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# Symbolic Dynamics and Collet-Eckmann Conditions

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## 1 Introduction

In this paper, we prove that unicritical polynomials with metrically generic combinatorics of the critical orbit satisfy Collet-Eckmann conditions. Here metrically generic means except for a set of Hausdorff dimension zero, and combinatorics can be understood in either of the following senses: Markov partition itineraries, kneading sequences, external angles, or harmonic measure on the Mandelbrot set.

Particularly, except for a set of Hausdorff dimension zero of angles, all external rays for the degree $d$ Mandelbrot set land at parameters $c$ such that polynomial $z^d + c$ is Collet-Eckmann. This statement is in a sense the best possible, and it is much stronger than saying that almost every $c$ with respect to harmonic measure corresponds to a Collet-Eckmann polynomial.

Some of the theorems can be generalized to polynomials with many critical points and rational functions.

### 1.1 Historical perspective

The perspective we take in this paper is to search for properties of a typical nonhyperbolic polynomial or rational function. Many nice properties of hyperbolic polynomials can be generalized to various classes of "weakly hyperbolic" ones, for example, critically preperiodic, Misiurewicz, Collet-Eckmann, etc. For polynomials from all classes mentioned above, Julia sets enjoy regular geometry and have Hausdorff dimension strictly less than 2 (see [5], [10], [25]). On the other hand, there are many nonhyperbolic polynomials

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with "bad" properties, and it makes sense to ask which kind of behavior (nice or bad) is typical.

Of course, there are different ways to define the class of "good" rational functions. We study rational functions satisfying the Collet-Eckmann condition and its variations (see Section 1.3 for the definitions and discussion). It makes sense to study this particular class, since such rational functions enjoy many nice properties (see [10], [11], [23], [25], and [26]) and are typical in many senses. On the other hand, smaller classes of functions, like Misiurewicz maps, turn out to be nontypical in the situations under consideration.

There are also different ways to interpret the word "typical." One possible way is to look at the metrically generic polynomials or rational functions. It was successfully exploited in the following two cases.

### Real quadratics

The theorem of M. Jakobson [14] states that there is a positive Lebesgue measure set of real parameters $c$ such that $z^2 + c$ admits an invariant measure absolutely continuous with respect to Lebesgue measure on the real line. It is interesting that it was shown that there are many nonhyperbolic polynomials by exhibiting a lot of them with nice properties. Moreover, by a theorem of M. Lyubich [18], there is no other way to do it, since for Lebesgue almost every real $c$ quadratic map $z^2 + c$ is either hyperbolic or stochastic (i.e., like those in Jakobson's theorem).

Later, M. Benedicks and L. Carleson extended results of M. Jakobson, proving in [1], [2] that there is a positive Lebesgue measure set of real $c$ such that $z^2 + c$ satisfies the Collet-Eckmann condition. Note that parameters $c$ corresponding to Misiurewicz maps have zero length by a result of D. Sands [28].

### Rational functions

A parallel result for the rational functions is due to M. Rees [27], who showed that there is a positive Lebesgue measure set of rational functions (for a fixed degree, we can parameterize them by $\mathbb{C}^N$), admitting an invariant measure absolutely continuous with respect to Lebesgue area on $\widehat{\mathbb{C}}$, and whose Julia set is the whole complex sphere $\widehat{\mathbb{C}}$. It seems plausible that a positive Lebesgue measure set of those rational functions actually satisfies the Collet-Eckmann condition.

Another possible way is to look at topologically generic polynomials.

### Complex quadratics

In [32], M. Shishikura showed that for topologically generic $c$ in the boundary of the

Mandelbrot set, the Hausdorff dimension of the Julia set $J_{z^2+c}$ is 2, and he deduced that the Hausdorff dimension of the boundary of the Mandelbrot set is 2 as well.

This shows some difficulties in establishing analogues of Jakobson's theorem for complex quadratics: It is not clear what the canonical Hausdorff measure on the boundary of the Mandelbrot set is, and there appears to be many "bad" parameters $c$.

Our original idea was to establish an analogue of Jakobson's result for complex quadratics with respect to harmonic measure on the boundary of the Mandelbrot set. Harmonic measure in this context plays a special role, since a Riemann uniformization map has dynamical meaning (see [4]). Roughly speaking, it measures angles of external rays landing on the Mandelbrot set, and these angles carry information about topological structure of the corresponding Julia sets. These investigations resulted in Theorem 4, and later the project expanded to include an investigation of rational functions such that encoding of the critical orbit is generic with respect to some symbolic dynamics.

### 1.2 Results

Consider the space $\Sigma_p$ of all one-sided $p$-symbol sequences $x = x_1x_2, \ldots, x_j \in \{0, \ldots, p-1\}$ equipped with canonical shift dynamics $T : x_1x_2 \cdots \mapsto x_2x_3 \ldots$. We denote by $n(x, y)$ the minimal number $n$ such that $x_n \neq y_n$ (if no such number exists, we set $n(x, y) := \infty$), then $d(x, y) := p^{-n(x,y)}$ is the canonical $p$-adic metric on $\Sigma_p$.

One method of studying dynamics $F$ on the Julia set is to somehow (semi)conjugate it to shift dynamics and thus introduce symbolic dynamics on the Julia set. We show that three canonical constructions of such conjugation polynomials with a metrically generic itinerary of the critical point are Collet-Eckmann (see 1.3 for the definition). Namely, we consider symbolic dynamics arising from Markov partitions, kneading sequences, and the Riemann uniformization map. Note that all unicritical polynomials with disconnected Julia set are hyperbolic (and hence, Collet-Eckmann), and therefore connectedness conditions can be dropped from the theorems below.

