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APPROXIMATION AND EXTENSION PROBLEMS FOR SOME CLASSES OF VECTOR FIELDS

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ABSTRACT. The uniform approximability on a compact set $K \subset \mathbb{R}^n$ of an arbitrary vector field continuous on K by curl-free, solenoidal, and harmonic vector fields is studied. It is proved that the metric disconnectedness of K ensures "free approximation" by curl-free fields. A complete geometric description is given of the sets K on which any continuous field coincides with the gradient of a smooth function. Free approximation by jets of order one is considered. An example is constructed showing that the Bishop locality principle is not applicable to harmonic fields in $\mathbb{R}^3$. A direct proof of the presence of rectifiable arcs in the support of any solenoidal charge is given (this result was obtained in [4] by a different method).

By definition, a *vector field* in $\mathbb{R}^n$ is a mapping $v\colon E \longrightarrow \mathbb{R}^n$, where E is a subset of $\mathbb{R}^n$; the set E will be called the *domain of* v and denoted by $\operatorname{dom} v$. We denote by v_k the kth coordinate of $v = (v_1, \dots, v_n)$.

Suppose that a class $X(U)$ of continuous vector fields v with $\operatorname{dom} v = U$ is associated with every $U \in \mathcal{O}(\mathbb{R}^n)$, where $\mathcal{O}(\mathbb{R}^n)$ is the class of all open subsets of $\mathbb{R}^n$. Let X denote the family $\{X(U)\}_{U \in \mathcal{O}(\mathbb{R}^n)}$ (a "presheaf of vector fields").

For a compact set $K \subset \mathbb{R}^n$, we denote by $\vec{C}(K)$ the set of all continuous vector fields([1]) v with $\operatorname{dom} v = K$. If $v \in X(U)$ and $U \subset K$, we can consider $v|K$, the restriction of v to K. If the set of all such restrictions (that correspond to any $U \in \mathcal{O}(\mathbb{R}^n)$ containing K and to any $v \in X(U)$) is uniformly dense in $\vec{C}(K)$, then K will be called an X-*set.* If the set $X(\mathbb{R}^n)|K$ of all restrictions $v|K$ with $v \in X(\mathbb{R}^n)$ is uniformly dense in $\vec{C}(K)$, we say that K is a *strong* X-set. We call K a *perfect* X-set if

$$X(\mathbb{R}^n)|K = \vec{C}(K).$$

These definitions stem from the following well-known problems (solved in many particular cases): given a family X, describe (more or less explicitly) the X-sets (the strong X-sets, the perfect X-sets).

In this paper, we treat the following presheaves X.

1) $X(U) = \operatorname{grad} U$, where $\operatorname{grad} U$ denotes the set of all curl-free continuous vector fields on U, i.e.,

$$(*)\qquad \begin{aligned} X(U) &:= \{\, v \in \vec{C}(U) : \operatorname{curl} v = 0 \,\} \\ &= \left\{ v \in \vec{C}(U) : \frac{\partial v_j}{\partial x_k} \equiv \frac{\partial v_k}{\partial x_j} \text{ in } U,\ j, k = 1, \dots, n \right\} \end{aligned}$$

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([1])The arrow shows that we consider a space of vector fields. Symbols $C(K)$, $C^m(K)$ without arrows will denote usual classes of scalar functions.

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(the derivatives are understood as distributions). Such fields are locally exact. Denoting by $B(p,r)$ the open ball of radius r centered at p, we may redefine $\operatorname{grad} U$ as follows:

$$v \in \operatorname{grad} U \iff \forall p \in U\, \exists \varepsilon_p > 0, f_p \in C^1(B) : v|B = \nabla f_p \quad (B = B(p, \varepsilon_p)).$$

In this case, the X-sets will be called *grad-sets*. A compact set $K \subset \mathbb{R}^n$ is a strong grad-set if and only if

$$\forall v \in \vec{C}(K)\, \forall \varepsilon > 0\, \exists U \in \mathcal{O}(\mathbb{R}^n), f \in C^1(U) : \max_K |v - \nabla f| < \varepsilon.$$

(Clearly, U can be replaced by $\mathbb{R}^n$.)

The perfect grad-sets K can be characterized by the following property: on K, any field $v \in \vec{C}(K)$ coincides with the gradient of a C^1-function.

Replacing $\vec{C}(U)$ by $\vec{C}^1(U)$ or by $\vec{C}^\infty(U)$ in $(*)$, we get the same notion of a grad-set and of a strong grad-set. But to make the notion of a *perfect* grad-set meaningful we need precisely the class $\vec{C}(U)$.

2) $X(U) = \operatorname{curl}(U)$, where $\operatorname{curl}(U)$ denotes the set of all divergence-free continuous vector fields, i.e.,

$$X(U) := \{ v \in \vec{C}(U) : \operatorname{div} v = 0 \} = \Big\{ v \in \vec{C}(U) : \sum_{i=1}^n \frac{\partial v_i}{\partial x_i} = 0 \text{ in } U \Big\}.$$

The corresponding X-sets (strong X-sets, perfect X-sets) will be called *curl-sets* (respectively, *strong* or *perfect curl-sets*). It is easily seen that K is a curl-set if and only if for any $\varepsilon > 0$ and any $\vec{v} \in \vec{C}(K)$ there is an open set $O \supset K$ and a field $\vec{w} \in \vec{C}^\infty(O)$ such that $\operatorname{div} \vec{w}|_O = 0$ and $\max_K |\vec{v} - \vec{w}| < \varepsilon$; locally, $\vec{w}$ is the curl of a $\vec{C}^\infty$-field. Replacing O by $\mathbb{R}^n$ and $\vec{w}$ by $\operatorname{curl} \vec{w}$, we get an equivalent definition of a strong curl-set.

3) The following example is our main motivation. Put

$$h(U) := \operatorname{grad}(U) \cap \operatorname{curl}(U), \quad U \in \mathcal{O}(\mathbb{R}^n).$$

The elements of $h(U)$ are called *harmonic vector fields* (*in* U). Such a field can be characterized as follows: *locally*, it is the gradient of a harmonic function, i.e.,

$$\forall p \in U\, \exists \varepsilon_p > 0, f_p \in C^\infty(B) : \Delta f_p = 0 \text{ in } B,\ v = \nabla f_p|B \quad (B = B(p, \varepsilon_p)).$$

In other words, $v \in h(U)$ if and only if $v \in \vec{C}^1(U)$ and the Jacobi matrix of the mapping $v\colon U \to \mathbb{R}^n$ is symmetric and has zero trace. Yet another (equivalent) version of this definition can be given in terms of differential forms if we identify v with the form $\omega = v_1\, dx_1 + \cdots + v_n\, dx_n$; then

$$\begin{gathered} v \in h(U) \\ \Updownarrow \\ d\omega = 0,\ \delta\omega = 0 \text{ in } U, \end{gathered}$$

where $d\omega$ is the exterior differential and $\delta\omega$ the codifferential of ω. In this formulation, the definition makes sense for any Riemannian manifold (in place of $\mathbb{R}^n$) and for differential forms of any degree. However, in this paper we restrict ourselves to vector fields in $\mathbb{R}^n$.

The notions of a harmonic vector field and a harmonic differential form play a fundamental role in many areas of mathematics. The one-dimensional complex analysis can be regarded as a very particular case of the theory of harmonic vector fields, because $h(U)$ with $U \in \mathcal{O}(\mathbb{R}^2)$ $(= \mathcal{O}(\mathbb{C}))$ is precisely the set of all complex functions on U whose complex conjugates are holomorphic in U. Indeed, $v \in h(U)$ if and only if $v_1 - iv_2$ satisfies the Cauchy–Riemann equations in U.

If $X = (h(U))_{U \in \mathcal{O}(\mathbb{R}^n)}$, then an X-set will be called an *h-set*; the term "a strong h-set" will be understood accordingly (the perfect h-sets are precisely finite sets).

Let K be a compact subset of $\mathbb{C}$. We denote by $R(K)$ the uniform closure in $C(K)$ ($=$ the space of all complex functions continuous on K) of the set of rational functions with poles off K. Then the h-sets are precisely the compact sets $K \subset \mathbb{C}$ for which

$$R(K) = C(K).$$

At present, the nature of such sets is understood rather well. The Vitushkin theorem yields a complete characterization of such sets in terms of analytic capacity (see, e.g., [1, 2]). However, very little is known about h-sets for $n \geq 3$ (see [3]; the papers [4, 5] contain some results on strong h-sets). A lot of results are known that generalize the theory of rational approximation (and, in particular, the Vitushkin theorem) in various directions. (A very general theory embracing many specific examples can be found in [5].) These results pertain to the case where the role of $X(U)$ is played by the space of solutions (in $U \in \mathcal{O}(\mathbb{R}^n)$) of a homogeneous elliptic system with constant coefficients. But the symbols of these systems are assumed to be surjective; this condition fails for the system $\operatorname{curl} v = 0$, $\operatorname{div} v = 0$ (except for $n = 2$).

Let (U_i) be an open covering of a compact subset $K \subset \mathbb{R}^2$. If $\overline{U_i \cap K}$ is an h-set for any i, then K is an h-set (the Bishop theorem; see [1, 2]). This means that the property to be an h-set in $\mathbb{R}^2$ is *local*. In this paper we show, in particular, that this is no longer true in $\mathbb{R}^3$.

We start (in §1) with the study of strong and perfect curl-sets, obtaining geometric characterizations which are not quite useful in practice (for $n \geq 3$), but are easy to prove and illustrate our approach: decomposing orthogonal vector measures in an integral of simple geometric objects. The decomposition of this kind used in §1 is well known; this is the Fleming–Rishel formula for the gradient of a function of bounded variation (a BV-function). The criteria obtained in §1 become fairly transparent for $n = 2$. For instance, we show that a plane compact set is a strong curl-set if and only if it contains no nontrivial rectifiable loops. But the case $n = 2$ is degenerate (in a sense), since in this case the curl-sets do not differ from the grad-sets, and the nature of the latter is much harder to understand for $n \geq 3$. The vector measures $\vec{\mu}$ orthogonal to the gradients are solenoids (i.e., they satisfy the equation $\operatorname{div} \vec{\mu} = 0$). The corresponding "elementary" vector measures that yield satisfactory decomposition formulas for solenoids are not so easy to describe (unlike in the case of $\mathbb{R}^2$, for $n \geq 3$ it does not suffice to use oriented rectifiable loops). We employ the decomposition results for solenoids proved in [4] and a more elementary result proved below in §2. In our opinion, the proof of the latter is of independent interest. In particular, this result guarantees that the support of any solenoid in $\mathbb{R}^n$ contains a nontrivial simple rectifiable arc (not necessarily closed).

In §3 we discuss strong grad-sets. From §2 it follows immediately that if a compact set $K \subset \mathbb{R}^n$ contains no (nondegenerate) rectifiable arcs, then K is a strong grad-set. We show that, moreover, the absence of nondegenerate rectifiable arcs in K is equivalent to the following approximation property: for any couple $(\varphi, \vec{v}) \in C(K) \times \vec{C}(K)$ and any $\varepsilon > 0$ there is a function $u \in C^1(\mathbb{R}^n)$ such that

$$\max_K |\varphi - u| + \max_K |\vec{v} - \nabla u| < \varepsilon$$

("uniform approximation by jets"). In §3 we also give a simple and complete geometric description of the perfect grad-sets. In §4 we construct a counterexample showing that the "naive" 3-dimensional version of the Bishop principle fails.

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§1. DIVERGENCE-FREE FIELDS AND CURLS

Notation. $\mathcal{B}_n$ will denote the Borel σ-algebra in $\mathbb{R}^n$, and $\vec{M}(\mathbb{R}^n)$ $(=\vec{M})$ the set of all countably additive $\mathbb{R}^n$-valued functions defined on $\mathcal{B}_n$; we call $\vec{\mu}\in\vec{M}$ a *vector charge*. The term "measure" will mean a positive (possibly, infinite) countably additive set function. The closed support of a distribution T will be denoted by $\operatorname{spt} T$; we put $\vec{M}(K) := \{\vec{\mu}\in\vec{M}(\mathbb{R}^n) : \operatorname{spt}\vec{\mu}\subset K\}$. Lebesgue measure in $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$.

