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Universality in the 2D Ising model and conformal invariance of fermionic observables

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Abstract It is widely believed that the celebrated 2D Ising model at criticality has a universal and conformally invariant scaling limit, which is used in deriving many of its properties. However, no mathematical proof has ever been given, and even physics arguments support (a priori weaker) Möbius invariance. We introduce discrete holomorphic fermions for the 2D Ising model at criticality on a large family of planar graphs. We show that on bounded domains with appropriate boundary conditions, those have universal and conformally invariant scaling limits, thus proving the universality and conformal invariance conjectures.

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1 Introduction

1.1 Universality and conformal invariance in the Ising model

1.1.1 Historical background

The celebrated Lenz-Ising model is one of the simplest systems exhibiting an order–disorder transition. It was introduced by Lenz in [32], and his student Ising proved [19] in his PhD thesis the absence of phase transition in dimension one, wrongly conjecturing the same picture in higher dimensions. This belief was widely shared, and motivated Heisenberg to introduce his model [17]. However, some years later Peierls [38] used estimates on the length of interfaces between spin clusters to disprove the conjecture, showing a phase transition in the two dimensional case. After Kramers and Wannier [30] derived the value of the critical temperature and Onsager [36] analyzed behavior of the partition function for the Ising model on the two-dimensional square lattice, it became an archetypical example of the phase transition in lattice models and in statistical mechanics in general, see [34, 35] for the history of its rise to prominence.

Over the last six decades, thousands of papers were written about the Ising model, with most of the literature, including this paper, restricted to the two dimensional case (similar behavior is expected in three dimensions, but for now the complete description remains out of reach). The partition function and other parameters were computed exactly in several different ways, usually on the square lattice or other regular graphs. It is thus customary to say that *the 2D Ising model is exactly solvable*, though one should remark that most of the derivations are non-rigorous, and moreover many quantities cannot be derived by traditional methods.

Arrival of the *renormalization group* formalism (see [16] for a historical exposition) led to an even better physical understanding, albeit still non-rigorous. It suggests that block-spin renormalization transformation (coarse-graining, i.e., replacing a block of neighboring sites by one) corresponds to appropriately changing the scale and the temperature. The Kramers-Wannier critical point arises then as a fixed point of the renormalization transformations, with the usual picture of stable and unstable directions.

In particular, under simple rescaling the Ising model at the critical temperature should converge to a *scaling limit*—a “continuous” version of the originally discrete Ising model, which corresponds to a quantum field theory. This leads to the idea of *universality*: the Ising models on different regular lattices or even more general planar graphs belong to the same renormalization space, with a unique critical point, and so at criticality the scaling limit and the scaling dimensions of the Ising model should be independent of the lattice (while the critical temperature depends on it). Being unique, the scaling limit at the critical point is *translation* and *scale invariant*, which allows to deduce some information about correlations [21, 37]. By additionally postulating invariance under inversions, one obtains *Möbius invariance*, i.e. invariance under global conformal transformations of the plane, which allows [39] to deduce more. In seminal papers [3, 4] Belavin, Polyakov and Zamolodchikov suggested much stronger *full conformal invariance* (under all conformal transformations of subregions), thus generating an explosion of activity in conformal field theory, which allowed to explain non-rigorously many phenomena, see [20] for a collection of the founding papers of the subject. Note that in the physics literature there is sometimes confusion between the two notions, with Möbius invariance often called conformal invariance, though the latter is a much stronger property.

Over the last 25 years our physical understanding of the 2D critical lattice models has greatly improved, and the universality and conformal invariance are widely accepted by the physics community. However, supporting arguments are largely non-rigorous and some even lack physical motivation. This is especially awkward in the case of the Ising model, which indeed admits many exact calculations.

1.1.2 Our results

The goal of this paper is to construct lattice holomorphic fermions and to show that they have a universal conformally invariant scaling limit. We give unambiguous (and mathematically rigorous) arguments for the existence of the scaling limit, its universality and conformal invariance for some observables for the 2D Ising model at criticality, and provide the framework to establish the same for all observables. By conformal invariance we mean not the Möbius invariance, but rather the *full conformal invariance*, or invariance under conformal transformations of subregions of $\mathbb{C}$. This is a much stronger property, since conformal transformations form an infinite dimensional pseudogroup, unlike the Möbius ones. Working in subregions necessarily leads us to consider the Ising model in domains with appropriate boundary conditions.

At present we cannot make rigorous the renormalization approach, but we hope that the knowledge gained will help to do this in the future. Rather, we use the integrable structure to construct *discrete holomorphic fermions* in the Ising model. For simplicity we work with discrete holomorphic functions, defined e.g. on the graph edges, which when multiplied by the $\sqrt{dz}$ field become fermions or spinors. Those functions turn out to be discrete holomorphic solutions of a discrete version of the Riemann-Hilbert boundary value problem, and we develop appropriate tools to show that they converge to their continuous counterparts, much as Courant, Friedrichs and Lewy have done in [13] for the Dirichlet problem. The continuous versions of our boundary value problems are $\sqrt{dz}$-covariant, and conformal invariance and universality then follow, since different discrete conformal structures converge to the same universal limit.

Starting from these observables, one can construct new ones, describe interfaces by the Schramm's SLE curves, and prove and improve many predictions originating in physics. Moreover, our techniques work off criticality, and lead to *massive* field theories and SLEs. Several possible developments will be the subject of our future work [9, 18, 24].

We will work with the family of *isoradial graphs* or equivalently *rhombic lattices*. The latter were introduced by Duffin [15] in late sixties as (perhaps) the largest family of graphs for which the Cauchy-Riemann operator admits a nice discretization. They reappeared recently in the work of Mercat [33] and Kenyon [28], as isoradial graphs—possibly the largest family of graphs were the Ising and dimer models enjoy the same integrability properties as on the square lattice: in particular, the critical point is well defined, with weights depending only on the local structure. More recently, Boutilier and de Tilière [6, 7] used the Fisher representation of the Ising model by dimers and Kenyon's techniques to calculate, among other things, free energy for the Ising model on isoradial graphs. While their work is closely related to ours (we can too use the Fisher representation instead of the vertex operators to construct holomorphic fermions), they work in the full plane and so do not address conformal invariance. Note that earlier eight vertex and Ising models were considered by Baxter [2] on Z-invariant graphs, arising from planar line arrangements. Those graphs are topologically the same as the isoradial graphs, and the choice of weights coincides with ours, so quantities like partition function would coincide. Kenyon and Schlenker [29] have shown that such graphs admit isoradial embeddings, but those change the conformal structure, and one does not expect conformal invariance for the Ising model on general Z-invariant graphs.

So there are two reasons for our choice of this particular family: firstly it seems to be the largest family where the Ising model we are about to study is nicely defined, and secondly (and perhaps not coincidentally) it seems to be the largest family of graphs where our main tools, the discrete complex analysis, works well. It is thus natural to consider this family of graphs in the context of *conformal invariance* and *universality* of the 2D Ising model scaling limits.

The fermion we construct for the random cluster representation of the Ising model on domains with two marked boundary points is roughly speaking given by the probability that the interface joining those points passes through a given edge, corrected by a complex weight. The fermion was proposed in [46, 47] for the square lattice (see also independent [41] for its physical connections, albeit without discussion of the boundary problem and covariance). The fermion for the spin representation is somewhat more difficult to construct, it corresponds to the partition function of the Ising model with a $\sqrt{z}$ monodromy at a given edge, again corrected by a complex weight. We describe it in terms of interfaces, but alternatively one can use a product of order and disorder operators at neighboring site and dual site, or work with the inverse Kasteleyn's matrix for the Fisher's dimer representation. It was introduced in [47], although similar objects appeared earlier in Kadanoff and Ceva [22] (without complex weight and boundary problem discussions) and in Mercat [33] (again without discussion of boundary problem and covariance).

Complex analysis on isoradial graphs is more complicated then on the square grid, and less is known a priori about the Ising model there. As a result parts of our paper are quite technical, so we would recommend reading the much easier square lattice proofs [46], as well as the general exposition [47, 48] first.

1.1.3 Other lattice models

Over the last decade, conformal invariance of the scaling limit was established for a number of critical lattice models. An up-to-date introduction can be found in [47], so we will only touch the question of universality here.

Spectacular results of Kenyon on conformal invariance of the dimer model, see e.g. [25, 27], were originally obtained on the square lattice. Some were extended to the isoradial case by de Tilière [14], but the questions of boundary conditions and hence conformal invariance were not addressed yet.

Kenyon's dimer results had corollaries [26] for the Uniform Spanning Tree (and the Loop Erased Random Walk). Those used the Temperley bijection between dimer and tree configurations on two coupled graphs, so they would extend to the situations where boundary conditions can be addressed and Temperley bijection exists.

Lawler, Schramm and Werner used in [31] simpler observables to establish conformal invariance of the scaling limit of the UST interfaces and the LERW curves, and to identify them with Schramm's SLE curves. In both cases one can obtain observables using the Random Walk, and for the UST one can use the Kirchhoff circuit laws to obtain discrete holomorphic quantities. The original paper deals with the square lattice only, but it easily generalizes whenever boundary conditions can be addressed.

In all those cases we have to deal with convergence of solutions of the Dirichlet or Neumann boundary value problems to their continuous counterparts. While this is a standard topic on regular lattices, there are technical difficulties on general graphs; moreover functions are unbounded (e.g. observable for the LERW is given by the Poisson kernel), so controlling their norm is far from trivial. Tools developed by us in [10] for use in the current paper however resolve most of such difficulties.

Situation is somewhat easier with the observables for the Harmonic Explorer and Discrete Gaussian Free Field, as discussed by Schramm and Sheffield [42, 43]—both are harmonic and solving Dirichlet problem with bounded boundary values, so the generalization from the original triangular lattice is straight-forward. Note though that the key difficulty in the DGFF case is to establish the martingale property of the observable.

Unlike the observables above, the one used for percolation in [44, 45] is very specific to the triangular lattice, so the question of universality is far from being resolved.

All the observables introduced so far (except for the fermions from this paper) are essentially bosonic, either invariant under conformal transformations φ or changing like "pre-pre-Schwarzian" forms, i.e. by an addition of const $\cdot \varphi'$. They all satisfy Dirichlet or Neumann boundary conditions, when establishing convergence is a classical subject, dating back to Courant, Friedrichs and Lewy [13], albeit in the non-bounded case one meets serious difficulties.

In the Ising case we work with fermions, hence the Riemann-Hilbert boundary value problem (or rather its homogeneous version due to Riemann). Such problems turn out to be much more complicated already on regular lattices: near rough boundaries (which arise naturally since interfaces are fractal) our observables blow up fast. When working on general graphs, the main problem remains, but the tools become quite limited.

We believe that further progress in other models requires the study of holomorphic parafermions [47], so we expect even more need to address the Riemann boundary value problems in the future.

1.2 Setup and main results

Throughout the paper, we work with isoradial graphs or, equivalently, rhombic lattices. A planar graph Γ embedded in $\mathbb{C}$ is called δ*-isoradial* if each face is inscribed into a circle of a common radius δ. If all circle centers are inside the corresponding faces, then one can naturally embed the dual graph Γ^* in $\mathbb{C}$ isoradially with the same δ, taking the circle centers as vertices of Γ^*. The name *rhombic lattice* is due to the fact that all quadrilateral faces of the corresponding bipartite graph Λ (having $\Gamma \cup \Gamma^*$ as vertices and radii of the circles as edges) are rhombi with sides of length δ. We denote the set of rhombi centers by $\diamondsuit$ (example of an isoradial graph is drawn in Fig. 1A). We also require the following mild assumption:

the rhombi angles are uniformly bounded away from 0 *and* π

(in other words, all these angles belong to $[\eta, \pi - \eta]$ for some fixed $\eta > 0$). Below we often use the notation *const* for absolute positive constants that don't depend on the mesh δ or the graph structure but, in principle, may depend on η. We also use the notation $f \asymp g$ which means that a double-sided estimate $\mathrm{const}_1 \cdot f \le g \le \mathrm{const}_2 \cdot g$ holds true for some $\mathrm{const}_{1,2} > 0$ which are independent of δ.

It is known that one can define the (critical) Ising model on Γ^* so that

(a) the interaction constants $J_{w_i w_j}$ are local (namely, depend on the lengths of edges connecting $w_{i,j} \in \Gamma^*$, $w_i \sim w_j$, only) and
(b) the model is invariant under the star-triangle transform.

Such invariance is widely recognized as the crucial sign of the integrability. Note that the star-triangle transform preserves the isoradial graph/rhombic

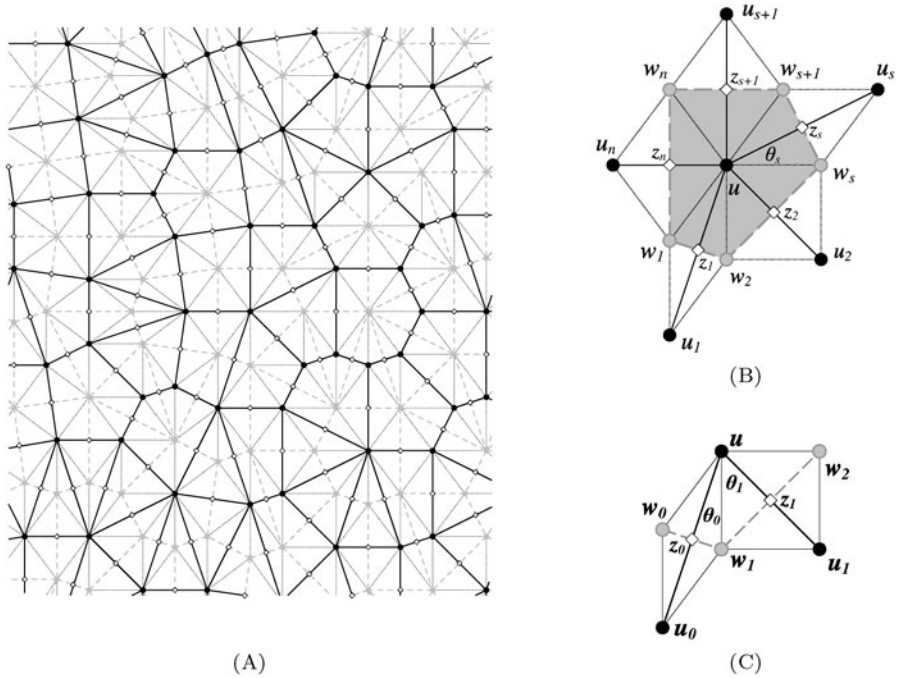


Fig. 1 (**A**) Example of an isoradial graph Γ (*black vertices*, *solid lines*), its dual isoradial graph Γ^* (*gray vertices*, *dashed lines*), the corresponding rhombic lattice or quad-graph (*vertices* $\Lambda = \Gamma \cup \Gamma^*$, *thin lines*) and the set $\diamond = \Lambda^*$ (*rhombi centers*, *white diamond-shaped vertices*). (**B**) Local notation near $u \in \Gamma$, with neighbors of u enumerated counterclockwise by $1, 2 \ldots, s, s+1, \ldots, n$. The weight $\mu_\Gamma^\delta(u)$ is equal to the shaded polygon area. (**C**) Definition of s-holomorphic functions: $F(z_0)$ and $F(z_1)$ have the same projections on the direction $[i(w_1 - u)]^{-\frac{1}{2}}$. Thus, we have one *real* identity for each pair of neighboring z_0, z_1

lattice structure. Moreover, isoradial graphs form the largest family of planar graphs (embedded into $\mathbb{C}$) satisfying these properties (see [12] and references therein). At the same time, discrete holomorphic functions on isoradial graphs provide the simplest example of a discrete integrable system in the so-called "consistency approach" to the (discrete) integrable systems theory (see [5]).

Recently, it was understood (see [33, 41, 47]) that some objects coming from the theoretical physics approach to the Ising model (namely, products of order and disorder operators with appropriate complex weights) can be considered and dealt with as discrete holomorphic *functions* (see Sect. 3.2 for further discussion). These functions (which we call *basic observables* or *holomorphic fermions*) provide a powerful tool for rigorous proofs of several results concerning the conformal invariance of the critical 2D Ising model. Implementing the program proposed and started in [46, 47] for the square grid, in this paper we mainly focus our attention on the topologically simplest case, when the model is defined in the simply-connected (discrete) domain Ω^δ

having two marked boundary points a^δ, b^δ (but see Sect. 6 for more involved setup). We have to mark some boundary points so that the conformal modulus is non-trivial, allowing us to construct conformal invariants.

We will work with two representations of the Ising model: the usual spin, as well as the random cluster (Fortuin-Kasteleyn). The observables are similar, but do not directly follow from each other, and require slightly different approaches. In either case there is an interface (between spin clusters or random clusters)—a discrete curve γ^δ running from a^δ to b^δ inside Ω^δ (see Sects. 2.1, 2.2 for precise definitions). In both cases the basic observables are *martingales* with respect to (filtration induced by) the interface grown progressively from a^δ, which opens the way to identify its scaling limit as a Schramm's SLE curve, cf. [47].

The interface in the random cluster representation can in principle pass through some point twice, but with our setup we move apart those passages, so that the curve becomes simple and when arriving at the intersection it is always clear how to proceed. This setup is unique where the martingale property holds, so there is only one conformally invariant way to address this problem. Note that the resulting scaling limit, the SLE$(16/3)$ curve, will have double points. A similar ambiguity arises in the spin model (when, e.g. a vertex is surrounded by four spins "$-+-+$"), but regardless of the way to address it (e.g. deterministic, like always turning right, or probabilistic, like tossing a coin every time) the martingale property always holds, and so the SLE(3) is the scaling limit. The latter is almost surely simple, so we conclude that the double points in the discrete case produce only very small loops, disappearing in the scaling limit.

The first two results of our paper say that, in both representations, the holomorphic fermions are uniformly close to their continuous *conformally invariant* counterparts, independently of the structure of Γ^δ (or $\diamondsuit^\delta$) and the shape of Ω^δ (in particular, we don't use any smoothness assumptions concerning the boundary). Namely, we prove the following two theorems, formulated in detail as Theorems 4.3 and 5.6:

Theorem A (FK-Ising fermion) *Let discrete domains* $(\Omega^\delta; a^\delta, b^\delta)$ *with two marked boundary points* a^δ, b^δ *approximate some continuous domain* $(\Omega; a, b)$ *as* $\delta \to 0$. *Then, uniformly on compact subsets of* Ω *and independently of the structure of* $\diamondsuit^\delta$,

$$F^\delta(z) \rightrightarrows \sqrt{\Phi'(z)},$$

where $F^\delta(z) = F^\delta(z; \Omega^\delta, a^\delta, b^\delta)$ *is the discrete holomorphic fermion and* Φ *denotes the conformal mapping from* Ω *onto the strip* $\mathbb{R} \times (0, 1)$ *such that* a, b *are mapped to* $\mp\infty$.

and

Theorem B (spin-Ising fermion) *Let discrete domains* $(\Omega^\delta; a^\delta, b^\delta)$ *approximate some continuous domain* $(\Omega; a, b)$ *as* $\delta \to 0$. *Then, uniformly on compact subsets of* Ω *and independently of the structure of* Γ^δ,

$$F^\delta(z) \rightrightarrows \sqrt{\Psi'(z)},$$

where $F^\delta = F^\delta(z; \Omega^\delta, a^\delta, b^\delta)$ *is the discrete holomorphic fermion and* $\Psi : \Omega \to \mathbb{C}_+$ *is the conformal mapping such that a and b are mapped to* ∞ *and* 0, *appropriately normalized at* b.

Because of the aforementioned martingale property, these results are sufficient to prove the *convergence of interfaces to conformally invariant Schramm's SLE curves* (in our case, SLE(3) for the spin representation and SLE(16/3) for the FK representation) in the weak topology given by the convergence of driving forces in the Loewner equation, cf. [31, 47]. A priori this is very far from establishing convergence of curve themselves, but our proof together with techniques of [23] implies stronger convergence, see also [11] for a simplified account.

The third result shows how our techniques can be used to find the (conformally invariant) limit of some macroscopic quantities, "staying on the discrete level", i.e. without consideration of the limiting curves. Namely, we prove a *crossing probability formula* for the critical FK-Ising model on isoradial graphs, analogous to Cardy's formula [44, 45] for critical percolation and formulated in detail as Theorem 6.1:

Theorem C (FK-Ising crossing probability) *Let discrete domains* $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$ *with alternating* (*wired/free/wired/free*) *boundary conditions on four sides approximate some continuous topological quadrilateral* $(\Omega; a, b, c, d)$ *as* $\delta \to 0$. *Then the probability of an FK cluster crossing between two wired sides has a scaling limit*, *which depends only on the conformal modulus of the limiting quadrilateral*, *and is given for the half-plane by*

$$p(\mathbb{H}; 0, 1-u, 1, \infty) = \frac{\sqrt{1-\sqrt{1-u}}}{\sqrt{1-\sqrt{u}} + \sqrt{1-\sqrt{1-u}}}, \quad u \in [0, 1]. \tag{1.1}$$

The version of this formula for multiple SLEs was derived by Bauer, Bernard and Kytölä in [1], see page 1160, their notation for the modulus related to ours by $x = 1 - u$. Besides being of an independent interest, this result together with [23] is needed to improve the topology of convergence of FK-Ising interfaces. Curiously, the (macroscopic) answer for a unit disc $(\mathbb{D}; -e^{i\phi}, e^{-i\phi}, e^{i\phi}, -e^{-i\phi})$ formally coincides with the relative weights corresponding to two possible crossings inside (microscopic) rhombi (see Fig. 2A) in the critical model (see Remark 6.2).

1.3 Organization of the paper

We begin with the definition of Fortuin-Kasteleyn (random cluster) and spin representations of the critical Ising model on isoradial graphs in Sect. 2. From the outset we work with critical interactions, but in principle one can introduce a temperature parameter, which would lead to *massive holomorphic fermions*. We also introduce the basic discrete holomorphic observables (holomorphic fermions) satisfying the martingale property with respect to the growing interface and, essentially, show that they satisfy discrete version of the Cauchy-Riemann equation (Proposition 2.2 and Proposition 2.5) using some simple combinatorial bijections between the sets of configurations. Actually, we show that our observables satisfy the stronger "two-points" equation which we call *spin* or *strong holomorphicity*, or simply *s-holomorphicity*.

We discuss the properties of s-holomorphic functions in Sect. 3. The main results are:

(a) The (rather miraculous) possibility to define naturally the discrete version of $h(z) = \operatorname{Im} \int (f(z))^2 \, dz$, see Proposition 3.6. Note that the square $(f(z))^2$ of a discrete holomorphic function $f(z)$ is not discrete holomorphic anymore, but unexpectedly it satisfies "half" of the Cauchy-Riemann equations, making its imaginary part a closed form with a well-defined integral;
(b) The sub- and super-harmonicity of h on the original isoradial graph Γ and its dual Γ^*, respectively, and the a priori comparability of the components $h|_\Gamma$ and $h|_{\Gamma^*}$ which allows one to deal with h as with a harmonic function: e.g. nonnegative h's satisfy a version of the Harnack Lemma (see Sect. 3.4);
(c) The uniform (w.r.t. δ and the structure of the isoradial graph/rhombic lattice $\Gamma, \Gamma^*/\diamondsuit$) boundedness and, moreover, uniform Lipschitzness of s-holomorphic functions inside their domains of definition Ω^δ, with the constants depending on $M = \max_{v \in \Omega^\delta} |h(v)|$ and the distance $d = \operatorname{dist}(z; \partial\Omega^\delta)$ only (Theorem 3.12, these results should be considered as discrete analogous of the standard estimates from the classical complex analysis);
(d) The combinatorial trick (see Sect. 3.6) that allows us to transform the discrete version of the Riemann-type boundary condition $f(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}$ into the Dirichlet condition for $h|_{\partial\Omega}$ on both Γ and Γ^*, thus completely avoiding the reference to Onsager's magnetization estimate used in [46, 47] to control the difference $h|_\Gamma - h|_{\Gamma^*}$ on the boundary.

We prove the (uniform) convergence of the basic observable in the FK-Ising model to its continuous counterpart in Sect. 4. The main result here is Theorem 4.3, which is the technically simplest of our main theorems, so the reader should consider the proof as a basic example of the application of our techniques. Besides the results from [10] and the previous Sections, the important idea (exactly as in [46]) is to use some compactness arguments (in the set of all simply-connected domains equipped with the Carathéodory topology) in order to derive the uniform (w.r.t. to the shape of Ω^δ and the structure of $\diamondsuit^\delta$) convergence from the "pointwise" one.

In Sect. 5 we prove analogous convergence result for the holomorphic fermion defined for the spin representation of the critical Ising model (Theorem 5.6). There are two differences from the preceding section: the unboundedness of the (discrete) integral $h = \operatorname{Im}\int (f(z))^2 dz$ (this prevents us from the immediate use of compactness arguments) and the need to consider the normalization of our observable at the target point b^δ (this is crucial for the martingale property). In order to handle the normalization at b^δ, we assume that our domains Ω^δ contain a (macroscopic) rectangle near b^δ and their boundaries $\partial\Omega^\delta$ approximate the corresponding straight segment as $\delta \to 0$. Making this technical assumption, we don't lose much generality, since the growing interface, though fractal in the limit, doesn't change the shape of the domain near b^δ. Then, we use a version of the boundary Harnack principle (Proposition 5.3) in order to control the values of h in the bulk through the fixed value $f(b^\delta)$. Another important technical ingredient is the universal (w.r.t. to the structure of $\diamondsuit^\delta$) multiplicative normalization of our observable. Loosely speaking, we define it using the value at b^δ of the discrete holomorphic fermion in the discrete half-plane (see Theorem 5.4 for further details).

Section 6 is devoted to the crossing probability formula for the FK-Ising model on discrete quadrilaterals $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$ (Theorem 6.1, see also Remark 6.2). The main idea here is to construct some discrete holomorphic in Ω^δ function whose boundary values reflect the conformal modulus of the quadrilateral. Namely, in our construction, discrete functions $h^\delta = \operatorname{Im}\int (f^\delta(z))^2 dz$ approximate the imaginary part of the conformal mapping from Ω^δ onto the slit strip $[\mathbb{R} \times (0,1)] \setminus (-\infty + i\varkappa; i\varkappa]$ such that a^δ is mapped to the "lower" $-\infty$, b^δ to $+\infty$; c^δ to the "upper" $-\infty$ and d^δ to the tip $i\varkappa$. The respective crossing probabilities are in the 1-to-1 correspondence with values $\varkappa^\delta$ which approximate $\varkappa$ as $\delta \to 0$. Since $\varkappa$ is uniquely determined by the limit of conformal moduli of $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$, we obtain (1.1) (see further details in Sect. 6). Finally, Appendix contains several auxiliary lemmas: estimates of the discrete harmonic measure, discrete version of the Cauchy formula, and technical estimates of the Green function in the disc. We refer the reader interested in a more detailed presentation of the discrete complex analysis on isoradial graphs to our paper [10].