We say that a polynomial $z^d + c$ has critical symbolic dynamics $\sigma$ if there exists a Markov partition (similar to the first level of Yoccoz puzzle, see Section 3 for details) such that itinerary of the critical orbit coincides with the sequence $\sigma$.

**Theorem 1 (Markov partitions).** Let $E_\sigma$ be the set of symbolic sequences $\sigma$ such that there exists a non-Collet-Eckmann polynomial $z^d+c$ with connected Julia set and critical symbolic dynamics $\sigma$. Then the Hausdorff dimension of $E_\sigma$ is equal to zero. □

Kneading sequence is the itinerary of the critical orbit with respect to the partition of the Julia set by the critical point (see Section 2 for details).

**Theorem 2 (Kneading sequences).** Let $E_\kappa$ be the set of kneading sequences $\kappa$ such that there exists a non-Collet-Eckmann polynomial $z^d + c$ with connected Julia set and kneading sequence $\kappa$. Then the Hausdorff dimension of $E_\kappa$ is equal to zero. $\square$

There is an easily described map, carrying an external angle of the critical value (see Section 4) to the corresponding kneading sequence. Its (multi-valued) inverse does not increase Hausdorff dimensions (see Proposition 1), and hence we arrive at the following theorem.

**Theorem 3 (External angles).** Let $E_\theta$ be the set of angles $\theta$ such that there exists a non-Collet-Eckmann polynomial $z^d + c$ with connected Julia set and external ray with an angle $\theta$ landing at the critical value $c$. Then the Hausdorff dimension of $E_\theta$ is equal to zero. $\square$

Let $\mathcal{M}_d$ denote the Mandelbrot set (locus of connectivity, i.e., the set of all $c$ with a Julia set of $z^d + c$ connected) for the family $z^d + c$. There is an intuitive principle, according to which the angle of a (dynamical plane) external ray landing at the critical value $c$ in the Julia set coincides with the angle of a (parameter plane) external ray landing at the point $c$ in the Mandelbrot set. It can be made precise in many situations, and in Section 4 we show that it can be made rigorous enough to conclude the following.

**Theorem 4 (Mandelbrot set).** Let $E_{\mathcal{M}}$ be the set of angles $\theta$ such that a corresponding external ray $R_\theta$ does not land at a Collet-Eckmann point $c \in \mathcal{M}_d$ (i.e., corresponding to a Collet-Eckmann polynomial $z^d + c$). Then Hausdorff dimension of $E_{\mathcal{M}}$ is equal to zero. $\square$

The intuitive meaning of this theorem is that looking at the Mandelbrot set from outside, one sees only Collet-Eckmann parameters. This statement is stronger than saying that almost all with respect to harmonic measure points on the boundary of the Mandelbrot set are Collet-Eckmann. (A similar theorem is also proven by J. Graczyk and G. Świątek by different methods in [12]. Earlier, J. Graczyk, G. Świątek, and the author showed that almost all points satisfy the weaker summability condition.) Our theorem is in a sense the best possible: infinitely renormalizable polynomials cannot satisfy the Collet-Eckmann condition by [11], and it was shown in [19] that the corresponding set of angles has positive logarithmic capacity (and even positive Hausdorff measure with gauge function $|\log r|^{-\omega}$ for $\omega \in (1, \log 5/\log 4)$).

Also note that in all the models under consideration, Misiurewicz (i.e., critically nonrecurrent) polynomials are not generic, which is easy to see since the corresponding sequences are also nonrecurrent (i.e., some block of digits appears only in the beginning).

Remark 1 (Multicritical case). We abstain from working with multicritical polynomials and rational functions in order to obtain more elegant formulations of our theorems. However, the same methods would prove the analogues of Theorems 1 and 2 in the multicritical case as well.

Roughly speaking, we can show that there is a set $E$ of symbolic sequences of Hausdorff dimension zero, such that if there is a partition of a Julia set by repelling periodic points, critical points, and finitely many of their preimages, and the corresponding rational function fails the topological Collet-Eckmann condition, then itinerary of *at least one* of the critical points should belong to $E$.

### 1.3 Variations of the Collet-Eckmann condition

The classical Collet-Eckmann condition asserts that expansion on the critical orbits grows exponentially.

Definition 1 (CE). We say that a rational function satisfies *the Collet-Eckmann condition* if there exist constants $K, Q > 1$ and $C > 0$ such that for every positive integer $n$ and every critical point $c$ belonging or accumulating to the Julia set, we have

$$\left|(F^n)'(F^K c)\right| \geq CQ^n.$$

We mostly work with the following related definition, which involves only topological (not metrical) considerations and repeats one in Section 4 of [24]. Denote by $\mathrm{Comp}_a\, F^{-j}B$ the component of connectivity of the preimage $F^{-j}B$ containing the point $a$, and denote by Crit the set of the critical points of $F$.

Definition 2 (Topological Collet-Eckmann condition). We say that a rational function satisfies the *topological Collet-Eckmann condition* (*TCE*), if for some $P > 1$, there exist $M > 0$ and $r > 0$ such that for every $x \in J$, there is a sequence of increasing integers $\{n_j\}$ with $n_j \leq Pj$ and

$$\#\left\{i : 0 \leq i < n_j, \mathrm{Comp}_{F^i(x)}\, F^{-n_j - i} B\left(F^{n_j}(x), r\right) \text{ intersects Crit}\right\} \leq M.$$

This condition is also called *finite criticality*, and it means that for large (i.e., $\geq 1/P$) proportion of the iterates $F^i(x)$, a ball of the radius $r$, when pulled back along the corresponding branch of $F^{-i}$, hits critical points at most $M$ times. Sometimes (e.g., in [26]), it is formulated by requiring a fixed proportion of times $i$ to be "good," that is, to have criticality bounded by $M$.