1.1. A compact set $K\subset\mathbb{R}^n$ is a *curl-set* (a *strong curl-set*) if and only if there is no nonzero linear functional continuous on $\vec{C}(K)$ that vanishes on any restriction of a solenoidal C^1-vector field (respectively, of a curl) to K. We identify $(\vec{C}(K))^*$, the dual space of $\vec{C}(K)$, with $\vec{M}(K)$. Any $\vec{\mu}\in\vec{M}(K)$ gives rise to an element $f_{\vec{\mu}}\in(\vec{C}(K))^*$:

$$f_{\vec{\mu}}(\vec{v}) = \int_K \langle \vec{v}, d\vec{\mu}\rangle = \int v_1\,d\mu_1+\dots+v_n\,d\mu_n,\quad \vec{v}\in\vec{C}(K),$$

where the μ_j are scalar charges (the coordinates of the vector charge $\vec{\mu}$). The correspondence $\vec{\mu}\mapsto f_{\vec{\mu}}$ is an isometric isomorphism of $\vec{M}(K)$ onto $(\vec{C}(K))^*$ (i.e., $\|f_{\vec{\mu}}\| = \operatorname{var}\vec{\mu}$; by definition, the norm of $\vec{v}\in\vec{C}(K)$ is $\max_K\|\vec{v}\|$). Sometimes we do not distinguish between $f_{\vec{\mu}}$ and $\vec{\mu}$ and write $\vec{\mu}[\vec{v}]$ instead of $f_{\vec{\mu}}(\vec{v})$.

In order to describe the strong curl-sets, we need to characterize the vector charges $\vec{\mu}\in\vec{M}(K)$ orthogonal to the curls.

1.2. We recall that, by definition, a function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a *function of bounded variation* (a BV-function; we write $f\in BV(\mathbb{R}^n)$) if the (distributional) gradient of f is in $\vec{M}(\mathbb{R}^n)$. In other words, $f\in BV(\mathbb{R}^n)$ if and only if $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and there is a charge $\vec{\mu}\in\vec{M}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f\operatorname{div}\vec{\varphi}\,d\mathcal{L}^n = -\vec{\mu}[\vec{\varphi}] \tag{1}$$

for any "test" C^∞-vector field $\vec{\varphi}$ in $\mathbb{R}^n$ with compact support. It is easily seen that if $f\in BV$ and $\operatorname{spt}\nabla f$ is compact, then (1) is true for any C^∞-field $\vec{\varphi}$ in $\mathbb{R}^n$. The theory of BV-functions can be found in [6–8].

1.3. Let χ_E denote the characteristic function of a set $E\in\mathcal{B}_n$. If $\chi_E\in BV$, then E is said to have *finite perimeter*; the quantity $\operatorname{var}(\nabla\chi_E)$ is called the *perimeter of* E and is denoted by $\mathcal{P}_n(E)$. In place of $\nabla\chi_E$ we often write ∂E.

1.4. Suppose that $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$ $(= \vec{M})$. We put $\|\vec{\mu}\|(E) = \sup \sum_{e\in\tau} |\vec{\mu}(e)|$, where $E \in \mathcal{B}_n$ and the supremum is taken over all finite Borel partitions τ of E. Observe that if E is a domain with "nice" boundary $\operatorname{Fr} E$, then $\|\partial E\|(B) = \mathcal{H}^{n-1}(B \cap \operatorname{Fr} E)$, where $B \in \mathcal{B}_n$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

1.5. Dealing with $\nabla\chi_E$ and $\mathcal{P}_n(E)$, we may assume without loss of generality that the boundary $\operatorname{Fr} E$ of E possesses the following property:

$$0 < \mathcal{L}^n(B(x,\rho) \cap E) < \mathcal{L}^n(B(x,\rho)), \quad x \in \operatorname{Fr} E, \ \rho > 0. \tag{2}$$

Indeed, for any $F \in \mathcal{B}_n$ there is a set $E \in \mathcal{B}_n$ satisfying (2) and such that

$$\int_{\mathbb{R}^n} |\chi_E - \chi_F| \, d\mathcal{L}^n = 0;$$

clearly, $\nabla\chi_E = \nabla\chi_F$ as distributions (see [7]).

Let C be a subset of $\mathbb{R}^n$. If there exists $E \in \mathcal{B}_n$ satisfying (2) and such that $\mathcal{L}^n(E) > 0$, $\mathcal{P}_n(E) < +\infty$, and $C = \operatorname{Fr} E$, then we call C a *generalized border*. Such a set C coincides with $\operatorname{spt}\|\partial E\|$ and with the closure of $\operatorname{Fr}^* E$, the *reduced boundary* of E. The definition and geometric analysis of the set $\operatorname{Fr}^* E$ can be found in [7, 8] where it is shown that

$$\|\partial E\|(B) = \|\partial E\|(B \cap \operatorname{Fr}^* E) = \mathcal{H}^{n-1}(B \cap \operatorname{Fr}^* E), \quad B \in \mathcal{B}_n,$$

and that $\operatorname{Fr}^* E = \bigcup_{k=1}^{+\infty} C_k \cup N$, where $\|\partial E\|(N) = 0$ and the C_k are compact parts of C^1-smooth hypersurfaces (more precisely, there are real functions $\varphi_k \in C^1(O_k)$ defined on open subsets O_k of $\mathbb{R}^n$ such that $C_k \subset \{x \in O_k : \varphi_k(x) = 0, \ \nabla\varphi_k(x) \neq 0\}$; see [7, Chapters 3, 4]).

1.6. Let f be a real function on $\mathbb{R}^n$, and let

$$\mathcal{E}_t^f := \{x \in \mathbb{R}^n : f(x) > t\} \quad (t \in \mathbb{R}).$$

The following result can be found in [6, 7].

If $f \in BV$, then $\mathcal{P}_n(\mathcal{E}_t^f) < \infty$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, and

$$\text{(3)} \qquad \begin{aligned} &\text{(a)} \quad \nabla f = \int_{-\infty}^{+\infty} \partial\mathcal{E}_t^f \, dt, \\ &\text{(b)} \quad \|\nabla f\| = \int_{-\infty}^{+\infty} \|\partial\mathcal{E}_t^f\| \, dt. \end{aligned}$$

Identity (3a) means that $\nabla f[\vec{\varphi}] = \int_{-\infty}^{+\infty} (\partial\mathcal{E}_t^f)[\vec{\varphi}] \, dt$ for any test field $\vec{\varphi}$; (3b) means that $\|\nabla f\|(B) = \int_{-\infty}^{+\infty} \|\partial\mathcal{E}_t^f\|(B) \, dt$ for $B \in \mathcal{B}_n$. Clearly, from (3) it follows that $\operatorname{spt} \partial\mathcal{E}_t^f \subset \operatorname{spt} \nabla f$ for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$ (for the details, see, e.g., [4]).

1.7. Now we can describe the vector charges orthogonal to the curls. Let $\vec{\mu} \in \vec{M}$. *The following assertions are equivalent*: (a) $\vec{\mu}[\operatorname{curl}\vec{\varphi}] = 0$ *for any test field* $\vec{\varphi}$; (b) $\vec{\mu} = \nabla f$, *where* f is a *BV-function.* Clearly, $\vec{\mu}[\operatorname{curl}\vec{\varphi}] = -(\operatorname{curl}\vec{\mu})[\vec{\varphi}]$ (in the sense of distributions). Hence, (a) implies that $\operatorname{curl}\vec{\mu} = 0$, and $\vec{\mu} = \operatorname{grad} f$ for some distribution f. It is easily seen ([7, Chapter 1]) that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, which proves the implication (a) $\Longrightarrow$ (b); the reverse implication follows from the identity $\nabla f[\operatorname{curl}\vec{\varphi}] = -\int f \cdot \operatorname{div}\operatorname{curl}\vec{\varphi} \, d\mathcal{L}^n = 0$. $\square$

1.8. We also need a characterization of the vector charges $\vec{\mu} \in M(K)$ (K is a compact part of $\mathbb{R}^n$) orthogonal to every C^∞-field solenoidal near K.

Let $\vec{\mu} \in \vec{M}(K)$. The following assertions are equivalent:

(a) *if $\vec{\varphi} \in \vec{C}^\infty(\mathbb{R}^n)$ and $\operatorname{div} \vec{\varphi} = 0$ in a neighborhood of K, then $\vec{\mu}[\vec{\varphi}] = 0$;*

(b) *$\vec{\mu} = \nabla f$, where $f \in BV$ and $f(x) \equiv 0$ off K.*

The implication (a) $\Longrightarrow$ (b) was already proved in Subsection 1.7: from (a) it follows that $\vec{\mu}$ is orthogonal to the curl of any test field; hence, $\vec{\mu} = \nabla f$, $f \in BV$. We take a function $\alpha \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \alpha \cap K = \varnothing$ and put

$$\vec{\varphi}(x) = c\nabla_x \int \Delta\alpha(y) \cdot |y - x|^{2-n}\, d\mathcal{L}^n(y).$$

Then $\vec{\varphi} \in \vec{C}^\infty(\mathbb{R}^n)$, and for a suitable choice of c we have $\operatorname{div} \vec{\varphi} = \alpha$, i.e., $\operatorname{div} \vec{\varphi} = 0$ near K,

$$0 = \vec{\mu}[\vec{\varphi}] = \nabla f[\vec{\varphi}] = -f[\operatorname{div} \vec{\varphi}] = -f[\alpha] = -\int f \cdot \alpha\, d\mathcal{L}^n.$$

We see that $f = 0$ $\mathcal{L}^n$-a.e. off K, and we can modify f on a set of $\mathcal{L}^n$-measure zero to obtain $f \equiv 0$ on $\mathbb{R}^n \setminus K$. Clearly, in (a) the space $\vec{C}^\infty$ can be replaced by $\vec{C}^1$.

1.9. Now we can completely describe the curl-sets and the strong curl-sets. Let $\operatorname{sol} K$ denote the set of all restrictions to K of the C^∞-fields solenoidal near K.

Theorem. *Let K be a compact set in $\mathbb{R}^n$.*

(a) *K is a curl-set if and only if K contains no Borel subset E satisfying*

$$\mathcal{L}^n(E) > 0, \qquad \mathcal{P}_n(E) < +\infty. \tag{4}$$

(b) *K is a strong curl-set if and only if K contains no generalized border.*

Proof. (a) Suppose that $E \in \mathcal{B}_n$ satisfies (4) and $E \subset K$. Clearly, $\operatorname{spt} \partial E \subset \operatorname{Fr} E \subset K$, so that $\partial E \in \vec{M}(K)$; moreover, $\partial E[\vec{\varphi}] = -\int_E \operatorname{div} \vec{\varphi}\, d\mathcal{L}^n = 0$ for any test field $\vec{\varphi} \in \operatorname{sol} K$. At the same time, $\partial E \neq 0$ because we may take a test function $\alpha \in C_0^\infty(\mathbb{R}^n)$ equal to 1 identically near K and then find a field $\vec{\varphi} \in \vec{C}^\infty(\mathbb{R}^n)$ such that $\operatorname{div} \vec{\varphi} = \alpha$. Then $\partial E[\vec{\varphi}] = -\int_E \operatorname{div} \vec{\varphi}\, d\mathcal{L}^n = -\mathcal{L}^n(E) \neq 0$. Hence, the nonzero linear functional $f_{\partial E}$ (see Subsection 1.1) vanishes on $\operatorname{sol} K$, whence K is not a curl-set.

