + \beta(-t) = P(t\theta),$$

where

$$\theta = \log g + h \circ T - h,$$

and

$$h = \left(1 - \frac{1}{\kappa}\right) \log |\varphi - a|.$$

Let $\lambda(t)$ denote the spectral radius of the Perron–Frobenius operator L_{g^t} and let ν_t denote the corresponding probability measure (see §3). By Lemma 3.3, for any interval of rank n we have

$$e^{t(\log g_n)_I} \asymp \lambda(t)^n \nu_t(I).$$

Therefore,

$$P(t\theta) = \log_d \lambda + \lim_{n \to \infty} \frac{1}{n} \log_d X_n(t),$$

where

$$X_n(t) = \sum_{\operatorname{rank} I = n} e^{-th_I} \nu_t(I). \tag{36}$$

To compute $X_n(t)$, we fix a small neighbourhood of the (finite) set $\varphi^{-1}a$. It is clear that the contribution to the sum (36) coming from the intervals which do not intersect this neighbourhood is $\asymp 1$. Let us estimate the contribution from the intervals that are close to some point $\zeta_o \in \varphi^{-1}a$.

Fix n and let $\mathcal{J}_k$ denote the family of rank n intervals that intersect the d^{-k}-neighbourhood of ζ_o. Define the intervals

$$I_n, I_{n-1}^{\pm}, \ldots, I_{k_o}^{\pm} \tag{37}$$

by

$$\begin{aligned} I_n &= \bigcup\nolimits_{I \in \mathcal{J}_n} I, \\ I_k^+ \sqcup I_k^- &= \bigcup\nolimits_{I \in \mathcal{J}_k \setminus \mathcal{J}_{k+1}} I, \quad (k_o \le k < n). \end{aligned}$$

If $\tilde{I}$ is one of the intervals (37) and $I \subset \tilde{I}$, then by Lemma 2.1,

$$|h_{\tilde{I}} - h_I| \le \text{const}.$$

Consequently,

$$\sum_{I \subset \operatorname{nbhd}(\zeta_o)} e^{-th_I} \nu_t(I) \asymp \sum_{\tilde{I} \in (37)} e^{-th_{\tilde{I}}} \nu_t(\tilde{I}).$$

Futhermore, if $|\tilde{I}| \asymp d^{-k}$, then

$$\begin{aligned} e^{-th_{\tilde{I}}} \nu_t(\tilde{I}) &\asymp \lambda(t)^{-k} e^{t(S_k\theta)_{\tilde{I}}} \\ &\asymp \lambda(t)^{-k} |\varphi'(z_{\tilde{I}})|^{-t} \end{aligned}$$

(see §1.2). We also have

$$\begin{aligned} |\varphi'(z_{\tilde{I}})| &\asymp |\varphi'(r_k\zeta_o)|, \quad (r_k = 1 + d^{-k}), \\ &\asymp \frac{\operatorname{dist}(\varphi(r_k\zeta_o), J)}{r_k - 1} \\ &\overset{\text{(John)}}{\asymp} \frac{|\varphi_k(\zeta_o) - a|}{r_k - 1} \\ &\overset{\text{(Lemma 2.1)}}{\asymp} \left(\frac{d}{\mu}\right)^k. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\tilde{I} \in (37)} e^{-th_{\tilde{I}}} \nu_t(\tilde{I}) &\asymp \sum_{k=k_o}^{n} \lambda(t)^{-k} \left(\frac{\mu}{d}\right)^{kt} \\ &\asymp \max\left\{1, \left[\left(\frac{\mu}{d}\right)^t \frac{1}{\lambda(t)}\right]^n\right\}, \end{aligned}$$

and

$$P(t\theta) = \max\left\{\log_d \lambda(t), t \log_d \frac{\mu}{d}\right\}.$$

□

Theorem 6.3 makes it possible to state some conditions for the phase transition in terms of the multipliers of periodic points. Recall that for $b \in \operatorname{Fix} F^p$ we denote

$$\mu(b) = |(F^p)'(b)|^{1/p}.$$

6.4.
COROLLARY 1. *A subhyperbolic polynomial with connected, non-smooth Julia set has a phase transition if and only if there is a fixed point a and positive number δ such that*

$$b \in \operatorname{Per} F, \quad b \neq a \Rightarrow \mu(b) \leq \mu(a) - \delta.$$

Proof. Suppose F has a phase transition. Then F satisfies the hypothesis of Theorem 6.3 and we can use the same notations. We have

$$\left.\frac{dP(t \log g)}{dt}\right|_{t=\infty} < \log_d \frac{\mu(a)}{d},$$

i.e.

$$P(t \log g) \leq t\left(\log_d \frac{\mu(a)}{d} - \delta\right)$$

for some $\delta > 0$ and all $t \geq t_o$. This implies

$$\lambda(t) \leq d^{-\delta t}\left(\frac{\mu(a)}{d}\right)^t, \quad (t \geq t_o).$$

On the other hand, if $b \neq a$ is a fixed point of F^p and $b = \varphi(\eta)$, then

$$d^{-pk}|(F^p)'(b)|^k = g_{pk}(\eta),$$

and

$$\begin{aligned}\lambda(t) &\asymp \left(\sum_{\operatorname{rank} I = kp} \max g_{kp}^t\right)^{1/kp} \\ &\geq (g_{kp}^t(\eta))^{1/kp} = d^{-t}\mu(b)^t.\end{aligned}$$

Hence, $\mu(b) \leq d^{-\delta}\mu(a)$. The converse statement follows from Corollary 2.5. □