In [25], it is shown that the Collet-Eckmann condition always implies TCE, and Julia sets of TCE rational functions enjoy nice geometry. Furthermore, for unicritical polynomials, TCE and Collet-Eckmann conditions are equivalent, as was shown by F. Przytycki in [24]. But it is not the case for multicritical ones (see [26]).

## 2 Kneading sequences

When the Julia set $J$ of a polynomial $F(z) = z^d + c$ with no attracting or indifferent cycles is connected and locally connected, there is an external ray landing at the critical value $c$ (consult Section 4 for the definition). A priori there can be several of them, generating different kneading sequences, which all correspond to our polynomial. Preimages of this ray are $d$ external rays landing at the critical point 0 and cutting the Julia set (without 0) in $d$ “kneading components” $K_1, K_2, \dots, K_d$. For a point $x \in J$, the *kneading sequence* $\kappa(x)$ is defined as the itinerary of the orbit of $x$ with respect to this partition.

We want to be able to work with non-locally connected Julia sets, so we generalize the definition above, employing terminology of *fibers*, as defined in [30]. Consider a polynomial $F(z) = z^d + c$ with no (super)attracting or indifferent cycles and connected Julia set. Fiber of a point $a \in J$ is the union of all points in $J$ that cannot be separated from $a$ by external rays landing at periodic or preperiodic points. (Note that in our case, all periodic points are repelling and there is only one Fatou component.) Denote by $G$ the fiber of the critical point 0. The Julia set $J$ has $d$-fold rotational symmetry around 0, and removing $G$ from $J$ splits the latter in components of connectivity in a symmetric way. Moreover, any point $x \in J \setminus G$ and its $(d-1)$-symmetric counterparts belong to different components. Hence, we can construct $d$ rotationally symmetric disjoint kneading components $K_1, K_2, \dots, K_d$ that are closed inside $J \setminus G$ and whose union is $J \setminus G$.

Then for every $x \in J$, we can write the itinerary $\kappa(x) = \kappa_F(x) \in \Sigma_d$, where $n$th term is equal to $i$ if and only if $F^n(x) \in K_i$. There is a possibility (realized only for points in the preimages of $G$) that one of the iterates of $x$ is inside $G$ (or coincides with 0 in the locally connected situation), and then we just put an arbitrary digit in the corresponding position of $x$ (traditionally the symbol $*$ is written)—it does not affect our reasoning, since for any of the exceptional $x$'s, it happens only once. In fact, otherwise $G$ would be periodic (images of fibers are fibers, and intersecting fibers coincide), which cannot happen since by [30, Lemma 3.8], the polynomial $F$ would have an indifferent or (super)attracting cycle in this case.

The sequence $\kappa(F) := \kappa_F(0)$ is called the kneading sequence of the polynomial $F$. (A priori it depends on the choice of the partition, but this does not affect our statements.) Note also that for a fixed polynomial $F$, the kneading map $\kappa : x \mapsto \kappa_F(x)$ conjugates dynamics $F$ on the Julia set with the shift dynamics $T$ on $\Sigma_d$: $\kappa(F(z)) = T(\kappa(z))$.

We do not consider polynomials $F$ with attracting or indifferent cycles so that the critical point belongs to the Julia set, and our definition of the kneading sequence works. Nevertheless, there is a logical way to define the kneading sequence for such polynomials as well, but such kneading sequences are periodic (see the discussion in Section 4). The set of periodic sequences is countable and of Hausdorff dimension zero, so it does not affect our statements.

Proof of Theorem 2. The proof has two parts: First, we show that if $F$ fails the TCE condition for some $P > 1$ (with any $M > 0$, $r > 0$), then any kneading sequence of $F$ satisfies the strongly recurrent condition (SR). In the second part of the proof, we estimate the size of the set of SR sequences in $\Sigma_d$.

We say that $u = u_1u_2 \cdots \in \Sigma_d$ is SR if the proportion of $u_1 \cdots u_N$ that duplicates long beginnings of $u$ (with the exception of a rare sequence of digits) can be arbitrarily close to 1 as $N$ grows, namely, if for every $D > 0$ there is a $D$-rare set $\mathcal{R} \subset \mathbb{N}$ such that

$$\limsup_{n\to\infty} \#\{\text{duplicating } i : 0 < i \leq n\}/n = 1.$$

Here, for a fixed $n$, the number $i$ is called *duplicating* if for some $l < i$ and $l' \geq D$ with $i \in (l, l + l']$, the digits of $u$ in positions $(l + 1), (l + 2), \ldots, (l + l')$ duplicate first $l'$ digits of sequence $u$ with the possible exception of positions in the $D$-rare set $\mathcal{R}$. We say that a set $\mathcal{R}$ is $D$-*rare* if for every $j \in \mathbb{N}$, we have $\#\mathcal{R} \cap [j, j + D] \leq 2$.

Part 1: Non-TCE kneading sequences satisfy SR

The set of periodic or eventually periodic kneading sequences has zero Hausdorff dimension since there are only countably many of them; thus, we can exclude them from consideration. Moreover, periodic kneading sequences obviously satisfy SR and if $\kappa(F)$ is eventually periodic, but not periodic, then the corresponding polynomial $F$ is nonrecurrent and hence TCE. Therefore, we can assume that $\kappa(F)$ is not eventually periodic. In particular, this means that $G$ is wandering (i.e., not periodic or eventually periodic).

First, we show that if orbits of two points are close, then so are their itineraries. Namely, we prove the following lemma.

**Lemma 2.1.** Let $F$ be a polynomial as above. For every $L > 0$, there is $r > 0$ such that if points $x, y \in J$ belong to one component of connectivity of $F^{-n}(B_r)$ for some ball $B_r$ of radius $r$, then first $n$ digits of their itineraries coincide except for a sequence $\{k_j\}$ of positions with $k_{j+1} - k_j > L$. □

Remark 2. If the Julia set is locally connected, this lemma is trivial modulo *backward Lyapunov stability*, which holds by [17 , Corollary 1].