Suppose that no Borel part E of K satisfies (4). Taking $\vec{\mu} \in \vec{M}(K)$ orthogonal to $\operatorname{sol} K$, we prove that $\vec{\mu} = 0$. By Subsection 1.6, we have $\vec{\mu} = \nabla f$, where

$$f \in BV, \qquad f = 0 \text{ on } \mathbb{R}^n \setminus K. \tag{5}$$

We show that (5) implies that $f^+ := \max(f, 0) = 0$ $\mathcal{L}^n$-a.e. As noted in Subsection 1.6, $\mathcal{P}(\mathcal{E}_t^f) = \mathcal{P}(\mathcal{E}_t^{f^+}) < +\infty$ for $\mathcal{L}^1$-a.e. positive t, and $\int_{\mathbb{R}^n} f^+ d\mathcal{L}^n = \int_0^{+\infty} \mathcal{L}^n(\mathcal{E}_t^{f^+})\, dt$. Hence, there is $t > 0$ such that $E := \mathcal{E}_t^{f^+}$ satisfies (4) provided that $\int_{\mathbb{R}^n} f^+ d\mathcal{L}^n > 0$. But $\mathcal{E}_t^{f^+} \subset K$, a contradiction. Replacing f by $-f$, we conclude that $f = 0$ $\mathcal{L}^n$ -a.e., and $\vec{\mu} = 0$.

1.10. We prove (b). Suppose that K contains a generalized border C. Then $\operatorname{spt} \partial E \subset C \subset K$ (see Subsection 1.5), and $(\partial E)(\operatorname{curl} \vec{v}) = 0$ for any $\vec{v} \in \vec{C}^\infty(\mathbb{R}^n)$. Consequently, ∂E is a nonzero vector charge supported on K and orthogonal to all curls. Thus, K is not a strong curl-set.

Now, suppose that K contains no generalized border and that $\vec{\mu} \in \vec{M}(K)$ is orthogonal to all curls. As in Subsection 1.7, we can write $\vec{\mu} = \nabla f$, $f \in BV(\mathbb{R}^n)$. Applying (3) to $\vec{\mu}$, we see that

$$\text{spt}\, \partial\mathcal{E}_t^f \subset \text{spt}\, \vec{\mu} \subset K, \quad \text{and} \quad \mathcal{P}(\mathcal{E}_t^f) < +\infty \quad \text{for } \mathcal{L}^1\text{-a.e. } t. \tag{6}$$

If $\vec{\mu} \neq 0$, then, by (3), $\partial\mathcal{E}_t^f \neq 0$ for all t in some set of positive $\mathcal{L}^1$-measure. Thus, for one of such values of t, $\partial\mathcal{E}_t^f$ is a generalized border contained in K (see Subsection 1.5), and we arrive at a contradiction. □

1.11. The same approach leads to a description of the perfect curl-sets. By Subsection 1.5, with any generalized border $C = \operatorname{Fr} E$ we can associate its "reduced part" $\operatorname{Fr}^* E$, which we denote by C^*. By $\vec{\mu} \llcorner A$, where $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$ and $A \in \mathcal{B}_n$, we denote the vector charge $\chi_A \vec{\mu}$.

Theorem. *A compact set $K \subset \mathbb{R}^n$ is a perfect curl-set if and only if there is a positive number $\lambda(K)$ such that*

$$\mathcal{H}^{n-1}(C^* \setminus K) \geq \lambda(K)\mathcal{H}^{n-1}(C^* \cap K) \tag{7}$$

for any generalized border C.

Proof. Let $\vec{C}_0(\mathbb{R}^n)$ be the space of all continuous vector fields $\vec{\varphi}$ vanishing at infinity (with the usual norm $\|\vec{\varphi}\| = \max |\vec{\varphi}|$). We put sol $:= \{\, \vec{\varphi} \in \vec{C}_0(\mathbb{R}^n) : \operatorname{div} \vec{\varphi} = 0 \,\}$. Clearly, sol is a closed subspace of $\vec{C}_0(\mathbb{R}^n)$.

Consider the operator $r_K\colon$ sol $\longrightarrow \vec{C}(K)$ transforming any $\vec{\varphi} \in$ sol to the restriction of $\vec{\varphi}$ to K, $r_K(\vec{\varphi}) := \vec{\varphi}|K$. Clearly, K is a perfect curl-set if and only if r_K is surjective. By the Banach theorem, this means that $r_K^*(\vec{C}(K))^* \longrightarrow (\text{sol})^*$ admits a lower estimate:

$$\exists\lambda(K) > 0\colon \|r_K^*(\vec{\mu})\|_{(\text{sol})^*} \geq \lambda(K) \operatorname{var} \vec{\mu}, \quad \vec{\mu} \in \vec{M}(K). \tag{8}$$

The same argument as in Subsection 1.7 shows that the set

$$\text{sol}^\perp := \{\, \vec{\mu} \in \vec{M}(\mathbb{R}^n) : \vec{\mu}[\vec{\varphi}] = 0,\ \vec{\varphi} \in \text{sol} \,\}$$

coincides with $\{\, \nabla f : f \in BV(\mathbb{R}^n) \,\}$ (observe that if $\vec{\varphi} \in$ sol, $f \in BV$, then $\nabla f[\vec{\varphi}] = \lim_{j \to \infty} \nabla f[\vec{\varphi}_j]$, where the $\vec{\varphi}_j$ are divergence-free test fields tending to $\vec{\varphi}$ uniformly on $\mathbb{R}^n$; hence, $\nabla f[\vec{\varphi}] = 0$). Using the standard isometry $(\text{sol})^* \simeq M/(\text{sol})^\perp$, we may rewrite (8) as follows:

$$\exists\lambda(K) > 0\colon \operatorname{var}(\vec{\mu} - \nabla f) \geq \lambda(K) \operatorname{var} \vec{\mu}, \quad \vec{\mu} \in M(K),\ f \in BV(\mathbb{R}^n). \tag{9}$$

For a generalized border C, we put $\vec{\mu} := \partial E \llcorner K$, where E is related to C by the formula $C = \operatorname{Fr} E$ (see Subsection 1.5). Then $\vec{\mu} \in \vec{M}(K)$; if K is a perfect curl-set, then (9) implies the inequality

$$\lambda(K) \operatorname{var} \vec{\mu} \leq \operatorname{var}(\vec{\mu} - \nabla \chi_E). \tag{10}$$

But

$$\operatorname{var}(\vec{\mu} - \nabla \chi_E) = \operatorname{var}(\partial E \llcorner K^c) = \|\partial E\|(K^c \cap C^*) = \mathcal{H}^{n-1}(C^* \setminus K)$$

(see Subsection 1.5; here $K^c := \mathbb{R}^n \setminus K$). On the other hand, $\operatorname{var} \mu = \|\partial E\|(K) = \mathcal{H}^{n-1}(K \cap C^*)$. Thus, (10) implies (7).

To finish the proof, suppose that K satisfies (7). We take $f \in BV$ and put $\operatorname{spt} \partial \mathcal{E}_t^t =: C_t$; clearly, C_t is a generalized border (for $\mathcal{L}^1$ -a.e. t). From (7) it follows that

$$\begin{aligned}\|\partial \mathcal{E}_t^f\|(K^c) &= \|\partial \mathcal{E}_t^f\|(K^c \cap C_t^*) = \mathcal{H}^{n-1}(K^c \cap C_t^*)\\ &\geq \lambda(K)\mathcal{H}^{n-1}(K^c \cap C_t^*) = \lambda(K)\|\partial \mathcal{E}_t^f\|(K).\end{aligned}$$

Integrating over $t \in \mathbb{R}$ and applying (3), we get

$$\|\nabla f\|(K^c) \geq \lambda(K)\|\nabla f\|(K).$$

Consider a vector charge $\vec{\mu} \in \vec{M}(K)$. We have

$$\begin{aligned}\operatorname{var}(\vec{\mu} - \nabla f) &= \operatorname{var}((\vec{\mu} - \nabla f)\llcorner K) + \operatorname{var}(\nabla f \llcorner K^c)\\ &\geq \operatorname{var}((\vec{\mu} - \nabla f)\llcorner K) + \lambda(K)\operatorname{var}(\nabla f \llcorner K)\\ &\geq \lambda(K)\operatorname{var}((\vec{\mu} - \nabla f)\llcorner K) + \lambda(K)\operatorname{var}(\nabla f \llcorner K)\\ &\geq \lambda(K)\operatorname{var}((\vec{\mu} - \nabla f + \nabla f)\llcorner K) = \lambda(K)\operatorname{var}(\vec{\mu})\end{aligned}$$

(without loss of generality we may assume that $\lambda(K) < 1$). Hence, K satisfies (9), which means that K is a perfect curl-set. □

1.12. The following remark is a consequence of Subsection 1.11. *Suppose that $K \subset \mathbb{R}^n$ is a compact set and that*

(11) $$\mathcal{H}^{n-1}(K \cap \Gamma) = 0 \quad \textit{for any } C^1\textit{-hypersurface in } \mathbb{R}^n.$$

Then K is a perfect curl-set.

Proof. Let C be a generalized border; then $C^* = \bigcup_{j=1}^{\infty} C_j \cup N$, where the C_j are compact parts of C^1-hypersurfaces ($j = 1, 2, \dots$), and $\mathcal{H}^{n-1}(N) = 0$ (see Subsection 1.5). By (11), $\mathcal{H}^{n-1}(C_j \cap K) = 0$ for any $j = 1, 2, \dots$, whence $\mathcal{H}^{n-1}(C^* \cap K) \leq \sum_j \mathcal{H}^{n-1}(C_j \cap K) + \mathcal{H}^{n-1}(N) = 0$. Thus, K satisfies (7). □

1.13. The criteria proved in Subsections 1.9–1.11 reduce the approximation and extension properties of divergence-free vector fields to some *geometric* properties of the compact set K in question. Unfortunately, these geometric properties are not easy to verify, because the "test-objects" they involve are very complicated (we mean generalized borders and sets satisfying (4)). However, for $n = 2$ the description of the strong and the perfect curl-sets can be simplified: our "test-objects" become simple rectifiable loops. In Subsections 1.14–1.19 below we discuss the plane case.

1.14. For $\vec{\mu} = (\mu_1, \mu_2) \in \vec{M}(\mathbb{R}^2)$ we put $\vec{\mu}^{\perp} := (-\mu_2, \mu_1)$. Clearly, $(\vec{\mu}^{\perp})^{\perp} = -\vec{\mu}$, and

(12) $$\operatorname{curl} \vec{\mu} = 0 \iff \operatorname{div} \vec{\mu}^{\perp} = 0.$$

Therefore, for $n = 2$, the curl-sets, the strong curl-sets, and the perfect curl-sets coincide, respectively, with the grad-sets, the strong grad-sets, and the perfect grad-sets.

1.15. A vector charge $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$ will be called a *rectifiable curve* if there exists a vector-valued function $\vec{f}\colon [0, S] \longrightarrow \mathbb{R}^n$ such that

(a) $|\vec{f}(s) - \vec{f}(s')| \leq |s - s'|$, $s, s' \in [0, S]$;

(b) $\vec{\mu}[\vec{\varphi}] = \int_0^S \langle \vec{\varphi}(\vec{f}(s)), \vec{f}'(s) \rangle \, ds$ for any test field $\vec{\varphi}$.

If $\vec{f}(0) = \vec{f}(S)$, then the curve $\vec{\mu}$ is said to be *closed*. If $\operatorname{var} \vec{\mu} = S$, we say that $\vec{\mu}$ is a *curve of length S*. If the function $\vec{f}|(0, S)$ is one-to-one, then $\vec{\mu}$ is a *simple curve.* If a curve $\vec{\mu}$ is closed and simple, we say that $\operatorname{spt} \vec{\mu} = \vec{f}([0, S])$ is a *rectifiable loop.* It is easily seen that for any rectifiable closed curve $\vec{\mu}$ we have $\operatorname{div} \vec{\mu} = 0$; if $n = 2$, then $\operatorname{curl}(\vec{\mu}^{\perp}) = 0$.