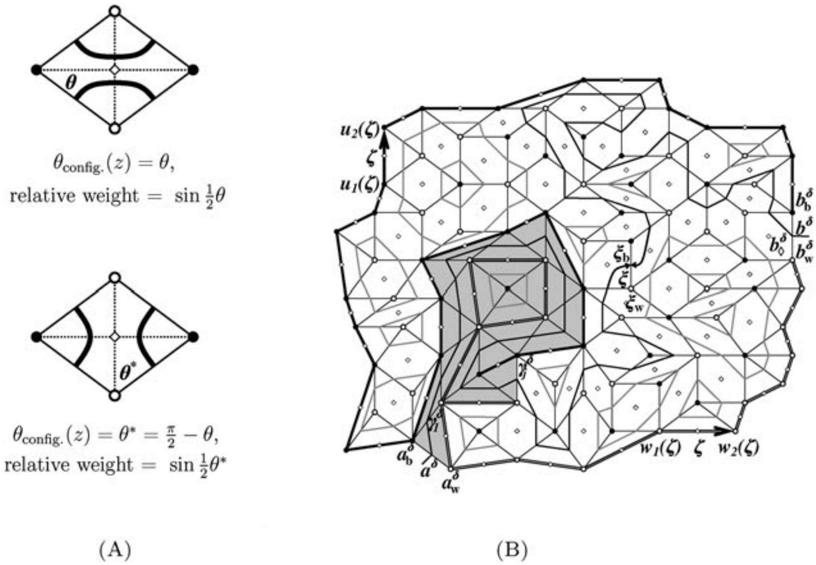


Fig. 2 (**A**) Loop representation of the critical FK-Ising model on isoradial graphs: the relative weights corresponding to two possible choices of connections inside the inner rhombus z (the partition function is given by (2.1)). (**B**) Discrete domain $\Omega^\delta_\diamond$ with a sample configuration. Besides loops, there is an interface γ^δ connecting a^δ to b^δ. Calculating the winding$(\gamma^\delta;\, b^\delta \rightsquigarrow \xi)$, we draw γ^δ so that it intersects the edge $\xi = [\xi_b \xi_w]$ orthogonally. As γ^δ grows, it separates some part of $\Omega^\delta_\diamond$ (*shaded*) from b^δ. We denote by $\Omega^\delta_\diamond \setminus [a^\delta \gamma_1^\delta .. \gamma_j^\delta]$ the connected component containing b^δ (*unshaded*)

2 Critical spin- and FK-Ising models on isoradial graphs. Basic observables (holomorphic fermions)

2.1 Critical FK-Ising model

2.1.1 *Loop representation of the model, holomorphic fermion, martingale property*

We will work with a graph domain which can be thought of as a discretization of a simply-connected planar domain with two marked boundary points. Let $\Omega^\delta_\diamond \subset \diamond$ be a simply-connected discrete domain composed of inner rhombi $z \in \operatorname{Int}\Omega^\delta_\diamond$ and boundary half-rhombi $\zeta \in \partial\Omega^\delta_\diamond$, with two marked boundary points a^δ, b^δ and Dobrushin boundary conditions (see Fig. 2B): $\partial\Omega^\delta_\diamond$ consists of the "white" arc $a^\delta_w b^\delta_w$, the "black" arc $b^\delta_b a^\delta_b$, and two edges $[a^\delta_b a^\delta_w]$, $[b^\delta_b b^\delta_w]$ of Λ. Without loss of generality, we assume that

$$b^\delta_b - b^\delta_w = i\delta, \textit{ i.e., the edge } b^\delta = [b^\delta_b b^\delta_w] \textit{ is oriented vertically.}$$

For each inner rhombus $z \in \operatorname{Int}\Omega^\delta_\diamond$ we choose one of two possibilities to connect its sides (see Fig. 2A, there is only one choice for boundary half-rhombi), thus obtaining the set of configurations (whose cardinality is $2^{\#(\mathrm{Int}\,\Omega^\delta_\diamond)}$). The *partition function* of the critical FK-Ising model is given by

$$Z=\sum_{\text{config.}}\sqrt{2}^{\#(\text{loops})}\prod_{z\in\mathrm{Int}\,\Omega^\delta_\diamond}\sin\frac{1}{2}\theta_{\text{config.}}(z),\tag{2.1}$$

where $\theta_{\text{config.}}(z)$ is equal to either θ or $\theta^*=\frac{\pi}{2}-\theta$ depending on the choice of connections inside rhombus z (see Fig. 2A).

We described the loop representation, since at criticality it is easier to work with, than the usual random cluster one. The loops trace the perimeters of random clusters, and the curve joining the two marked boundary points is the interface between a cluster and a dual cluster wired on two opposite boundary arcs (the so-called Dobrushin boundary conditions).

Let $\xi=[\xi_{\mathrm{b}}\xi_{\mathrm{w}}]$ be some inner edge of $\Omega^\delta_\diamond$ (where $\xi_{\mathrm{b}}\in\Gamma$, $\xi_{\mathrm{w}}\in\Gamma^*$). Due to the boundary conditions chosen, each configuration consists of (a number of) loops and one interface γ^δ running from a^δ to b^δ. The *holomorphic fermion* is defined as

$$F^\delta(\xi)=F^\delta_{(\Omega^\delta;\,a^\delta,b^\delta)}(\xi):=(2\delta)^{-\frac{1}{2}}\cdot\mathbb{E}\big[\chi(\xi\in\gamma^\delta)\cdot e^{-\frac{i}{2}\,\mathrm{winding}(\gamma^\delta;\,b^\delta\rightsquigarrow\xi)}\big],\tag{2.2}$$

where $\chi(\xi\in\gamma^\delta)$ is the indicator function of the event that the interface intersects ξ and

$$\mathrm{winding}(\gamma^\delta;\,b^\delta\rightsquigarrow\xi)=\mathrm{winding}(\gamma^\delta;\,a^\delta\rightsquigarrow\xi)-\mathrm{winding}(\gamma^\delta;\,a^\delta\rightsquigarrow b^\delta)\tag{2.3}$$

denotes the total turn of γ^δ measured (in radians) from b^δ to ξ. Note that, for all configurations and edges ξ one has (see Fig. 2B),

$$e^{-\frac{i}{2}\,\mathrm{winding}(\gamma^\delta;\,b^\delta\rightsquigarrow\xi)}\parallel[i(\xi_{\mathrm{w}}-\xi_{\mathrm{b}})]^{-\frac{1}{2}}.$$

Remark 2.1 (Martingale property) For each ξ, $F^\delta_{(\Omega^\delta\setminus[a^\delta\gamma^\delta_1..\gamma^\delta_j];\gamma^\delta_j,b^\delta)}(\xi)$ is a martingale with respect to the growing interface $(a^\delta=\gamma^\delta_0,\gamma^\delta_1,\dots,\gamma^\delta_j,\dots)$ (till the stopping time when γ^δ hits ξ or ξ becomes separated from b^δ by the interface, see Fig. 2B).

Proof Since the winding$(\gamma^\delta;\,b^\delta\rightsquigarrow\xi)$ doesn't depend on the beginning of the interface, the claim immediately follows from the total probability formula. □

2.1.2 Discrete boundary value problem for F^δ

We start with the extension of F^δ to the centers of rhombi $z\in\Omega^\delta_\diamond$. Actually, the (rather fortunate) opportunity to use the definition given below reflects the discrete holomorphicity of F^δ (see discussion in Sect. 3.2).

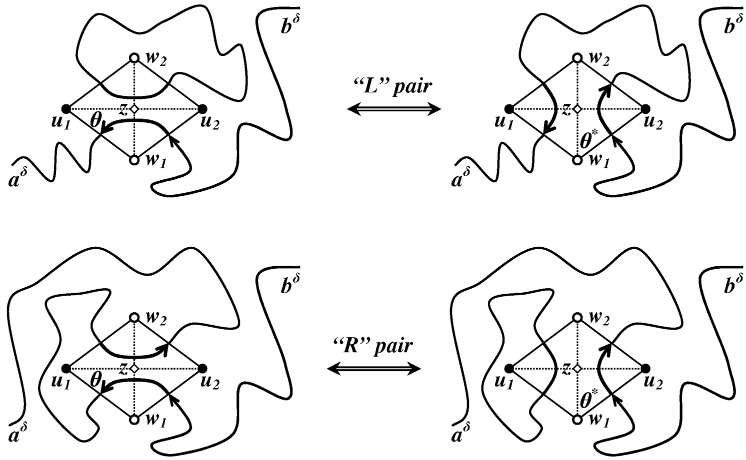


Fig. 3 Local rearrangement at z: the bijection between configurations. Without loss of generality, we may assume that the (reversed, i.e., going from b^δ) interface enters the rhombus z through the edge $[u_2w_1]$ (the case $[u_1w_2]$ is completely similar). There are two possibilities: either γ^δ (finally) leaves z through $[u_1w_1]$ ("L", left turn) or through $[u_2w_2]$ ("R", right turn). In view of (2.1), the relative weights of configurations are $\sqrt{2}\sin\frac{1}{2}\theta$, $\sin\frac{1}{2}\theta^*$ ("L" pairs), and $\sin\frac{1}{2}\theta$, $\sqrt{2}\sin\frac{1}{2}\theta^*$ ("R" pairs)

Proposition 2.2 *Let $z \in \operatorname{Int}\Omega^\delta_\diamondsuit$ be the center of some inner rhombus $u_1w_1u_2w_2$. Then, there exists a complex number $F^\delta(z)$ such that*

$$F^\delta([u_jw_k]) = \operatorname{Proj}\big[\,F^\delta(z)\,;\ [i(w_k-u_j)]^{-\frac{1}{2}}\big], \quad j,k=1,2. \tag{2.4}$$

The proposition essentially states that F^δ is *spin holomorphic* as specified in Definition 3.1 below. By $\operatorname{Proj}[X;\nu]$ we denote the orthogonal projection of the vector X on the vector ν, which is parallel to ν and equal to

$$\operatorname{Proj}[X;\nu] = \operatorname{Re}\left(X\frac{\overline{\nu}}{|\nu|}\right)\frac{\nu}{|\nu|} = \frac{X}{2} + \frac{\overline{X}\nu^2}{2|\nu|^2}$$

(here we consider complex numbers as vectors). Because of the latter rewriting, the choice of the sign in the square root in (2.4) does not matter.

Proof As on the square grid (see [46]), the proof is based on the bijection between configurations which is produced by their local rearrangement at z. It is sufficient to check that the contributions to $F^\delta([u_jw_k])$'s of each pair of configurations drawn in Fig. 3 are the specified projections of the same complex number. The relative contributions of configurations (up to the same real factor coming from the structure of the configuration away from z) to the values of F on four edges around z for the pairs "L" and "R" are given by

	$e^{i\frac{\varphi}{2}}\cdot F^\delta([u_2w_1])$	$e^{i\frac{\varphi}{2}}\cdot F^\delta([u_2w_2])$	$e^{i\frac{\varphi}{2}}\cdot F^\delta([u_1w_2])$	$e^{i\frac{\varphi}{2}}\cdot F^\delta([u_1w_1])$
L	$\sqrt{2}\sin\frac{\theta}{2}+\sin\frac{\theta^*}{2}$	$e^{i\theta}\sin\frac{\theta^*}{2}$	$-i\sin\frac{\theta^*}{2}$	$e^{-i\theta^*}[\sqrt{2}\sin\frac{\theta}{2}+\sin\frac{\theta^*}{2}]$
R	$\sin\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta^*}{2}$	$e^{i\theta}[\sin\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta^*}{2}]$	$i\sin\frac{\theta}{2}$	$e^{-i\theta^*}\sin\frac{\theta}{2}$

where φ denotes the total turn of the interface traced from b^δ to $[u_2w_1]$. An easy trigonometric calculation using that $\theta+\theta^*=\pi/2$ shows that

$$\left[\sqrt{2}\sin\frac{\theta}{2}+\sin\frac{\theta^*}{2}\right]-i\sin\frac{\theta^*}{2}=e^{i\theta}\sin\frac{\theta^*}{2}+e^{-i\theta^*}\left[\sqrt{2}\sin\frac{\theta}{2}+\sin\frac{\theta^*}{2}\right].$$

Denoting the common value of the two sides by $e^{i\frac{\varphi}{2}}\cdot F^\delta(z)$ and observing that $1\perp i$ and $e^{i\theta}\perp e^{-i\theta^*}$, we conclude that the first row ("L") describes the four projections of $e^{i\frac{\varphi}{2}}\cdot F^\delta(z)$ onto the lines $\mathbb{R}$, $e^{i\theta}\mathbb{R}$, $i\mathbb{R}$ and $e^{-i\theta^*}\mathbb{R}$, respectively. Multiplying by the common factor $e^{-i\frac{\varphi}{2}}$ (which is always parallel to $[i(w_1-u_2)]^{-\frac{1}{2}}$), we obtain the result for "L" pairs of configurations. The interchanging of θ and θ^* yields the result for "R" pairs. $\square$

Remark 2.3 For $\zeta\in\partial\Omega^\delta_\diamondsuit$ we define $F^\delta(\zeta)$ so that (2.4) holds true (in this case, only two projections are meaningful, so $F^\delta(\zeta)$ is easily and uniquely defined). Note that all interfaces passing through the half-rhombus ζ intersect both its sides. Moreover, since the winding of the interface at ζ is independent of the configuration chosen (and coincides with the winding of the corresponding boundary arc) for topological reasons, we have

$$F^\delta(\zeta)\parallel(\tau(\zeta))^{-\frac{1}{2}},\quad \zeta\in\partial\Omega^\delta_\diamondsuit, \tag{2.5}$$

where (see Fig. 2B)

$$\tau(\zeta)=w_2(\zeta)-w_1(\zeta),\quad \zeta\in(a^\delta b^\delta),\ w_{1,2}(\zeta)\in\Gamma^*,$$
$$\tau(\zeta)=u_2(\zeta)-u_1(\zeta),\quad \zeta\in(b^\delta a^\delta),\ u_{1,2}(\zeta)\in\Gamma,$$

is the "discrete tangent vector" to $\partial\Omega^\delta_\diamondsuit$ directed from a^δ to b^δ on both boundary arcs.

Thus, we arrive at

Discrete Riemann boundary value problem for F^δ (FK-case) The function F^δ is defined in $\Omega^\delta_\diamondsuit$ and for each pair of neighbors $z_0,z_1\in\Omega^\delta_\diamondsuit$, $z_0\sim z_1$, the discrete holomorphicity condition holds:

$$\mathrm{Proj}\big[F^\delta(z_0)\,;[i(w-u)]^{-\frac{1}{2}}\big]=\mathrm{Proj}\big[F^\delta(z_1)\,;[i(w-u)]^{-\frac{1}{2}}\big]. \tag{2.6}$$

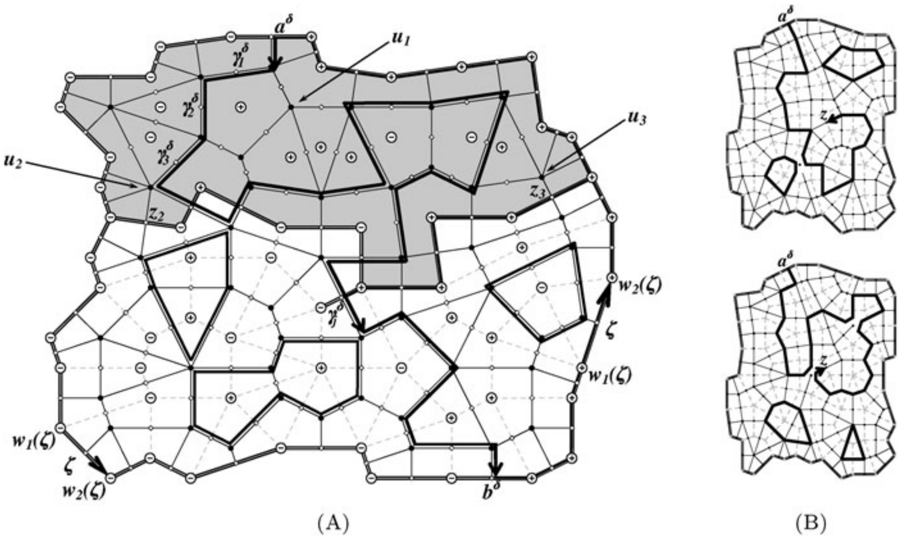


Fig. 4 (**A**) Ising model on isoradial graphs: discrete domain $\Omega^\delta_\diamond$ and a sample configuration (the partition function is given by (2.7)). By our choice of the "turning rule," loops and the interface γ^δ separate clusters of "+" spins connected through edges and clusters of "−" spins connected through vertices. To illustrate this, γ^δ and loops are drawn slightly closer to "+" spins. The component of Ω^δ not "swallowed" by the path $[a^\delta\gamma_1^\delta..\gamma_j^\delta]$ is unshaded. The vertex u_1 is shaded since it is not connected to b^δ anymore. The vertices u_2 and u_3 are shaded too since each of them is connected to the bulk by a single edge which contradicts our definition of connected discrete domains. Moreover, since the "interface" $a^\delta \rightsquigarrow \gamma_j^\delta \rightsquigarrow z$ could arrive at each of these points only in a single way, our observable certainly satisfies boundary condition (2.10) at z_2 and z_3, thus not distinguishing them from the other boundary points. (**B**) Two samples of "interface pictures" composed from a number of loops and a single interface γ^δ running from a^δ to z. To define the winding$(\gamma^\delta;\, a^\delta \rightsquigarrow z)$ unambiguously, we draw γ^δ so that, if there is a choice, it turns to the left (for $z \in \partial\Omega^\delta_\diamond$, this corresponds to the edge-connectivity of "+" clusters)

Moreover, F^δ satisfies the boundary conditions (2.5) and, since all interfaces pass through b^δ, satisfies the normalization $F^\delta(b^\delta) = \operatorname{Re} F^\delta(b^\delta_\diamond) = (2\delta)^{-\frac{1}{2}}$.

2.2 Critical spin-Ising model

2.2.1 Definition of the model, holomorphic fermion, martingale property

Let $\Omega^\delta_\diamond \subset \diamond$ be a simply-connected discrete domain composed of inner rhombi $z \in \operatorname{Int}\Omega^\delta_\diamond$ and boundary half-rhombi $\zeta \in \partial\Omega^\delta_\diamond$, with two marked boundary points $a^\delta, b^\delta \in \partial\Omega^\delta_\diamond$, such that $\partial\Omega^\delta_\diamond$ contains only "white" vertices (see Fig. 4A) and there is no edge of Γ^* breaking $\Omega^\delta_\diamond$ into two non-connected pieces.

To each inner "white" vertex $w \in \operatorname{Int}\Omega^\delta_{\Gamma^*}$, we assign the spin $\sigma(w)$ (+ or −), thus obtaining the set of configurations (whose cardinality is $2^{\#(\operatorname{Int}\Omega^\delta_{\Gamma^*})}$).

We also impose Dobrushin boundary conditions assigning the $-$ spins to vertices on the boundary arc $(a^\delta b^\delta)$ and the $+$ spins on the boundary arc $(b^\delta a^\delta)$ (see Fig. 4A). The *partition function* of the critical spin-Ising model is given by

$$\widetilde{Z}_{(\Omega^\delta;\, a^\delta\rightsquigarrow b^\delta)} = \left[\sin\frac{1}{2}\theta(b^\delta)\right]^{-1} \sum_{\text{spin config.}} \prod_{w_1\sim w_2:\sigma(w_1)\neq\sigma(w_2)} x_{w_1w_2},$$
$$x_{w_1w_2} = \tan\frac{1}{2}\theta(z), \tag{2.7}$$

where $\theta(z)$ is the half-angle of the rhombus $u_1w_1u_2w_2$ having center at z (i.e., $\tan\theta_{w_1w_2} = |w_2-w_1|/|u_2-u_1|$). The first factor $\sin^{-1}$ doesn't depend on the configuration, and is introduced for technical reasons.

Due to Dobrushin boundary conditions, for each configuration, there is an interface γ^δ running from a^δ to b^δ and separating $+$ spins from $-$ spins. If Γ is not a trivalent graph, one needs to specify the algorithm of "extracting γ^δ from the picture", if it can be done in different ways. Below we assume that,

if there is a choice, the "interface" takes the left-most possible route (see Fig. 4).

With this choice the interface separates clusters of "+" spins connected through edges and clusters of "−" spins connected through vertices. Any other choice would do for the martingale property and eventual conformal invariance, as discussed in the Introduction. For example, one can toss a coin at each vertex to decide whether "+" or "−" spin clusters connect through it. Note that, drawing all the edges separating $+$ spins from $-$ spins, one can rewrite the partition function as a sum over all configurations ϖ of edges which consist of a single interface running from a^δ to b^δ and a number of loops. Namely,

$$\widetilde{Z}_{(\Omega^\delta;\, a^\delta\rightsquigarrow b^\delta)} = \left[\sin\frac{1}{2}\theta(b^\delta)\right]^{-1} \sum_{\varpi=\{\text{interface+loops}\}} \times \prod_{w_1\sim w_2:[w_1;w_2]\text{ intersects }\omega} x_{w_1w_2}. \tag{2.8}$$

For $z\in\Omega^\delta_\diamondsuit$, the *holomorphic fermion* is defined as (cf. (2.2), (2.3))

$$F^\delta(z) = F^\delta_{(\Omega^\delta;\, a^\delta, b^\delta)}(z) := \mathcal{F}^\delta(b^\delta)\cdot\frac{\widetilde{Z}_{(\Omega^\delta;\, a^\delta\rightsquigarrow z)}\cdot\mathbb{E}_{(\Omega^\delta;\, a^\delta\rightsquigarrow z)}e^{-\frac{i}{2}\,\text{winding}(\gamma^\delta;\, a^\delta\rightsquigarrow z)}}{\widetilde{Z}_{(\Omega^\delta;\, a^\delta\rightsquigarrow b^\delta)}\cdot e^{-\frac{i}{2}\,\text{winding}(a^\delta\rightsquigarrow b^\delta)}}. \tag{2.9}$$

As in (2.8), $\widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow z^\delta)}$ denotes the partition function for the set of "interfaces pictures" containing (besides loops) one interface γ^δ running from a^δ to z (see Fig. 4B),

for each configuration we count all weights corresponding to the drawn edges, including $\tan\frac{1}{2}\theta(a^\delta)$ *and* $[\cos\frac{1}{2}\theta(z)]^{-1} = [\sin\frac{1}{2}\theta(z)]^{-1}\cdot\tan\frac{1}{2}\theta(z)$ *for the first and the last.*

Then $\mathbb{E}_{(\Omega^\delta;\, a^\delta \rightsquigarrow z^\delta)}$ will stand for the expectation with respect to the corresponding probability measure. Equivalently, one can take the partition function, multiplying the weight of each configuration by the complex factor $e^{-\frac{i}{2}\,\mathrm{winding}(\gamma^\delta;\, a^\delta \rightsquigarrow z^\delta)}$ (which, for $z \in \operatorname{Int}\Omega^\delta_\diamondsuit$, may be equal to one of the four different complex values α, $i\alpha$, $-\alpha$, $-i\alpha$ depending on the particular configuration). Finally,

$\mathcal{F}^\delta(b^\delta) \parallel (\tau(b^\delta))^{-1/2}$, *where* $\tau(b^\delta) = w_2(b^\delta) - w_1(b^\delta)$, *is a normalizing factor that depends only on the structure of* $\diamondsuit^\delta$ *near* b^δ *(see Sect. 5.1).*

Here and below, for $\zeta \in \partial\Omega^\delta_\diamondsuit$, $\tau(\zeta) = w_2(\zeta) - w_1(\zeta)$ denotes "discrete tangent vector" to $\partial\Omega^\delta_\diamondsuit$ oriented counterclockwise (see Fig. 4A for notation). For any $\zeta \in \partial\Omega^\delta_\diamondsuit$, the winding$(\gamma^\delta;\, a^\delta \rightsquigarrow \zeta)$ is fixed due to topological reasons, and so

$$F^\delta(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}, \quad \zeta \in \partial\Omega^\delta_\diamondsuit \setminus \{a^\delta\}, \quad \text{while} \quad F^\delta(a^\delta) \parallel i(\tau(a^\delta))^{-\frac{1}{2}}. \tag{2.10}$$

Remark 2.4 (Martingale property) For each $z \in \operatorname{Int}\Omega^\delta_\diamondsuit$, $F^\delta_{(\Omega^\delta\setminus[a^\delta\gamma^\delta_j];\,\gamma^\delta_j, b^\delta)}(z)$ is a martingale with respect to the growing interface $(a^\delta = \gamma^\delta_0, \gamma^\delta_1, \ldots, \gamma^\delta_j, \ldots)$ (till the stopping time when γ^δ hits ξ or z becomes separated from b^δ by the interface, see Fig. 4A).