Proof. First consider $L$ iterates of the critical point: $\{F^j(0)\}_{j=0}^L$. We assume that fiber $G$ of the critical point is wandering; therefore, these iterates belong to different fibers and we can separate them by a finite collection $\Gamma$ of external rays landing at preperiodic points. Adding several more rays, we can assume $\Gamma$ to be forward-invariant. Note that $\Gamma$ depends on $L$, and all further parameters, including $r$, depend on $\Gamma$.

By our construction intersection of the closures of two different kneading components, $K_i$, $K_j$ lies inside $G$, and $\Gamma$ does not intersect the latter set. Hence, if $\delta > 0$ is the distance from $G$ to $\Gamma$, then $\delta$-neighborhood of any point in $\Gamma$ intersects at most one kneading component.

Next we choose $r$ so small that for any ball $B_r$ intersecting the Julia set, if for some positive integer $l$ a component of connectivity $f^{-l}(B_r)$ of its preimage intersects $\Gamma$, then $\operatorname{diam}(f^{-l}(B_r)) < \delta$. To do so, denote by $\Gamma'$ the union of (repelling) cycles in $\Gamma$, and note that a ball $B_r$, one of whose preimages intersects $\Gamma$, also must intersect $\Gamma$. There are finitely many branches of $F^i$ mapping points of $\Gamma \setminus \Gamma'$ into $\Gamma'$, and we can choose $r'$ so small that pulling back $B_{r'}$, intersecting $\Gamma'$ along one of these branches produces a set of diameter less than $\delta$. Without loss of generality (consider an iterate of $F$), all cycles in $\Gamma'$ are fixed points. Then we can choose $r'' < r'$ so small that pulling back by a branch of $F$, preserving fixed points $a \in \Gamma'$ decreases the diameter of the ball $B_{r''}(a)$. Finally, choose $r < r''$ so small that any ball of radius $r$ intersecting $\Gamma$ and the Julia set must be within distance $r$ from the preperiodic points in $\Gamma$. This $r$ is clearly the desired one.

Now suppose that the lemma fails for this choice of $r$, some ball $B_r$, two points $x, y \in F^{-n}(B_r)$, and positions $i, j$ with $j < i < j + L$. Then $F^{i-n}(B_r)$ must intersect $\Gamma$ and the Julia set, hence its diameter is smaller than $\delta$, it intersects only one kneading component, and $i$th positions of the itineraries of $x$ and $y$ coincide, thus proving the lemma. ∎

Now we show that if a polynomial $F$ fails TCE, its kneading sequence is SR. To do so, we must prove the following lemma.

**Lemma 2.2.** For every non-TCE polynomial $F$ with connected Julia set, and arbitrary $D > 0$, $\tau < 1$, there are time $n \geq D$ and a $D$-rare set $\mathcal{R} \subset \mathbb{N}$ so that for the kneading sequence $\kappa(F)$,

$$\#\{\text{duplicating } i : 0 < i \leq n\} > \tau n.$$

Note that we actually obtain arbitrarily big $n$. (Applying lemma with larger and larger $D \gg D_0$, we get arbitrarily large $n \geq D \gg D_0$.) □

Proof. To prove the lemma, fix a polynomial $F$ with connected Julia set (failing TCE with a fixed $P > 1$) and constants $D > 0$, $\tau < 1$.

Choose $r > 0$ according to the Lemma 2.1 applied with $L = 2D$ and $M > 2D$ so large that $(1-\tau)(M/2)((P-1)/P) > 1$. Since $F$ fails TCE (with parameters $P$, $M$, $r$), there exist $x \in J$ and $N > 0$ such that at least $((P-1)/P)N$ integers $l \in [1, N]$ satisfy

$$\#\left\{i : 0 \le i < l, \mathrm{Comp}_{F^i(x)} F^{i-l} B\left(F^l(x), r\right) \ni 0\right\} > M.$$

For every $j \le N$, define $j'$ as the largest number $j' \le N$, such that

$$\mathrm{Comp}_{F^j(x)} F^{j-j'} B\left(F^{j'}(x), r\right) \ni 0.$$

If such a number exists, we add the interval $[j, j']$ to the collection $\mathcal{I}$.

For every $j$, there is at most one interval $I$ with such a beginning, and the fact that $F$ fails TCE as specified above means that at least $((P-1)/P)N$ integers in $[1, N]$ are covered by the collection $\mathcal{I}$ at least $M$ times. Removing all intervals of length less than $D$ from $\mathcal{I}$, we obtain the new collection $\mathcal{I}'$, which covers at least $((P-1)/P)N$ integers in $[1, N]$ at least $(M - D) > M/2$ times, and therefore,

$$\sum_{I \in \mathcal{I}'} |I| \ge \frac{P-1}{P} N \frac{M}{2} > \frac{N}{(1-\tau)}, \tag{1}$$

where $|I|$ denotes the number of integers in $I$.

Note that for every interval $I = [i, i'] \in \mathcal{I}'$, we can apply Lemma 2.1 to points $F^i(x)$ and $0$, deducing that first $(i' - i)$ digits in the itinerary of $0$ duplicate digits in the $(i+1), (i+2), \ldots, i'$th positions of the itinerary of $x$, except for a set $\mathcal{S}_I$. Note that the difference of any two numbers in the latter set is bigger than $2D$ by the choice of $L$ and Lemma 2.1. It follows that for any two intervals $[i, i']$, the $[j, j'] \in \mathcal{I}'$ part of the itinerary of $0$ starting at $|j - i|$ duplicates the beginning of the same itinerary, except for positions in $\mathcal{S}_I \cap \mathcal{S}_J$.