1.16. Suppose that C is a generalized border of a set E in $\mathbb{R}^2$ ($C = \mathrm{Fr}^* E$; see Subsection 1.5). The following expansion explains why Theorems 1.9 and 1.11 simplify for $n = 2$:

$$(\partial E)^\perp = \sum_{j=1}^{\infty} c_j, \qquad \|\partial E\| = \sum_{j=1}^{\infty} \|c_j\|, \tag{13}$$

where the c_j are simple closed rectifiable curves. Indeed, $(\partial E)^\perp$ is a divergence-free vector charge (because $\partial E = \nabla \chi_E$ is curl-free), and it may be interpreted as an integral one-dimensional current in the sense of [6, pp. 381, 384]; (13) follows from [6, p. 421].

1.17. Any solenoid in $\mathbb{R}^2$ can be decomposed into simple rectifiable loops. Indeed, if $\vec{\mu} \in \vec{M}(\mathbb{R}^2)$ and $\operatorname{div} \vec{\mu} = 0$, then $\operatorname{curl} \vec{\mu}^\perp = 0$, and

$$\vec{\mu} = \nabla f^\perp = \int_{-\infty}^{+\infty} (\partial \mathcal{E}_t^f)^\perp \, dt, \qquad \|\vec{\mu}\| = \|\nabla f\| = \int_{-\infty}^{+\infty} \|\partial \mathcal{E}_t^f\| \, dt \tag{14}$$

for some BV-function f (see (3)). Combining (14) and (13), we arrive at the following representation:

$$\vec{\mu} = \int_J c \, d\rho(c), \qquad \|\mu\| = \int_J \|c\| \, d\rho(c),$$

where J is the space of simple rectifiable curves and ρ a positive measure on J. No such representation is possible for $n \geq 3$; see [4].

1.18. The following statement is a simplified version of Theorem 1.9 valid for $n = 2$.

A compact set $K \subset \mathbb{R}^n$ is a strong curl-set if and only if K contains no rectifiable loop of positive length.

A similar theorem is true for the strong grad-sets; see Subsection 1.14.

Proof. If K contains a nondegenerate rectifiable loop and if C is the corresponding simple rectifiable curve, then $C^\perp$ is a nonzero charge orthogonal to any divergence-free continuous field, $C^\perp \in \vec{M}(K)$, and K is not a strong curl-set. Conversely, if K is not a strong curl-set, then, by Theorem 1.9, there is a generalized border $C \subset K$. Applying (13), we get a simple closed rectifiable curve c_j with $\operatorname{var} c_j > 0$ and $\operatorname{spt} c_j \subset C \subset K$ (this inclusion follows from (13) for all j); therefore, $\operatorname{spt} c_j \subset C$ is a nondegenerate rectifiable loop. □

1.19. The description of the perfect curl-sets (perfect grad-sets for $n = 2$) obtained in Subsection 1.11 can be reshaped as follows.

A compact set $K \subset \mathbb{R}^n$ is a perfect curl-set if and only if there is a positive number $\lambda(K)$ such that

$$\mathcal{H}^1(l \setminus K) \geq \lambda(K) \mathcal{H}^1(l \cap K) \quad \textit{for any rectifiable loop } l. \tag{15}$$

Proof. (7) $\Longrightarrow$ (15): any nondegenerate rectifiable loop is a generalized border, and $\mathcal{H}^1(l^*) = \mathcal{H}^1(l)$.

(15) $\Longrightarrow$ (7): if C is a generalized border, then, by (13),

$$\begin{aligned}\mathcal{H}^{n-1}(C^* \setminus K) &= \|\partial E\|(K^c) = \sum \|c_j\|(K^c) = \sum \mathcal{H}^1(K^c \cap c_j) \\ &\geq \lambda(K) \sum \mathcal{H}^1(K \cap c_j) = \lambda(K) \sum \|c_j\|(K) = \lambda(K) \|\partial E\|(K) \\ &= \lambda(K) \mathcal{H}^{n-1}(C^* \cap K). \qquad \square\end{aligned}$$

As we shall see later on, relation (15) characterizes perfect grad-sets in all dimensions (not only for $n = 2$).

§2. The support of a divergence-free vector charge contains a simple rectifiable arc

2.1. In order to apply the same approach as in §1 to grad-sets, we need an analog of the decomposition formulas (3) for divergence-free (= solenoidal) vector charges. Theorems of this kind (and stronger ones) were obtained in [4] (one of them is reproduced at the end of this section). Here we suggest another approach to this problem which seems interesting in itself and proves the claim made in the section title. We establish the following theorem.

Theorem. *Suppose that*
(a) $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$, $\operatorname{spt}\vec{\mu}$ *is compact*, $\vec{\mu} \neq 0$;
(b) $\operatorname{div}\vec{\mu} \in M(\mathbb{R}^n)$ (*i.e.*, $\operatorname{div}\vec{\mu}$ *is a finite scalar charge*).
Then $\operatorname{spt}\vec{\mu}$ *contains a nondegenerate simple rectifiable arc* (*i.e., a set of the form* $\operatorname{spt} c$, *where* c *is a simple rectifiable curve of positive length*; *see Subsection* 1.15).

2.2. Any charge $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$ can be written as follows:

$$\vec{\mu} = \vec{\nu} m, \quad m := \|\vec{\mu}\|, \tag{16}$$

where $\vec{\nu}$ is a Borel measurable vector field in $\mathbb{R}^n$, and $|\vec{\nu}| = 1$ m-a.e. The field $\vec{\nu}$ of unit vectors gives rise to a differential operator A:

$$(Af)(x) := \langle \nabla f(x), \vec{\nu}(x)\rangle = \frac{\partial f}{\partial \vec{\nu}(x)}(x) \quad (f \in C^1(\mathbb{R}^n)). \tag{17}$$

The right-hand side of (17) is defined m-almost everywhere, namely, for any x where $|\vec{\nu}(x)| = 1$. We may view A as an (algebraically) linear transformation of $C^1(\mathbb{R}^n)$ to $L^\infty(m)$; the ess-sup-norm in $L^\infty(m)$ will be denoted by $\|\cdot\|_\infty$. The proof of Theorem 2.1 is based on the study of the operator A.

2.3.1. We put
$$\mathcal{A} := \{\, f \in C^1(\mathbb{R}^n) : \|A(f)\|_\infty \le 1\,\}.$$
If $f \in \mathcal{A}$ *and* $\varphi \in C^1(\mathbb{R}^n)$, $\sup|\varphi'| \le 1$, *then* $\varphi \circ f \in \mathcal{A}$.

Indeed,
$$|\langle \nabla(\varphi\circ f), \nu\rangle| = |\langle (\varphi'\circ f)\nabla f, \nu\rangle| \le |(\nabla f, \nu)|. \qquad \square$$

2.3.2. *For any* $f_1, \dots, f_n \in \mathcal{A}$ *and any* $\delta > 0$ *there is* $h \in \mathcal{A}$ *such that*

$$\|h - \max(f_1, \dots, f_n)\|_\infty < \delta$$

(max *may be replaced by* min).

Proof. It suffices to consider the case $n = 2$. Let $g := f_1 - f_2$; then $\max(f_1, f_2) = g^+ + f_2$. We take $\varphi \in C^1(\mathbb{R})$ such that $\varphi(t) = t^+$ for $|t| \ge \delta$ and $0 \le \varphi'(t) \le 1$ for all $t \in \mathbb{R}$. Then $|t^+ - \varphi(t)| < \delta$ everywhere in $\mathbb{R}$. We put $h := \varphi\circ g + f_2$. Clearly, $\|h - \max(f_1, f_2)\|_\infty = \|\varphi\circ g - g^+\|_\infty \le \delta$, and for m-almost every x we have

$$\begin{aligned}|(Ah)(x)| &= |\varphi'(g(x))(Af_1)(x) - \varphi'(g(x))(Af_2)(x) + (Af_2)(x)| \\ &\le \varphi'(g(x)) + (-\varphi'(g(x)) + 1) = 1. \qquad \square\end{aligned}$$

2.4. Now we define a quasidistance ρ in $\mathbb{R}^n$ related to A:

$$\rho(x,y) := \sup\{\,|f(x)-f(y)| : f \in \mathcal{A}\,\} \tag{18}$$

("quasi" refers to the fact that $\rho(x,y)$ may be infinite). The triangle inequality and the symmetry of ρ are obvious, and $\rho(x,y)=0 \implies x=y$ (see Subsection 2.4.3 below).

2.4.1. Fixing $x, y \in \mathbb{R}^n$, we consider the set

$$\mathcal{A}_{x,y} = \{\, f \in \mathcal{A} : f(x) \le f(u) \le f(y),\ u \in \mathbb{R}^n\,\}.$$

We put $\tilde\rho(x,y) := \sup\{\,|f(x)-f(y)| : f \in \mathcal{A}_{x,y}\,\}$ and prove that $\tilde\rho \equiv \rho$. Indeed, it suffices to prove that $\tilde\rho \ge \rho$. This inequality is implied by the following fact:

$$\begin{aligned} &\forall f \in \mathcal{A}\, \forall \varepsilon \in (0, |f(y)-f(x)|)\, \exists g \in \mathcal{A}_{x,y} : \\ &\qquad |g(y)-g(x)| \ge |f(y)-f(x)| - \varepsilon. \end{aligned} \tag{19}$$

In the proof of (19) we may assume that $f(y) \ne f(x)$, and, moreover, $f(y) > f(x)$ (otherwise we replace f by $-f$). We take a function $\varphi \in C^1(\mathbb{R})$ such that $0 \le \varphi'(t) \le 1$ for all $t \in \mathbb{R}$, $\varphi(t) \equiv 0$ on $(-\infty, f(x)]$, and $\varphi(t) \equiv f(y)-f(x)-\varepsilon$ on $[f(y), +\infty)$. Then $g := \varphi \circ f$ satisfies (19), because $g \in \mathcal{A}$ by the statement proved in Subsection 2.3.1, and $f(y)-f(x)-\varepsilon = g(y) \ge g(u) \ge g(x) = 0$ for all $u \in \mathbb{R}^n$; hence, $g \in \mathcal{A}_{x,y}$. Moreover, $|g(y)-g(x)| = |f(y)-f(x)| - \varepsilon$. □

We observe that $f \in \mathcal{A}_{x,y} \implies f + \mathrm{const} \in \mathcal{A}_{x,y}$. Therefore, we may prescribe any value of $f(x)$ in the definition of $\mathcal{A}_{x,y}$.

2.4.2. *If $x \notin \operatorname{spt}\mu$, $y \in \mathbb{R}^n$, $y \ne x$, then $\rho(x,y) = +\infty$.* Indeed, for any $N > 0$ there is a function $f \in C^1(\mathbb{R}^n)$ vanishing on $\operatorname{spt}\mu$ and at y and such that $f(x) > N$; clearly, $f \in \mathcal{A}$. □

2.4.3. *The Euclidean metric does not exceed ρ:* $|x-y| \le \rho(x,y)$ for $x, y \in \mathbb{R}^n$. Indeed, let $x \ne y$; we put $f(z) := \langle z, \frac{x-y}{|x-y|}\rangle$. Then $|\nabla f| \equiv 1$, and $f \in \mathcal{A}$; but $|f(x)-f(y)| = |x-y|$.