Proof It is sufficient to check that F^δ has the martingale property when γ^δ makes one step. Let $a^\delta_{\mathrm{L}}, \ldots, a^\delta_{\mathrm{R}}$ denote all possibilities for the first step. Then,

$$\begin{aligned}\widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow b^\delta)} = \tan\frac{1}{2}\theta(a^\delta)\cdot\big[\widetilde{Z}_{(\Omega^\delta\setminus[a^\delta a^\delta_{\mathrm{L}}];\, a^\delta_{\mathrm{L}} \rightsquigarrow b^\delta)} + \cdots \\ + \widetilde{Z}_{(\Omega^\delta\setminus[a^\delta a^\delta_{\mathrm{R}}];\, a^\delta_{\mathrm{R}} \rightsquigarrow b^\delta)}\big],\end{aligned} \tag{2.11}$$

and so

$$\mathbb{P}(\gamma^\delta_1 = a^\delta_{\mathrm{L}}) = \frac{\widetilde{Z}_{(\Omega^\delta\setminus[a^\delta a^\delta_{\mathrm{L}}];\, a^\delta_{\mathrm{L}} \rightsquigarrow b^\delta)}}{[\tan\frac{1}{2}\theta(a^\delta)]^{-1}\widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow b^\delta)}}, \quad \ldots,$$

$$\mathbb{P}(\gamma_1^\delta = a_{\mathrm{R}}^\delta) = \frac{\widetilde{Z}_{(\Omega^\delta \setminus [a^\delta a_{\mathrm{R}}^\delta];\, a_{\mathrm{R}}^\delta \rightsquigarrow b^\delta)}}{[\tan\frac{1}{2}\theta(a^\delta)]^{-1}\widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow b^\delta)}}.$$

Taking into account that the difference

$$\mathrm{winding}(a^\delta \rightsquigarrow b^\delta) - \mathrm{winding}(a_{\mathrm{L}}^\delta \rightsquigarrow b^\delta) = \mathrm{winding}(a^\delta \to a_{\mathrm{L}}^\delta),$$

we obtain

$$\begin{aligned}
&\mathbb{P}(\gamma_1^\delta = a_{\mathrm{L}}^\delta) \cdot \frac{F^\delta_{(\Omega^\delta \setminus [a^\delta a_{\mathrm{L}}^\delta];\, a_{\mathrm{L}}^\delta, b)}(z)}{F^\delta_{(\Omega^\delta;\, a^\delta, b)}(z)} \\
&\quad = \frac{e^{-\frac{i}{2}\mathrm{winding}(a^\delta \to a_{\mathrm{L}}^\delta)}}{[\tan\frac{1}{2}\theta(a^\delta)]^{-1}} \cdot \frac{\widetilde{Z}_{(\Omega^\delta \setminus [a^\delta a_{\mathrm{L}}^\delta];\, a_{\mathrm{L}}^\delta \rightsquigarrow z)} \cdot \mathbb{E} e^{-\frac{i}{2}\mathrm{winding}(\gamma^\delta;\, a_{\mathrm{L}}^\delta \rightsquigarrow z)}}{\widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow z)} \cdot \mathbb{E} e^{-\frac{i}{2}\mathrm{winding}(\gamma^\delta;\, a^\delta \rightsquigarrow z)}}
\end{aligned}$$

and so on. On the other hand, counting "interface pictures" depending on the first step as in (2.11), we easily obtain

$$\begin{aligned}
&\left[\tan\frac{1}{2}\theta(a^\delta)\right]^{-1} \widetilde{Z}_{(\Omega^\delta;\, a^\delta \rightsquigarrow z)} \cdot \mathbb{E} e^{-\frac{i}{2}\mathrm{winding}(\gamma^\delta;\, a^\delta \rightsquigarrow z)} \\
&\quad = e^{-\frac{i}{2}\mathrm{winding}(a^\delta \to a_{\mathrm{L}}^\delta)} \cdot \widetilde{Z}_{(\Omega^\delta \setminus [a^\delta a_{\mathrm{L}}^\delta];\, a_{\mathrm{L}}^\delta \rightsquigarrow z)} \cdot \mathbb{E} e^{-\frac{i}{2}\mathrm{winding}(\gamma^\delta;\, a_{\mathrm{L}}^\delta \rightsquigarrow z)} + \cdots \\
&\qquad + e^{-\frac{i}{2}\mathrm{winding}(a^\delta \to a_{\mathrm{R}}^\delta)} \cdot \widetilde{Z}_{(\Omega^\delta \setminus [a^\delta a_{\mathrm{R}}^\delta];\, a_{\mathrm{R}}^\delta \rightsquigarrow z)} \cdot \mathbb{E} e^{-\frac{i}{2}\mathrm{winding}(\gamma^\delta;\, a_{\mathrm{R}}^\delta \rightsquigarrow z)},
\end{aligned}$$

which gives the result. □

2.2.2 Discrete boundary value problem for F^δ

Proposition 2.5 *For each pair of neighbors $z_0, z_1 \in \Omega^\delta_\diamondsuit$ separated by an edge $(w_1 u)$, we have*

$$\mathrm{Proj}\big[F^\delta(z_0)\,;\,[i(w_1 - u)]^{-\frac{1}{2}}\big] = \mathrm{Proj}\big[F^\delta(z_1)\,;\,[i(w_1 - u)]^{-\frac{1}{2}}\big]. \tag{2.12}$$

The proposition amounts to saying that F^δ is *spin holomorphic* as in Definition 3.1 below (see Fig. 1C for notation).

Proof The proof is based on the bijection between the set of "interface pictures" $(\Omega^\delta;\, a^\delta \rightsquigarrow z_0)$ and the similar set $(\Omega^\delta;\, a^\delta \rightsquigarrow z_1)$, which is schematically drawn in Fig. 5. The relative contributions of the corresponding pairs

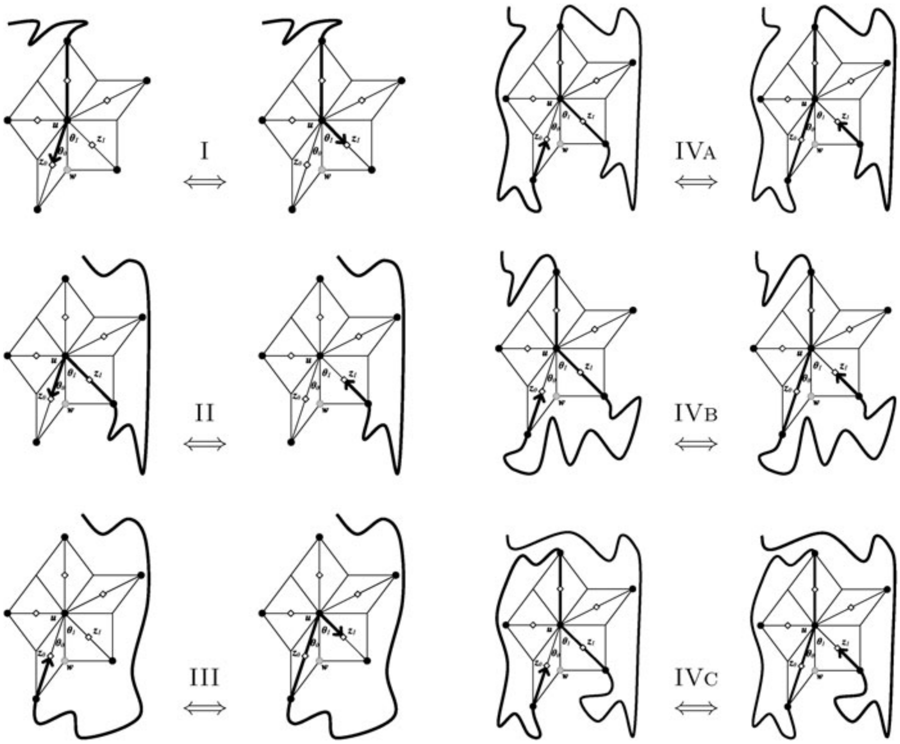


Fig. 5 Bijection between the sets $(\Omega^\delta;\, a^\delta \rightsquigarrow z_0)$ and $(\Omega^\delta;\, a^\delta \rightsquigarrow z_1)$ (local notations $z_{0,1}$, $\theta_{0,1}$, u and w_1 are given in Fig. 1C). In cases *I–III*, the winding $\varphi_1 = \text{winding}(\gamma_1^\delta;\, a^\delta \rightsquigarrow z_1)$ is unambiguously defined by $\varphi_0 = \text{winding}(\gamma_0^\delta;\, a^\delta \rightsquigarrow z_0)$. In case *IV*, there are two possibilities: φ_1 is equal to either $\varphi_0 - 2\pi + \theta_0 + \theta_1$ (*IVa*, *IVc*) or $\varphi_0 + 2\pi + \theta_0 + \theta_1$ (*IVb*)

to $F^\delta(z_0)$ and $F^\delta(z_1)$ (up to the same real factor) are given in the following table:

	$F^\delta(z_0)$	$F^\delta(z_1)$
I	$[\cos\frac{1}{2}\theta_0]^{-1}\cdot e^{-\frac{i}{2}\varphi}$	$[\cos\frac{1}{2}\theta_1]^{-1}\cdot e^{-\frac{i}{2}(\varphi+\theta_0+\theta_1)}$
II	$[\cos\frac{1}{2}\theta_0]^{-1}\tan\frac{1}{2}\theta_1\cdot e^{-\frac{i}{2}\varphi}$	$[\cos\frac{1}{2}\theta_1]^{-1}\cdot e^{-\frac{i}{2}(\varphi-\pi+\theta_0+\theta_1)}$
III	$[\cos\frac{1}{2}\theta_0]^{-1}\cdot e^{-\frac{i}{2}\varphi}$	$[\cos\frac{1}{2}\theta_1]^{-1}\tan\frac{1}{2}\theta_0\cdot e^{-\frac{i}{2}(\varphi-\pi+\theta_0+\theta_1)}$
IV	$[\cos\frac{1}{2}\theta_0]^{-1}\tan\frac{1}{2}\theta_1\cdot e^{-\frac{i}{2}\varphi}$	$[\cos\frac{1}{2}\theta_1]^{-1}\tan\frac{1}{2}\theta_0\cdot e^{-\frac{i}{2}(\varphi\pm2\pi+\theta_0+\theta_1)}$

where $\varphi = \text{winding}(\gamma^\delta;\, a^\delta \rightsquigarrow z_0) - \text{winding}(a^\delta \rightsquigarrow b^\delta) + \arg\tau(b^\delta)$. Note that

$$e^{-\frac{i}{2}\varphi} \parallel [w_1 - w_0]^{-\frac{1}{2}}, \qquad e^{-\frac{i}{2}(\varphi+\theta_0+\theta_1)} \parallel [w_2 - w_1]^{-\frac{1}{2}}, \quad \text{in cases I \& II,}$$

$$e^{-\frac{i}{2}\varphi} \parallel [w_0 - w_1]^{-\frac{1}{2}}, \qquad e^{-\frac{i}{2}(\varphi+\theta_0+\theta_1)} \parallel [w_1 - w_2]^{-\frac{1}{2}}, \quad \text{in cases III \& IV.}$$

A simple trigonometric calculation then shows that the (relative) contributions to *both* projections $\mathrm{Proj}[F^\delta(z_j); [i(w-u)]^{-\frac{1}{2}}]$ for $j=0,\ 1$ are equal to 1, $\tan\frac{1}{2}\theta_1$, $\tan\frac{1}{2}\theta_0$ and $\tan\frac{1}{2}\theta_0\tan\frac{1}{2}\theta_1$ in cases I–IV, respectively. □

Summing it up, we arrive at

Discrete Riemann boundary value problem for F^δ (spin-case) The function F^δ is defined in $\Omega^\delta_\diamondsuit$ so that discrete holomorphicity (2.12) holds for every pair of neighbors $z_0, z_1 \in \Omega^\delta_\diamondsuit$. Furthermore, F^δ satisfies the boundary conditions (2.10) and is normalized at b^δ.

3 S-holomorphic functions on isoradial graphs

3.1 Preliminaries. Discrete harmonic and discrete holomorphic functions on isoradial graphs

We start with basic definitions of the discrete complex analysis on isoradial graphs, more details can be found in Appendix and our paper [10], where a "toolbox" of discrete versions of continuous results is provided.

Let Γ be an isoradial graph, and H be defined on some vertices of Γ. We define its discrete Laplacian whenever possible by

$$[\Delta^\delta H](u) := \frac{1}{\mu^\delta_\Gamma(u)} \sum_{u_s\sim u} \tan\theta_s \cdot [H(u_s)-H(u)], \qquad (3.1)$$

where $\mu^\delta_\Gamma(u) = \frac{1}{2}\delta^2\sum_{u_s\sim u}\sin 2\theta_s$ (see Fig. 1B for notation). Function H is called (*discrete*) *harmonic* in some discrete domain Ω^δ_Γ if $\Delta^\delta H = 0$ at all interior vertices of Ω^δ_Γ. It is worthwhile to point out that, on isoradial graphs, as in the continuous setup, harmonic functions satisfy some (uniform w.r.t. δ and the structure of $\diamondsuit$) variant of the Harnack's Lemma (see Proposition A.4).

Let $\omega^\delta(u; E; \Omega^\delta_\Gamma)$ denote the *harmonic measure* of $E \subset \partial\Omega^\delta_\Gamma$ viewed from $u \in \mathrm{Int}\,\Omega^\delta_\Gamma$, i.e., the probability that the random walk generated by (3.1) (i.e. such that transition probabilities at u are proportional to $\tan\theta_s$'s) on Γ started from u exits Ω^δ_Γ through E. As usual, ω^δ is a probability measure on $\partial\Omega^\delta_\Gamma$ and a harmonic function of u. If Ω^δ_Γ is bounded, then we have

$$H(u) = \sum_{a\in\partial\Omega^\delta_\Gamma} \omega^\delta(u; \{a\}; \Omega^\delta_\Gamma)\cdot H(a), \qquad u \in \mathrm{Int}\,\Omega^\delta_\Gamma,$$

for any discrete harmonic in Ω^δ_Γ function H. Below we use some uniform (w.r.t. δ and the structure of Γ) estimates of ω^δ from [10], which are quoted in Appendix.

Let H be defined on some part of Γ or Γ^* or $\Lambda = \Gamma \cup \Gamma^*$ and z be the center of the rhombus $v_1v_2v_3v_4$. We set

$$[\partial^\delta H](z) := \frac{1}{2}\left[\frac{H(v_1)-H(v_3)}{v_1-v_3} + \frac{H(v_2)-H(v_4)}{v_2-v_4}\right], \quad z \in \diamondsuit.$$

Furthermore, let F be defined on some subset of $\diamondsuit$. We define its discrete $\overline{\partial}^\delta$-derivative by setting

$$[\overline{\partial}^\delta F](u) = -\frac{i}{2\mu_\Gamma^\delta(u)} \sum_{z_s \sim u} (w_{s+1}-w_s)F(z_s), \quad u \in \Lambda = \Gamma \cup \Gamma^* \tag{3.2}$$

(see Fig. 1B for notation when $u \in \Gamma$). Function F is called (*discrete*) *holomorphic* in some discrete domain $\Omega_\diamondsuit^\delta \subset \diamondsuit$ if $\overline{\partial}^\delta F = 0$ at all interior vertices. It is easy to check that $\Delta^\delta = 4\overline{\partial}^\delta \partial^\delta$, and so $\partial^\delta H$ is holomorphic for any harmonic function H. Conversely, in simply connected domains, if F is holomorphic on $\diamondsuit$, then there exists a harmonic function $H = \int^\delta F(z)\, d^\delta z$ such that $\partial^\delta H = F$. Its components $H|_\Gamma$ and $H|_{\Gamma^*}$ are defined uniquely up to additive constants by

$$H(v_2) - H(v_1) = F\left(\frac{1}{2}(v_2+v_1)\right) \cdot (v_2 - v_1), \quad v_2 \sim v_1,$$

where both $v_1, v_2 \in \Gamma$ or both $v_1, v_2 \in \Gamma^*$, respectively.

It is most important that discrete holomorphic (on $\diamondsuit$) functions are Lipschitz continuous in an appropriate sense, see Corollary A.7. For the sake of the reader, we quote all other necessary results in Appendix.

3.2 S-holomorphic functions and the propagation equation for spinors

In this section we investigate the notion of s-holomorphicity which appears naturally for holomorphic fermions in the Ising model (see (2.6), (2.12)). We discuss its connections to spinors defined on the double-covering of $\diamondsuit$ edges (see [33]) and essentially equivalent to the introduction of disorder operators (see [22, 40] and the references therein). We don't refer to this discussion (except Definition 3.1 and elementary Lemma 3.2) in the rest of our paper.

Definition 3.1 Let $\Omega_\diamondsuit^\delta \subset \diamondsuit$ be some discrete domain and $F : \Omega_\diamondsuit^\delta \to \mathbb{C}$. We call function F *strongly* or *spin holomorphic*, or *s-holomorphic* for short, if for each pair of neighbors $z_0, z_1 \in \Omega_\diamondsuit^\delta$, $z_0 \sim z_1$, the following projections of two values of F are equal:

$$\mathrm{Proj}[F(z_0); [i(w_1-u)]^{-\frac{1}{2}}] = \mathrm{Proj}[F(z_1); [i(w_1-u)]^{-\frac{1}{2}}] \tag{3.3}$$

or, equivalently,

$$\overline{F(z_1)} - \overline{F(z_0)} = -i(w_1 - u)\delta^{-1} \cdot (F(z_1) - F(z_0)), \tag{3.4}$$

where $(w_1 u)$, $u \in \Gamma$, $w_1 \in \Gamma^*$, is the common edge of rhombi z_0, z_1 (see Fig. 1C).

Recall that orthogonal projection of X on ν satisfies

$$\mathrm{Proj}[X; \nu] = \mathrm{Re}\left(X \frac{\overline{\nu}}{|\nu|} \right) \frac{\nu}{|\nu|} = \frac{X}{2} + \frac{\overline{X}\nu^2}{2|\nu|^2},$$

which we used above.

It's easy to check that the property to be s-holomorphic is stronger than the usual discrete holomorphicity:

Lemma 3.2 *If $F : \Omega^\delta_\diamond \to \mathbb{C}$ is s-holomorphic, then F is holomorphic in $\Omega^\delta_\diamond$, i.e., $[\overline{\partial}^\delta F](v) = 0$ for all $v \in \mathrm{Int}\,\Omega^\delta_\Lambda$.*

Proof Let $v = u \in \Gamma$ (the case $v = w \in \Gamma^*$ is essentially the same) and $F_s = F(z_s)$ (see Fig. 1B for notation). Then

$$-i \sum_{s=1}^{n} (w_{s+1} - w_s) F_s = -i \sum_{s=1}^{n} (w_{s+1} - u)(F_s - F_{s+1})$$
$$= -\delta \cdot \sum_{s=1}^{n} (\overline{F}_s - \overline{F}_{s+1}) = 0.$$

Thus, $[\overline{\partial}^\delta F](u) = 0$. □

Conversely, in a simply-connected domain every discrete holomorphic function can be decomposed into the sum of two s-holomorphic functions (one multiplied by i):

Lemma 3.3 *Let $\Omega^\delta_\diamond$ be a simply connected discrete domain and $F : \Omega^\delta_\diamond \to \mathbb{C}$ be a discrete holomorphic function. Then there are (unique up to an additive constant) s-holomorphic functions $F_1, F_2 : \Omega^\delta_\diamond \to \mathbb{C}$ such that $F = F_1 + iF_2$.*

Proof Let Ω^δ_Υ denote the set of all oriented edges $\xi = [\xi_{\mathrm{b}}\xi_{\mathrm{w}}]$ of the rhombic lattice $\diamond$ connecting neighboring vertices $\xi_{\mathrm{b}} \in \Omega^\delta_\Gamma$, $\xi_{\mathrm{w}} \in \Omega^\delta_{\Gamma^*}$. For a function $F : \Omega^\delta_\diamond \to \mathbb{C}$, we define its differential on edges (more precisely, dF a 1-form on the edges of the dual graph, but there is no difference) by

$$dF([uw_1]) := F(z_1) - F(z_0), \quad dF : \Omega^\delta_\Upsilon \to \mathbb{C},$$

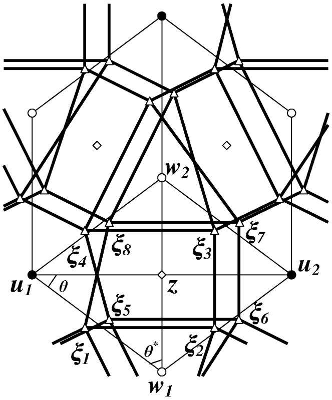


Fig. 6 Double covering $\widehat{\Upsilon}$ of Υ (= edges of $\diamondsuit$)

The spinor $\mathcal{S}(\xi) := [i(w_\xi - u_\xi)]^{-\frac{1}{2}}$ is naturally defined on $\widehat{\Upsilon}$ ("continuously" around rhombi and vertices). In particular,

$$\mathcal{S}(\xi_1) = -\mathcal{S}(\xi_5) =: \alpha,$$
$$\mathcal{S}(\xi_2) = -\mathcal{S}(\xi_6) = e^{i\theta^*}\alpha,$$
$$\mathcal{S}(\xi_3) = -\mathcal{S}(\xi_7) = i\alpha,$$
$$\mathcal{S}(\xi_4) = -\mathcal{S}(\xi_8) = ie^{i\theta^*}\alpha.$$

(see Fig. 1C for notation). Then, a given antisymmetric function G defined on Ω^δ_Υ is the differential of some discrete holomorphic function F (uniquely defined on $\Omega^\delta_\diamondsuit$ up to an additive constant) if for each (black or white) vertex $u \in \Omega^\delta_\Lambda$ (see Fig. 1B for notation when $u \in \Gamma$) two identities hold:

$$\sum_{s=1}^{n} G([uw_s]) = 0 \quad \text{and} \quad \sum_{s=1}^{n} G([uw_s])(w_s - u) = 0. \tag{3.5}$$

Indeed, the first identity means that G is an exact form and so a differential of some function F, and the second ensures that F is holomorphic by (3.2):

$$\sum_{s=1}^{n} G([uw_s])(w_s - u) = \sum_{s=1}^{n}(F(z_s) - F(z_{s-1}))(w_s - u)$$
$$= -\sum_{s=1}^{n} F(z_s)(w_{s+1} - w_s).$$

Note that identities (3.5) are invariant under the antilinear involution $G \mapsto G^\sharp$, where

$$G^\sharp([uw]) := \overline{G([uw])} \cdot i(\overline{w} - \overline{u})\delta^{-1}.$$

On the other hand, from (3.4) we see that F is s-holomorphic iff $[dF]^\sharp = dF$. Thus, the functions F_j defined by $dF_1 = \frac{1}{2}(dF + (dF)^\sharp)$ and $dF_2 = \frac{1}{2i}(dF - (dF)^\sharp)$ are s-holomorphic and do the job. Uniqueness easily follows from (3.3): If $0 = F_1 + iF_2$, and both functions are s-holomorphic, then (3.3) implies that $F_1(z_0) = F_2(z_0)$, and uniqueness follows. $\square$

Following Ch. Mercat [33], we denote by $\widehat{\Upsilon}$ the *double-covering* of the set Υ of edges $\diamondsuit$ which is connected around each $z \in \diamondsuit$ and each $v \in \Lambda$ (see [33, p. 209]). A function S defined on $\widehat{\Upsilon}$ is called a *spinor* if it changes the sign between the sheets. The simplest example is the square root $[i(w-u)]^{-\frac{1}{2}}$ naturally defined on $\widehat{\Upsilon}$ (see Fig. 6). We say that a spinor S satisfies the *propagation equation* (see [33, p. 210] for historical remarks and Fig. 6 for notation) if, when walking around the edges $\xi_1, \ldots, \xi_8$ of a doubly-covered rhombus, for any three consecutive edges spinor values satisfy

$$S(\xi_{j+2}) = (\cos\theta_j)^{-1} \cdot S(\xi_{j+1}) - \tan\theta_j \cdot S(\xi_j), \tag{3.6}$$

where θ_j denotes the half-angle "between" ξ_j and ξ_{j+1}, i.e., $\theta_j = \theta$, if j is odd, and $\theta_j = \frac{\pi}{2} - \theta$, if j is even.

Lemma 3.4 *Let $\Omega^\delta_\diamondsuit$ be a simply connected discrete domain. If a function $F : \Omega^\delta_\diamondsuit \to \mathbb{C}$ is s-holomorphic then the real spinor*

$$F_{\widehat{\Upsilon}}([uw]) := \mathrm{Re}[F(z_0) \cdot [i(w-u)]^{\frac{1}{2}}] = \mathrm{Re}[F(z_1) \cdot [i(w-u)]^{\frac{1}{2}}] \tag{3.7}$$

satisfies the propagation equation (3.6). *Conversely, if $F_{\widehat{\Upsilon}}$ is a real spinor satisfying* (3.6), *then there exists a unique s-holomorphic function F such that* (3.7) *holds.*

Proof Note that (3.6) is nothing but the relation between the projections of the same complex number onto three directions $\alpha = [i(w_1-u_1)]^{-\frac{1}{2}}$, $e^{i(\frac{\pi}{2}-\theta)}\alpha = [i(w_2-u_1)]^{-\frac{1}{2}}$ and $i\alpha = [i(w_2-u_2)]^{-\frac{1}{2}}$. Thus, s-holomorphicity of F implies (3.6). Conversely, starting with some real spinor $F_{\widehat{\Upsilon}}$ satisfying (3.6), one can construct a function F such that

$$\mathrm{Proj}[F(z); [i(w-u)]^{-\frac{1}{2}}] = F_{\widehat{\Upsilon}}([uw]) \cdot [i(w-u)]^{-\frac{1}{2}}$$

(which is equivalent to (3.7)) for any z, and this function is s-holomorphic by the construction. □

3.3 Integration of F^2 for s-holomorphic functions

Lemma 3.5 *If $F, \widetilde{F} : \Omega^\delta_\diamondsuit \to \mathbb{C}$ are s-holomorphic, then $[\overline{\partial}^\delta(F\widetilde{F})](v) \in i\mathbb{R}$, $v \in \Omega^\delta_\Lambda$.*

Proof Let $v = u \in \Gamma$ (the case $v = w \in \Gamma^*$ is essentially the same) and denote $F_s := F(z_s)$ (see Fig. 1B for notation). Using (3.4), we infer that $\overline{\partial}^\delta F^2$

satisfies:

$$-i\sum_{s=1}^{n}(w_{s+1}-w_s)F_s^2$$
$$=-\delta\cdot\sum_{s=1}^{n}(F_s+F_{s+1})(\overline{F}_s-\overline{F}_{s+1})=2i\delta\cdot\mathrm{Im}\sum_{s=1}^{n}\overline{F}_sF_{s+1}\in i\mathbb{R}.$$

Therefore, $4[\overline{\partial}^\delta(F\widetilde{F})](u)=[\overline{\partial}^\delta(F+\widetilde{F})^2](u)-[\overline{\partial}^\delta(F-\widetilde{F})^2](u)\in i\mathbb{R}$. □

In the continuous setup, the condition $\mathrm{Re}[\overline{\partial}G](z)\equiv 0$ allows one to define the function $\mathrm{Im}\int G(z)\,dz$ (i.e., in simply connected domains, the integral doesn't depend on the path). It is easy to check that the same holds in the discrete setup. Namely, if $\Omega^\delta_\diamondsuit$ is simply connected and $G:\Omega^\delta_\diamondsuit\to\mathbb{C}$ is such that $\mathrm{Re}[\overline{\partial}^\delta G]\equiv 0$ in $\Omega^\delta_\diamondsuit$, then the discrete integral $H=\mathrm{Im}\int^\delta G(z)\,d^\delta z$ is well-defined (i.e., doesn't depend on the path of integration) on both $\Omega^\delta_\Gamma\subset\Gamma$ and $\Omega^\delta_{\Gamma^*}\subset\Gamma^*$ up to two (different for Γ and Γ^*) additive constants.

It turns out that if $G=F^2$ for some s-holomorphic F, then $H=\mathrm{Im}\int^\delta (F(z))^2\,d^\delta z$ can be defined *simultaneously* on Γ and Γ^* (up to *one* additive constant) in the following way:

$$H(u)-H(w_1):=2\delta\cdot|\mathrm{Proj}[F(z_j);[i(w_1-u)]^{-\frac{1}{2}}]|^2,\quad u\sim w_1,\tag{3.8}$$

where (uw_1), $u\in\Gamma$, $w_1\in\Gamma^*$, is the common edge of two neighboring rhombi $z_0,z_1\in\diamondsuit$ (see Fig. 1C), and taking $j=0,1$ gives the same value.