COROLLARY 2. *Suppose F satisfies the hypothesis of Theorem 6.3. Then for the phase-transition case, it is:*
(1) necessary *that $\mu(a) > \mu(b)$ for every fixed point $b \neq a$, and*
(2) sufficient *that $\|g\|_\infty < \mu(a)/d$.*

COROLLARY 3. *Let F be a subhyperbolic polynomial with connected, non-smooth Julia set. Then*
(1) *for any $a \in \operatorname{Per} F \setminus \operatorname{Fix} F$ and any $\varepsilon > 0$, there is a periodic point b such that*

$$\mu(b) > \mu(a) - \varepsilon,$$

(2) *for any $a \in \operatorname{Per} F \cap J$ and any $\varepsilon > 0$, there is a periodic point b such that*

$$\mu(b) < \mu(a) + \varepsilon.$$

6.5. *Cubic polynomials.*

THEOREM. *If F is a subhyperbolic polynomial of degree 3 with connected, non-smooth Julia set, then $\beta(t)$ is real analytic.*

Proof. We know that the phase-transition case can occur only within the family

$$F_c(z) = (z + 2c)(z - c)^2 - 2c.$$

In this case the critical points are $\pm c$, the fixed points are

$$a = -2c, \quad b_+ = c + 1, \quad b_- = c - 1,$$

and

$$c \twoheadrightarrow a \rightarrow a.$$

The multipliers of the fixed points are

$$|F'(a)| = 9|c|^2,$$
$$|F'(b_\pm)| = 3|1 \pm 2c|.$$

By Corollary 2, it is sufficient to show that if the Julia set J of F_c is connected, then either

$$|1 + 2c| \geq 3|c|^2 \quad \text{or} \quad |1 - 2c| \geq 3|c|^2.$$

If J is connected, then $\operatorname{diam} J \leq n$ (because the logarithmic capacity of J is one). Since J contains the points $a = -2c$ and

$$F_c^2(-c) = 4c^5(4c^2 - 3)^2 - 2c,$$

we have

$$\operatorname{diam} J \geq 4|c|^5|4c^2 - 3|^2.$$

The assertion now follows from Figure 3 where the inner curve represents the locus

$$\{c : |c|^5|4c^2 - 3|^2 = 1\},$$

and the outer curve is the boundary of the domain

$$\{c : |1 \pm 2c| < 3|c|^2\}.$$

□

6.6. *An example in degree 4.*

Claim. The polynomial

$$F(z) = (z - c)^3(z + 3c) - 3c$$

with

$$c = \frac{1 + i\sqrt{2}}{3} \tag{38}$$

is critically finite (hence, subhyperbolic with connected, non-smooth Julia set), and the spectrum $\beta(t)$ has a phase-transition point.

Proof. The polynomial F has two critical points: $c_1 = c$ and $c_2 = -2c$, and the fixed points are:

$$a \stackrel{\text{def}}{=} -3c \text{ and } c + \sqrt[3]{1} \quad \text{(three points)}.$$

We chose (38) to satisfy the equation $F^2 c = 1 + c$. Thus we have

$$c_1 \twoheadrightarrow a \circlearrowleft,$$
$$c_2 \twoheadrightarrow\!\!\!\rightarrow 1 + c \circlearrowleft,$$

and so F is critically finite.

To prove that we have a phase-transition case, we will apply Corollary 2 (see §6.4). Since

$$\frac{|F'(a)|}{d} = 16|c|^3 = \frac{16}{3\sqrt{3}},$$

and

$$\|g\|_\infty = \max_{z \in J} |z + 2c|$$

(see Theorem 6.3), we must show that

$$J \subset B\left(-2c, \frac{16}{3\sqrt{3}}\right).$$

Denote $x = z + 2c$. Then

$$F(z) + 2c = (x - 3c)^3(x + c) - c,$$

and it is sufficient to show (since $16/3\sqrt{3} > 3$) that

$$|x| = 3 + \varepsilon,\ \varepsilon > 0 \Rightarrow \left|(x - 3c)^3(x + c) - c\right| \geq 3 + 2\varepsilon.$$

In fact, we have

$$\begin{aligned} \left|(x - 3c)^3(x + c) - c\right| &\geq \left||x| - 3|c|\right|^3 \left||x| - |c|\right| \\ &\geq \left|3 + \varepsilon - 3 \cdot \frac{1}{\sqrt{3}}\right|^3 \left|3 + \varepsilon - \frac{1}{\sqrt{3}}\right| \geq 3 + 2\varepsilon. \end{aligned}$$

□

REFERENCES

[B] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics 470.* Springer, Berlin, 1975.

[CJ] L. Carleson and P. W. Jones. On the coefficient problems for univalent functions and conformal dimension. *Duke Math. J.* **66** (1992), 169–206.

[HK] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** (1982), 119–140.

[Ma] N. G. Makarov. Probability methods in the theory of conformal mappings. *Leningrad Math. J.* **1** (1990), 1–56.

[M] J. Milnor. Dynamics in one complex variable: introductory lectures SUNY. *Stony Brook Institute for Mathematical Sciences, Preprints* **5** (1990).

[P] M. Pollicott. A complex Ruelle–Perron–Frobenius theorem and two counterexamples. *Ergod. Th. & Dynam. Sys.* **4** (1984), 135–146.

[Pom] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps.* Springer, Berlin,1992.

[R1] D. Ruelle. *Thermodynamic Formalism.* Addison-Wesley, Reading, MA, 1978.

[R2] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.* **2** (1982), 99–107.

[Ry] M. Rychlik. Bounded variation and invariant measures. *Studia Math.* **LXXVI** (1983), 69–80.