2.4.4. *The function $z \mapsto \rho(x,z)$ is lower semicontinuous for any $x \in \mathbb{R}^n$, i.e., $\rho(x,y) \le \underline{\lim}_{|z-y|\to 0}\, \rho(x,z)$ $(x, y \in \mathbb{R}^n)$.*

Indeed, we have $|f(x)-f(z)| \le \rho(x,z)$ for any $f \in \mathcal{A}$, whence $\underline{\lim}_{|z-y|\to 0}\, \rho(x,z) \ge \lim_{|z-y|\to 0} |f(x)-f(z)| = |f(x)-f(y)|$. The same argument yields

$$\rho(x,y) \le \underline{\lim}_{|z'-x|\to 0,\, |z''-y|\to 0} \rho(z', z'').$$

Corollary. *Any closed ρ-ball $\{\,x \in \mathbb{R}^n : \rho(x,a) \le T\,\}$, where $a \in \mathbb{R}^n, T > 0$, is closed with respect to the Euclidean topology of $\mathbb{R}^n$.*

2.5. *Let K_1, K_2 be compact subsets of $\mathbb{R}^n$, and let T be a positive number, $T < \rho$ $(K_1, K_2) := \inf\{\,\rho(x_1, x_2) : x_j \in K_j\,\}$. Then there exists $f \in \mathcal{A}$ such that $0 \le f \le T$, $f|_{K_1} = 0$, $f|_{K_2} = T$.*

Proof. We take a positive number ε satisfying $T + 8\varepsilon < \rho(K_1, K_2)$ and fix a point $y \in K_2$. Then $\rho(x,y) > T + 8\varepsilon$ for any $x \in K_1$. Hence, there exists a function $h_x \in \mathcal{A}$ such that $0 = h_x(x) \le h_x(u) \le h_x(y)$ for $u \in \mathbb{R}^n$, and $h_x(y) > T + 8\varepsilon$ (see Subsection 2.4.1). We put $U_x := \{u \in \mathbb{R}^n : h_x(u) < \varepsilon\}$. There is a finite set $\{x_1, \dots, x_N\}$ such that $K_1 \subset \bigcup_{k=1}^N U_{x_k}$. Then

$$0 \le h(z) := \min_{1 \le j \le N} h_{x_j}(z) < \varepsilon$$

if $z \in K_1$, and $h(y) > T + 8\varepsilon$. By the statement proved in Subsection 2.3.2, we can find $g_y \in \mathcal{A}$ satisfying $|g_y - h| \le \varepsilon$ everywhere in $\mathbb{R}^n$. Hence,

$$g_y < 2\varepsilon \text{ on } K_1, \qquad g_y(y) > T + 7\varepsilon.$$

Putting $V_y := \{ u \in \mathbb{R}^n : g_y(u) > T + 6\varepsilon \}$, we find a finite set $\{y_1, \dots, y_M\}$ such that $K_2 \subset \bigcup_{k=1}^M V_{y_k}$. Let $g := \max_{1 \le k \le M} g_{y_k}$. Clearly, $g(z) < 2\varepsilon$ if $z \in K_1$ and $g(z) > T + 7\varepsilon$ if $z \in K_2$. Using Subsection 2.3.2 again, we find $\tilde{f} \in \mathcal{A}$ such that $|\tilde{f} - g| < \varepsilon$ everywhere in $\mathbb{R}^n$. Then $\tilde{f}(z) < 3\varepsilon$ if $z \in K_1$ and $\tilde{f}(z) > T + 5\varepsilon$ if $z \in K_2$. There exists a function $\varphi \in C^1(\mathbb{R})$ such that $0 \le \varphi'(t) \le 1$ for all $t \in \mathbb{R}$, $\varphi|_{(-\infty, 3\varepsilon]} = 0$, $\varphi|_{[T+5\varepsilon, +\infty)} = T$. Then $f := \varphi \circ \tilde{f} \in \mathcal{A}$ is the required function. □

2.6. The distance $\rho(x, y)$ can always be halved: *for* $x, y \in \mathbb{R}^n$, *if* $\rho(x, y) < +\infty$, *then there is a point* $z \in \mathbb{R}^n$ *such that* $\rho(x, z) = \rho(z, y) = \rho(x, y)/2$.

Proof. Let $T := \rho(x, y)$. It suffices to find a point z satisfying $\rho(x, z) \le T/2$, $\rho(z, y) \le T/2$ (the rest follows from the triangle inequality). Put $B_\rho(x, a) := \{ y \in \mathbb{R}^n : \rho(x, y) \le a \}$. We must show that

$$B_\rho(x, T/2) \cap B_\rho(y, T/2) \ne \varnothing.$$

By the said in Subsections 2.4.2 and 2.4.4, $B^x := B_\rho(x, T/2)$ and $B^y := B_\rho(y, T/2)$ are closed subsets of $K := \operatorname{spt} \mu$; consequently, these sets are compact. Since they are disjoint, there is a positive number Δ such that $0 < \Delta < \inf\{ |u - v| : u \in B^x,\ v \in B^y \}$. Let E_σ denote the open (Euclidean) σ-neighborhood of the set $E \subset \mathbb{R}^n$:

$$E_\sigma := \{ p \in \mathbb{R}^n : \exists q \in E,\ |p - q| < \sigma \}.$$

We put

$$B'_x := \overline{K_{\Delta/3} \setminus (B^x)_{\Delta/3}}, \qquad B'_y := \overline{K_{\Delta/3} \setminus (B^y)_{\Delta/3}}.$$

The sets B'_x, B'_y are compact, and

(a) $K \subset \operatorname{Int} B'_x \cup \operatorname{Int} B'_y$;

(b) $d(B^x, B'_x) \ge \Delta/3$, $d(B^y, B'_y) \ge \Delta/3$, where d denotes the Euclidean distance between sets;

(c) $B^x \subset B'_y$, $B^y \subset B'_x$.

Since $z \mapsto \rho(x, z)$ is a lower semicontinuous function (see Subsection 2.4.4), $\inf\{\rho(x, t) : t \in B'_x \} =: J$ is attained at some point $x' \in B'_x$. From (b) it follows that $x' \notin B^x$, whence $J = \rho(x, x') > T/2 + \varepsilon$ for some $\varepsilon > 0$. Putting $K_1 := B'_x$, $K_2 := \{x\}$, we observe that $\rho(K_1, K_2) = \rho(x, B'_x) > T/2 + \varepsilon$. Hence, we can apply the statement proved in Subsection 2.5 to get a function $f \in \mathcal{A}$ that vanishes on B'_x and is equal to $T/2 + \varepsilon$ at x. Interchanging B'_x and B'_y, we get another function $g \in \mathcal{A}$ such that $g|_{B'_y} = 0$ and $g(y) = T/2 + \delta$ for some $\delta > 0$. Let $h = f - g$. Then $h(z) = f(z)$ if $z \in B'_y$, and $h(z) = -g(z)$ if $z \in B'_x$. Hence, $(Ah)(z) = (Af)(z)$ m-a.e. on B'_y and $(Ah)(z) = -(Ag)(z)$ m-a.e. on B'_x. By (a), we have $|(Ah)(z)| \le 1$ m-a.e., so that $h \in \mathcal{A}$. But

$$\begin{aligned} T = \rho(x, y) &\ge |h(x) - h(y)| = |f(x) + g(y)| \\ &= (T/2 + \varepsilon) + (T/2 + \delta) = T + \varepsilon + \delta > T, \end{aligned}$$

a contradiction. □

2.7. Now we prove that ρ is a geodesic metric.

Let $x, y \in \mathbb{R}^n$, and let $0 < \rho(x,y) < +\infty$; then there is a mapping $\psi\colon [0, \rho(x,y)] \longrightarrow \mathbb{R}^n$ such that

(a) $\psi(0) = x$, $\psi(\rho(x,y)) = y$;

(b) *for any $a, b \in [0, \rho(x,y)]$ we have $|a-b| = \rho(\psi(a), \psi(b))$.*

Subsections 2.5.3 and 2.4.2 show that ψ is a compression, i.e., $|\psi(a) - \psi(b)| \le |a-b|$, and that all values of ψ are in $\operatorname{spt}\mu$. Hence, x and y can be joined in $\operatorname{spt}\mu$ by a simple rectifiable arc of length not exceeding $\rho(x,y)$.

Proof. Let $T := \rho(x,y)$. It suffices to construct a mapping ψ satisfying (a) and the following condition (b$'$): $|a-b| \ge \rho(\psi(a), \psi(b))$ for any $a, b \in [0, T]$. (Indeed, then $T \le \rho(x, \psi(a)) + \rho(\psi(a), \psi(b)) + \rho(\psi(b), y) \le |0-a| + |a-b| + |b-T| = T$, and (b) follows.) We put $\psi(0) := x$, $\psi(T) := y$. By the statement proved in Subsection 2.6, there is a point $z \in \mathbb{R}^n$ such that $\rho(\psi(0), z) = \rho(z, \psi(T)) = T/2$. Let $\psi(T/2) := z$. Thus, ψ is already defined on the set $E_1 := \{0, T/2, T\}$. We denote $c_{j,k} := jT \cdot 2^{-k}$, $j = 0, 1, \dots, 2^k$. Continuing this procedure by induction, after the kth step we get a function ψ defined on $E_k := \{c_{j,k}\}_{j=0}^{2^k}$ and satisfying (b$'$) on E_k: $\rho(C_{j,k}, C_{j+1,k}) = T/2^k$, where $C_{j,k} := \psi(c_{j,k})$. Using Subsection 2.6, we can extend ψ to E_{k+1} in such a way that

$$\rho(C_{j,k}, C_{2j+1,k+1}) = \rho(C_{2j+1,k+1}, C_{j+1,k}) = T/2^{k+1},$$

and ψ satisfies (b$'$) on E_{k+1}: if $0 \le p \le q \le 2^{k+1}$, then

$$\begin{aligned}\rho(C_{p,k+1}, C_{q,k+1}) &\le \sum_{j=p}^{q-1} \rho(C_{j,k+1}, C_{j+1,k+1}) = (q-p)T/2^{k+1} \\ &= |c_{p,k+1} - c_{q,k+1}|, \quad 0 \le p \le q \le 2^{k+1}.\end{aligned}$$

This process yields a mapping $\psi\colon E \longrightarrow \mathbb{R}^n$, where $E = \bigcup E_k \{jT/2^k\}_{j,k}$. This mapping satisfies (a) and (b$'$) on E. It can be extended by continuity to $[0, T]$, and condition (b$'$) will persist (see Subsection 2.4.4). $\square$

2.7.1. So far, we have not used property (b) in Theorem 2.1; we need it only at this stage of the proof. Now we assume that $\vec{\mu}$ satisfies both conditions (a) and (b) of Theorem 2.1 and put $\lambda := \operatorname{div}\vec{\mu}$; λ is a *scalar* charge ($\lambda \in M(\mathbb{R}^n)$).

2.7.2. Clearly, $L^\infty(m) \subset L^2(m)$. Let $[\ ,\ ]$ denote the scalar product in $L^2(m)$. The operator A possesses the following properties:

(a) $A(uv) = (Au) \cdot v + u \cdot (Av)$ m-a.e.;

(b) $[Au, v] + [u, Av] = -\int uv\, d\lambda$ for any $u, v \in C^1(\mathbb{R}^n)$.

Proof. Indeed, (a) is true at any point x where $|\vec{\nu}(x)| = 1$; (b) follows from the definition of λ. $\square$

2.7.3. We put $K := \operatorname{spt}\vec{\mu}$, $\mathbb{N}(K) := \{\vec{\xi} \in \vec{M}(K) : \operatorname{div}\vec{\xi} \in M(K)\}$, $\mathbb{N}_1(K) := \{\vec{\xi} \in \mathbb{N}(K) : \operatorname{var}\vec{\xi} \le 1,\ \operatorname{var}(\operatorname{div}\vec{\xi}) \le 1\}$, $\vec{M}_1(K) := \{\zeta \in \vec{M}(K) : \operatorname{var}\zeta \le 1\}$. *The set $\mathbb{N}_1(K)$ is weakly compact* (i.e., is compact in the weak topology defined by the pairing $(\vec{f}, \vec{\xi}) \mapsto \vec{\xi}[\vec{f}]$, $\vec{f} \in \vec{C}(K)$, $\vec{\xi} \in \vec{M}(K)$).