Proposition 3.6 *Let $\Omega^\delta_\diamondsuit$ be a simply connected discrete domain. If $F:\Omega^\delta_\diamondsuit\to\mathbb{C}$ is s-holomorphic, then*

(i) *function $H:\Omega^\delta_\Lambda\to\mathbb{C}$ is well-defined (up to an additive constant) by* (3.8);
(ii) *for any neighboring $v_1,v_2\in\Omega^\delta_\Gamma\subset\Gamma$ or $v_1,v_2\in\Omega^\delta_{\Gamma^*}\subset\Gamma^*$ the identity*

$$H(v_2)-H(v_1)=\mathrm{Im}\left[(v_2-v_1)\left(F\left(\frac{1}{2}(v_1+v_2)\right)\right)^2\right]$$

holds (and so $H=\mathrm{Im}\int^\delta(F(z))^2\,d^\delta z$ on both Γ and Γ^*);
(iii) *H is (discrete) subharmonic on Γ and superharmonic on Γ^*, i.e.,*

$$[\Delta^\delta H](u)\geq 0\quad\textit{and}\quad[\Delta^\delta H](w)\leq 0$$

for all $u\in\mathrm{Int}\,\Omega^\delta_\Gamma\subset\Gamma$ and $w\in\mathrm{Int}\,\Omega^\delta_{\Gamma^*}\subset\Gamma^*$;

Proof (i), (ii) Let z be the center of the rhombus $u_1w_1u_2w_2$. For $j=1,2$ we have

$$\begin{aligned}
&[H(u_2)-H(w_j)]+[H(w_j)-H(u_1)]\\
&\quad=2|\operatorname{Re}[[i(w_j-u_2)]^{1/2}F(z)]|^2-2|\operatorname{Re}[[i(w_j-u_1)]^{1/2}F(z)]|^2\\
&\quad=\frac{1}{2}[[i(w_j-u_2)]^{1/2}F(z)+[-i(\overline{w_j}-\overline{u_2})]^{1/2}\overline{F(z)}]^2\\
&\qquad-\frac{1}{2}[[i(w_j-u_1)]^{1/2}F(z)+[-i(\overline{w_j}-\overline{u_1})]^{1/2}\overline{F(z)}]^2\\
&\quad=\frac{1}{2}[i(u_1-u_2)(F(z))^2-i(\overline{u}_1-\overline{u}_2)(\overline{F(z)})^2]=\operatorname{Im}[(u_2-u_1)(F(z))^2].
\end{aligned}$$

The computations for $H(w_2)-H(w_1)$ are similar.

(iii) Let $u\in\Gamma$ and set $F_s:=F(z_s)$ (see Fig. 1B for notation). Denote

$$t_s:=\operatorname{Proj}[F_s\,;[i(w_s-u)]^{-\frac{1}{2}}]=\operatorname{Proj}[F_{s-1}\,;[i(w_s-u)]^{-\frac{1}{2}}].$$

Knowing two projections of F_s uniquely determines it's value:

$$F_s=\frac{i(t_se^{-i\theta_s}-t_{s+1}e^{i\theta_s})}{\sin\theta_s}$$

and so

$$\begin{aligned}
&\mu_\Gamma^\delta(u)\cdot[\Delta^\delta H](u)\\
&\quad=-\operatorname{Im}\left[\sum_{s=1}^n\tan\theta_s\cdot\frac{(t_se^{-i\theta_s}-t_{s+1}e^{i\theta_s})^2\cdot(u_s-u)}{\sin^2\theta_s}\right]\\
&\quad=-2\delta\cdot\operatorname{Im}\left[\sum_{s=1}^n\frac{(t_s^2e^{-2i\theta_s}-2t_st_{s+1}+t_{s+1}^2e^{2i\theta_s})\cdot e^{i\arg(u_s-u)}}{\sin\theta_s}\right].
\end{aligned}$$

Let $t_s=x_s\cdot\exp[i\arg([i(w_s-u)]^{-\frac{1}{2}})]$, where $x_s\in\mathbb{R}$ and the argument of the square root changes "continuously" when we move around u, taking $s=1,\dots,n$. Then

$$\begin{aligned}
&(2\delta)^{-1}\mu_\Gamma^\delta(u)\cdot[\Delta^\delta H](u)\\
&\quad=-\operatorname{Im}\left[-i\sum_{s=1}^n\frac{x_s^2e^{-i\theta_s}\mp 2x_sx_{s+1}+x_{s+1}^2e^{i\theta_s}}{\sin\theta_s}\right]
\end{aligned}$$

$$= \sum_{s=1}^{n} \frac{\cos\theta_s \cdot (x_s^2 + x_{s+1}^2) \mp 2x_s x_{s+1}}{\sin\theta_s} =: Q^{(n)}_{\theta_1;\ldots;\theta_n}(x_1, \ldots, x_n) \ge 0,$$

where "$\mp$" is "$-$" for all terms except $x_n x_1$ (these signs come from our choice of arguments). The non-negativity of the quadratic form $Q^{(n)}$ (for arbitrary $\theta_1, \ldots, \theta_n > 0$ with $\theta_1 + \cdots + \theta_n = \pi$) can be easily shown by induction. Indeed, the identity

$$\begin{aligned} &Q^{(n)}_{\theta_1;\ldots;\theta_n}(x_1, \ldots, x_n) - Q^{(n-1)}_{\theta_1;\ldots;\theta_{n-2}\,;\,\theta_{n-1}+\theta_n}(x_1, \ldots, x_{n-1}) \\ &\quad = Q^{(3)}_{\pi-\theta_{n-1}-\theta_n\,;\,\theta_{n-1}\,;\,\theta_n}(x_1, x_{n-1}, x_n), \end{aligned}$$

reduces the problem to the non-negativity of $Q^{(3)}$'s. But, if $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi$, then

$$\begin{aligned} &Q^{(3)}_{\alpha\,;\,\beta\,;\,\gamma}(x, y, z) \\ &\quad = \left[\frac{\sin^{\frac12}\beta}{\sin^{\frac12}\alpha \cdot \sin^{\frac12}\gamma} \cdot x - \frac{\sin^{\frac12}\gamma}{\sin^{\frac12}\alpha \cdot \sin^{\frac12}\beta} \cdot y + \frac{\sin^{\frac12}\alpha}{\sin^{\frac12}\beta \cdot \sin^{\frac12}\gamma} \cdot z \right]^2. \end{aligned}$$

This finishes the proof for $v = u \in \Gamma$, the case $v = w \in \Gamma^*$ is similar. The opposite sign of $[\Delta^\delta H](w)$ comes from the invariance of definition (3.3) under the (simultaneous) multiplication of F by i and the transposition of Γ and Γ^*. □

Remark 3.7 As it was shown above, the quadratic form $Q^{(n)}_{\theta_1;\ldots;\theta_n}(x_1, \ldots, x_n)$ can be represented as a sum of $n-2$ perfect squares $Q^{(3)}$. Thus, its kernel is (real) two-dimensional. Clearly, this corresponds to the case when all the (complex) values $F(z_s)$ are equal to each other, since in this case $[\Delta^\delta H] = 0$ by definition. Thus,

$$\delta \cdot |[\Delta^\delta H](u)| \asymp Q^{(n)}_{\theta_1;\ldots;\theta_n}(x_1, \ldots, x_n) \asymp \sum_{s=1}^{n} |F(z_{s+1}) - F(z_s)|^2, \qquad (3.9)$$

since both sides considered as real quadratic forms in $x_1, \ldots, x_n$ (with coefficients of order 1) are nonnegative and have the same two-dimensional kernel.

3.4 Harnack lemma for the integral of F^2

As it was shown above, using (3.8), for any s-holomorphic function F, one can define a function $H = \operatorname{Im} \int^\delta (F^\delta(z))^2 \, d^\delta z$ on both Γ and Γ^*. Moreover, $H|_\Gamma$ is subharmonic while $H|_{\Gamma^*}$ is superharmonic. It turns out, that H, despite not being a harmonic function, *a priori* satisfies a version of the Harnack

Lemma (cf. Proposition A.4(i)). In Sect. 5.1 we also prove a version of the boundary Harnack principle which compares the values of H in the bulk with its normal derivative on a straight part of the boundary.

We start by showing that $H|_\Gamma$ and $H|_{\Gamma^*}$ cannot differ too much.

Proposition 3.8 *Let $w \in \mathrm{Int}\,\Omega^\delta_{\Gamma^*}$ be an inner face surrounded by inner vertices $u_s \in \mathrm{Int}\,\Omega^\delta_\Gamma$ and faces $w_s \in \Omega^\delta_{\Gamma^*}$, $s = 1, \dots, n$. If H is defined by* (3.8) *for some s-holomorphic function $F : \Omega^\delta_\diamondsuit \to \mathbb{C}$, then*

$$\max_{s=1,\dots,n} H(u_s) - H(w) \le \mathrm{const} \cdot \Big(H(w) - \min_{s=1,\dots,n} H(w_s) \Big).$$

Remark 3.9 Since definition (3.3) is invariant under the (simultaneous) multiplication of F by i and the transposition of Γ and Γ^*, one also has

$$H(u) - \min_{s=1,\dots,n} H(w_s) \le \mathrm{const} \cdot \Big(\max_{s=1,\dots,n} H(u_s) - H(u) \Big)$$

for any inner vertex $u \in \mathrm{Int}\,\Omega^\delta_\Gamma$ surrounded by $w_s \in \mathrm{Int}\,\Omega^\delta_{\Gamma^*}$ and $u_s \in \Omega^\delta_\Gamma$, $s = 1, \dots, n$.

Proof By subtracting a constant, we may assume that $\min_{s=1,\dots,n} H(w_s) = 0$. Since $H|_{\Gamma^*}$ is superharmonic at w, it is non-negative there and moreover

$$H(w) \ge \mathrm{const} \cdot H(w_s) \quad \text{for all } s = 1, \dots, n.$$

Since $H|_\Gamma(w_s) \ge 0$, by (3.8), it is non-negative at all points of Γ which are neighbors of w_s's. From subharmonicity of $H|_\Gamma$ we similarly deduce that $H(u_s) \le \mathrm{const} \cdot H(u_{s+1})$. Therefore,

$$H(u_s) \asymp M := \max_{s=1,\dots,n} H(u_s) \quad \text{for all } s = 1, \dots, n.$$

We need to prove that $K := M/H(w) \le \mathrm{const}$. Assume the opposite, i.e. $K \gg 1$. Then, for any $s = 1, \dots, n$, one has

$$\frac{H(u_s) - H(w_s)}{H(u_s) - H(w)} = 1 + O(K^{-1}).$$

By (3.8), these increments of H are derived from projections of $F(z_s)$ on two directions, and so

$$\frac{|\mathrm{Proj}[F(z_s); [i(w_s - u_s)]^{-\frac{1}{2}}]|}{|\mathrm{Proj}[F(z_s); [i(w - u_s)]^{-\frac{1}{2}}]|} = 1 + O(K^{-1}).$$

Here z_s is a center of the rhombus $wu_sw_su_{s+1}$, and we infer

$$\arg F(z_s) \bmod \pi = \big[\text{either } \arg[(w_s-w)^{-\frac{1}{2}}] \text{ or } \arg[(w-w_s)^{-\frac{1}{2}}]\big] + O(K^{-1}).$$

Moreover, $\delta \cdot |F(z_s)|^2 \asymp H(u_s) - H(w) \asymp M$ for all $s = 1, \ldots, n$, if K is big enough. Since $F(z_s)$ and $F(z_{s+1})$ have very different arguments, using (3.9) we obtain

$$M \asymp \delta \cdot \sum_{s=1}^{n} |F(z_{s+1}) - F(z_s)|^2 \asymp \delta^2 \cdot |[\Delta^\delta H](w)| \le \text{const} \cdot H(w),$$

which contradicts to our assumption $K = M/H(w) \gg 1$ and completes the proof. $\square$

Remark 3.10 (Uniform comparability of H_Γ^δ and $H_{\Gamma^*}^\delta$) Suppose $H|_{\Gamma^*} \ge 0$ on $\partial\Omega_{\Gamma^*}^\delta$ and hence, due to its superharmonicity, everywhere inside $\Omega_{\Gamma^*}^\delta$. Then, definition (3.8) and Proposition 3.8 give

$$H_{\Gamma^*}^\delta(w) \le H_\Gamma^\delta(u) \le \text{const} \cdot H_{\Gamma^*}^\delta(w)$$

for all neighboring $u \sim w$ in Ω^δ, where the constant is independent of Ω^δ and δ.

Proposition 3.11 (Harnack Lemma for $\operatorname{Im} \int^\delta (F(z))^2 d^\delta z$) *Take $v_0 \in \Lambda = \Gamma \cup \Gamma^*$ and let $F : B_\diamondsuit^\delta(v_0, R) \to \mathbb{C}$ be an s-holomorphic function. Define*

$$H := \operatorname{Im} \int^\delta (F(z))^2 d^\delta z : \Omega_\Lambda^\delta \to \mathbb{R}$$

by (3.8) *so that $H \ge 0$ in $B_\Lambda^\delta(v_0, R)$. Then,*

$$H(v_1) \le \text{const} \cdot H(v_0) \quad \textit{for any } v_1 \in B_\Lambda^\delta\left(v_0, \frac{1}{2}r\right).$$

Proof Due to Remark 3.10, we may assume that $v_0 \in \Gamma^*$ while $v_1 \in \Gamma$. Set $M := \max_{u \in B_\Gamma^\delta(v_0, \frac{1}{2}r)} H(u)$. Since the function $H|_\Gamma$ is subharmonic, one has

$$M = H(v_1) \le H(v_2) \le H(v_3) \le \cdots$$

for some path of consecutive neighbors $K_\Gamma^\delta = \{v_1 \sim v_2 \sim v_3 \sim \cdots\} \subset \Gamma$ running from v_1 to the boundary $\partial B_\Gamma^\delta(v_0, R)$. Take a nearby path of consecutive neighbors $K_{\Gamma^*}^\delta = \{w_1^\delta \sim w_2^\delta \sim w_3^\delta \sim \cdots\} \subset \Gamma^*$ starting in $\partial B_{\Gamma^*}^\delta(v_0, \frac{1}{2}R)$ and running to $\partial B_{\Gamma^*}^\delta(v_0, R)$. By Remark 3.10, $H|_{\Gamma^*} \ge \text{const} \cdot M$ on $K_{\Gamma^*}^\delta$, and so

$$H(v_0) \ge \text{const} \cdot M \cdot \omega^\delta(v_0; K_{\Gamma^*}^\delta; B_{\Gamma^*}^\delta(v_0, R) \setminus K_{\Gamma^*}^\delta) \ge \text{const} \cdot M,$$

since $H|_{\Gamma^*}$ is superharmonic and the discrete harmonic measure of the path $K^\delta_{\Gamma^*}$ viewed from v_0 is uniformly bounded from below due to standard random walk arguments (cf. [10] Proposition 2.11). □

3.5 Regularity of s-holomorphic functions

Theorem 3.12 *For a simply connected $\Omega^\delta_\diamondsuit$ and an s-holomorphic $F:\Omega^\delta_\diamondsuit\to\mathbb{C}$ define $H=\int^\delta (F(z))^2 d^\delta z:\Omega^\delta_\Lambda\to\mathbb{C}$ by (3.8) in accordance with Proposition 3.6. Let point $z_0\in\operatorname{Int}\Omega^\delta_\diamondsuit$ be at definite distance from the boundary: $d=\operatorname{dist}(z_0;\partial\Omega^\delta_\diamondsuit)\geq \mathrm{const}\cdot\delta$ and set $M=\max_{v\in\Omega^\delta_\Lambda}|H(v)|$. Then*

$$|F(z_0)|\leq \mathrm{const}\cdot\frac{M^{1/2}}{d^{1/2}} \tag{3.10}$$

and, for any neighboring $z_1\sim z_0$,

$$|F(z_1)-F(z_0)|\leq \mathrm{const}\cdot\frac{M^{1/2}}{d^{3/2}}\cdot\delta. \tag{3.11}$$

Remark 3.13 Estimates (3.10) and (3.11) have exactly the same form as if H was harmonic. Due to (3.8) and (3.9), a posteriori this also means that the subharmonic function $H|_\Gamma$ and the superharmonic function $H|_{\Gamma^*}$ should be uniformly close to each other inside Ω^δ, namely $H|_\Gamma - H|_{\Gamma^*}=O(\delta M/d)$, and, moreover, $|\Delta^\delta H|=O(\delta M/d^3)$.

The proof consists of four steps:

Step 1. Let $B^\delta_\Gamma(z_0;r)\subset\Gamma$ denote the discrete disc centered at z_0 of radius r. Then the discrete L_1 norm (as defined below) of the Laplacian of H satisfies

$$\|\Delta^\delta H\|_{1;\,B^\delta_\Gamma(z_0;\frac34 d)}:=\sum\nolimits_{u\in B^\delta_\Gamma(z_0;\frac34 d)}|[\Delta^\delta H](u)|\mu^\delta_\Gamma(u)\leq \mathrm{const}\cdot M \tag{3.12}$$

and the same estimate for H restricted to Γ^* holds.

Proof of Step 1 Represent H on $B^\delta_\Gamma(z_0;d)$ as a sum of a harmonic function with the same boundary values and a subharmonic one:

$$H|_\Gamma=H_{\mathrm{harm}}+H_{\mathrm{sub}},\qquad \Delta^\delta H_{\mathrm{harm}}=0,\qquad \Delta^\delta H_{\mathrm{sub}}\geq 0\quad\text{in } B^\delta_\Gamma(z_0;d)$$

and

$$H_{\mathrm{harm}}=H,\qquad H_{\mathrm{sub}}=0\quad\text{on } \partial B^\delta_\Gamma(z_0;d).$$

Note that the negative function H_{sub} satisfies

$$H_{\mathrm{sub}}(\cdot)=\sum_{u\in\operatorname{Int}B^\delta_\Gamma(z_0;d)}G(\cdot\,;u)[\Delta^\delta H_{\mathrm{sub}}](u)\mu^\delta_\Gamma(u)$$

$$\leq \sum_{u \in B_\Gamma^\delta(z_0; \frac{3}{4}d)} G(\cdot\,; u)[\Delta^\delta H_{\mathrm{sub}}](u)\mu_\Gamma^\delta(u),$$

since the Green's function $G(\cdot\,; u)$ in $B_\Gamma^\delta(z_0; d)$ is negative. By the maximum principle, $|H_{\mathrm{harm}}| \leq M$ and so $|H_{\mathrm{sub}}| \leq 2M$ in $B_\Gamma^\delta(z_0; d)$. Therefore,

$$\begin{aligned} \mathrm{const} \cdot Md^2 &\geq \|H_{\mathrm{sub}}\|_{1\,; B_\Gamma^\delta(z_0; d)} \\ &\geq \|\Delta^\delta H_{\mathrm{sub}}\|_{1\,; B_\Gamma^\delta(z_0; \frac{3}{4}d)} \cdot \min_{u \in B_\Gamma^\delta(z_0; \frac{3}{4}d)} \|G(\cdot\,; u)\|_{1\,; B_\Gamma^\delta(z_0; d)}. \end{aligned}$$

Since $\Delta^\delta H_{\mathrm{sub}} = \Delta^\delta H$, the inequality (3.12) follows from the (uniform) estimate

$$\|G(\cdot\,; u)\|_{1\,; B_\Gamma^\delta(z_0; d)} \geq \mathrm{const} \cdot d^2 \quad \text{for all } u \in B_\Gamma^\delta\left(z_0; \frac{3}{4}d\right)$$

which we prove in Appendix (Lemma A.8). □

Step 2. The estimate

$$\|F\|^2_{2\,; B_\diamondsuit^\delta(z_0; \frac{1}{2}d)} := \sum_{z \in B_\diamondsuit^\delta(z_0; \frac{1}{2}d)} |F(z)|^2 \mu_\diamondsuit^\delta(z) \leq \mathrm{const} \cdot Md \tag{3.13}$$

holds.

Proof of Step 2 Since $H = \mathrm{Im} \int^\delta (F(z))^2 \, d^\delta z$ on both Γ and Γ^*, it is sufficient to prove

$$\|\partial^\delta [H|_\Gamma]\|_{1\,; B_\diamondsuit^\delta(z_0; \frac{1}{2}d)} \leq \mathrm{const} \cdot Md$$

and a similar estimate for $\partial^\delta [H|_{\Gamma^*}]$. Represent H on $B_\Gamma^\delta(z_0; \frac{3}{4}d)$ as a sum:

$$H|_\Gamma = H_{\mathrm{harm}} + H_{\mathrm{sub}}, \qquad \Delta^\delta H_{\mathrm{harm}} = 0, \qquad \Delta^\delta H_{\mathrm{sub}} \geq 0 \quad \text{in } B_\Gamma^\delta\left(z_0; \frac{3}{4}d\right)$$

and

$$H_{\mathrm{harm}} = H, \qquad H_{\mathrm{sub}} = 0 \quad \text{on } \partial B_\Gamma^\delta\left(z_0; \frac{3}{4}d\right).$$

It follows from the discrete Harnack's Lemma (see Corollary A.5) and the estimate $|H_{\mathrm{harm}}| \leq M$ that

$$|\partial^\delta H_{\mathrm{harm}}(z)| \leq \mathrm{const} \cdot M/d$$

for all $z \in B_\Gamma^\delta(z_0; \frac{1}{2}d)$ and hence

$$\|\partial^\delta H_{\mathrm{harm}}\|_{1\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)} \le \mathrm{const}\cdot Md.$$

Furthermore,

$$[\partial^\delta H_{\mathrm{sub}}](z) \;=\; \sum_{u\in \mathrm{Int}\, B_\Gamma^\delta(z_0;\frac{3}{4}d)} [\partial^\delta G](z\,;u)[\Delta^\delta H](u)\mu_\Gamma^\delta(u),$$

where $G(\cdot\,;u)\le 0$ denotes the Green's function in $B_\Gamma^\delta(z_0;\frac{3}{4}d)$. We infer that

$$\begin{aligned}&\|\partial^\delta H_{\mathrm{sub}}\|_{1\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)}\\ &\quad\le \|\Delta^\delta H\|_{1\,;\,B_\Gamma^\delta(z_0;\frac{3}{4}d)}\cdot \max_{u\in \mathrm{Int}\, B_\Gamma^\delta(z_0;\frac{3}{4}d)} \|[\partial^\delta G](\cdot\,;u)\|_{1\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)}.\end{aligned}$$

The first factor is bounded by const$\cdot M$ (Step 1) and the second, as in the continuous setup, is bounded by const$\cdot d$ (see Lemma A.9), which concludes the proof. □

Step 3. The uniform estimate

$$|F(z)| \le \mathrm{const}\cdot \frac{M^{1/2}}{d^{1/2}} \tag{3.14}$$

holds for all $z \in B_\diamondsuit^\delta(z_0; \frac{1}{4}d)$.

Proof of Step 3 Applying the Cauchy formula (see Lemma A.6(i)), to the discs

$$B_\diamondsuit^\delta(z;5\delta k)\subset B_\diamondsuit^\delta\Big(z_0;\frac{1}{2}d\Big),\quad k:\frac{1}{8}d<5\delta k<\frac{1}{4}d,\; k\in\mathbb{Z},$$

whose boundaries don't intersect each other and summing over all such k, we estimate

$$\frac{d}{\delta}\cdot|F(z)|\le \frac{\mathrm{const}}{d}\cdot\frac{\|F\|_{1\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)}}{\delta}.$$

Thus, by the Cauchy-Schwarz inequality and (3.13)

$$|F(z)|\le \frac{\mathrm{const}}{d^2}\cdot\|F\|_{1\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)}\le \frac{\mathrm{const}}{d}\cdot\|F\|_{2\,;\,B_\diamondsuit^\delta(z_0;\frac{1}{2}d)}\le \mathrm{const}\cdot\frac{M^{1/2}}{d^{1/2}}.$$

□

Step 4. The estimate

$$|F(z_1) - F(z_0)| \le \mathrm{const} \cdot \frac{M^{1/2}}{d^{3/2}} \cdot \delta \tag{3.15}$$

holds for all $z_1 \sim z_0$.

Proof of Step 4 It follows from the discrete Cauchy formula (see Lemma A.6(ii)) applied to the disc $B^{\delta}_{\diamond}(z_0; \frac{1}{4}d)$ and Step 3 for its boundary that there exist $A, B \in \mathbb{C}$ such that, for any neighboring $z_1 \sim z_0$ (see Fig. 1C for notation), we have for both points the identity

$$\begin{aligned} F(z_{0,1}) &= \mathrm{Proj}[A\,;\, \overline{u_{0,1} - u}] + \mathrm{Proj}[B\,;\, \overline{w_{0,2} - w_1}] + O(\epsilon) \\ &= A + \mathrm{Proj}[B - A\,;\, \overline{w_{0,2} - w_1}] + O(\epsilon), \end{aligned}$$

where

$$\epsilon = \frac{\max_{z \in \partial B^{\delta}_{\diamond}(z_0; \frac{1}{4}d)} |F(z)| \cdot \delta}{d} \le \frac{M^{1/2} \cdot \delta}{d^{3/2}}.$$

The definition of an s-holomorphic function stipulates that

$$\mathrm{Proj}[F(z_0) - F(z_1);\ [i(w_1 - u)]^{-\frac{1}{2}}] = 0$$

(for all $z_1 \sim z_0$), which implies $B - A = O(\epsilon)$, and hence $F(z_i) = A + O(\epsilon)$ for $i = 0, 1$. □

3.6 The "$(\tau(z))^{-\frac{1}{2}}$" boundary condition and the "boundary modification trick"

Throughout the paper, we often deal with s-holomorphic in $\Omega^{\delta}_{\diamond}$ functions F^{δ} satisfying the Riemann boundary condition

$$F(\zeta) \parallel (\tau(z))^{-\frac{1}{2}} \tag{3.16}$$

on, say, the "white" boundary arc $L^{\delta}_{\Gamma *} \subset \partial\Omega^{\delta}_{\diamond}$ (see Fig. 7A), where $\tau(z) := w_2(z) - w_1(z)$ is the "discrete tangent vector" to $\partial\Omega^{\delta}_{\diamond}$ at ζ. Being s-holomorphic, these functions possess discrete primitives

$$H^{\delta} := \mathrm{Im} \int^{\delta} (F^{\delta}(z))^2 \, d^{\delta}z.$$

Due to the boundary condition (3.16), an additive constant can be fixed so that

$$H^{\delta}_{\Gamma *} = 0 \quad \text{on } L^{\delta}_{\Gamma *}.$$

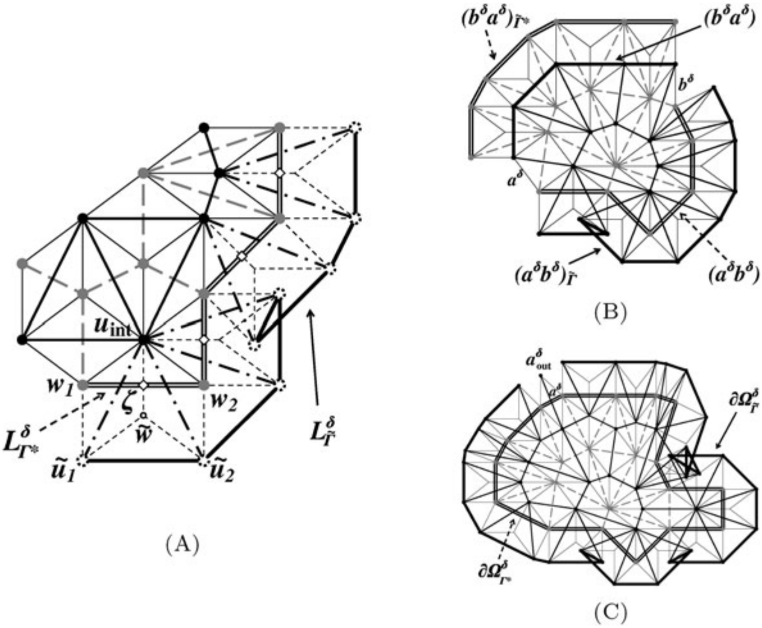


Fig. 7 "Boundary modification trick": (**A**) local modification of the boundary; (**B**) FK-Ising model: example of a modified domain; (**C**) spin-Ising model: example of a modified domain

The boundary condition for H^δ_Γ is more complicated. Fortunately, one can reformulate it exactly in the same way, using the following "*boundary modification trick*":

> *For each half-rhombus $u_{\mathrm{int}}w_1w_2$ touching the boundary arc $L^\delta_{\Gamma^*}$, we draw two new rhombi $u_{\mathrm{int}}w_1\widetilde{u}_1\widetilde{w}$ and $u_{\mathrm{int}}\widetilde{w}\widetilde{u}_2w_2$ so that the corresponding angles $\widetilde{\theta}_1=\widetilde{\theta}_2$ are equal to $\frac{1}{2}\theta$ (see Fig. 7A).*

In general, these new edges $(u_{\mathrm{int}}\widetilde{u}_{1,2})$ constructed for neighboring inner vertices u_{int}, may intersect each other but it is not important for us (one can resolve the problem of a locally self-overlapping domain by placing it on a Riemann surface).