Therefore, if there is an interval $I = [i, i'] \in \mathcal{I}'$, such that at least $\tau|I|$ integers in $I$ are covered by further right intervals in $\mathcal{I}'$ (which is canonically ordered by left ends), then we can finish the first part of the proof. In fact, from such a cover, we can choose a subcover $\mathcal{J}$ that covers every point at most twice (by excluding redundant intervals). We take $(i' - i)$ to be the desired $n$. The set $\mathcal{R}$ can then be defined as the union $\cup_{J \in \mathcal{J}} \mathcal{S}_J \cup \mathcal{S}_I$. The set $\mathcal{R}$ is $D$-rare, since the difference of any two numbers in any set $\mathcal{S}_J$ is bigger than $2D$, as was noted above.

Suppose that this is not so, and for every $I \in \mathcal{I}'$, at most the $\tau|I|$ part of it is covered by further right intervals. But if some $l \in [1, N]$ is covered by $m(l)$ intervals from $\mathcal{I}'$, then for only one of them (the rightmost one), this number $l$ is not covered by further right intervals. So we arrive at

$$\begin{aligned}\sum_{I\in\mathcal{J}'}|I| &= \sum_{l=1}^{N} m(l) = \sum_{l=1}^{N}\big(m(l)-1\big)+N\\ &\le \sum_{I\in\mathcal{J}'}\#\{i\in I : \text{it is covered by an interval further right of } I\}+N\\ &\le \sum_{I\in\mathcal{J}'}\tau|I|+N,\end{aligned}$$

and we deduce that $(1-\tau)\sum_{I\in\mathcal{J}'}|I|\le N$. The latter contradicts (1), thus completing the first part of the proof. ∎

Part 2: SR kneading sequences have zero Hausdorff dimension

In this part, we forget about complex dynamics and estimate the Hausdorff dimension of the (combinatorially described) set of SR kneading sequences.

Denote by $E_n(\tau, D)$ the set of all $u \in \Sigma_d$ such that there exists a D-rare set $\mathcal{R}$ and

$$\#\{\text{duplicating } i : 0 < i \le n\} \ge \tau n, \tag{2}$$

as in the definition of SR. Clearly,

$$\mathrm{SR} = \bigcap_{\tau<1,\, D>0} \limsup_{n\to\infty} E_n(\tau, D).$$

For $\tau < 1$, $D > 0$, define $\delta = \delta(\tau, D)$ by

$$9\frac{\log D}{D} + (1-\tau)\log d = \delta\frac{\log d}{2}.$$

It is easy to see that $\lim_{\tau\to1,\, D\to\infty}\delta(\tau, D) = 0$, and hence it is sufficient to prove the following lemma.

**Lemma 2.3.** For any $\tau < 1$, $D > 0$, we have $\mathrm{HD}(\limsup_{n\to\infty} E_n(\tau, D)) \le \delta(\tau, D)$. □

Proof. To prove the lemma, fix $\tau < 1$ and $D > 0$, and denote $\delta := \delta(\tau, D)$, $E_n := E_n(\tau, D)$. First, study the size of $E_n$ for some particular $n$. The inequality (2) means that we can choose a collection $\mathcal{J}$ of intervals $[j, j'] \subset [1, n]$ of length greater than $D$, which cover at least $\tau n$ integers in $[1, n]$. Also for $i \in [j, j']$, $i$th and $(i-j)$th positions in $u$ coincide, unless $i$ belongs to a rare set $\mathcal{R}$.

First, we estimate the number of cylinders $\nu = \nu_1 \cdots \nu_n$ of length $n$ needed to cover $E_n$.

Take some element $u$ satisfying the condition above, and choose (by the standard Besikovitch covering argument) from $\mathcal{I}$ a nice subcollection $\mathcal{I}'$ having the same property and such that no integer is covered more than twice (e.g., no interval from $\mathcal{I}'$ being covered by a union of others is sufficient). Then at most two beginnings and at most two ends of intervals from $\mathcal{I}'$ belong to any interval $[i, i + D)$. (If three intervals of lengths more than $D$ have beginnings in the same interval of length $D$, one of them is contained in the union of two others.) Observe that choosing beginnings and ends uniquely determines $\mathcal{I}'$ (easy induction shows that $j$th beginning from the left and $j$th end from the left must correspond to the same interval, otherwise there are two intervals with $I \subset J$), and there are less than $D^4$ possibilities to choose at most two beginnings and at most two ends in each of the $n/D + 1$ intervals $[jD, jD + D)$ for $j \in [0, n/D)$. Thus, there are at most $(D^4)^{n/D+1} < \exp(n5 \log D/D)$ ways to choose $\mathcal{I}'$. Also, in every interval $[i, i + D)$, there are at most 2 numbers of the rare set $\mathcal{R}$, so there are less than $(D + 1)^{2n/D+1} < \exp(n4 \log D/D)$ ways to choose $\mathcal{R}$. Hence, there are at most $\exp(n9 \log D/D)$ ways to choose $\mathcal{I}'$ and $\mathcal{R}$.

Once the nice collection $\mathcal{I}'$ and rare set $\mathcal{R}$ are chosen, the sequence $u$ duplicates itself so extensively that $(1 - \tau - (1/D))n$ digits fully determine the first $n$ digits of $u$. (In fact, for any integer $k$ covered by $\mathcal{I}'$, but not by $\mathcal{R}$, there is an integer $j = j(k, \mathcal{I}', \mathcal{R}) < k$ independent of $u$ such that $u_j = u_k$.) Thus, there are at most

$$\exp\left(n9\frac{\log D}{D}\right) d^{(1-\tau)n} = \exp\left(n\left(9\frac{\log D}{D} + (1-\tau)\log d\right)\right)$$

different ways to choose the first $n$ digits of $u$. Each of these has a corresponding cylinder of diameter $\exp(-n \log d)$.