Proof. We have $\mathbb{N}_1(K) \subset \vec{M}_1(K)$; the latter set, endowed with the weak topology, is a metrizable space. Therefore, the weak compactness of $\mathbb{N}_1(K)$ will be proved if we take an arbitrary sequence $\{\vec{\xi}_k\}$ of elements of $\mathbb{N}_1(K)$ and find its subsequence weakly convergent to an element of $\mathbb{N}_1(K)$. We have

$$\vec{\xi}_k[\nabla u] = -\int u\,d\lambda_k, \quad \lambda_k := \operatorname{div}\vec{\xi}_k, \quad u \in C^1(\mathbb{R}^n), \quad k \geq 1. \tag{20}$$

Since $\operatorname{var}\vec{\xi}_k \leq 1$, $\operatorname{var}\lambda_k \leq 1$, we can find a sequence $k_j \nearrow +\infty$ such that $\vec{\xi}_{k_j} \longrightarrow \vec{\xi}$, $\lambda_{k_j} \longrightarrow \lambda$ weakly; passing to the limit in (20) (with $k = k_j$), we get $\vec{\xi}[\nabla u] = -\int u\,d\lambda$ ($u \in C^1(\mathbb{R}^n)$). Hence, $\operatorname{div}\vec{\xi} = \lambda \in M_1(K)$, and $\vec{\xi} \in \mathbb{N}(K)$. □

2.7.4. The Krein–Milman theorem applied to the convex set $\mathbb{N}_1(K) \subset \vec{M}(K)$ guarantees the existence of an extreme point of $\mathbb{N}_1(K)$. Consequently, in the proof of our theorem it may be assumed that

$$\vec{\mu} \text{ is an extreme point of } \mathbb{N}_1(K). \tag{21}$$

2.7.5. *If $\vec{\mu}$ satisfies* (21), *then there exists no Borel function w on $\mathbb{R}^n$ satisfying the following conditions*:

$$m\{w = 0\} > 0, \qquad m\{w = 1\} > 0, \qquad 0 \leq w \leq 1 \quad m\text{-a.e.}; \tag{22}$$

$$[Au, w] = -\int uw\,d\lambda \quad \textit{for any } u \in C^1(\mathbb{R}^n). \tag{23}$$

Proof. The left-hand side in (23) is equal to $\int \langle \nabla u, \vec{\nu}\rangle w\,dm = (w\vec{\mu})[\nabla u]$. Therefore, (23) means that $\operatorname{div}(w\vec{\mu}) = w \operatorname{div}\vec{\mu}$, and $w\vec{\mu} \in \mathbb{N}_1(K)$. Let $\vec{\mu}_1 := w\vec{\mu}$, $\vec{\mu}_2 := (1-w)\vec{\mu}$. Then $\vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2$, $\operatorname{var}\vec{\mu}_1 + \operatorname{var}\vec{\mu}_2 = \int w\,dm + \int (1-w)\,dm = \operatorname{var}\vec{\mu}$, whereas $\vec{\mu} \neq \operatorname{const}\vec{\mu}$, $\vec{\mu}_2 \neq \operatorname{const}\vec{\mu}$, which contradicts (21). □

2.7.6. We denote by $\operatorname{sol}_1(K)$ the set $\{\,\vec{\mu} \in M(K) : \operatorname{div}\vec{\mu} = 0,\ \operatorname{var}\vec{\mu} \leq 1\,\}$. Assume that

$$\vec{\mu} \text{ is an extreme point of } \operatorname{sol}_1(K). \tag{21$'$}$$

Then the arguments of Subsection 2.7.5 can be repeated with $\lambda = 0$.

2.8. The following assertion is the key point of the proof.

Suppose that $\vec{\mu}$ satisfies the assumptions of Theorem 2.1 *and condition* (21). *If K_1, K_2 are compact sets in $\mathbb{R}^n$ and $m(K_j) > 0$, $j = 1, 2$, then $\rho(K_1, K_2) < +\infty$ (i.e., there exist $x_j \in K_j$ such that $\rho(x_1, x_2) < +\infty$)).*

Proof. Suppose the contrary: let $\rho(K_1, K_2) = +\infty$. Then $\rho(K_1, K_2) > N$ for any $N > 0$. By the statement proved in Subsection 2.5, there is a function $u_N \in \mathcal{A}$ such that $u_N|_{K_1} = 0$, $u_N|_{K_2} = N$, and $0 \leq u_N \leq N$ everywhere in $\mathbb{R}^n$. We put $w_N = N^{-1}u_N$; then $w_N \in C^1(\mathbb{R}^n)$, $w_N|_{K_1} = 0$, $w_N|_{K_2} = 1$, $0 \leq w_N \leq 1$, and $\|Aw_N\|_\infty \leq 1/N$ (here $\|\cdot\|_\infty := \|\cdot\|_{\infty,m}$). Let $l = \|\lambda\|$. The closed unit balls in $L^\infty(m)$ and $L^\infty(l)$ are compact with respect to the weak topologies defined by the natural pairings $(L^1(m), L^\infty(m))$, $(L^1(l), L^\infty(l))$. Hence, there is a sequence of integers $N_j \uparrow +\infty$ and a function $w \in L^\infty(m)$ such that

$$\int w_{N_j} U\,dm \xrightarrow[j\to\infty]{} \int wU\,dm, \qquad \int w_{N_j} U\,d\lambda \xrightarrow[j\to\infty]{} \int wU\,d\lambda$$

for any $U \in L^1(m) \cap L^1(l)$. Clearly, $0 \leq w \leq 1$, $w = 0$ m-a.e. on K_1, and $w = 1$ m-a.e. on K_2. In accordance with Subsection 2.7.2, we have

$$[Au, w_{N_j}] + [u, Aw_{N_j}] = -\int w_{N_j} u \, d\lambda, \quad u \in C^1(\mathbb{R}^n). \tag{24}$$

Recalling the estimate $|[u, Aw_{N_j}]| \leq N_j^{-1} \cdot \int |u| \, dm$ and passing to the limit in (24), we get

$$[Au, w] = -\int wu \, d\lambda, \quad u \in C'(\mathbb{R}^n),$$

which contradicts the statement proved in Subsection 2.7.5. □

2.9. Now we can finish the proof of Theorem 2.1. We show that

If $\vec{\mu}$ satisfies the assumptions of Theorem 2.1 *and* (21), *then for any $x, y \in \operatorname{spt} \vec{\mu}$ and any $\varepsilon > 0$ there exist points $x', y' \in \operatorname{spt} \vec{\mu}$ such that x' and y' can be joined by a simple rectifiable arc in* $\operatorname{spt} \vec{\mu}$, *and $|x - x'| < \varepsilon$, $|y - y'| < \varepsilon$.*

Proof. Let K_1 and K_2 be disjoint (Euclidean) closed balls of radius less than ε and centered, respectively, at x and y. Since $x, y \in \operatorname{spt} \mu$, we have $m(K_j) > 0$ ($j = 1, 2$). Hence, we can apply the assertion proved in Subsection 2.8 to find points $x' \in K_1$, $y' \in K_2$ such that $0 < \rho(x', y') < +\infty$. Now, the existence of the desired arc follows from Subsection 2.7. Theorem 2.1 is proved.

2.9.1. Using the statement in Subsection 2.7.5 and putting $\lambda = 0$, we can repeat the argument given in Subsections 2.8–2.9 to get the following conclusion: *the assertion of Subsection* 2.9 *remains valid if we replace* (21) *by* (21′).

2.10. Again, suppose that $\vec{\mu}$ satisfies conditions (a), (b) of Theorem 2.1 and condition (21) (or (21′)). The above arguments result in the following alternative. *Either any two points $x, y \in \operatorname{spt} \vec{\mu}$ can be joined by a simple rectifiable arc in* $\operatorname{spt} \vec{\mu}$, *or* $\operatorname{spt} \vec{\mu}$ *contains arbitrarily long simple rectifiable arcs.*

Proof. Let C denote the supremum of the lengths of the simple rectifiable arcs contained in $\operatorname{spt} \vec{\mu}$. Suppose that $C < +\infty$. We take any pair (x, y) of points in $\operatorname{spt} \vec{\mu}$. Using the statement proved in Subsection 2.9, we can find a sequence $\{\gamma_j\}$ of simple rectifiable arcs that are contained in $\operatorname{spt} \vec{\mu}$ and have endpoints x_j, y_j tending, respectively, to x and y. Since the lengths $\mathcal{H}^1(\gamma_j)$ arc uniformly bounded, a simple compactness argument yields a rectifiable (not necessarily simple) arc $\gamma \subset \operatorname{spt} \mu$ joining x and y. It is easily seen that γ contains a simple arc with the same endpoints. □

2.11. A deeper analysis of the structure of vector charges of class $\mathbb{N}(K)$ (and, in particular, of divergence-free charges) can be found in [4]. We need the following result (see [4]).

Let $\vec{\mu} \in \vec{M}(\mathbb{R}^n)$, $\operatorname{div} \vec{\mu} = 0$, $S > 0$. Then

$$\vec{\mu} = \int_{C_S} R \, d\gamma_S(R), \quad \|\vec{\mu}\| = \int_{C_S} \|R\| \, d\gamma_S(R), \quad \frac{2}{S}\|\vec{\mu}\| \geq \int_{C_S} \|\operatorname{div} R\| \, d\gamma_S(R), \tag{25}$$

where C_S is the set of all curves of length S (see Subsection 1.15), and *γ_S is a positive measure on C_S (γ_S is a Borel measure relative to the natural topology on C_S).*

§3. Approximation and extension theorems for gradients. Approximation by jets

3.1. We say that a set $E \subset \mathbb{R}^n$ is *metrically disconnected* (E is an m.d. set) if E contains no nontrivial simple rectifiable arc. The following statement is an immediate consequence of Theorem 2.1:

Every compact m.d. set is a strong grad-set.

Proof. Suppose that $K \subset \mathbb{R}^n$ is a compact m.d. set. It suffices to show that any charge $\vec{\mu} \in \vec{M}(K)$ orthogonal to all gradients of C^1-functions is zero. But the identity $\vec{\mu}[\nabla u] = 0$ ($u \in C^1(\mathbb{R}^n)$) means that $\operatorname{div} \vec{\mu} = 0$; if $\vec{\mu} \neq 0$, then $\operatorname{spt} \vec{\mu} \subset K$ contains a nontrivial simple rectifiable arc by Theorem 2.1. □

3.1.1. The following remark will be used in §4. A vector field $\vec{v}$ defined on a set $E \subset \mathbb{R}^n$ is called a *quasigradient* (on E) if $\vec{v} = \lim_{j \to \infty} \nabla u_j$ uniformly on E for some sequence (u_j), $u_j \in C^1(\mathbb{R}^n)$, $j = 1, 2, \dots$.

Let K_0, K be compact sets in $\mathbb{R}^n$ such that $K_0 \subset K$, and let $\vec{v} \in \vec{C}(K)$. If

$$(26) \qquad \vec{v}|_{K_0} \quad \textit{is a quasigradient on } K_0$$

and $K \setminus K_0$ is an m.d. set, then $\vec{v}$ is a quasigradient on K.