Lemma 3.14 *Let $u_{\mathrm{int}}w_1w_2$ be the half-rhombus touching $L^\delta_{\Gamma^*}$ and $\zeta=\frac{1}{2}(w_2+w_1)$. Suppose that the function F^δ is s-holomorphic in $\Omega^\delta_\diamondsuit$ and $F^\delta(\zeta)\parallel(w_2-w_1)^{-\frac{1}{2}}$. Then, if we set*

$$H^\delta_\Gamma(\widetilde{u}_2)=H^\delta_\Gamma(\widetilde{u}_1)\;:=\;H^\delta_{\Gamma^*}(w_2)=H^\delta_{\Gamma^*}(w_1),$$

the function H^δ_Γ remains subharmonic at u_{int}.

Proof Definition (3.8) says that

$$H_\Gamma^\delta(u_{\mathrm{int}}) - H_{\Gamma^*}^\delta(w_{1,2}) = 2\delta|\mathrm{Proj}[F^\delta(\zeta)\,;\,[i(w_{1,2}-u_{\mathrm{int}})]^{-\frac{1}{2}}]|^2$$
$$= 2\delta\cos^2\frac{\theta}{2}\cdot|F^\delta(\zeta)|^2$$

because $i(w_{1,2}-u_{\mathrm{int}})$ has the same direction as $e^{\mp i\theta}(w_2-w_1)$ and $F^\delta(\zeta)\parallel(w_2-w_1)^{-\frac{1}{2}}$. Therefore

$$2\tan\frac{\theta}{2}\cdot(H_\Gamma^\delta(\widetilde{u}_{1,2}) - H_\Gamma^\delta(u_{\mathrm{int}})) = -2\delta\sin\theta\cdot|F^\delta(\zeta)|^2$$

and, if u denotes the fourth vertex of the rhombus $u_{\mathrm{int}}w_1uw_2$,

$$\tan\theta\cdot(H_\Gamma^\delta(u) - H_\Gamma^\delta(u_{\mathrm{int}})) = \tan\theta\cdot\mathrm{Im}[(F^\delta(\zeta))^2(u-u_{\mathrm{int}})]$$
$$= -2\delta\sin\theta\cdot|F^\delta(\zeta)|^2.$$

Thus, the standard definition of H_Γ^δ at u and the new definition at $\widetilde{u}_{1,2}$ give the same contributions to the (unnormalized) discrete Laplacian at u_{int} (see (3.1)). □

Remark 3.15 After this trick, an additive constant in the definition of H^δ can be chosen so that both

$$H_{\Gamma^*}^\delta = 0 \quad \textit{on } L_{\Gamma^*}^\delta \quad \textit{and} \quad H_\Gamma^\delta = 0 \quad \textit{on } \widetilde{L}_\Gamma^\delta,$$

where $L_{\widetilde{\Gamma}}^\delta$ denotes the set of newly constructed "black" vertices near $L_{\Gamma^*}^\delta$.

4 Uniform convergence for the holomorphic observable in the FK-Ising model

Recall that the discrete holomorphic fermion $F^\delta(z) = F^\delta_{(\Omega^\delta;a^\delta,b^\delta)}(z)$ constructed in Sect. 2.1 satisfies the following discrete boundary value problem:

(A) Holomorphicity: $F^\delta(z)$ is s-holomorphic in $\Omega_\diamond^\delta$.

(B) Boundary conditions: $F^\delta(\zeta)\parallel(\tau(\zeta))^{-\frac{1}{2}}$ for $\zeta\in\partial\Omega_\diamond^\delta$, where

$$\begin{aligned} \tau(\zeta) &= w_2(\zeta) - w_1(\zeta), \quad \zeta\in(a^\delta b^\delta),\ w_{1,2}(\zeta)\in\Gamma^*, \\ \tau(\zeta) &= u_2(\zeta) - u_1(\zeta), \quad \zeta\in(b^\delta a^\delta),\ u_{1,2}(\zeta)\in\Gamma, \end{aligned} \tag{4.1}$$

is the "discrete tangent vector" to $\partial\Omega_\diamond^\delta$ directed from a^δ to b^δ on both arcs (see Fig. 2B).

(C) Normalization at b^δ: $F^\delta(b^\delta) = \operatorname{Re} F^\delta(b^\delta_\diamondsuit) = (2\delta)^{-\frac{1}{2}}$.

Remark 4.1 For each discrete domain $(\Omega^\delta_\diamondsuit; a^\delta, b^\delta)$, the discrete boundary value problem (A)&(B)&(C) has a unique solution.

Proof Existence of the solution is given by the explicit construction of the holomorphic fermion in the FK-Ising model. Concerning uniqueness, let F_1^δ and F_2^δ denote two different solutions. Then the difference $F^\delta := F_1^\delta - F_2^\delta$ is s-holomorphic in $\Omega^\delta_\diamondsuit$, thus $H^\delta := \operatorname{Im} \int^\delta (F^\delta(z))^2 d^\delta z$ is well-defined (see Sect. 3.3, especially (3.8)). Due to condition (B), H^δ is constant on both boundary arcs $(a^\delta b^\delta) \subset \Gamma^*$ and $(b^\delta a^\delta) \subset \Gamma$. Moreover, in view of the same normalization of $F^\delta_{1,2}$ near b^δ, one can fix an additive constant so that $H^\delta_{\Gamma^*} = 0$ on $(a^\delta b^\delta)$ and $H^\delta_\Gamma = 0$ on $(b^\delta a^\delta)$.

The "boundary modification trick" described in Sect. 3.6 provides us the slight modification of $\Omega^\delta_\diamondsuit$ (see Fig. 7B) such that the Dirichlet boundary conditions $H^\delta_\Gamma = 0$, $H^\delta_{\Gamma^*} = 0$ hold true *everywhere* on $\partial\Omega^\delta_\Lambda$. Using sub-/super-harmonicity of H^δ on Γ / Γ^* and (3.8), we arrive at $0 \geq H^\delta_\Gamma \geq H^\delta_{\Gamma^*} \geq 0$ in Ω^δ. Thus, $H^\delta \equiv 0$ and $F_1^\delta \equiv F_2^\delta$. $\square$

Let

$$H^\delta = H^\delta_{(\Omega^\delta; a^\delta, b^\delta)} := \operatorname{Im} \int^\delta (F^\delta_{(\Omega^\delta; a^\delta, b^\delta)}(z))^2 d^\delta z. \tag{4.2}$$

It follows from the boundary conditions (B) that H^δ is constant on both boundary arcs $(a^\delta b^\delta) \subset \Gamma^*$ and $(b^\delta a^\delta) \subset \Gamma$. In view of the chosen normalization (C), we have

$$H^\delta_\Gamma|_{(b^\delta a^\delta)} - H^\delta_{\Gamma^*}|_{(a^\delta b^\delta)} = 1.$$

Remark 4.2 Due to the "boundary modification trick" (Sect. 3.6), one can fix an additive constant so that

$$\begin{aligned} H^\delta_\Gamma &= 0 \quad \text{on } (a^\delta b^\delta)_{\tilde\Gamma}, & H^\delta_{\Gamma^*} &= 0 \quad \text{on } (a^\delta b^\delta), \\ H^\delta_\Gamma &= 1 \quad \text{on } (b^\delta a^\delta), & H^\delta_{\Gamma^*} &= 1 \quad \text{on } (b^\delta a^\delta)_{\tilde\Gamma^*}, \end{aligned} \tag{4.3}$$

where $(a^\delta b^\delta)_{\tilde\Gamma}$ (and, in the same way, $(b^\delta a^\delta)_{\tilde\Gamma^*}$) denotes the set of newly constructed "black" vertices near the "white" boundary arc $(a^\delta b^\delta)$ (see Fig. 7B).

Let $f^\delta(z) = f^\delta_{(\Omega^\delta; a^\delta, b^\delta)}(z)$ denote the solution of the corresponding continuous boundary value problem inside the *polygonal* domain Ω^δ:

(a) holomorphicity: f^δ is holomorphic in Ω^δ;

(b) boundary conditions: $f^\delta(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}$ for $\zeta \in \partial\Omega^\delta$, where $\tau(\zeta)$ denotes the tangent vector to $\partial\Omega^\delta$ oriented from a^δ to b^δ (on both arcs);

(c) normalization: the function $h^\delta = h^\delta_{\Omega^\delta, a^\delta, b^\delta} := \operatorname{Im} \int (f^\delta(\zeta))^2 d\zeta$ is uniformly bounded in Ω^δ and

$$h^\delta|_{(a^\delta b^\delta)} = 0, \qquad h^\delta|_{(b^\delta a^\delta)} = 1.$$

Note that (a) and (b) guarantee that h^δ is harmonic in Ω^δ and constant on both boundary arcs $(a^\delta b^\delta)$, $(b^\delta a^\delta)$. In other words,

$$f^\delta = \sqrt{2i\partial h^\delta}, \quad h^\delta = \omega(\cdot\,; b^\delta a^\delta; \Omega^\delta),$$

where ω denotes the (continuous) harmonic measure in the (polygonal) domain Ω^δ. Note that $\partial h^\delta \neq 0$ in Ω^δ, since h^δ is the imaginary part of the conformal mapping from Ω^δ onto the infinite strip $(-\infty, \infty) \times (0, 1)$ sending a^δ and b^δ to $\mp\infty$, respectively. Thus, f^δ is well-defined (up to the sign).

Theorem 4.3 (convergence of FK-observable) *The solutions F^δ of the discrete Riemann-Hilbert boundary value problems* (A)&(B)&(C) *are uniformly close in the bulk to their continuous counterpart f^δ defined by* (a)&(b)&(c). *Namely, for all $0 < r < R$ there exists $\varepsilon(\delta) = \varepsilon(\delta, r, R)$ such that for all discrete domains $(\Omega^\delta_\diamond; a^\delta, b^\delta)$ and $z^\delta \in \Omega^\delta_\diamond$ the following holds true*:

$$\textit{if} \quad B(z^\delta, r) \subset \Omega^\delta \subset B(z^\delta, R), \quad \textit{then} \quad |F^\delta(z^\delta) - f^\delta(z^\delta)| \le \varepsilon(\delta) \to 0 \quad \textit{as } \delta \to 0$$

(*for a proper choice of f^δ's sign*), *uniformly with respect to the shape of Ω^δ and $\diamond^\delta$*.

Remark 4.4 Moreover, the sign of f^δ is the same for, at least, all $\tilde{z}^\delta$ lying in the same connected component of the r-interior of Ω^δ.

Proof Assume that neither f^δ nor $-f^\delta$ approximates F^δ well, and so for both signs $|F^\delta(z^\delta) \pm f^\delta(z^\delta)| \ge \varepsilon_0 > 0$ for some sequence of domains Ω^δ, $\delta \to 0$. Applying translations one can without loss of generality assume $z^\delta = 0$ for all δ's. The set of all simply-connected domains $\Omega : B(0, r) \subset \Omega \subset B(0, R)$ is compact in the Carathéodory topology (of convergence of conformal maps germs). Thus, passing to a subsequence, we may assume that

$$(\Omega^\delta; a^\delta, b^\delta) \xrightarrow{\text{Cara}} (\Omega; a, b) \quad \text{as } \delta \to 0$$

(with respect to $0 = z^\delta$). Let $h = h_{(\Omega; a,b)} := \omega(\,\cdot\,; ba; \Omega)$. Note that $h^\delta \rightrightarrows h$ as $\delta \to 0$, uniformly on compact subsets of Ω, since the harmonic measure is

Carathéodory stable. Moreover,

$$(f^\delta)^2 = 2i\partial h^\delta \rightrightarrows f^2 = 2i\partial h \quad \text{as } \delta \to 0.$$

We are going to prove that, at the same time,

$$H^\delta \rightrightarrows h \quad \text{and} \quad (F^\delta)^2 \rightrightarrows f^2 \quad \text{as } \delta \to 0,$$

uniformly on compact subsets of Ω, which gives a contradiction.

It easily follows from (4.3) and the sub-/super-harmonicity of H^δ on Γ/Γ^* that

$$0 \le H^\delta \le 1 \quad \text{everywhere in } \Omega^\delta_\Lambda.$$

In view of Theorem 3.12, this (trivial) uniform bound implies the uniform boundedness and the equicontinuity of functions F^δ on compact subsets K of Ω. Thus, both $\{H^\delta\}$ and $\{F^\delta\}$ are normal families on each compact subset of Ω. Therefore, taking a subsequence, we may assume that

$$F^\delta \rightrightarrows F \text{ and } H^\delta \rightrightarrows H \quad \text{for some } F:\Omega\to\mathbb{C},\ H:\Omega\to\mathbb{R},$$

uniformly on all compact subsets of Ω. The simple passage to the limit in (4.2) gives

$$H(v_2) - H(v_1) = \operatorname{Im}\int_{[v_1;v_2]} (F(\zeta))^2\, d\zeta,$$

for each segment $[v_1;v_2]\subset\Omega$. Thus, $F^2 = 2i\partial H$. Being a limit of discrete subharmonic functions H^δ_Γ, as well as discrete superharmonic functions $H^\delta_{\Gamma^*}$, the function H should be harmonic. The sub-/super-harmonicity of H^δ on Γ/Γ^* gives

$$\omega^\delta(\,\cdot\,;(b^\delta a^\delta)_{\widetilde{\Gamma}^*};\Omega^\delta_{\Gamma^*}) \le H^\delta_{\Gamma^*} \le H^\delta_\Gamma \le \omega^\delta(\,\cdot\,;(b^\delta a^\delta)_\Gamma;\Omega^\delta_\Gamma) \quad \text{in } \Omega^\delta_\Lambda,$$

where the middle inequality holds for any pair of neighbors $w\in\Gamma^*$, $u\in\Gamma$ due to (3.8). It is known (see [10] Theorem 3.12) that both discrete harmonic measures $\omega^\delta(\cdot)$ (as on Γ^*, as on Γ) are uniformly close in the bulk to the continuous harmonic measure $\omega(\cdot)=h$. Thus, $H^\delta \rightrightarrows h$ uniformly on compact subsets of Ω, and so $F^2 = 2i\partial h$. □

Proof of Remark 4.4 Consider simply-connected domains $(\Omega^\delta;a^\delta,b^\delta;z^\delta,\widetilde{z}^\delta)$, with $z^\delta,\widetilde{z}^\delta$ lying in the same connected components of the r-interiors Ω^δ_r. Assume that we have $|F^\delta(z^\delta)-f^\delta(z^\delta)|\to 0$ but $|F^\delta(\widetilde{z}^\delta)+f^\delta(\widetilde{z}^\delta)|\to 0$ as $\delta\to 0$. Applying translations to Ω^δ and taking a subsequence, we may assume that

$$(\Omega^\delta;a^\delta,b^\delta;\widetilde{z}^\delta) \xrightarrow{\text{Cara}} (\Omega;a,b;\widetilde{z}) \quad \text{w.r.t. } 0 = z^\delta,$$

for some $\tilde{z}$ connected with 0 inside the r-interior Ω_r of Ω. As it was shown above,

$$F^\delta \rightrightarrows F \quad \text{and} \quad f^\delta \rightrightarrows f \quad \text{uniformly on } \Omega_r,$$

where either $F \equiv f$ or $F \equiv -f$ (everywhere in Ω), which gives a contradiction. □

5 Uniform convergence for the holomorphic observable in the spin-Ising model

For the spin-Ising model, the discrete holomorphic fermion $F^\delta(z) = F^\delta_{(\Omega^\delta;a^\delta,b^\delta)}(z)$ constructed in Sect. 2.2 satisfies the following discrete boundary value problem:

(A°) Holomorphicity: $F^\delta(z)$ is s-holomorphic inside $\Omega^\delta_\diamondsuit$.

(B°) Boundary conditions: $F^\delta(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}$ for all $\zeta \in \partial\Omega^\delta_\diamondsuit$ except at a^δ, where $\tau(\zeta) = w_2(\zeta) - w_1(\zeta)$ is the "discrete tangent vector" to $\partial\Omega^\delta_\diamondsuit$ oriented in the counterclockwise direction (see Fig. 4A).

(C°) Normalization at b^δ: $F^\delta(b^\delta) = \mathcal{F}^\delta(b^\delta)$, where the normalizing constants $\mathcal{F}^\delta(b^\delta) \parallel (\tau(b^\delta))^{-\frac{1}{2}}$ are defined in Sect. 5.1.

Remark 5.1 For each discrete domain $(\Omega^\delta_\diamondsuit; a^\delta, b^\delta)$, the discrete boundary value problem (A°)&(B°)&(C°) has a unique solution.

Proof Existence is given by the holomorphic fermion in the spin-Ising model. Concerning uniqueness, let F^δ denote some solution. Then $H^\delta = \int^\delta (F^\delta(z))^2 d^\delta z$ is constant on $\partial\Omega^\delta_{\Gamma^*}$, so either $F^\delta(a^\delta) \parallel (\tau(a^\delta))^{-1/2}$ or $F^\delta(a^\delta) \parallel (-\tau(a^\delta))^{-1/2}$. In the former case, using the "boundary modification trick" (Sect. 3.6), we arrive at $H^\delta = 0$ on both $\partial\Omega^\delta_{\Gamma^*}$ and $\partial\Omega^\delta_{\tilde{\Gamma}}$. Then, sub-/super-harmonicity of H^δ on Γ/Γ^* and (3.8) imply that $0 \ge H^\delta_\Gamma \ge H^\delta_{\Gamma^*} \ge 0$ in Ω^δ. Therefore, $H^\delta \equiv 0$ and $F^\delta \equiv 0$, which is impossible. Thus, $F^\delta(a^\delta) \parallel (-\tau(a^\delta))^{-1/2}$ (for the holomorphic fermion this follows from the definition).

Let $F_{1,2}$ be two different solutions. Denote

$$F^\delta(z) := (-\tau(a^\delta))^{\frac{1}{2}} \cdot [F_1^\delta(a^\delta)F_2^\delta(z) - F_2^\delta(a^\delta)F_1^\delta(z)].$$

Then, F^δ is s-holomorphic and, since $(-\tau(a^\delta))^{\frac{1}{2}} F^\delta_{1,2}(a^\delta) \in \mathbb{R}$, satisfies the boundary condition (B°) on $\partial\Omega^\delta \setminus \{a^\delta\}$. Moreover, we also have $F^\delta(a^\delta) = 0 \parallel (\tau(a^\delta))^{-\frac{1}{2}}$. Arguing as above, we obtain $F^\delta \equiv 0$. The identity $F_1^\delta \equiv F_2^\delta$ then follows from (C°). □

Let

$$H^\delta = H^\delta_{(\Omega^\delta;a^\delta,b^\delta)} := \operatorname{Im}\int^\delta (F^\delta_{(\Omega^\delta;a^\delta,b^\delta)}(z))^2\, d^\delta z.$$

Remark 5.2 Using Sect. 3.6, one can fix an additive constant so that

$$\begin{aligned} &H^\delta_{\Gamma^*} = 0 \quad \text{everywhere on } \partial\Omega^\delta_{\Gamma^*} \\ &H^\delta_\Gamma = 0 \quad \text{everywhere on } \partial\Omega^\delta_{\tilde\Gamma} \text{ except } a^\delta_{\text{out}}, \end{aligned} \tag{5.1}$$

where $\partial\Omega^\delta_{\tilde\Gamma}$ denotes the modified boundary (everywhere except a^δ) and a^δ_{out} is the original outward "black" vertex near a^δ (see Fig. 7C). Then, $H^\delta_\Gamma \ge H^\delta_{\Gamma^*} \ge 0$ everywhere in Ω^δ.

5.1 Boundary Harnack principle and solution to (A°)&(B°) in the discrete half-plane

We start with a version of the Harnack Lemma (Proposition 3.11) which compares the values of $H^\delta = \operatorname{Im}\int^\delta (F^\delta)^2(z)\, d^\delta z$ in the bulk with its normal derivative at the boundary.

Let $R(s,t) := (-s;s)\times(0;t) \subset \mathbb{C}$ be an open rectangle, $R^\delta_\diamondsuit(s,t) \subset \Gamma$ denote its discretization, and $L^\delta(s)$, $U^\delta(s,t)$ and $V^\delta(s,t)$ be the lower, upper and vertical parts of the boundary ∂R^δ_Γ (see Fig. 8A).

Proposition 5.3 *Let $t \ge \delta$, F^δ be an s-holomorphic function in a discrete rectangle $R^\delta_\diamondsuit(2t,2t)$ satisfying the boundary condition* (B°) *on the lower boundary $L^\delta_\diamondsuit(2t)$ and $H^\delta = \operatorname{Im}\int^\delta (F^\delta(z))^2\, d^\delta z$ be defined by* (3.8) *so that $H = 0$ on $L^\delta_{\Gamma^*}$ and $H \ge 0$ everywhere in $R^\delta_{\Gamma^*}(2t,2t)$. Let $b^\delta \in \diamondsuit$ be the boundary vertex closest to* 0, *and $c^\delta \in \Gamma^*$ denote the inner face* (*dual vertex*) *containing the point $c = it$. Then, uniformly in t and δ,*

$$|F^\delta(b^\delta)|^2 \asymp \frac{H^\delta(c^\delta)}{t}.$$

Proof Let $t \ge \text{const}\cdot\delta$ (the opposite case is trivial). It follows from Remark 3.10 and Proposition 3.11 that all the values of H^δ on $U^\delta(t,\frac12 t)$ are uniformly comparable with $H(c^\delta)$. Then, the superharmonicity of $H^\delta|_{\Gamma^*}$ and simple estimates of the discrete harmonic measure in $R^\delta_{\Gamma^*}(t,\frac12 t)$ (see Lemma A.3) give

$$H^\delta(b^\delta_{\Gamma^*}) \ge \omega^\delta\left(b^\delta_{\Gamma^*}; U^\delta_{\Gamma^*}; R^\delta_{\Gamma^*}\left(t,\frac12 t\right)\right)\cdot \min_{w^\delta\in U^\delta_{\Gamma^*}} H^\delta(w^\delta) \ge \text{const}\cdot\delta/t\cdot H^\delta(c^\delta),$$

where $b^\delta_{\Gamma^*} \in \Gamma^*$ denotes the inner dual vertex closest to b^δ (see Fig. 8A). Therefore, $|F^\delta|^2 \geq \text{const} \cdot H^\delta(c^\delta)/t$ for a neighbor of b^δ. Due to the s-holomorphicity of F^δ, this is sufficient to conclude that

$$|F^\delta(b^\delta)|^2 \geq \text{const} \cdot H^\delta(c^\delta)/t.$$

On the other hand, since $H = 0$ on $L^\delta(2t)$, one has $H^\delta \leq \text{const} \cdot H^\delta(c^\delta)$ *everywhere* in $R^\delta_\Gamma(t, \frac{1}{2}t)$ (the proof mimics the proof of Proposition 3.11: if $H^\delta(v) \gg H^\delta(c^\delta)$ at some $v \in R^\delta_\Gamma(t, \frac{1}{2}t)$, then, since $H^\delta \equiv 0$ on $L^\delta(2t)$, the same holds true along some path running from v to $U^\delta(2t, 2t) \cup V^\delta(2t, 2t)$, which gives a contradiction). Thus, estimating the discrete harmonic measure in $R^\delta_\Gamma(t, \frac{1}{2}t)$ from the inner vertex $b^\delta_\Gamma \in \Gamma$ closest to b (see Fig. 8A), we arrive at

$$\begin{aligned} H^\delta(b^\delta_\Gamma) &\leq \omega^\delta\left(b^\delta_\Gamma; U^\delta_\Gamma \cup V^\delta_\Gamma; R^\delta_\Gamma\left(t, \frac{1}{2}t\right)\right) \cdot \max_{u^\delta \in U^\delta_\Gamma \cup B^\delta_\Gamma} H^\delta(u^\delta) \\ &\leq \text{const} \cdot \delta/t \cdot H^\delta(c^\delta). \end{aligned}$$

Since $H^\delta(b^\delta_\Gamma) \asymp \delta \cdot |F^\delta(b^\delta)|^2$, this means $|F^\delta(b^\delta)|^2 \leq \text{const} \cdot H^\delta(c^\delta)/t.$ ☐

Now we are able to construct a special solution $\mathcal{F}^\delta$ to the discrete boundary value problem (A°)&(B°) in the discrete half-plane. The value $\mathcal{F}^\delta(b^\delta)$ will be used later on for the normalization of the spin-observable at the target point b^δ.