By the choice of $\delta$, we conclude that it is possible to cover $E_n$ by a collection $\{C_j^n\}_j$ of cylinders with

$$\begin{aligned}\sum_j \operatorname{diam}\left(C_j^n\right)^\delta &< \exp\left(n\left(9\frac{\log D}{D} + (1-\tau)\log d\right)\right) \cdot \exp(-\delta n \log d)\\ &= \exp\left(-\delta\frac{\log d}{2}n\right),\end{aligned}$$

and a union $\cup_{n\geq m} E_n$ by a collection $\{C_j^n\}_{n\geq m, j}$ with

$$\sum_{n\geq m, j} \operatorname{diam}(C_j^n)^\delta < \sum_{n\geq m} \exp\left(-\delta\frac{\log d}{2}n\right) \leq \text{const}\exp\left(-\delta\frac{\log d}{2}m\right).$$

The desired conclusion $\mathrm{HD}(\limsup_{m\to\infty} E_m) \leq \delta$ readily follows. ■

## 3 Markov partitions

In this section, we outline the proof of the Theorem 1. The proof is very similar to that of Theorem 2, so we skip many details.

### Markov partitions

Suppose that the Julia set except for a set $X$ is covered by a union of disjoint, open, connected domains $A_j$, $1 \le j \le p$, so that for every $j$, every component of connectivity of the preimage $F^{-1}A_j$ is inside one of the domains $A_l$. We consider only *Markov partitions* of $J$ when $X$ is a finite set containing only repelling cycles or their preimages. Constructions of the Yoccoz puzzle (see [21], [13]), and the fixed point portraits (see [8], [9]) start with such a partition.

Then every point in the Julia set, except countably many ($X$ and its preimages), can be encoded by a sequence from $\Sigma_p$. Namely, we can define a map $\phi : J \setminus \cup_{j\ge 0} F^{-j}X \to \Sigma_p$ by $z \mapsto x_1 x_2 \ldots$, where $x_j = k$ when $F^j(z) \in A_k$. Clearly, when so constructed, $\phi$ conjugates dynamics $F$ to shift dynamics $T$.

The proof of Theorem 1 (like Theorem 2) has two steps: First, we show that if $f$ fails the TCE condition, then the symbolic sequence $\phi(0)$ satisfies the modified strongly recurrent condition (SR′). In the second part of the proof, we estimate the size of the set of SR′ parameters.

We say that $u \in \Sigma_p$ is *SR′* if the proportion of $u_1 \cdots u_N$ that duplicates long beginnings of $u$ or one of a few periodic sequences can be arbitrarily close to 1 as $N$ grows, namely, if there exist $T < \infty$ periodic sequences $v^1, \ldots, v^T$, such that for any $D > 0$,

$$\limsup_{n\to\infty} \#\{\text{duplicating } i : 0 < i \le n\}/n = 1.$$

Here the number $i$ is called *duplicating* if for some $l < i$ and $l' \ge D$ with $i \in (l, l+l']$, the digits of $u$ in positions $(l+1), (l+2), \ldots, (l+l')$ duplicate first $l'$ digits of one of the sequences $u, v^1, \ldots, v^T$.

### Part 1: Non-TCE parameters satisfy SR′

Near repelling points in $X$ dynamics $F$ just permutes $A_j$, so there exist periodic sequences $v^1, \ldots, v^T$ in $\Sigma_p$ such that while the orbit of a point $z$ stays in some small $R$-neighborhood $X_R$ of $X$, it visits $A_j$ in the corresponding order.

Taking $R$ to be small, we can assume that any component of $F^{-1}X_R$, which is disjoint from $X$, is contained in $A_j$ for some $j$, and that branch of $F^{-1}$ on $X_R$, which preserves $X$, is contracting.

Take $r$ small enough so that $z, z' \in J, |z-z'| < r$ implies that $z$ and $z'$ either belong with their $r$-neighborhood to the same domain $A_j$ for some $j \in [1,p]$, or are both inside $X_R$. Furthermore, we take $r$ so small that if a point $z$ gets inside $r$-neighborhood of $X$, its images under $D'$ iterations remain inside $X_R$.

After that, we can follow the proof of Theorem 2 once the following analog of Lemma 2.1 is proven.

**Lemma 3.1.** If $|F^j x - F^j y| < r$ and $x, y$ belong to the same component of connectivity of $F^{-j}B_r(x)$, then there exists $i \leq j$ such that first $i$ digits of the sequences $\phi(x)$ and $\phi(y)$ coincide, and their digits in the positions $(i+1), \dots, j$ duplicate first $(j-i)$-digits of the periodic sequences $\nu^A$ and $\nu^B$ correspondingly for some $A, B \in [1, T]$. □

Proof. Take the maximal $i \leq j$ so that $F^i(x), F^i(y)$ do not belong to the $r$-neighborhood of $X$. Then the second part of the statement is clear. To prove the first part, note that the component of connectivity of $F^{-1}X_R$, containing $F^i(x)$, contains also the corresponding component of $F^{i-j}B_r(f^j(x))$ and is inside $A_l$ for some $l$. Pulling back, we obtain the first claim. ■

Part 2: SR′ parameters have zero Hausdorff dimension

If sequences $\nu^1, \dots, \nu^T$ are fixed, we can repeat the analogous arguments from the proof of Theorem 2 and obtain that the Hausdorff dimension of the set of SR′ elements of $\Sigma_p$ with such periodic sequences is zero. But there are only countably many different ways to choose a positive integer $T$ and $T$ periodic sequences $\nu^1, \dots, \nu^T$, and zero dimension is preserved under countable unions, hence the Hausdorff dimension of the set of SR′ sequences is zero.