Proof. Suppose that $\vec{\mu} \in \vec{M}(K)$, $\vec{\mu}[\nabla u] = 0$ for any $u \in C^1(\mathbb{R}^n)$. By the Hahn–Banach theorem, it suffices to show that (26) implies the relation $\vec{\mu}[\vec{v}] = 0$. Clearly, $\operatorname{div} \vec{\mu} = 0$, so that $\vec{\mu} \in \mathbb{N}(K)$. By the Krein–Milman theorem, we may assume that $\vec{\mu}$ is an extreme point of $\operatorname{sol}_1(K)$. We prove that $\operatorname{spt} \vec{\mu} \subset K_0$. Suppose that $x \in \operatorname{spt} \vec{\mu} \setminus K_0$. Since the support of a divergence-free field cannot be a singleton, we can find a point $y \in \operatorname{spt} \vec{\mu} \setminus K_0$, $y \neq x$. In accordance with Subsection 2.9.1, there are points $x', y' \in \operatorname{spt} \vec{\mu}$ arbitrarily close, respectively, to x and y that can be joined by a simple rectifiable arc γ in $\operatorname{spt} \vec{\mu}$. The points x', y' can be chosen so that $x' \in \operatorname{spt} \vec{\mu} \setminus K_0$ (because $x \in \operatorname{spt} \vec{\mu} \setminus K_0$, and the latter set is relatively open in $\operatorname{spt} \vec{\mu}$) and $y' \neq x'$. Then a simple rectifiable arc joining x' and y' must have nontrivial intersection with $K \setminus K_0$, a contradiction. Thus, $\operatorname{spt} \vec{\mu} \setminus K_0 = \varnothing$, and we see that $\vec{\mu}[\vec{v}] = 0$ because $v|_{K_0}$ is a quasigradient. □

3.1.2. Here we prove a theorem yielding a *complete* description of the m.d. sets in terms of approximation properties. A compact set $K \subset \mathbb{R}^n$ will be called a *jet set* if for any $\varepsilon > 0$ and any couple $(\varphi, \vec{\psi}) \in C(K) \times \vec{C}(K) =: \vec{c}(K)$ there is a function $u \in C^\infty(\mathbb{R}^n)$ such that

$$\max_K |\varphi - u| + \max_K |\vec{\psi} - \nabla u| < \varepsilon.$$

Theorem. *The class of all jet sets coincides with the class of all compact m.d. sets.*

Proof. Let $J(K)$ denote the set of all jets or order 1 restricted to K:

$$J(K) := \{\, (\varphi, \vec{\psi}) \in \vec{c}(K) : \exists u \in C^1(\mathbb{R}^n),\ \varphi = u|_K,\ \vec{\psi} = \nabla u|_K \,\}.$$

If K is not an m.d. set, then there is a nontrivial simple rectifiable curve R with $\operatorname{spt} R \subset K$. Consider the linear functional $F_R \in (\vec{c}(K))^*$ defined by the formula

$$F_R(\varphi, \vec{\psi}) = \varphi(e(R)) - \varphi(b(R)) - R[\vec{\psi}], \quad (\varphi, \vec{\varphi}) \in \vec{c}(K)$$

($e(R)$ is the end and $b(R)$ the origin of R). Then $F_R|_{J(K)} = 0$, i.e., there is a nonzero linear functional orthogonal to J, and K is not a jet set.

Now, suppose that K is an m.d. set, and that

$$(27) \qquad \text{(a)} \quad F \in (\vec{c}(K))^*, \qquad \text{(b)} \quad F|_{J(K)} = 0.$$

By property (a) in Subsection 2.7, there is a couple $(\mu_1, \vec{\mu}_2) \in M(K) \times \vec{M}(K)$ such that

$$F(\varphi, \vec{\psi}) = \int \varphi \, d\mu_1 + \int \langle \vec{\varphi}, d\vec{\mu}_2 \rangle, \quad (\varphi, \vec{\psi}) \in \vec{c}(K).$$

Property (b) in the same subsection shows that $\int u \, d\mu_1 = -\int \langle \nabla u, d\mu_2 \rangle = \operatorname{div} \vec{\mu}_2[u]$ for any test function $u \in C^\infty(\mathbb{R}^n)$. Hence, $\operatorname{div} \vec{\mu}_2 = \vec{\mu}_1$, i.e., $\mu_2 \in \mathbb{N}(K)$. If $F \neq 0$, then $\mu_2 \neq 0$, and, by Theorem 2.1, K must contain a nondegenerate simple rectifiable arc. □

3.1.3. Using the result mentioned in Subsection 2.11, we can give an "individual" theorem characterizing the quasigradients on a given compact set $K \subset \mathbb{R}^n$. Suppose $\vec{\varphi} \in \vec{C}(K)$.

The following assertions are equivalent:

(a) *$\vec{\varphi}$ is a quasigradient on K*;

(b) *for any $\varepsilon > 0$ there is a number $M(\varepsilon) > 0$ such that*

$$(28) \qquad \vec{R}[\vec{\varphi}] \leq M(\varepsilon) + \varepsilon l(\vec{R}) \quad \textit{for any curve } \vec{R} \textit{ with } \operatorname{spt} \vec{R} \subset K,$$

where $l(\vec{R})$ is the length of $\vec{R}$ (see Subsection 1.15).

Proof. Let $\vec{\varphi}$ be a quasigradient. We put $M(\varepsilon) := 2 \max_K |u|$, where $u \in C^1(\mathbb{R}^n)$ satisfies $\max_K |\vec{\varphi} - \nabla u| < \varepsilon$. Then (28) follows from the identity

$$\vec{R}[\vec{\varphi}] = u(e_R) - u(b_R) + \vec{R}[\vec{\varphi} - \nabla u]$$

(b_R and c_R are the origin and the end of the curve R). In order to prove the implication (b) $\Longrightarrow$ (a), we take $\varepsilon > 0$ and put $S := M(\varepsilon)/\varepsilon$. Applying the result cited in Subsection 2.11 to an arbitrary divergence-free charge $\vec{\mu} \in \vec{M}(K)$, we get

$$\vec{\mu}[\vec{\varphi}] = \int_{\vec{R} \in C_S} \vec{R}[\vec{\varphi}] \, d\gamma_S(\vec{R}) \leq (M(\varepsilon) + \varepsilon S) \operatorname{var} \gamma_S = (M(\varepsilon) + \varepsilon S) \frac{\operatorname{var} \vec{\mu}}{S} = 2\varepsilon \operatorname{var} \vec{\mu}$$

(we have used the identity $\operatorname{var} \vec{\mu} = S \operatorname{var} \gamma_S$ and the inclusions $\operatorname{spt} \vec{R} \subset \operatorname{spt} \vec{\mu} \subset K$ for γ_S-a.e. $\vec{R}$; see (25)). Hence, $\vec{\mu}[\vec{\varphi}] \leq 0$. Replacing $\vec{\mu}$ by $-\vec{\mu}$, we conclude that $\vec{\mu}[\vec{\varphi}] = 0$ for any solenoidal $\vec{\mu} \in \vec{M}(K)$, whence $\vec{\varphi}$ is a quasigradient. □

It is natural to ask about a purely geometric description of the compact sets $K \subset \mathbb{R}^n$ on which "free" approximation by jets of higher order is possible (in the spirit of Theorem 3.1.2). No such description is known to us (even for jets of order 2 in $\mathbb{R}^2$).

3.3. Can Theorem 1.18 be generalized to $\mathbb{R}^n$ for $n \geq 3$? The answer is in the negative (see, e.g. [4]). There are compact sets $K \subset \mathbb{R}^3$ that contain no nondegenerate rectifiable loops, but are not strong grad-sets. Nevertheless, Theorem 1.19 can be generalized to $\mathbb{R}^n$ with any n.

Theorem. *Let $K \subset \mathbb{R}^n$ be a compact set. The following assertions are equivalent*:

(a) *K is a perfect grad-set*;

(b) *there is a positive number $\lambda(K)$ such that*

$$(29) \qquad \lambda(K) \mathcal{H}^1(l \cap K) \leq \mathcal{H}^1(l \setminus K)$$

for any rectifiable loop l.

Adding $\lambda(K) \cdot \mathcal{H}^1(l \setminus K)$ to both parts of inequality (29), we obtain an equivalent condition:

$$(29') \qquad \lambda'(K)\mathcal{H}^1(l) \leq \mathcal{H}^1(l \setminus K),$$

where $\lambda'(K) = \lambda(K)/(1 + \lambda(K))$.

Proof. We fix a large closed ball B the interior of which contains K. Let $\operatorname{Grad} B$ denote the set $\{ \nabla u : u \in C^1(B) \} = \{ \vec{v} \in \vec{C}(B) : \operatorname{curl} \vec{v} = 0|_{\operatorname{Int} B} \}$. Assertion (a) means that $r(\operatorname{Grad} B) = \vec{C}(K)$ where r is the restriction operator, $r(\vec{v}) := \vec{v}|_K$ ($\vec{v} \in \vec{C}(B)$). Clearly, $\operatorname{Grad} B$ is a closed subspace of $\vec{C}(B)$. Identifying $(\vec{C}(B))^*$ with $\vec{M}(B)$, we have

$$\begin{aligned}(\operatorname{Grad} B)^\perp &:= \{ \vec{\sigma} \in \vec{M}(B) : \vec{\sigma}[\nabla u] = 0,\ u \in C^1(B) \} \\ &= \{ \vec{\sigma} \in \vec{M}(B) : \operatorname{div} \vec{\sigma} = 0 \} =: s(B).\end{aligned}$$

Hence, by the Banach theorem, condition (a) is equivalent to the following condition:

(c) $\exists \lambda(K) > 0$ such that $\operatorname{var}(\vec{\mu} - \vec{\sigma}) \leq \lambda(K) \operatorname{var} \vec{\mu}, \quad \vec{\mu} \in \vec{M}(K),\ \vec{\sigma} \in s(B)$.

The implication (a) $\Longrightarrow$ (b) is fairly easy to prove. We take a simple rectifiable closed curve $\vec{\sigma}$ contained in B. Clearly, $\vec{\sigma} \in s(B)$. Putting $\vec{\mu} = \chi_K \vec{\sigma}$ and applying (c), we get (29) because $\|\vec{\sigma}(E)\| \equiv \mathcal{H}^1(E \cap \operatorname{spt} \vec{\sigma})$ for any $E \in \mathcal{B}_n$. It remains to observe that for every simple rectifiable curve l we can find a simple rectifiable curve $l^* \subset B$ such that $l \cap K = l^* \cap K$ and $\mathcal{H}^1(l^* \setminus K) \leq \mathcal{H}^1(l \setminus K)$.

3.4. We start the proof of the implication (b) $\Longrightarrow$ (a) with the following remark.

Suppose that K satisfies (b), *and that C is a closed* (*not necessarily simple*) *rectifiable curve. Then*

$$(30) \qquad \|C\|(K^c) \geq \lambda(K) \cdot \|C\|(K).$$

Proof. As already mentioned, we can write

$$C = \sum_{j=1}^{\infty} C_j, \qquad \|C\| = \sum_{j=1}^{\infty} \|C_j\|,$$

where the C_j are simple closed rectifiable curves (see the references in Subsection 1.1.6). Putting $l_j := \operatorname{spt} C_j$, we observe that $\|C_j\|(E) = \mathcal{H}^1(l_j \cap E)$ ($E \in \mathcal{B}_n$), whence

$$\begin{aligned}\|C\|(K^c) &= \sum_{j=1}^{\infty} \|C_j\|(K^c) = \sum_{j=1}^{\infty} \mathcal{H}^1(l_j \setminus K) \geq \lambda(K) \sum_{j=1}^{\infty} \mathcal{H}^1(l_j \cap K) \\ &= \lambda(K) \sum_{j=1}^{\infty} \|C_j\|(K) = \lambda(K)\|C\|(K). \qquad \square\end{aligned}$$

3.5. For a large number $S > 0$, consider the set $C_{S,B}$ of all curves of length S contained in B. First we check that *inequality* (30) *is fulfilled* (*possibly*, *with a smaller positive constant* $\lambda(K)$) *for any* $R \in C_{S,B}$.

Proof. If $R \in C_{S,B}$ is not closed, then we can join the endpoints $b(R)$ and $e(R)$ of R by a suitably oriented segment j to get a closed rectifiable curve R^*. We note that

$$\operatorname{var}(R - R^*) \leq |b(R) - e(R)| \leq \operatorname{diam} B =: d.$$

By the remark in Subsection 3.4, $\|R^*\|(K^c) \ge \lambda'(K)\operatorname{var}(R^*)$ (this analog of (29′) follows from (30)). Hence, for any $R \in C_{S,B}$ we have

$$\begin{aligned}\|R\|(K^c) &\ge \|R^*\|(K^c) - \|j\|(K^c) \ge \lambda'(K)\operatorname{var} R^* - d\\ &\ge \lambda'(K)(\operatorname{var} R - \operatorname{var} j) - d = \lambda'(K)(S-d) - d\\ &= \lambda'(K)S(1 - S^{-1}d(1+\lambda^1(K)))\\ &\ge \frac{\lambda'(K)}{2}S = \frac{\lambda'(K)}{2}\operatorname{var} R \ge \frac{\lambda'(K)}{2}\|R\|(K),\end{aligned}$$

provided that $S > 2d(1+\lambda'(K))$.