Theorem 5.4 *Let $\mathbb{H}^\delta$ denote some discretization of the upper half-plane* (*see Fig.* 8*A*). *Then, there exist a unique s-holomorphic function $\mathcal{F}^\delta : \mathbb{H}^\delta_\diamondsuit \to \mathbb{C}$ satisfying boundary conditions $\mathcal{F}^\delta(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}$ for $\zeta \in \partial\mathbb{H}^\delta_\diamondsuit$, such that*

$$\mathcal{F}^\delta(z) = 1 + O(\delta^{\frac{1}{2}} \cdot (\operatorname{Im} z)^{-\frac{1}{2}}),$$

uniformly with respect to $\diamondsuit^\delta$. Moreover, $|\mathcal{F}^\delta| \asymp 1$ on the boundary $\partial\mathbb{H}^\delta_\diamondsuit$.

Remark 5.5 If $\partial\mathbb{H}^\delta_\diamondsuit$ is a straight line (e.g., for the proper oriented square or triangular/hexagonal grids), then $\mathcal{F}^\delta \equiv 1$ easily solves the problem.

Proof Uniqueness. Let F^δ_1, F^δ_2 be two different solutions. Clearly, $F^\delta := F^\delta_1 - F^\delta_2$ is s-holomorphic and satisfies the same boundary conditions on $\partial\mathbb{H}^\delta_\diamondsuit$. Thus we can set $H^\delta := \operatorname{Im} \int^\delta (F^\delta(z))^2 d^\delta z$, where $H^\delta = 0$ on both $\partial\mathbb{H}^\delta_{\Gamma^*}$ and $\partial\mathbb{H}^\delta_\Gamma$ (see Sect. 3.6).

Since $(F^\delta(z))^2 = O(\delta \cdot (\operatorname{Im} z)^{-1})$, the integration over "vertical" paths gives

$$H^\delta(v) = O(\delta \cdot \log(\delta^{-1} \operatorname{Im} v)) \quad \text{as } \operatorname{Im} v \to \infty,$$

so H^δ grows sublinearly as $\operatorname{Im} v \to \infty$ which is impossible. Indeed, using simple estimates of the discrete harmonic measure (see Lemma A.3) in big rectangles $R(2n, n)$, $n \to \infty$, and sub-/super-harmonicity of H^δ on Γ/Γ^* together with the Dirichlet boundary conditions on the boundary $\partial\mathbb{H}^\delta_\Lambda$, we conclude that

$$H^\delta_\Gamma(u) \le \lim_{n\to\infty} O(\delta \cdot \log(\delta^{-1} n)) \cdot (\operatorname{Im} u + 2\delta) n^{-1} = 0 \quad \text{for any } u \in \mathbb{H}^\delta_\Gamma,$$

and, similarly, $H^\delta_{\Gamma^*}(w) \ge 0$ for any $w \in \mathbb{H}^\delta_{\Gamma^*}$. Thus, $H^\delta \equiv 0$, and so $F^\delta_1 \equiv F^\delta_2$.

Existence. We construct $\mathcal{F}^\delta$ as a (subsequential) limit of holomorphic fermions in *increasing* discrete rectangles. Let δ *be fixed*, $b^\delta \in \diamondsuit$ denote the closest to 0 boundary vertex, and R^δ_n denote discretizations (see Fig. 8A) of the rectangles

$$R_n = R(4n, 2n) := (-4n; 4n) \times (0; 2n).$$

Let $F^\delta_n : R^\delta_{n,\diamondsuit} \to \mathbb{C}$ be the discrete s-holomorphic fermion solving the boundary value problem (A°)&(B°) in R^δ_n with a^δ_n being the discrete approximations of the points $2ni$. For the time being, we normalize F^δ_n by the condition

$$|F^\delta(b^\delta_n)| = 1.$$

Having this normalization, it follows from the discrete Harnack principle (Propositions 3.11 and 5.3) that $H^\delta_n \asymp n$ everywhere near the segment $[-2n + in; 2n + in]$. Moreover, since $H = 0$ on the lower boundary, one also has $H \le \text{const} \cdot n$ everywhere in the smaller rectangle $R^\delta_\diamondsuit(2n, n)$ (the proof mimics the proof of Proposition 3.11). Thus, estimating the discrete harmonic measure of $U^\delta(2n, n)_\Gamma \cup V^\delta_\Gamma(2n, n)$ in $R^\delta_\Gamma(2n, n)$ from any *fixed* vertex $v^\delta \in \mathbb{H}^\delta_\Lambda$ and using the subharmonicity of $H|_\Gamma$, one obtains

$$H^\delta_n(v^\delta) \le \frac{\operatorname{Im} v^\delta + 2\delta}{n} \cdot \text{const} \cdot n \le \text{const} \cdot (\operatorname{Im} v^\delta + 2\delta),$$

if $n = n(v^\delta)$ is big enough. Moreover, since $H^\delta|_{\Gamma^*}$ is superharmonic, one also has the inverse estimate for v^δ near the imaginary axis $i\mathbb{R}_+$:

$$H^\delta_n(v^\delta) \ge \text{const} \cdot (\operatorname{Im} v^\delta + 2\delta), \quad \text{if } |\operatorname{Re} v^\delta| \le \delta.$$

Further, Theorem 3.12 applied in $(\operatorname{Re} z - \frac{1}{2}\operatorname{Im} z; \operatorname{Re} z + \frac{1}{2}\operatorname{Im} z) \times (\frac{1}{2}\operatorname{Im} z; \frac{3}{2}\operatorname{Im} z)$ gives $|F^\delta_n(z^\delta)| \le \text{const}$ for any $z^\delta \in H^\delta_\diamondsuit$, if $n = n(z^\delta)$ is big enough.

Note that there are only countably many points $v^\delta \in \mathbb{H}^\delta_\Lambda$ and $z^\delta \in \mathbb{H}^\delta_\diamondsuit$. Since for any fixed vertex the values $H^\delta_n(v^\delta)$ and $F^\delta_n(z^\delta)$ are bounded, we may choose a subsequence $n = n_k \to \infty$ so that

$$H^\delta_n(v^\delta) \to \mathcal{H}^\delta(v^\delta) \quad \text{and} \quad F^\delta_n(z^\delta) \to \mathcal{F}^\delta(z^\delta)$$
$$\text{for each } v^\delta \in \mathbb{H}^\delta_\Lambda \text{ and } z^\delta \in \mathbb{H}^\delta_\diamondsuit,$$

It's clear that $\mathcal{F}^\delta : \mathbb{H}^\delta_\diamondsuit \to \mathbb{C}$ is s-holomorphic, $\mathcal{H}^\delta = \operatorname{Im} \int^\delta (\mathcal{F}^\delta(z))^2 d^\delta z \geq 0$, $\mathcal{H}^\delta$ and $\mathcal{F}^\delta$ satisfy the same boundary conditions as H^δ_n, F^δ_n, and $\mathcal{F}^\delta(b^\delta) = 1$. Moreover,

$$\begin{aligned} &\mathcal{H}^\delta(v^\delta) = O(\operatorname{Im} v^\delta + 2\delta), \qquad \mathcal{F}^\delta(z^\delta) = O(1) \quad \text{uniformly in } \mathbb{H}^\delta, \\ &\text{and} \quad \mathcal{H}^\delta(v^\delta) \asymp (\operatorname{Im} v^\delta + 2\delta) \quad \text{for } v^\delta \text{ near } i\mathbb{R}_+. \end{aligned} \tag{5.2}$$

Now we are going to improve this estimate and show that, uniformly in $\mathbb{H}^\delta_\Lambda$, $\mathcal{H}^\delta(v) = \mu \cdot (\operatorname{Im} v + O(\delta))$ for some $\mu > 0$. For this purpose, we *re-scale our lattice and functions* by a small factor $\varepsilon \to 0$. Let

$$\begin{aligned} &v^{\varepsilon\delta} := \varepsilon v^\delta, \qquad z^{\varepsilon\delta} := \varepsilon z^\delta, \quad \text{and} \quad \mathcal{H}^{\varepsilon\delta}(v^{\varepsilon\delta}) := \varepsilon \mathcal{H}^\delta(v^\delta), \\ &\mathcal{F}^{\varepsilon\delta}(z^{\varepsilon\delta}) := \mathcal{F}^\delta(z^\delta). \end{aligned}$$

Note that the uniform estimates (5.2) remains valid for the re-scaled functions. Therefore, Theorem 3.12 guarantees that the functions $\mathcal{H}^{\varepsilon\delta}$ and $\mathcal{F}^{\varepsilon\delta}$ are uniformly bounded and equicontinuous on compact subsets of $\mathbb{H}$, so, taking a subsequence, we may assume

$$\mathcal{H}^{\varepsilon\delta}(v) \rightrightarrows h(v) \quad \text{uniformly on compact subsets of } \mathbb{H}.$$

Being a limit of discrete subharmonic functions $H^{\varepsilon\delta}_\Gamma$ as well as discrete superharmonic functions $H^{\varepsilon\delta}_{\Gamma*}$, the function h is harmonic. Moreover, it is non-negative and, due to (5.2), has zero boundary values everywhere on $\mathbb{R}$. Thus, $h(v) \equiv \mu v$ for some $\mu > 0$ (the case $\mu = 0$ is excluded by the uniform double-sided estimate of $\mathcal{H}^{\varepsilon\delta}$ near $i\mathbb{R}_+$).

Thus, for any fixed $s \gg t > 0$ one has

$$\mathcal{H}^{\varepsilon\delta} \rightrightarrows \mu t \quad \text{on } [-s + it; s + it] \text{ as } \varepsilon \to 0.$$

For the original function H^δ, this means

$$\mathcal{H}^\delta(v^\delta) = (\mu + o_{k\to\infty}(1)) \cdot \operatorname{Im} v^\delta \quad \text{uniformly on } U^\delta(ks, kt) \text{ as } k = \varepsilon^{-1} \to \infty.$$

Estimating the discrete harmonic measure in (big) rectangles $R^\delta(ks, kt)$ from a fixed vertex v^δ (see Lemma A.3) and using subharmonicity of $H|_\Gamma$

and superharmonicity of $H|_{\Gamma^*}$, we obtain

$$\begin{aligned}\mathcal{H}^\delta(v^\delta) = \left[\frac{\operatorname{Im} v^\delta + O(\delta)}{kt} + O\left(\frac{|v^\delta| \cdot kt}{(ks)^2}\right)\right] \cdot (\mu + o_{k\to\infty}(1)) \cdot kt \\ + O\left(\frac{|v^\delta| \cdot kt}{(ks)^2}\right) \cdot O(kt),\end{aligned}$$

where the O-bounds are uniform in v^δ, t and s, if $k = k(v^\delta)$ is big enough. Passing to the limit as $k \to \infty$, we arrive at

$$\mathcal{H}^\delta(v^\delta) = \mu \cdot (\operatorname{Im} v^\delta + O(\delta)) + O(|v^\delta| \cdot t^2/s^2),$$

where the O-bound is uniform in v^δ and t, s. Therefore,

$$\mathcal{H}^\delta(v^\delta) = \mu \cdot (\operatorname{Im} v^\delta + O(\delta)) \quad \text{uniformly in } \mathbb{H}^\delta_\Lambda.$$

It follows from (5.2) and Theorem 3.12 that both $\mathcal{F}^\delta$ and $(\mathcal{F}^\delta)^2$ are uniformly Lipschitz in each strip $\beta \le \operatorname{Im}\zeta \le 2\beta$ with a Lipschitz constant bounded by $O(\beta^{-1})$. Taking some $v \in \mathbb{H}^\delta_\Lambda$ near z and $v' \in \mathbb{H}^\delta_\Lambda$ such that $|v' - v| \asymp \delta^{1/2}(\operatorname{Im} z)^{1/2}$, we obtain

$$\begin{aligned}\mathcal{H}^\delta(v') - \mathcal{H}^\delta(v) &= \operatorname{Im} \int_{[v,v']}^{\delta} (F^\delta(\zeta))^2 \, d^\delta\zeta \\ &= \operatorname{Im}[(\mathcal{F}^\delta(z))^2(v' - v)] + O\left(\frac{|v' - v|^2}{\operatorname{Im} z}\right),\end{aligned}$$

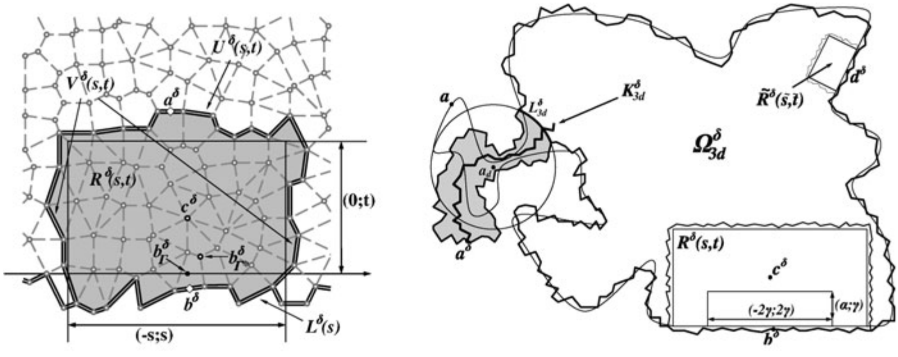
i.e., $\operatorname{Im}[(\mathcal{F}^\delta(z))^2(v' - v)] = \mu \cdot \operatorname{Im}(v' - v) + O(\delta)$ for all v'. Thus,

$$(\mathcal{F}^\delta(z))^2 = \mu + O(\delta^{\frac{1}{2}} \cdot (\operatorname{Im} z)^{-\frac{1}{2}}) \quad \text{uniformly in } \mathbb{H}^\delta_\diamond.$$

Since $\mathcal{F}^\delta$ is Lipschitz with a Lipschitz constant bounded by $O((\operatorname{Im} z)^{-1})$ (see above), this allows us to conclude that

$$\pm\mathcal{F}^\delta(z) = \mu^{\frac{1}{2}} + O(\delta^{\frac{1}{2}} \cdot (\operatorname{Im} z)^{-\frac{1}{2}}) \quad \text{uniformly in } \mathbb{H}^\delta_\diamond$$

(for some choice of the sign). Thus, the function $\widetilde{\mathcal{F}}^\delta := \pm\mu^{-1/2}\mathcal{F}^\delta$ satisfies the declared asymptotics and boundary conditions. Moreover, $|\widetilde{\mathcal{F}}^\delta(b^\delta)| = \mu^{-1/2} \asymp 1$. Since such a function $\widetilde{\mathcal{F}}^\delta$ is unique, all other values $|\widetilde{\mathcal{F}}^\delta|$ on the boundary $\partial\mathbb{H}^\delta_\diamond$ are $\asymp 1$ too. □



(A) (B)

Fig. 8 (**A**) Discretizations of the upper half-plane $\mathbb{H}$ and the rectangle $R(s,t) = (-s;s) \times (0;t)$. Boundary points $a^\delta, b^\delta \in \diamondsuit$ approximate the points it and 0, while $c^\delta \in \Gamma^*$ approximate the center $\frac{1}{2}it$. We denote by $L^\delta(s)$, $U^\delta(s,t)$ and $V^\delta(s,t)$ the lower, upper and vertical parts of $\partial R^\delta(s,t)$, respectively. (**B**) To perform the passage to the limit under the normalization condition at b^δ, we assume that $\Omega^\delta_\diamondsuit \supset R^\delta_\diamondsuit(s,t)$ and $\partial\Omega^\delta_\diamondsuit \setminus \{a^\delta\} \supset L^\delta_\diamondsuit(s)$. For δ small enough, the discrete harmonic measure from c^δ of any path $K^\delta = K^\delta_{3d}$ going from Ω^δ_{3d} to a^δ is uniformly bounded from below. We can similarly use any point d^δ lying on the "straight" part of the boundary for the (different) normalization of the observable

5.2 Main convergence theorem

To handle the normalization at b^δ of our discrete observable, from now on we assume that, for some $s, t > 0$,

$$\begin{aligned} &\Omega^\delta_\diamondsuit \text{ contains the discrete rectangle } R^\delta_\diamondsuit(s,t), \\ &\partial\Omega^\delta_\diamondsuit \setminus \{a^\delta\} \text{ contains the lower side } L^\delta_\diamondsuit(s) \text{ of } R^\delta_\diamondsuit(s,t), \\ &\text{and } b^\delta \text{ is the closest to 0 vertex of } \partial\Omega^\delta_\diamondsuit \text{ (see Fig. 8).} \end{aligned} \tag{5.3}$$

Some assumption of a kind is certainly necessary: one can imagine continuous domain with such an irregular approach to b, that any approximation is forced to have many "bottlenecks," ruining the estimates.

Let $f^\delta(z) = f^\delta_{(\Omega^\delta;a^\delta,b^\delta)}(z)$ denote the solution of the following boundary value problem inside the *polygonal* domain $\widetilde{\Omega}^\delta$ (here the tilde means that *we slightly modify the original polygonal domain* Ω^δ *near* b^δ, *replacing the polyline* $L^\delta(s)$ *by the straight real segment* $[-s;s]$, cf. [10] Theorem 3.20):

(a°) holomorphicity: f^δ is holomorphic in $\widetilde{\Omega}^\delta$;

(b°) boundary conditions: $f^\delta(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}}$ for $\zeta \in \partial\widetilde{\Omega}^\delta$, where $\tau(\zeta)$ is the tangent to $\partial\widetilde{\Omega}^\delta$ vector oriented in the counterclockwise direction, f^δ is bounded away from a^δ;

(c°) normalization: the function $h^\delta = h^\delta_{\Omega^\delta,a^\delta,b^\delta} := \operatorname{Im}\int (f^\delta(\zeta))^2\,d\zeta$ is non-negative in $\widetilde{\Omega}^\delta$, bounded away from a^δ, and

$$f^\delta(0) = [\partial_y h^\delta(0)]^{1/2} = 1.$$

As in Sect. 4, (a°) and (b°) guarantee that h^δ is harmonic in Ω^δ and constant on $\partial\widetilde{\Omega}^\delta$. Thus,

$$f^\delta = \sqrt{2i\,\partial h^\delta}, \quad \text{where } h^\delta = P_{(\Omega^\delta;a^\delta,0)},$$

is the Poisson kernel in Ω^δ having mass at a^δ and normalized at 0. In other words, h^δ is the imaginary part of the conformal mapping (normalized at 0) from Ω^δ onto the upper half-plane $\mathbb{H}$ sending a^δ and 0 to ∞ and 0, respectively. Note that $\partial h^\delta \neq 0$ everywhere in Ω^δ, thus f^δ is well-defined in Ω^δ up to a sign, which is fixed by $f^\delta(0) = +1$.

Theorem 5.6 (convergence of the spin-observable) *The discrete solutions of the Riemann-Hilbert boundary value problems* (A°)&(B°)&(C°) *are uniformly close in the bulk to their continuous counterparts f^δ, defined by* (a°)&(b°)&(c°). *Namely, there exists $\varepsilon(\delta) = \varepsilon(\delta, r, R, s, t)$ such that for all discrete domains $(\Omega^\delta_\diamond; a^\delta, b^\delta) \subset B(0,R)$ satisfying* (5.3) *and for all $z^\delta \in \Omega^\delta_\diamond$ lying in the same connected component of the r-interior of Ω^δ as the neighborhood of b^δ* (*see Fig.* 8) *the following holds true*:

$$|F^\delta(z^\delta) - f^\delta(z^\delta)| \le \varepsilon(\delta) \to 0 \quad \textit{as } \delta \to 0$$

(*uniformly with respect to the shape of Ω^δ and the structure of $\diamondsuit^\delta$*).

Moreover, it is easy to conclude from this theorem that the convergence also should hold true at any boundary point d^δ such that $\partial\Omega^\delta_\diamond$ has a "straight" part near d^δ. Namely, as in (5.3), let (see Fig. 8B)

> *$d^\delta \in \partial\Omega^\delta_\diamond$ be a boundary point, the boundary $\partial\Omega^\delta_\diamond$ is "straight" near d^δ and oriented in the (macroscopic) direction $\tau_d : |\tau_d| = 1$, i.e., $\Omega^\delta_\diamond$ contains the discretization $\widetilde{R}^\delta_\diamond(\widetilde{s},\widetilde{t})$ of the rectangle $d + \tau_d \cdot R(\widetilde{s},\widetilde{t})$ and $\partial\Omega^\delta_\diamond \setminus \{a^\delta\}$ contains the discretization $\widetilde{L}^\delta_\diamond(\widetilde{s})$ of the segment $d + \tau_d \cdot [-\widetilde{s};\widetilde{s}]$.*

Further, let $\widetilde{\mathcal{F}}^\delta$ denotes the solution of the boundary value problem (A°)&(B°) in the discrete half-plane $(d + \tau_d \cdot \mathbb{H})_\diamond$ which is asymptotically equal to $(\tau_d)^{-1/2}$ (again, $\widetilde{\mathcal{F}} \equiv (\tau_d)^{-1/2}$, if one deals with, e.g., the properly oriented square grid).

Corollary 5.7 (convergence of spin-observable on the boundary) *If centers of $R^\delta_\diamond(s,t)$ and $\widetilde{R}^\delta_\diamond(\widetilde{s},\widetilde{t})$ are connected in the r-interior of $\Omega^\delta \subset B(0,R)$,* *then*

$$|F^\delta(d^\delta) - [(\tau_d)^{1/2}\widetilde{\mathcal{F}}^\delta(d^\delta)] \cdot f^\delta(d)| \le \varepsilon(\delta) \to 0 \quad \textit{as } \delta \to 0$$

(*one should replace both* $L^\delta_{\Gamma^*}(s)$ *and* $\widetilde{L}^\delta_{\Gamma^*}(\widetilde{s})$ *by the corresponding straight segments to define properly the value of the "continuous" solution* $f^\delta(d)$ *on the boundary*).

Proof of Corollary 5.7 Let c^δ be a discrete approximation of the point $c := \frac{1}{2}it$ and $\widetilde{c}^\delta$ be a discrete approximation of the point $\widetilde{c} := d + \frac{1}{2}i\tau_d\widetilde{t}$. It follows from the discrete Harnack principle (Proposition 5.3 and Proposition A.4) that

$$|F^\delta(d^\delta)| \asymp |H^\delta(\widetilde{c}^\delta)| \asymp |H^\delta(c^\delta)| \asymp |F^\delta(b^\delta)| \asymp 1,$$

uniformly in Ω^δ and δ, if all the parameters $r, R, s, t, \widetilde{s}, \widetilde{t}$ are fixed. Denote by $\widetilde{F}^\delta$ the discrete observable F^δ renormalized at d^δ:

$$\widetilde{F}^\delta := \frac{\widetilde{\mathcal{F}}^\delta(d^\delta)}{F^\delta(d^\delta)} \cdot F^\delta,$$

and by $\widetilde{f}^\delta$ the corresponding continuous function renormalized at d:

$$\widetilde{f}^\delta := \frac{(\tau_d)^{-1/2}}{f^\delta(d)} \cdot f^\delta.$$

Again, $|f^\delta(d)| \asymp |h^\delta(\widetilde{c})| \asymp |h^\delta(c)| \asymp |f^\delta(0)| = 1$ due to the Harnack principle. Moreover, since h^δ is equal to the imaginary part of the conformal mapping from Ω^δ onto the upper half-plane, the Koebe Distortion Theorem gives

$$|f^\delta(c^\delta)|^2 = 2|\partial h^\delta(c^\delta)| \asymp |h^\delta(c^\delta)| \asymp 1.$$

One needs to prove that the ratio

$$\frac{F^\delta(d^\delta)}{[(\tau_d)^{1/2}\widetilde{\mathcal{F}}^\delta(d^\delta)] \cdot f^\delta(d)} = \frac{F^\delta}{\widetilde{F}^\delta} \cdot \frac{\widetilde{f}^\delta}{f^\delta} = \frac{F^\delta(c^\delta)}{f^\delta(c^\delta)} \cdot \frac{\widetilde{f}^\delta(c^\delta)}{\widetilde{F}^\delta(c^\delta)}$$

is uniformly close to 1. This follows from Theorem 5.6, since $F^\delta(c^\delta)$ is uniformly close to $f^\delta(c^\delta)$, $\widetilde{F}^\delta(c^\delta)$ is uniformly close to $\widetilde{f}^\delta(c^\delta)$, and $|\widetilde{f}^\delta(c^\delta)| \asymp |f^\delta(c^\delta)| \asymp 1$. □

Proof of Theorem 5.6 Assume that

$$|F^\delta(z^\delta) - f^\delta(z^\delta)| \ge \varepsilon_0 > 0$$

for some sequence of domains Ω^δ with $\delta \to 0$. Passing to a subsequence, we may assume that $z^\delta \to z$. The set of all simply-connected domains $B(z,r) \subset \Omega \subset B(0,R)$ is compact in the Carathéodory topology, so, passing to a subsequence once more, we may assume that

$$(\Omega^\delta; a^\delta, b^\delta) \xrightarrow{\text{Cara}} (\Omega; a, b) \quad \text{with respect to } z^\delta \to z \in \Omega \text{ as } \delta \to 0.$$

Note that $\Omega \supset R(s,t) = (-s;s) \times (0;t)$, $\partial\Omega \supset [-s;s]$, and $b^\delta \to b = 0$. Let $h = h_{(\Omega;a,b)}$ be the continuous Poisson kernel in Ω having mass at a and normalized at 0 (i.e., the imaginary part of the properly normalized conformal mapping from Ω onto $\mathbb{H}$). Then,

$$h^\delta \rightrightarrows h \quad \text{as } \delta \to 0,$$

uniformly on compact subsets of Ω, since this kernel can be easily constructed as a pullback of the Poisson kernel in the unit disc. Moreover, it gives

$$f^\delta = \sqrt{2i\partial h^\delta} \rightrightarrows f = \sqrt{2i\partial h} \quad \text{as } \delta \to 0$$

uniformly on compact subsets of Ω (here and below the sign of the square root is chosen so that $f^\delta(0) = f(0) = +1$). We are going to prove that, at the same time,

$$H^\delta \rightrightarrows h \quad \text{and} \quad F^\delta \rightrightarrows \sqrt{2i\partial h} \quad \text{as } \delta \to 0.$$

We start with the proof of the *uniform boundedness of H^δ away from a^δ*. Denote by $c := \frac{1}{2}it$ the center of the rectangle $R(s,t)$ and by $c^\delta \in \Gamma^*$ the dual vertex closest to c (see Fig. 8B). Let $d > 0$ be small enough and $\gamma_d^a \subset B(a_d, \frac{1}{2}d)$ be some crosscut in Ω separating a from c in Ω. Further, let $L_{3d}^\delta \subset \Omega^\delta \cap \partial B(a_d, 3d)$ be an arc separating a^δ from c^δ in Ω^δ (such an arc exists, if δ is small enough), and Ω_{3d}^δ denote the connected component of $\Omega^\delta \setminus L_{3d}^\delta$, containing c^δ.