## 4 External angles

For a connected Julia set and an angle $\theta$, we introduce an *external ray* (or *dynamical ray* in order to distinguish from parameter rays, defined below) $R_\theta$, which is Green's line starting at infinity with the angle $\theta$, or, equivalently, the image of the radius $\{re^{i\theta} : 0 < r < 1\}$ under the Riemann uniformization map $\phi$ from the unit disc to the domain of attraction to infinity $A_\infty$. The latter is chosen so that it conjugates dynamics $z \mapsto z^d$ on the unit disc to $F$ on $A_\infty$. Note that wherever $\phi$ extends to the boundary, it semiconjugates $z \mapsto z^d$ on the unit circle to $F$ on the Julia set.

Similarly, we define external rays in the parameter space, or simply *parameter rays*, as Green's lines for the set $\mathcal{M}_d$, the latter being the set of all $c \in \mathbb{C}$ for which the Julia set of $z^d + c$ is connected.

### 4.1 From kneading to angles

Consider the unit circle (which we parameterize by angles and identify with the real line modulo 1) with dynamics $T : \theta \mapsto d\theta$. Points $(\theta + j)/d$, $j = 1, \dots, d$ split the circle in $d$ arcs. Hence, we can prescribe to an angle $\theta$ its kneading sequence $\Psi(\theta) \in \Sigma_d$, which is the itinerary of the orbit $\{T^n(\theta)\}_{n=1}^{\infty}$. If $T^n(\theta)$ coincides with one of the separation points, we write $*$ in the corresponding position. The latter can happen only for countably many $\theta$, which we can exclude from consideration without affecting dimension estimates.

Suppose that critical value $c$ of a polynomial $F(z) = z^d + c$ is in the accumulation set of an external ray $R_\theta$ (or, more generally, if $R_\theta$ accumulates inside the fiber of $c$). Then for any $n$, its image $F^n(c)$ is in the accumulation set of the external ray $R_{T^k\theta}$. Removing from the plane $d$ external rays with angles $(\theta + j)/d$, $j = 1, \dots, d$ and fiber of the critical point $0$ (which contains accumulation sets of those rays, see [30]) splits the plane into $d$ domains in a $d$-fold, rotationally symmetric way, and it is easy to see that for such a choice of the kneading components, we obtain $\kappa(F) = \kappa_F(0) = \Psi(\theta)$. Let $E_\theta \subset \mathbb{T}$ denote the set of angles $\theta$ such that there is a non-TCE polynomial for which an external ray $R_\theta$ accumulates at the critical value. By Theorem 2, the set $\Psi(E_\theta) \subset \Sigma_d$ has Hausdorff dimension zero. Then Theorem 3 follows from the following proposition.

**Proposition 1.** Pulling back by the map $\Psi : \mathbb{T} \to \Sigma_d$ does not increase the Hausdorff dimension. □

Proof. If we increase $\theta \in [0, 1]$, then it is easy to see that points $k/(d^n - 1), k \in \mathbb{Z}$ are precisely the places where the $n$th digit in the sequence $\Psi(\theta) \in \Sigma_d$ changes, following the periodic pattern $0, 1, \dots, (d-1), 0, 1, \dots$, and that this digit is equal to $(d^n - 1)\theta \mod d$. Let $\sigma$ be an element of $\Sigma_d$ and let $S_n$ denote a cylinder of depth $n$ in $\Sigma_d$, which contains sequences, such that their first $n$ digits coincide with those of $\sigma$. Denote by $\mathcal{I}(n)$ the collection (and sometimes the union) of intervals in $\Psi^{-1}(S_n)$. The diameter of $S_n$ (in the $d$-adic metric on $\Sigma_d$) is $1/d^n$, so we need to prove that for any positive $\alpha$ and $\beta$, we have

$$\sum_{I \in \mathcal{I}(n)} |I|^\alpha \le \text{const}\, d^{-\alpha n + \beta n}.$$

But intervals in $\mathcal{I}(n)$ have length at most $1/(d^n - 1)$, so it is sufficient to show that for any positive $\epsilon$ and $\gamma$, we have

$$\sum_{I \in \mathcal{I}(n)} |I|^\epsilon \le \text{const}\, d^{\gamma n}. \tag{3}$$

The latter follows from the fact that for any $\epsilon > 0$ and positive integer $m$, we have for

$k > k_0 = \mathrm{const}(\epsilon, m)$,

$$\sum_{I \in \mathcal{I}(k+m)} |I|^{\epsilon} \leq 2m \sum_{I \in \mathcal{I}(k)} |I|^{\epsilon}. \tag{4}$$

Indeed, repeatedly applying (4), by induction, we obtain

$$\sum_{I \in \mathcal{I}(n)} |I|^{\epsilon} \leq \mathrm{const}(2m)^{n/m} = \mathrm{const}\, d^{n \log_d(2m)/m},$$

and (3) follows if $m$ is large enough to satisfy $\gamma \geq \log_d(2m)/m$.

To prove (4), first consider an easier model map $\widetilde{\Psi}$. Change the mesh of the partitions from $1/(d^n - 1)$ to $1/d^n$. Namely, take some sequence $\{c_n\}$, consider the partition given by points $(c_n + k)/d^n$, and define a map $\widetilde{\Psi} : [0,1] \to \Sigma_d$ so that the $n$th digit in the sequence $\widetilde{\Psi}(\theta) \in \Sigma_d$ is equal to $(d^n\theta - c_n) \mod d$. For the model map, we can easily see by induction that $\widetilde{\Psi}^{-1}(S_n) \mod 1/d^n$ covers univalently interval $[0, 1/d^n)$. It also consists of at most $(n+1)$ components, because there are only $n$ partition points modulo $1/d^n$, and these points are exactly $\{c_j/d^j\}$. Therefore, $\widetilde{\mathcal{I}}(n) := \widetilde{\Psi}^{-1}(S_n)$ consists of at most $(n+1)$ intervals with total length $1/d^n$ (and thus inequality (3) for the model map is trivial).