3.6. To finish the proof, we take $\vec{\mu} \in \vec{M}(K)$ and $\vec{\sigma} \in s(B)$. Applying the statement proved above in Subsection 3.5 and Theorem 2.11, we obtain

$$\begin{aligned}\|\vec{\sigma}\|(K^c) &= \int_{C_{S,B}} \|R\|(K^c)\,d\gamma_S(R)\\ &\ge \frac{\lambda'(K)}{2}\int_{C_{S,B}} \|R\|(K)\,d\gamma_S(R) = \frac{\lambda'(K)}{2}\|\vec{\sigma}\|(K)\end{aligned}$$

(we recall that γ_S-almost all curves occurring in (25) are contained in $\operatorname{spt}\vec{\mu}$). Consequently,

$$\begin{aligned}\operatorname{var}(\vec{\mu} - \vec{\sigma}) &= \operatorname{var}(\vec{\mu} - \chi_K\vec{\sigma} + \|\vec{\sigma}\|(K^c)\\ &\ge \frac{\lambda'(K)}{2}(\operatorname{var}(\vec{\mu} - \chi_K\vec{\sigma}) + \operatorname{var}(\chi_K\vec{\sigma})) = \frac{\lambda'(K)}{2}\operatorname{var}(\vec{\mu}),\end{aligned}$$

which proves (c) and, with it, Theorem 3.3. □

3.7. We apply Theorem 3.3 to the graph $K = K_f \subset \mathbb{R}^2$ of a continuous function $f \in C([0;1]$ ($K_f := \{(x, f(x)) \in \mathbb{R}^2 : x \in [0;1]\}$). Clearly, K_f is a perfect grad-set if f is sufficiently smooth, e.g., $f \in C^1([0;1])$. This can be shown directly, by extending a given field $\vec{v} \in \vec{C}(K_f)$ to a gradient "by hands", or by applying Theorem 3.3. (Both ways are not difficult, but not so easy as it may seem at first glance, even if $f \equiv 0$.) However, if f is not so good, K_f may fail to be a perfect grad-set. Suppose, for instance, that there is a sequence $\{\Delta_j\}$ of disjoint segments $\Delta_j = [a_j, b_j] \subset (0,1)$ such that

$$\begin{gathered}f(a_j) = f(b_j) = 0,\\ f(c_j) = h_j > 0, \quad \text{where } c_j = \frac{a_j+b_j}{2},\\ f|_{[a_j,c_j]}, f|_{[c_j,b_j]} \quad \text{are linear},\\ \lim a_j = \lim b_j = 0.\end{gathered}$$

Then $\lim h_j = 0$. If $b_j - a_j = o(h_j)$, then condition (29) fails (it suffices to consider the triangles formed by the graphs of $f|_{\Delta_j}$ and by Δ_j). In this situation we can explicitly construct a field $\vec{v} \in \vec{C}(K_f)$ coinciding with no gradient on K_f. Namely, consider very small disks $B = B(p_j, \varepsilon_j)$, $B = B(q_j, \varepsilon_j)$, and $B = B(r_j, \varepsilon_j)$ (here $p_j = (c_j, h_j)$, $q_j = (a_j, 0)$, $r_j = (b_j, 0)$) with $\varepsilon_j \ll b_j - a_j$. Suppose that $\vec{v} \in \vec{C}(K_f)$, $\vec{v}(0) = 0$, $\vec{v} = \delta_j\vec{\tau}$ on $K_f \setminus \{0 \cup \bigcup B\}$, where the δ_j are positive constants and $\vec{\tau}$ is the unit vector tangent to K_f. Let Γ_j denote the oriented graph of $f|_{\Delta_j}$. Then

$$\int_{\Gamma_j} \langle\vec{v}, \vec{\tau}\rangle\,ds \asymp \delta_j h_j. \tag{31}$$

But if $u \in C^1(\mathbb{R}^2)$, then

$$\left|\int_{\Gamma_j} \langle \nabla u, \vec{\tau}\rangle\, ds\right| = \left|\int_{\Delta_j} \frac{\partial u}{\partial x}\, dx\right| = O(b_j - a_j). \tag{32}$$

Estimates (31) and (32) contradict each other if the δ_j tend to zero sufficiently slowly.

This example shows that the notions of a perfect grad-set and of a strong grad-set are different (any graph K_f with $f \in C([0,1])$ is a strong grad-set because it contains no rectifiable loops).

3.8. The following consequence of Theorem 3.3 should be mentioned.

The area of any plane perfect grad-set is zero.

Proof. Let $K \subset \mathbb{R}^2$ be a compact set with $\mathcal{L}^2(K) > 0$. Then K contains a density point p. We denote by γ_R the circle of radius R centered at p. It is easy to check that

$$\varliminf_{R\to 0} \mathcal{H}^1(\gamma_R \setminus K)/\mathcal{H}^1(\gamma_R) = 0,$$

so that K is not a perfect grad-set (by Theorem 3.3). □

In particular, let K be a totally disconnected compact set of positive area. Any vector field continuous on K can be approximated uniformly on K by fields locally constant near K. Hence, such a set K yields yet another example of a strong but not perfect grad-set.

§4. Nonlocality of h-sets in $\mathbb{R}^3$

4.1. We need the following result.

Let $K \subset \mathbb{R}^3$ be a compact set of zero volume (i.e., the 3-dimensional Lebesgue measure of K is zero). Then for any function $u \in C^1(\mathbb{R}^3)$ and any $\varepsilon > 0$ there exists a function H harmonic in a neighborhood of K and such that $|\nabla u - \nabla H| < \varepsilon$.

This fact was proved in [9] and generalized to harmonic differential forms in [3] (see also [10], [11]).

4.2. We denote by U the closed unit disk in $\mathbb{R}^2$,

$$U = \{\,(x,y) : x^2 + y^2 \le 1\,\},$$

and put $C = \{\,(x,y) : x^2 + y^2 = 1\,\}$. Consider a function $F \in C(U)$ with the following properties:

(a) $F = 0$ on C;

(b) the graph of $F|_{U\setminus C}$ is metrically disconnected.

(A method for constructing such functions is described below in Subsection 4.6.) Fixing a number $\sigma \in (0,1)$, we put

$$K := \text{the graph of } F = \{\,(x,y,z) \in \mathbb{R}^3 : (x,y) \in U,\ z = F(x,y)\,\},$$
$$K_- = K \cap \{x \le \sigma\}, \qquad K_+ := K \cap \{x \ge -\sigma\}.$$

Clearly, the $K_\pm$ are the closures of relatively open subsets of K whose union is K. We shall prove that

(a) K is not an h-set, although

(b) K_+ and K_- are strong h-sets.

Therefore, no analog of the Bishop localization principle for $R(K)$ (see the Introduction) is valid in $\mathbb{R}^3$.

4.3. First, we prove (a). Suppose K is an h-set. Then any field $\vec{v} \in \vec{C}(K)$ is the uniform limit (on K) of some fields $\vec{v}_j \in h(O_j)$, where O_j is a neighborhood of K. We may assume that the O_j are simply connected. Then $\vec{v}_j = \nabla H_j$ (because curl $\vec{v}_j = 0$). This means that K is a strong grad-set, which is impossible because K contains a rectifiable loop C.

4.4. Now we prove (b). It suffices to show that K_+ and K_- are strong grad-sets. Indeed, the volume of K is zero (since K is the graph of a continuous function). By the result cited in Subsection 4.1, any field $\vec{v} \in \vec{C}(K)$ admitting approximation in $\vec{C}(K)$ by gradients of C^1-functions can also be approximated in $\vec{C}(K)$ by gradients of functions harmonic in a neighborhood of K. We may assume that this neighborhood is simply connected. A standard application of the Runge theorem yields a sequence of functions h_j harmonic in $\mathbb{R}^3$ and such that $\nabla h_j \to \vec{v}$ in $\vec{C}(K)$.

4.5. In order to show that K_+ is a strong grad-set, we apply the statement proved in Subsection 3.1.1 to the couple K'_+, K_+, where $K'_+ := K_+ \cap C$. Let $\vec{v} \in \vec{C}(K_+)$; we show that $\vec{v}$ is a quasigradient (on K_+). Since the set $K_+ \setminus K'_+$ is metrically disconnected, it suffices to show that $\vec{v}|_{K'_+}$ is a quasigradient. But this is obvious, because K'_+ is a strong grad-set (and even a perfect grad-set, by Theorem 3.3). □

4.6. For constructing a function F satisfying (a) and (b) (see Subsection 4.2), we start with a Weierstrass-type function $W\colon \mathbb{R} \to \mathbb{R}$ (i.e., W is continuous and nowhere differentiable). Clearly, the total variation $\bigvee_a^b W$ is infinite whenever $a < b$. We may assume that $|W| < 1$. Putting

$$F_1(x, y) := W(W(x) - y)), \quad (x, y) \in \mathbb{R}^2,$$

we prove that

The graph γ of F_1 is metrically disconnected.

Proof. Let γ be a continuous path in $\mathbb{R}^2$,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)), \ t \in [0, 1], \quad \text{where } \gamma_j \in C([0; 1]).$$

We put $g(t) := (\gamma_1(t), \gamma_2(t), F_1(\gamma(t)))$. We must prove that $\bigvee_0^1 g = +\infty$ unless $\gamma(t) \equiv$ const. There are two possibilities: 1) $w := w(\gamma_1) - \gamma_2 \not\equiv$ const; 2) $w \equiv$ const. In the first case we can find α, β such that $0 \le \alpha < \beta \le 1$ and $w(\alpha) \ne w(\beta)$. Denoting by J the segment with endpoints $w(\alpha)$, $w(\beta)$, we have

$$\bigvee_0^1 g \ge \bigvee_0^1 F_1(\gamma) = \bigvee_0^1 W(w) \ge \bigvee_\alpha^\beta W(w) \ge \bigvee_J W = +\infty.$$

In the second case $\gamma_1 \equiv$ const (hence, $\gamma \equiv$ const). Indeed, if $0 \le \alpha < \beta \le 1$ and $\gamma_1(\alpha) \ne \gamma_1(\beta)$, then

$$\bigvee_0^1 g \ge \bigvee_\alpha^\beta \gamma_2 = \bigvee_\alpha^\beta W(\gamma_1) \ge \bigvee_I W = +\infty,$$

where I is the segment with endpoints $\gamma_1(\alpha)$, $\gamma_1(\beta)$. Thus, we have constructed a metrically disconnected graph Γ over $\mathbb{R}^2$.

4.6.1. Consider the cylinder $\mathbf{Z} = U \times [-2, 2]$ containing the graph Γ_1 of $F_1|_U$. It is easy to construct a continuous mapping $T : \mathbf{Z} \to B = \{x^2 + y^2 + z^2 \le 1\}$ such that $T(C \times [-2, 2]) = C$ and $T_{\mathbf{Z}\setminus(C\times[-2,2])}$ is a diffeomorphism of the set $\mathbf{Z} \setminus (C \times [-2, 2])$ onto $B \setminus C$ preserving the projection to the plane (x, y). Then $T(\Gamma_1)$ is the graph of a function F enjoying properties (a) and (b) in Subsection 4.2.

4.7. In conclusion, we briefly discuss the relationship between grad-sets, curl-sets, and h-sets. F. L. Nazarov (personal communication) constructed an elegant example of a plane compact set K with the following properties: 1) K is a curl-set (hence a grad-set); 2) K is not an h-set.

Note that the situation changes drastically if we replace uniform approximation by L^p-approximation. Namely, let $K \subset \mathbb{R}^2$ be a compact set, and let $p > 1$.

The following statements are equivalent:

1) *any field of class* $\vec{L}^p(K)$ *can be approximated in* $\vec{L}^p(K)$ *by fields that are solenoidal near* K;

2) *any field of class* $\vec{L}^p(K)$ *can be approximated in* $\vec{L}^p(K)$ *by fields that are harmonic near* K (see [12]).

For more information about $\vec{L}^p$-approximation of vector fields, see also [13, 14].

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