The Harnack principle (Propositions 5.3 and 3.11) immediately give

$$H^\delta(c^\delta) \asymp 1 \quad \text{uniformly in } \delta,$$

if s and t are fixed. Let

$$M_{3d}^\delta := \max\{H_\Gamma^\delta(u^\delta),\ u^\delta \in (\Omega_{3d}^\delta)_\Gamma\}.$$

Because of the subharmonicity of $H|_\Gamma$, $M_{3d}^\delta = H_\Gamma^\delta(u_0^\delta) \le H_\Gamma^\delta(u_1^\delta) \le H_\Gamma^\delta(u_2^\delta) \le \cdots$ for some path of consecutive neighbors $K_\Gamma^\delta = \{u_0^\delta \sim u_1^\delta \sim u_2^\delta \sim \cdots\} \subset \Gamma$. Since the function H_Γ^δ vanishes everywhere on $\partial\Omega_{\tilde{\Gamma}}^\delta$ except a^δ, this path necessarily ends at a^δ. Taking on the dual graph a path

$K^\delta_{\Gamma^*}=\{w^\delta_0\sim w^\delta_1\sim w^\delta_2\sim\cdots\}\subset\Gamma^*$ close to K^δ_Γ, starting near u^δ_0 and ending near a^δ, we deduce from Remark 3.10 that

$$H^\delta_{\Gamma^*}(w^\delta_k)\geq \mathrm{const}\cdot M^\delta_{3d}\,.$$

Then,

$$H^\delta_{\Gamma^*}(c^\delta)\geq \omega^\delta(c^\delta;K^\delta_{\Gamma^*};\Omega^\delta_{\Gamma^*}\setminus K^\delta_{\Gamma^*})\cdot\mathrm{const}\cdot M^\delta_{3d}\geq \mathrm{const}((\Omega;a),d)\cdot M^\delta_{3d},$$

since $H|_{\Gamma^*}$ is superharmonic and

$$\omega^\delta(c^\delta;K^\delta_{\Gamma^*};\Omega^\delta_{\Gamma^*}\setminus K^\delta_{\Gamma^*})\geq \frac{1}{2}\omega(c^\delta;K^\delta_{\Gamma^*};\Omega^\delta\setminus K^\delta_{\Gamma^*})\geq \mathrm{const}((\Omega;a),d)>0$$

for all sufficiently small δ's (see Fig. 8B and [10] Lemma 3.14).

Thus, the functions H^δ are uniformly bounded away from a^δ. Due to Theorem 3.12, we have

$$F^\delta=O(1)\quad\text{uniformly on compact subsets of }\Omega. \tag{5.4}$$

Moreover, using uniform estimates of the discrete harmonic measure in rectangles (Lemma A.3) exactly in the same way as in the proof of Theorem 5.4, we arrive at

$$H^\delta(v)=O(\operatorname{Im} v),\qquad F^\delta=O(1)\quad\text{uniformly in }R^\delta\left(\frac{1}{2}s,\frac{1}{2}t\right). \tag{5.5}$$

Taking a subsequence, we may assume that

$$F^\delta\rightrightarrows F\quad\text{and}\quad H^\delta\rightrightarrows H\quad\text{for some }F:\Omega\to\mathbb{C},\ H:\Omega\to\mathbb{R},$$

uniformly on all compact subsets of Ω. The simple passage to the limit in (4.2) gives $H(v_2)-H(v_1)=\operatorname{Im}\int_{[v_1;v_2]}(F(\zeta))^2\,d\zeta$, for each segment $[v_1;v_2]\subset\Omega$. So, $F^2=2i\partial H$, and it is sufficient to show that $H=P_{(\Omega;a,b)}$. Being a limit of discrete subharmonic functions H^δ_Γ, as well as discrete superharmonic functions $H^\delta_{\Gamma^*}$, the function H is harmonic. The next step is the *identification of the boundary values of* H.

Let $u\in\Omega$ and $d>0$ be so small that $u\in\Omega^\delta_{4d}$. Recall that the functions $H^\delta|_\Gamma$ are subharmonic, uniformly bounded away from a^δ, and $H^\delta_\Gamma=0$ on the (modified) boundary $\partial\Omega^\delta_{\tilde\Gamma}$, except at a^δ. Thus, the weak Beurling-type estimate of the discrete harmonic measure (Lemma A.2) easily gives

$$H(u)=\lim_{\delta\to 0}H^\delta_\Gamma(u)\leq \mathrm{const}(\Omega,d)\cdot\lim_{\delta\to 0}\left[\frac{\mathrm{dist}(u\,;\partial\Omega^\delta_{3d}\setminus\partial B^\delta_\Gamma(a_d,3d))}{\mathrm{dist}_{\Omega^\delta_\Gamma}(u\,;\partial B^\delta_\Gamma(a_d,3d))}\right]^\beta$$

$$\le \mathrm{const}(\Omega, d) \cdot (\mathrm{dist}(u\,; \partial\Omega_{3d} \setminus B(a_d, 3d))^{\beta} \quad \text{for all } u \in \Omega_{5d}$$

(since, if δ is small enough, $u \in \Omega^{\delta}_{4d}$). Thus, for each $d > 0$, $H(u) \to 0$ as $u \to \partial\Omega$ inside Ω_{5d}, i.e., $H = 0$ on $\partial\Omega \setminus \{a\}$. Clearly, H is nonnegative because H^{δ} are nonnegative. Therefore, H should be proportional to the Poisson kernel in Ω having mass at a, i.e.,

$$H = \mu^2 P_{(\Omega;a,b)} \quad \text{and} \quad F = \mu\sqrt{2i\partial P_{(\Omega;a,b)}} \quad \text{for some } \mu \in \mathbb{R}.$$

Note that $|\mu|$ is uniformly bounded from ∞ and 0, since $H^{\delta}(c^{\delta}) \asymp 1$ uniformly in δ.

To finish the proof, we need to show that $\mu = 1$. For each $0 < \alpha \ll \gamma \ll t$, we have

$$F^{\delta}(z) \rightrightarrows \mu \cdot (1 + O(\gamma)) \quad \text{uniformly for } z \in [-2\gamma, 2\gamma] \times [\alpha, \gamma],$$

as $\delta \to 0$. Recall that $\Omega^{\delta}_{\diamondsuit} \supset R^{\delta}_{\diamondsuit}(s,t)$ and $\partial\Omega^{\delta}_{\diamondsuit} \supset L^{\delta}_{\diamondsuit}(s)$ for all δ (see (5.3) and Fig. 8B). Set $F^{\delta}_0 := F^{\delta} - \mu\mathcal{F}^{\delta}$, where the function $\mathcal{F}^{\delta}$ is defined in Theorem 5.4. Then F^{δ}_0 is s-holomorphic in $R^{\delta}_{\diamondsuit}(s,t)$, satisfies the boundary condition (B°) on the lower boundary, and

$$F^{\delta}_0(z) \rightrightarrows O(\gamma) \quad \text{uniformly for } z \in [-2\gamma, 2\gamma] \times [\alpha, \gamma],$$

since $\mathcal{F}^{\delta} \rightrightarrows 1$. Moreover, due to (5.4), we have $F^{\delta}_0(z) = O(1)$ everywhere in the rectangle $R^{\delta}_{\diamondsuit}(\frac{1}{2}s, \frac{1}{2}t)$. Let $H^{\delta}_0 := \int^{\delta}(F^{\delta}_0(z))^2\, d^{\delta}z$, where the additive constant is chosen so that $H^{\delta}_0 = 0$ on the boundary $L^{\delta}(s)$. Then

$$H^{\delta}_0 = O(\alpha + \gamma^3) + o_{\delta\to 0}(1) \quad \text{uniformly on the boundary of } R^{\delta}_{\Gamma}(2\gamma, \gamma).$$

Since the subharmonic function $H^{\delta}_0|_{\Gamma}$ vanishes on $\widetilde{L}^{\delta}_{\Gamma}(s)$, Lemma A.3 gives

$$\begin{aligned} H^{\delta}_0(b^{\delta}_{\mathrm{int}}) &\le O(\delta\gamma^{-1}) \cdot [O(\alpha + \gamma^3) + o_{\delta\to 0}(1)] \\ &= \delta \cdot O(\gamma^{-1}[\alpha + o_{\delta\to 0}(1)] + \gamma^2), \end{aligned}$$

where $b^{\delta}_{\mathrm{int}} \in \Gamma$ denotes the inner vertex near b^{δ}. On the other hand,

$$H^{\delta}_0(b^{\delta}_{\mathrm{int}}) \asymp \delta|F^{\delta}_0(b^{\delta})|^2 = \delta|(1-\mu)\mathcal{F}^{\delta}(b^{\delta})|^2 \asymp \delta|1-\mu|^2$$

which doesn't depend on α and γ. Successively passing to the limit as $\delta \to 0$, $\alpha \to 0$ and $\gamma \to 0$, we obtain $\mu = 1$. Thus, $F^{\delta} \rightrightarrows \sqrt{2i\partial P_{(\Omega;a,b)}}$ as $\delta = \delta_k \to 0$. □

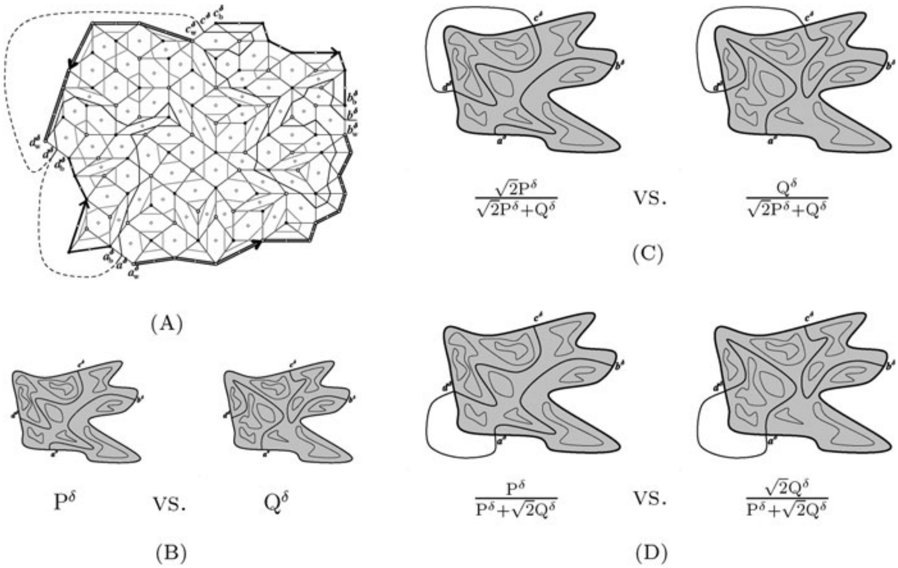


Fig. 9 (**A**) Discrete quadrilateral $\Omega^\delta_\diamond$ with four marked boundary points and Dobrushin-type boundary conditions. If we draw the additional external edge $[c^\delta d^\delta]$ (or $[a^\delta d^\delta]$), there is one interface γ^δ, going from a^δ (or d^δ, respectively) to b^δ. Since γ^δ has black vertices to the left and white vertices to the right, its winding on all boundary arcs is defined uniquely due to topological reasons. (**B**) Besides loops, there are two interfaces: either $a^\delta \leftrightarrow b^\delta$ and $c^\delta \leftrightarrow d^\delta$, or $a^\delta \leftrightarrow d^\delta$ and $b^\delta \leftrightarrow c^\delta$. We denote the probabilities of these events by P^δ and Q^δ, respectively. (**C**) The external edge $[c^\delta d^\delta]$ changes the probabilities: there is one additional loop (additional factor $\sqrt{2}$), if b^δ is connected directly with a^δ. (**D**) The external edge $[a^\delta d^\delta]$ changes the probabilities differently

6 4-point crossing probability for the FK-Ising model

Let $\Omega^\delta_\diamond \subset \diamond$ be a discrete quadrilateral, i.e. simply-connected discrete domain composed of inner rhombi $z \in \operatorname{Int}\Omega^\delta_\diamond$ and boundary half-rhombi $\zeta \in \partial\Omega^\delta_\diamond$, with four marked boundary points $a^\delta, b^\delta, c^\delta, d^\delta$ and alternating Dobrushin-type boundary conditions (see Fig. 9): $\partial\Omega^\delta_\diamond$ consists of two "white" arcs $a^\delta_{\mathrm{w}} b^\delta_{\mathrm{w}}$, $c^\delta_{\mathrm{w}} d^\delta_{\mathrm{w}}$ and two "black" arcs $b^\delta_{\mathrm{b}} c^\delta_{\mathrm{b}}$, $d^\delta_{\mathrm{b}} a^\delta_{\mathrm{b}}$. In the random cluster language it means that the four arcs are wired/free/wired/free, and in the loop representation this creates two interfaces that end at the four marked points and can connect in two possible ways. As in Sect. 2.1, we assume that $b^\delta_{\mathrm{b}} - b^\delta_{\mathrm{w}} = i\delta$.

Due to Dobrushin-type boundary conditions, each configuration (besides loops) contains *two interfaces*, either connecting a^δ to b^δ and c^δ to d^δ, or vice versa. Let

$$\mathrm{P}^\delta = \mathrm{P}^\delta(\Omega^\delta_\diamond; a^\delta, b^\delta, c^\delta, d^\delta) := \mathbb{P}(a^\delta \leftrightarrow b^\delta; c^\delta \leftrightarrow d^\delta)$$

denote the probability of the first event, and $\mathrm{Q}^\delta = 1 - \mathrm{P}^\delta$.

Theorem 6.1 *For all $r, R, t > 0$ there exists $\varepsilon(\delta) = \varepsilon(\delta, r, R, t)$ such that if $B(0, r) \subset \Omega^\delta \subset B(0, R)$ and either both $\omega(0; \Omega^\delta; a^\delta b^\delta), \omega(0; \Omega^\delta; c^\delta d^\delta)$ or both $\omega(0; \Omega^\delta; b^\delta c^\delta), \omega(0; \Omega^\delta; d^\delta a^\delta)$ are $\geq t$ (i.e., quadrilateral Ω^δ has no neighboring small arcs), then*

$$|\mathrm{P}^\delta(\Omega^\delta_\diamondsuit; a^\delta, b^\delta, c^\delta, d^\delta) - p(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)| \leq \varepsilon(\delta) \to 0 \quad \text{as } \delta \to 0$$

(*uniformly with respect to the shape of Ω^δ and $\diamondsuit^\delta$*), *where p depends only on the conformal modulus of the quadrilateral $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$. In particular, for $u \in [0, 1]$*,

$$p(\mathbb{H}; 0, 1-u, 1, \infty) = \frac{\sqrt{1-\sqrt{1-u}}}{\sqrt{1-\sqrt{u}} + \sqrt{1-\sqrt{1-u}}}.$$

Remark 6.2 This formula is a special case of a hypergeometric formula for crossings in a general FK model. In the Ising case it becomes algebraic and furthermore can be rewritten in several ways. It has an especially simple form for the crossing probabilities

$$p(\phi) := p(\mathbb{D}; -e^{i\phi}, e^{-i\phi}, e^{i\phi}, -e^{-i\phi}) \quad \text{and} \quad p\left(\frac{\pi}{2} - \phi\right) = 1 - p(\phi)$$

in the unit disc $\mathbb{D}$ (clearly, the cross-ratio u is equal to $\sin^2 \phi$). Namely,

$$\frac{p(\phi)}{p(\frac{\pi}{2} - \phi)} = \frac{\sin\frac{\phi}{2}}{\sin(\frac{\pi}{4} - \frac{\phi}{2})} \quad \text{for } \phi \in \left[0, \frac{\pi}{2}\right].$$

Curiously, this macroscopic formula formally coincides with the relative weights corresponding to two different possibilities of crossings inside microscopic rhombi (see Fig. 2A) in the critical FK-Ising model on isoradial graphs.

Proof We start with adding to our picture the "external" edge connecting c^δ and d^δ (see Fig. 9). Then, exactly as in Sect. 2.1, (2.2) and (2.4) allow us to define the s-holomorphic in $\Omega^\delta_\diamondsuit$ function $F^\delta_{[cd]} : \Omega^\delta_\diamondsuit \to \mathbb{C}$ such that

$$F^\delta_{[cd]}(\zeta) \parallel (\tau(\zeta))^{-\frac{1}{2}} \tag{6.1}$$

on $\partial\Omega^\delta_\diamondsuit$, where

$$\begin{aligned} \tau(\zeta) &= w_2(\zeta) - w_1(\zeta), \quad \zeta \in (a^\delta b^\delta) \cup (c^\delta d^\delta),\ w_{1,2}(\zeta) \in \Gamma^*, \\ \tau(\zeta) &= u_2(\zeta) - u_1(\zeta), \quad \zeta \in (b^\delta c^\delta) \cup (d^\delta a^\delta),\ u_{1,2}(\zeta) \in \Gamma, \end{aligned} \tag{6.2}$$

is the "discrete tangent vector" to $\partial\Omega^\delta_\diamondsuit$ oriented from a^δ/c^δ to b^δ/d^δ (see Fig. 9).

Note that $F^\delta_{[cd]}(b^\delta)=(2\delta)^{-\frac12}$ and $F^\delta_{[cd]}(a^\delta)=(2\delta)^{-\frac12}\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow a^\delta)}$, but

$$F^\delta_{[cd]}(d^\delta)=(2\delta)^{-\frac12}\cdot\frac{\mathrm{Q}^\delta}{\sqrt2\cdot\mathrm{P}^\delta+\mathrm{Q}^\delta}\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow(c^\delta\rightsquigarrow d^\delta))}$$

(and similarly for $F^\delta_{[cd]}(c^\delta)$, since the interface passes through d^δ if and only if b^δ is connected with c^δ, see Fig. 9).

Similarly, we can add an external edge $[a^\delta d^\delta]$ and construct another s-holomorphic in $\Omega^\delta_\diamondsuit$ function $F^\delta_{[ad]}$ satisfying the same boundary conditions (6.1). Arguing in the same way, we deduce that $F^\delta_{[ad]}(b^\delta)=(2\delta)^{-\frac12}$, $F^\delta_{[ad]}(c^\delta)=(2\delta)^{-\frac12}\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow c^\delta)}$, and

$$F^\delta_{[ad]}(d^\delta)=(2\delta)^{-\frac12}\cdot\frac{\mathrm{P}^\delta}{\mathrm{P}^\delta+\sqrt2\cdot\mathrm{Q}^\delta}\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow(a^\delta\rightsquigarrow d^\delta))}$$

(and similarly for $F^\delta_{[ad]}(a^\delta)$). Note that

$$e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow(a^\delta\rightsquigarrow d^\delta))}=-e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow(c^\delta\rightsquigarrow d^\delta))}. \tag{6.3}$$

Let

$$F^\delta:=\frac{\mathrm{P}^\delta(\sqrt2\mathrm{P}^\delta+\mathrm{Q}^\delta)\cdot F^\delta_{[cd]}+\mathrm{Q}^\delta(\mathrm{P}^\delta+\sqrt2\mathrm{Q}^\delta)\cdot F^\delta_{[ad]}}{\mathrm{P}^\delta(\sqrt2\mathrm{P}^\delta+\mathrm{Q}^\delta)+\mathrm{Q}^\delta(\mathrm{P}^\delta+\sqrt2\mathrm{Q}^\delta)}.$$

Then, F^δ also satisfies boundary conditions (6.1), (6.2) and, in view of (6.3),

$$\begin{aligned}&F^\delta(a^\delta)=(2\delta)^{-\frac12}\cdot\mathrm{A}^\delta\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow a^\delta)}, &&F^\delta(b^\delta)=(2\delta)^{-\frac12},\\ &F^\delta(c^\delta)=(2\delta)^{-\frac12}\cdot\mathrm{C}^\delta\cdot e^{-\frac{i}{2}\,\mathrm{winding}(b^\delta\rightsquigarrow c^\delta)}, &&F^\delta(d^\delta)=0,\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}\mathrm{A}^\delta&=\frac{\mathrm{P}^\delta(\sqrt2\mathrm{P}^\delta+\mathrm{Q}^\delta)+\mathrm{Q}^\delta\mathrm{P}^\delta}{\mathrm{P}^\delta(\sqrt2\mathrm{P}^\delta+\mathrm{Q}^\delta)+\mathrm{Q}^\delta(\mathrm{P}^\delta+\sqrt2\mathrm{Q}^\delta)},\\ \mathrm{C}^\delta&=\frac{\mathrm{P}^\delta\mathrm{Q}^\delta+\mathrm{Q}^\delta(\mathrm{P}^\delta+\sqrt2\mathrm{Q}^\delta)}{\mathrm{P}^\delta(\sqrt2\mathrm{P}^\delta+\mathrm{Q}^\delta)+\mathrm{Q}^\delta(\mathrm{P}^\delta+\sqrt2\mathrm{Q}^\delta)}.\end{aligned}$$

Since F^δ is s-holomorphic and satisfies the boundary conditions (6.1), (6.2), we can define $H^\delta:=\int^\delta(F^\delta(z))^2\,d^\delta z$ and use the "boundary modification trick" (see Sect. 3.6). Then, (6.4) implies

$$\begin{aligned} &H^\delta_\Gamma = 0 \quad \text{on } (a^\delta b^\delta)_{\widetilde{\Gamma}}, && H^\delta_{\Gamma^*} = 0 \quad \text{on } (a^\delta b^\delta), \\ &H^\delta_\Gamma = 1 \quad \text{on } (b^\delta c^\delta), && H^\delta_{\Gamma^*} = 1 \quad \text{on } (b^\delta c^\delta)_{\widetilde{\Gamma}^*}, \\ &H^\delta_\Gamma = \varkappa^\delta \quad \text{on } (c^\delta d^\delta) \cup (d^\delta a^\delta)_{\widetilde{\Gamma}}, && H^\delta_{\Gamma^*} = \varkappa^\delta \quad \text{on } (c^\delta d^\delta)_{\widetilde{\Gamma}^*} \cup (d^\delta a^\delta), \end{aligned} \tag{6.5}$$

where

$$\varkappa^\delta = (\mathrm{A}^\delta)^2 = 1 - (\mathrm{C}^\delta)^2 = \left[\frac{(t^\delta)^2 + \sqrt{2}t^\delta}{(t^\delta)^2 + \sqrt{2}t^\delta + 1}\right]^2, \quad t^\delta = \frac{\mathrm{P}^\delta}{\mathrm{Q}^\delta} = \frac{\mathrm{P}^\delta}{1 - \mathrm{P}^\delta}. \tag{6.6}$$

Suppose that $|\mathrm{P}^\delta(\Omega^\delta_\diamondsuit) - p(\Omega^\delta)| \ge \varepsilon_0 > 0$ for some domains $\Omega^\delta_\diamondsuit = (\Omega^\delta_\diamondsuit; a^\delta, b^\delta, c^\delta, d^\delta)$ with $\delta \to 0$. Passing to a subsequence (exactly as in Sect. 4), we may assume that

$$(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta) \xrightarrow{\mathrm{Cara}} (\Omega; a, b, c, d), \qquad \varkappa^\delta \to \varkappa \in [0, 1],$$

$$H^\delta \rightrightarrows H, \quad \text{and} \quad F^\delta \rightrightarrows F = \sqrt{2i\partial H},$$

uniformly on compact subsets, for some harmonic function $H : \Omega \to \mathbb{R}$. It follows from our assumptions that $B(0, r) \subset \Omega \subset B(0, R)$ and either $a \ne b$, $c \ne d$ or $b \ne c$, $d \ne a$.

We begin with the main case, when the limiting quadrilateral $(\Omega; a, b, c, d)$ is non-degenerate and $0 < \varkappa < 1$. As in Sect. 4, we see that

$$H = 0 \quad \text{on } (ab), \qquad H = 1 \quad \text{on } (bc) \quad \text{and} \quad H = \varkappa \quad \text{on } (cd) \cup (da). \tag{6.7}$$

Consider the conformal mapping Φ from Ω onto the slit strip $[\mathbb{R} \times (0; 1)] \setminus (i\varkappa - \infty; i\varkappa]$ such that a is mapped to "lower" $-\infty$, b to $+\infty$ and c to "upper" $-\infty$ (note that such a mapping is uniquely defined). Then, the imaginary part of Φ is harmonic and has the same boundary values as H, so we conclude that $H = \operatorname{Im} \Phi$. We prove that

d is mapped exactly to the tip $i\varkappa$.

Then, $\varkappa$ can be uniquely determined from the conformal modulus of $(\Omega; a, b, c, d)$.

Suppose that above is not the case, and d is mapped, say, on the lower bank of the cut. It means that $H < \varkappa$ near some (close to d) part of the boundary arc (cd). Then, there exists a (small) contour $C = [p; q] \cup [q; r] \cup [r; s] \subset \Omega$ such that $H < \varkappa$ everywhere on C,

$$s, p \in (cd) \subset \partial\Omega, \quad \operatorname{dist}(q; \partial\Omega) = |q - p| \quad \text{and} \quad \operatorname{dist}(r; \partial\Omega) = |r - s|$$

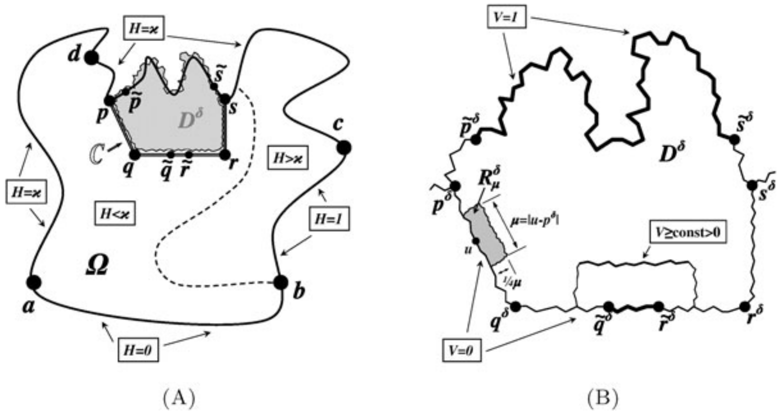


Fig. 10 (**A**) If d is mapped on the lower bank of the cut, then $H < \varkappa$ somewhere near (cd). The contour $C = [p;q] \cup [q;r] \cup [r;s]$ is chosen so that $H < \varkappa$ on C, $\operatorname{dist}(q;\partial\Omega) = |q-p|$ and $\operatorname{dist}(r;\partial\Omega) = |r-s|$. (**B**) Since $V^\delta = 0$ on C^δ, we have $\partial_n^\delta V^\delta \le 0$ everywhere on C^δ. Moreover, $\partial_n^\delta V^\delta \le \text{const} < 0$ on $(\widetilde{q}^\delta \widetilde{r}^\delta)$ and $\partial_n^\delta V^\delta = O([\operatorname{dist}(u;\partial\Omega^\delta)]^{\beta-1})$ near p^δ, s^δ (here ∂_n^δ denotes the discrete derivative in the *outer* normal direction)

(see Fig. 10). Denote by $D \subset \Omega$ the part of Ω lying inside C. For technical purposes, we also fix some intermediate points $\widetilde{s}, \widetilde{p} \in (sp) \subset (cd)$ and $\widetilde{q}, \widetilde{r} \in [q;r]$ (see Fig. 10). For sufficiently small δ, we can find discrete approximations $s^\delta, \widetilde{s}^\delta, \widetilde{p}^\delta, p^\delta \in (c^\delta d^\delta) \subset \partial\Omega^\delta_{\widetilde{\Gamma}}$ and $q^\delta, \widetilde{q}^\delta, \widetilde{r}^\delta, r^\delta \in \Omega^\delta_\Gamma$ to these points such that the contour $[p^\delta; q^\delta] \cup [q^\delta; r^\delta] \cup [r^\delta; s^\delta]$ approximates C. Denote by $D^\delta \subset \Omega^\delta$ the part of Ω^δ lying inside C and by $D^\delta_\Gamma \subset \Omega^\delta_\Gamma$ the set of all "black" vertices lying in D^δ and their neighbors.