Returning to the original map $\Psi$, let $k_0$ be so large that $8d^{m+1-k_0\epsilon} < m - 1$. First, fix $k > k_0$ and an interval $J \in \mathcal{I}(k)$. We view the map $\Psi$ as a perturbation of $\widetilde{\Psi}$ with properly chosen parameters (depending on $J$). Denote by $a$ the left endpoint of $J$ and let $\theta_n$ be the smallest number $j/(d^k - 1)$ that is bigger than $a$. Take $c_n := \theta_n$. Then the partitions of depth $n$ for $\widetilde{\Psi}$ and $\Psi$ coincide at $\theta_n$ and hence do not differ much inside $J$. Namely, for $n < k$, partitions for $\Psi$ and $\widetilde{\Psi}$ would coincide inside $J$, while for $n \geq k$ any partition point for $\Psi$ inside $J$ and its counterpart for $\widetilde{\Psi}$ would differ by at most

$$\left(\frac{1}{d^n - 1} - \frac{1}{d^n}\right) |J| / d^{-n} < 2d^{-2k}.$$

Separate intervals from $J \cap \mathcal{I}(k+m)$ into two kinds: "long," that is, of length $> 4d^{-2k}$, and "short," that is, of length $\leq 4d^{-2k}$. Since partition points of $\Psi$ and $\widetilde{\Psi}$ inside $J$ differ by at most $2d^{-2k}$, every long interval corresponds to some interval from $J \cap \widetilde{\mathcal{I}}(k+m)$ (and intersects it). By rescaling, we see that $J \cap \widetilde{\mathcal{I}}(k+m)$ consists of at most $(m+1)$ intervals. Thus, there are at most $(m+1)$ long intervals, and they are contained in $J$, so we can write

$$\sum_{\mathrm{long}\, I} |I|^{\epsilon} \leq (m+1)|J|^{\epsilon}.$$

Intervals in $\mathcal{I}(k+m)$ can occur at most one per interval of length $1/(d^{k+m-1} - 1)$, so there can be at most $|J|(d^{k+m-1}-1) < |J|d^{k+m}$ of them inside J. Noting that $|J| < 2d^{1-k}$, we can write

$$\sum_{\text{short} I} |I|^{\epsilon} \leq \left(4d^{-2k}\right)^{\epsilon} |J|d^{m+k} = 4d^{m-k\epsilon}|J|^{\epsilon}\left(d^k|J|\right)^{1-\epsilon} \leq 8d^{m+1-k\epsilon}|J|^{\epsilon}.$$

Combining these observations and recalling that $k > k_0$, we obtain the estimate

$$\sum_{I\in\mathcal{J}(k+m),\ I\subset J} |I|^{\epsilon} \leq (m+1)|J|^{\epsilon} + 8d^{m+1-k\epsilon}|J|^{\epsilon} < 2m|J|^{\epsilon}.$$

When summed over all $J \in \mathcal{J}(k)$, this implies (4) and hence the proposition. ■

A different proof of this proposition is given in [3].

## 4.2 From dynamical plane to parameter space

Only a parameter ray such that the kneading sequence of its angle is periodic can accumulate at the point $c \in \mathcal{M}_d$ with Julia set $z^d + c$ having an indifferent cycle (see [16, Section 5], [29, Lemma 3.9], and [15, Theorem 14.4]). Since there are only countably many such kneading sequences, the corresponding set of angles has Hausdorff dimension zero by Proposition 1. See also [6] for a neat computation, explicitly showing that cardiod (the set of parameters $c \in \mathcal{M}_2$ with $z^2 + c$ having an indifferent fixed point) has harmonic measure zero.

Now we can exclude Julia sets with indifferent or (super)attracting cycles (the latter do not matter, since corresponding parameters are not on the boundary of $\mathcal{M}_d$ anyway), and [31, Lemma 2.1] implies that if a parameter ray $R_\theta$ accumulates in the fiber of point c, then a dynamical ray $R_\theta$ for the Julia set $z^d + c$ also accumulates in the fiber of the critical value c. By Beurling's theorem, nearly every (i.e., except for logarithmic capacity zero, and hence except for Hausdorff dimension zero (see [22, Theorem 9.19])), parameter ray lands, so combining this observation with Theorem 3, we arrive at Theorem 4.

In the quadratic case, we can give an alternative proof. C. McMullen shows in [20, Theorem 8.4] that every infinitely renormalizable polynomial is infinitely simply renormalizable. Every simply renormalizable polynomial can be obtained by tuning, as described in [7]. There is also a tuning procedure for angles, formally defined for their dyadic expansions and described in [6]. If $z^2 + c'$ is obtained by tuning from $z^2 + c$, apart from countably many exceptions (some rays landing at Misiurewicz points), angles of parameter rays landing at $c'$ can be obtained by tuning from those of rays landing at c. We can call an angle *renormalizable* if it can be obtained by tuning. Then, except for countably many possibilities, angles of all rays landing at points $c$ with infinitely renormalizable $z^2 + c$ are infinitely renormalizable. As A. Manning computed in [19], the set of infinitely renormalizable angles has Hausdorff dimension zero, and hence so does the set of angles of parameter rays landing at $c$ with $z^2 + c$ infinitely renormalizable. Combining the observations above with Beurling's theorem, we deduce that except for a set of angles of Hausdorff dimension zero, all parameter rays land at points corresponding to finitely renormalizable quadratic polynomials without indifferent cycles. But J.-C. Yoccoz has shown (see [13] and [21]), that for those, the Julia set is locally connected, and external rays with the same angles land at the point $c$ both in the Julia set and on the boundary of the Mandelbrot set. Invoking Theorem 3 completes the proof. ∎

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