Let $\widetilde{H}^\delta_\Gamma := H^\delta_\Gamma - \varkappa^\delta$ and $V^\delta := \omega^\delta(\cdot\,; (\widetilde{s}^\delta \widetilde{p}^\delta); D^\delta)$. Since H^δ_Γ is subharmonic and $V^\delta \ge 0$ is harmonic, the discrete Green's formula gives

$$\sum_{u\in\partial D^\delta} [(\widetilde{H}^\delta_\Gamma(u) - \widetilde{H}^\delta_\Gamma(u_{\mathrm{int}}))V^\delta(u) - (V^\delta(u) - V^\delta(u_{\mathrm{int}}))\widetilde{H}^\delta_\Gamma(u)]\tan\theta_{uu_{\mathrm{int}}} \ge 0.$$

Note that $\widetilde{H}^\delta_\Gamma \equiv 0$ on $(s^\delta p^\delta)_{\widetilde{\Gamma}}$ and $V^\delta(u) \equiv 0$ on $C^\delta := \partial D^\delta_\Gamma \setminus (s^\delta p^\delta)_{\widetilde{\Gamma}}$. Thus,

$$\begin{aligned}\sum_{u\in C^\delta} & V^\delta(u_{\mathrm{int}})\widetilde{H}^\delta_\Gamma(u)\tan\theta_{uu_{\mathrm{int}}} \\ &\ge \sum_{u\in(s^\delta p^\delta)} (\widetilde{H}^\delta_\Gamma(u_{\mathrm{int}}) - \widetilde{H}^\delta_\Gamma(u))V^\delta(u)\tan\theta_{uu_{\mathrm{int}}} \ge 0, \end{aligned} \tag{6.8}$$

since $H^\delta_\Gamma(u_{\mathrm{int}}) \ge H^\delta_\Gamma(u)$ everywhere on $(s^\delta p^\delta)$ due to the boundary conditions (6.1), (6.2). On the other hand, on most of C^δ, we have $\widetilde{H}^\delta_\Gamma < 0$ (since $H - \varkappa = \lim_{\delta\to 0} \widetilde{H}^\delta_\Gamma < 0$ on C by assumption), and $V^\delta(u_{\mathrm{int}}) \ge 0$ everywhere on C^δ, which gives a contradiction. Unfortunately, we cannot immediately claim that $\widetilde{H}_\Gamma^\delta < 0$ near the boundary, so one needs to prove that the neighborhoods of p^δ and s^δ cannot produce an error sufficient to compensate this difference of signs.

More accurately, it follows from the uniform convergence $V^\delta \rightrightarrows \omega(\,\cdot\,; (\widetilde{s}\widetilde{p}); D) > 0$ on compacts inside D and Lemma A.3 that $V^\delta(u_{\mathrm{int}}) \ge \mathrm{const}(D) \cdot \delta$ everywhere on $(\widetilde{q}^\delta \widetilde{r}^\delta) \subset C^\delta$, so, for small enough δ,

$$\sum_{u \in (\widetilde{q}^\delta \widetilde{r}^\delta)} V^\delta(u_{\mathrm{int}}) \widetilde{H}_\Gamma^\delta(u) \tan\theta_{uu_{\mathrm{int}}} \le -\mathrm{const}(D, H) < 0. \tag{6.9}$$

Thus, it is sufficient to prove that the neighborhoods of p^δ and s^δ cannot compensate this negative amount which is independent of δ. Let $u \in (p^\delta q^\delta) \subset C^\delta$ and $\mu = \mathrm{dist}(u; p^\delta) = \mathrm{dist}(u; \partial\Omega^\delta)$ be small. Denote by R_μ^δ the discretization of the $\mu \times \frac{1}{4}\mu$ rectangle near u (see Fig. 10). Due to Lemma A.3, we have

$$\omega^\delta(u_{\mathrm{int}}; \partial R_\mu^\delta \setminus (p^\delta q^\delta); R_\mu^\delta) = O(\delta\mu^{-1}).$$

Furthermore, for each $v \in \partial R_\mu^\delta \setminus (p^\delta q^\delta)$, Lemma A.2 gives

$$\omega^\delta(v; (\widetilde{s}^\delta \widetilde{p}^\delta); D^\delta) = O(\mu^\beta) \quad \text{uniformly on } \partial R_\mu^\delta.$$

Hence,

$$V^\delta(u_{\mathrm{int}}) = \omega^\delta(u_{\mathrm{int}}; (\widetilde{s}^\delta \widetilde{p}^\delta); D^\delta) \le \mathrm{const}(D) \cdot \delta\mu^{-(1-\beta)}.$$

Recalling that $H^\delta = O(1)$ by definition and summing, for any $p_\mu^\delta \in (p^\delta q^\delta) \subset C^\delta$ sufficiently close to p^δ we obtain

$$\begin{aligned}\sum_{u \in (p^\delta p_\mu^\delta)} V^\delta(u_{\mathrm{int}}) \widetilde{H}_\Gamma^\delta(u) \tan\theta_{uu_{\mathrm{int}}} &\le \mathrm{const}(D) \cdot \sum_{u \in (p^\delta p_\mu^\delta)} \delta \cdot (\mathrm{dist}(u; p^\delta))^{-(1-\beta)} \\ &\le \mathrm{const}(D) \cdot (\mathrm{dist}(p_\mu^\delta; p^\delta))^\beta \end{aligned} \tag{6.10}$$

(uniformly with respect to δ). The same estimate holds near s^δ. Taking into account $\widetilde{H}_\Gamma^\delta(u) < 0$ which holds true (if δ is small enough) for all $u \in C^\delta$ lying μ-away from p^δ, s^δ, we deduce from (6.8), (6.9) and (6.10) that $0 < \mathrm{const}(D, H) \le \mathrm{const}(D) \cdot \mu^\beta$ for any $\mu > 0$ (and sufficiently small $\delta \le \delta(\mu)$), arriving at a contradiction.

All "degenerate" cases can be dealt with in the same way:

- if the quadrilateral $(\Omega; a, b, c, d)$ is non-degenerate, then
  - if $\varkappa = 1$, then $H < \varkappa$ near some part of (cd), which is impossible;

– if d is mapped onto the upper bank of the slit or $\varkappa = 0$, then $H > \varkappa$ near some part of (ad), which leads to a contradiction via the same arguments as above;

- if $b = c$ (and so $c \neq d$), then $\varkappa = 0$ since otherwise $H < \varkappa$ everywhere near (cd) due to boundary conditions (6.7);
- if $d = a$ (and so $c \neq d$, $a \neq b$), then again $\varkappa = 0$ since otherwise (6.7) implies $H < \varkappa$ near some part of $(cd) = (ca)$ close to a;
- finally, $a = b$ or $c = d$ lead to $\varkappa = 1$ (otherwise $H > \varkappa$ near some part of (da)).

Thus, d is mapped to the tip and so $\varkappa = \varkappa(\Omega; a, b, c, d)$ is uniquely determined by the conformal modulus of the quadrilateral ($\varkappa$ is either 0 or 1 in degenerate cases). Recall that $\varkappa^\delta = \xi(P^\delta)$, where the bijection $\xi : [0, 1] \to [0, 1]$ is given by (6.6). Let

$$p(\Omega; a, b, c, d) := \xi^{-1}(\varkappa(\Omega; a, b, c, d)).$$

Then (since $\varkappa(\cdot)$ is Carathéodory stable) both $\mathrm{P}^\delta(\Omega^\delta_\diamondsuit) = \xi^{-1}(\varkappa^\delta)$ and $p(\Omega^\delta)$ tend to $p(\Omega) = \xi^{-1}(\varkappa)$ as $\delta = \delta_k \to 0$, which contradicts to $|\mathrm{P}^\delta(\Omega^\delta_\diamondsuit) - p(\Omega^\delta)| \geq \varepsilon_0 > 0$.

Finally, the simple calculation for the half-plane $(\mathbb{H}; 0, 1-u, 1, \infty)$ gives

$$H(z) \equiv u + \frac{1}{\pi}(-\arg[z-(1-u)] + u \arg z + (1-u)\arg[z-1]), \quad z \in \mathbb{H}.$$

Hence, $\varkappa(\mathbb{H}; 0, 1-u, 1, \infty) = u$ and $p = \xi^{-1}(u)$ which coincides with (1.1). □

Remark 6.3 In fact, above we have shown that the "$(\tau(z))^{-\frac{1}{2}}$" boundary condition (3.16) reformulated in the form $\partial_n^\delta H^\delta \leq 0$ remains valid in the limit as $\delta \to 0$. Namely, let a sequence of discrete domains Ω^δ converge to some limiting Ω in the Carathéodory topology, while s-holomorphic functions F^δ defined on Ω^δ satisfy (3.16) on arcs $(s^\delta p^\delta)$ converging to some boundary arc $(sp) \subset \partial\Omega$. Let their integrals $H^\delta = \operatorname{Im} \int^\delta (F^\delta(z))^2 d^\delta z$ are defined so that they are uniformly bounded near $(s^\delta p^\delta)$ and their (constant) values $\varkappa^\delta$ on $(s^\delta p^\delta)$ tends to some $\varkappa$ as $\delta \to 0$. Then, if H^δ converge to some (harmonic) function H inside Ω, one has $\partial_n H^\delta \leq 0$ on (sp) in the following sense: there is no point $\zeta \in (sp)$ such that $H < \varkappa$ in a neighborhood of ζ. The proof mimics the corresponding part of the proof of Theorem 6.1. Since the boundary conditions (3.16) are typical for holomorphic observables in the critical Ising model, this statement eventually can be applied to all such observables.

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Appendix

A.1 Estimates of the discrete harmonic measure

Here we formulate uniform estimates for the discrete harmonic measure on isoradial graphs which were used above.

Lemma A.1 (exit probabilities in the disc) *Let* $u_0 \in \Gamma$, $r \geq \delta$ *and* $a \in \partial B_\Gamma^\delta(u_0, r)$. *Then,*

$$\omega^\delta(u_0; \{a\}; B_\Gamma^\delta(u_0, r)) \asymp \delta/r.$$

Proof See [8] (or [10] Proposition A.1). The proof is based on the asymptotics (A.1) of the free Green's function. □

Lemma A.2 (weak Beurling-type estimate) *There exists an absolute constant* $\beta > 0$ *such that for any simply connected discrete domain* Ω_Γ^δ, *point* $u \in \operatorname{Int}\Omega_\Gamma^\delta$ *and some part of the boundary* $E \subset \partial\Omega_\Gamma^\delta$ *we have*

$$\omega^\delta(u; E; \Omega_\Gamma^\delta) \leq \text{const} \cdot \left[\frac{\operatorname{dist}(u; \partial\Omega_\Gamma^\delta)}{\operatorname{dist}_{\Omega_\Gamma^\delta}(u; E)}\right]^\beta.$$

Here $\operatorname{dist}_{\Omega_\Gamma^\delta}$ *denotes the distance inside* Ω_Γ^δ.

Proof See [10] Proposition 2.11. The proof is based on the uniform bound of the probability that the random walk on Γ crosses the annulus without making the full turn inside. □

Finally, let $R_\Gamma^\delta(s,t) \subset \Gamma$ denote the discretization of the open rectangle

$$R(s,t) = (-s; s) \times (0; t) \subset \mathbb{C}, \quad s, t > 0;$$

$b^\delta \in \partial\mathbb{H}_\Gamma^\delta$ be the boundary vertex closest to 0; and $L_\Gamma^\delta(s)$, $U_\Gamma^\delta(s,t)$, $V_\Gamma^\delta(s,t)$ be the lower, upper and vertical parts of the boundary $\partial R_\Gamma^\delta(s,t)$, respectively.

Lemma A.3 (exit probabilities in the rectangle) *Let* $s \geq 2t$ *and* $t \geq 2\delta$. *Then, for any* $v^\delta = x + iy \in R_\Gamma^\delta(s,t)$, *one has*

$$\frac{y+2\delta}{t+2\delta} \geq \omega^\delta(v^\delta; U_\Gamma^\delta(s,t); R_\Gamma^\delta(s,t)) \geq \frac{y}{t+2\delta} - \frac{x^2 + (y+2\delta)(t+2\delta-y)}{s^2}$$

and

$$\omega^\delta(v^\delta; V_\Gamma^\delta(s,t); R_\Gamma^\delta(s,t)) \le \frac{x^2+(y+2\delta)(t+2\delta-y)}{s^2}.$$

Proof See [10] Lemma 3.17. The claim easily follows from the maximum principle for discrete harmonic functions. □

A.2 Lipschitzness of discrete harmonic and discrete holomorphic functions

Proposition A.4 (discrete Harnack Lemma) *Let $u_0 \in \Gamma$ and $H : B_\Gamma^\delta(u_0, R) \to \mathbb{R}$ be a nonnegative discrete harmonic function. Then,*

(i) *for any $u_1, u_2 \in B_\Gamma^\delta(u_0, r) \subset \operatorname{Int} B_\Gamma^\delta(u_0, R)$,*

$$\exp\left[-\operatorname{const}\cdot\frac{r}{R-r}\right] \le \frac{H(u_2)}{H(u_1)} \le \exp\left[\operatorname{const}\cdot\frac{r}{R-r}\right];$$

(ii) *for any $u_1 \sim u_0$,*

$$|H(u_1)-H(u_0)| \le \operatorname{const}\cdot\delta H(u_0)/R.$$

Proof See [8] (or [10] Proposition 2.7). The proof is based on the asymptotics (A.1) of the free Green's function. □

Corollary A.5 (Lipschitzness of harmonic functions) *Let H be discrete harmonic in $B_\Gamma^\delta(u_0, R)$ and $u_1, u_2 \in B_\Gamma^\delta(u_0, r) \subset \operatorname{Int} B_\Gamma^\delta(u_0, R)$. Then*

$$|H(u_2)-H(u_1)| \le \operatorname{const}\cdot\frac{M|u_2-u_1|}{R-r}, \quad \text{where } M = \max_{B_\Gamma^\delta(u_0,R)} |H(u)|.$$

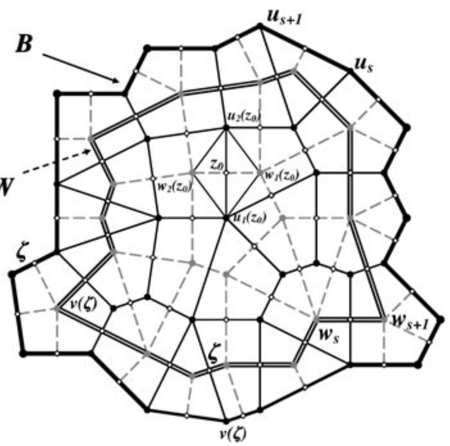
In order to formulate the similar result for discrete holomorphic functions we need some preliminary definitions. Let F be defined on some part of $\diamondsuit$. Taking the real and imaginary parts of $\overline{\partial}^\delta F$ (see (3.2)), it is easy to see that F is holomorphic if and only if both functions

$$[\mathcal{B}F](z) := \operatorname{Proj}[F(z); \overline{u_1(z)-u_2(z)}],$$
$$[\mathcal{W}F](z) := \operatorname{Proj}[F(z); \overline{w_1(z)-w_2(z)}]$$

are holomorphic, where $u_{1,2}(z) \in \Gamma$ and $w_{1,2}(z) \in \Gamma^*$ are the black and white neighbors of $z \in \diamondsuit$, respectively (note that $F = \mathcal{B}F + \mathcal{W}F$).

Let Ω_Γ^δ be a bounded simply connected discrete domain. For a function G defined on both "boundary contours" B,W (see Fig. 11), we set

$$\oint_{B\cup W}^{\delta} G(\zeta)\, d^\delta\zeta := \sum_{s=0}^{n-1} G\left(\frac{1}{2}(u_{s+1}+u_s)\right)(u_{s+1}-u_s)$$



Let Ω_Γ^δ be a bounded simply connected discrete domain. We denote by $B = u_0u_1..u_n$, $u_s \in \Gamma$, its closed polyline boundary enumerated in the counterclockwise order, and by $W = w_0w_1..w_m$, $w_s \in \Gamma^*$, the closed polyline path passing through the centers of all faces touching B enumerated in the counterclockwise order. In order to write down the discrete Cauchy formula (see Lemma A.6), one needs to "integrate" over both B and W.

Fig. 11 Discrete Cauchy formula, notations (see Lemma A.6)

$$+\sum_{s=0}^{m-1} G\left(\frac{1}{2}(w_{s+1}+w_s)\right)(w_{s+1}-w_s).$$

Lemma A.6 (Cauchy formula) (i) *There exists a function* (*discrete Cauchy kernel*) $K(\cdot;\cdot) : \Lambda \times \diamondsuit \to \mathbb{C}$, $K(v,z) = O(|v-z|^{-1})$, *such that for any discrete holomorphic function* $F : \Omega_\diamondsuit^\delta \to \mathbb{C}$ *and* $z_0 \in \Omega_\diamondsuit^\delta \setminus (B \cup W)$ *the following holds true*:

$$F(z_0) = \frac{1}{4i} \oint_{B\cup W}^{\delta} K(v(\zeta); z_0) F(\zeta)\, d^\delta\zeta,$$

where $\zeta \sim v(\zeta) \in W$, *if* $\zeta \in B$, *and* $\zeta \sim v(\zeta) \in B$, *if* $\zeta \in W$ (*see Fig.* 11*B*).

(ii) *Moreover, if* $F = \mathcal{B}F$, *then*

$$F(z_0) = \mathrm{Proj}\left[\frac{1}{2\pi i}\oint_{B\cup W}^{\delta} \frac{F(\zeta)\, d^\delta\zeta}{\zeta - z_0};\ \overline{u_1(z_0) - u_2(z_0)}\right] + O\left(\frac{M\delta L}{d^2}\right),$$

where $d = \mathrm{dist}(z_0, W)$, $M = \max_{z\in B\cup W}|F(z)|$ *and* L *is the length of* $B \cup W$. *The similar formula* (*with* $w_1(z_0) - w_2(z_0)$ *instead of* $u_1(z_0) - u_2(z_0)$) *holds true, if* $F = \mathcal{W}F$.

Proof See [10] Proposition 2.22 and Corollary 2.23. The proof is based on the discrete integration by parts and asymptotics of the discrete Cauchy kernel $K(\cdot;\cdot)$ proved by R. Kenyon in [28]. □

Corollary A.7 (Lipschitzness of holomorphic functions) *Let* $F : B_\diamondsuit^\delta(z_0, R) \to \mathbb{C}$ *be discrete holomorphic. Then there exist* $A, B \in \mathbb{C}$ *such that*

$$F(z) = \mathrm{Proj}[A;\ \overline{u_1(z) - u_2(z)}] + \mathrm{Proj}[B;\ \overline{w_1(z) - w_2(z)}] + O(Mr/(R-r)),$$

where $M = \max_{z\in B^\delta_\diamondsuit(z_0,R)} |F(z)|$, *for any* z *such that* $|z-z_0| \le r < R$.

Proof Namely,

$$A = \frac{1}{2\pi i}\oint_{B\cup W}^{\delta} \frac{[\mathcal{B}F](\zeta)d^\delta\zeta}{\zeta - z_0} \quad \text{and} \quad B = \frac{1}{2\pi i}\oint_{B\cup W}^{\delta} \frac{[\mathcal{W}F](\zeta)d^\delta\zeta}{\zeta - z_0}. \qquad \square$$

A.3 Estimates of the discrete Green's function

Here we prove two technical lemmas which were used in Sect. 3.5. Recall that the Green's function $G_{\Omega^\delta_\Gamma}(\cdot;u) : \Omega^\delta_\Gamma \to \mathbb{R}$, $u \in \Omega^\delta_\Gamma \subset \Gamma$, is the (unique) discrete harmonic in $\Omega^\delta_\Gamma \setminus \{u\}$ function such that $G_{\Omega^\delta_\Gamma} = 0$ on the boundary $\partial\Omega^\delta_\Gamma$ and $\mu^\delta_\Gamma(u)\cdot[\Delta^\delta G_{\Omega^\delta_\Gamma}](u) = 1$. Clearly,

$$G_{\Omega^\delta_\Gamma} = G_\Gamma - G^*_{\Omega^\delta_\Gamma},$$

where G_Γ is the free Green's function and $G^*_{\Omega^\delta_\Gamma}$ is the unique discrete harmonic in Ω^δ_Γ function that coincides with G_Γ on the boundary $\partial\Omega^\delta_\Gamma$. It is known that G_Γ satisfies ([28], see also [10] Theorem 2.5) the asymptotics

$$G_\Gamma(v;u) = \frac{1}{2\pi}\log|v-u| + O\left(\frac{\delta^2}{|v-u|^2}\right), \qquad v \ne u. \tag{A.1}$$

Lemma A.8 *Let* $B^\delta_\Gamma = B^\delta_\Gamma(z_0,r) \subset \Gamma$, $r \ge \text{const}\cdot\delta$, *be the discrete disc,* $u \in B^\delta_\Gamma$ *be such that* $|u-z_0| \le \frac{3}{4}r$ *and* $G = G_{B^\delta_\Gamma}(\cdot;u) : B^\delta_\Gamma \to \mathbb{R}$ *be the corresponding discrete Green's function. Then*

$$\|G\|_{1,B^\delta_\Gamma} = \sum_{v\in B^\delta_\Gamma} \mu^\delta_\Gamma(v)|G(v)| \ge \text{const}\cdot r^2.$$

Proof It immediately follows from (A.1) that

$$(2\pi)^{-1}\log\left(\frac{1}{4}r\right) + O(\delta^2/r^2) \le G^*_{B^\delta_\Gamma}(\cdot;u) \le (2\pi)^{-1}\log\left(\frac{7}{4}r\right) + O(\delta^2/r^2) \tag{A.2}$$

on the boundary ∂B^δ_Γ, and so inside B^δ_Γ. For $v \in B^\delta_\Gamma$ such that $|v-u| \le \frac{1}{8}r$, this gives

$$G(v) = G_\Gamma(v;u) - G^*_{B^\delta_\Gamma}(v;u) \le -(2\pi)^{-1}\log 2 + O(\delta^2/r^2) \le -\text{const}.$$

Thus, $\|G\|_{1,B^\delta_\Gamma} \ge \|G\|_{1,B^\delta_\Gamma(u,\frac{1}{8}r)} \ge \text{const}\cdot r^2$. $\square$

Lemma A.9 *Let* $B^\delta_\Gamma = B^\delta_\Gamma(z_0, r) \subset \Gamma$, $r \geq \mathrm{const}\cdot\delta$, *be the discrete disc,* $u \in B^\delta_\Gamma$ *and* $G = G_{B^\delta_\Gamma}(\cdot\,; u) : B^\delta_\Gamma \to \mathbb{R}$ *be the corresponding discrete Green's function. Then*

$$\|\partial^\delta G\|_{1, B^\delta_\diamond(z_0, \frac{2}{3}r)} = \sum_{z \in B^\delta_\diamond(z_0, \frac{2}{3}r)} \mu^\delta_\diamond(z) |[\partial^\delta G](z)| \leq \mathrm{const}\cdot r.$$

Proof Let $|u - z_0| \leq \frac{3}{4}r$. It easily follows from (A.1) and Corollary A.5 applied in the disc $B^\delta_\diamond(z, \frac{1}{2}|z-u|)$ that $|[\partial^\delta G_\Gamma](z; u)| \leq \mathrm{const}\cdot|z-u|^{-1}$. Therefore,

$$\|\partial^\delta G_\Gamma\|_{1, B^\delta_\diamond(z_0, \frac{2}{3}r)} \leq \|\partial^\delta G_\Gamma\|_{1, B^\delta_\diamond(u, 2r)} \leq \mathrm{const}\cdot r.$$

Furthermore, double-sided bound (A.2) and Corollary A.5 imply

$$[\partial^\delta G^*_{B^\delta_\Gamma}](z) = \partial^\delta [G^*_{B^\delta_\Gamma} - (2\pi)^{-1}\log r](z) = O(\mathrm{const}\cdot r^{-1}), \quad |z - z_0| \leq \frac{2}{3}r.$$

Thus,

$$\|\partial^\delta G^*_{B^\delta_\Gamma}\|_{1, B^\delta_\diamond(z_0, \frac{2}{3}r)} \leq \mathrm{const}\cdot r.$$

Otherwise, let $|u - z_0| > \frac{3}{4}r$. We have

$$G^*_{B^\delta_\Gamma}(\cdot\,; u) \leq (2\pi)^{-1}\log(2r) + O(1)$$

on ∂B^δ_Γ, and so on the boundary of the smaller disc $B^\delta_\Gamma(z_0, \frac{17}{24}r)$ which still contains $B^\delta_\diamond(z_0, \frac{2}{3}r)$. At the same time,

$$G_\Gamma(\cdot\,; u) \geq (2\pi)^{-1}\log\frac{1}{24}r + O(1) \quad \text{on } \partial B^\delta_\Gamma\left(z_0, \frac{17}{24}r\right).$$

Thus,

$$0 \geq G = G_\Gamma(\cdot\,; u) - G^*_{B^\delta_\Gamma}(\cdot\,; u) \geq -\mathrm{const}$$

on the boundary, and so inside $B^\delta_\Gamma(z_0, \frac{17}{24}r)$. Due to Corollary A.5, this gives

$$[\partial^\delta G](z) = O(\mathrm{const}\cdot r^{-1}), \quad \text{for } |z - z_0| \leq \frac{2}{3}r.$$

Hence, $\|\partial^\delta G\|_{1, B^\delta_\diamond(z_0, \frac{2}{3}r)} \leq \mathrm{const}\cdot r$. $\square$

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