Conformal Invariance of Ising model correlations

Clément Hongler

Stas Smirnov

Conformally invariant processes: invariance of probability measures

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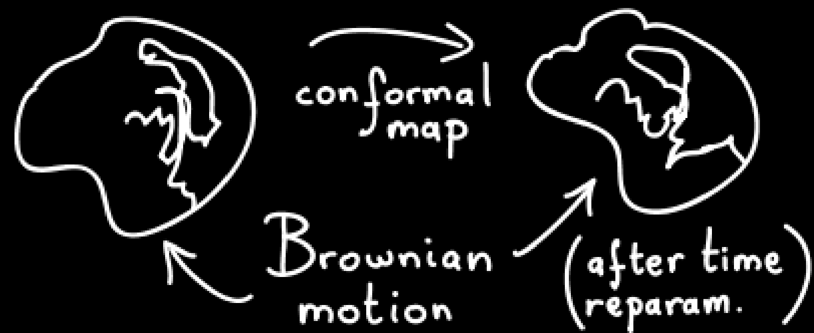

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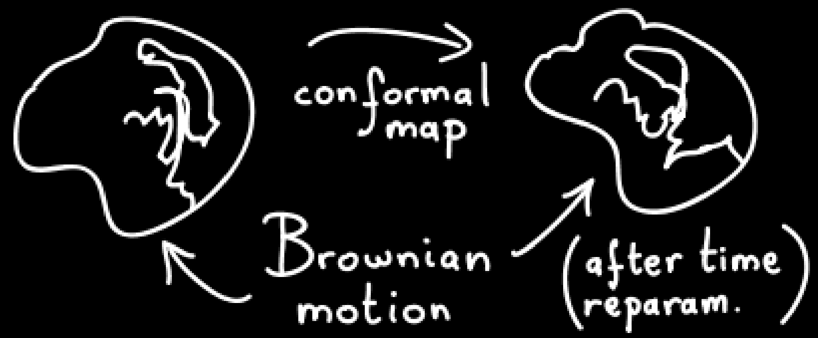

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Conformally invariant critical systems:

- Percolation
- Ising model
- Potts models with 3 or 4 states
- Gaussian free field
- Uniform spanning tree
- Dimer model
- Self-avoiding polymer
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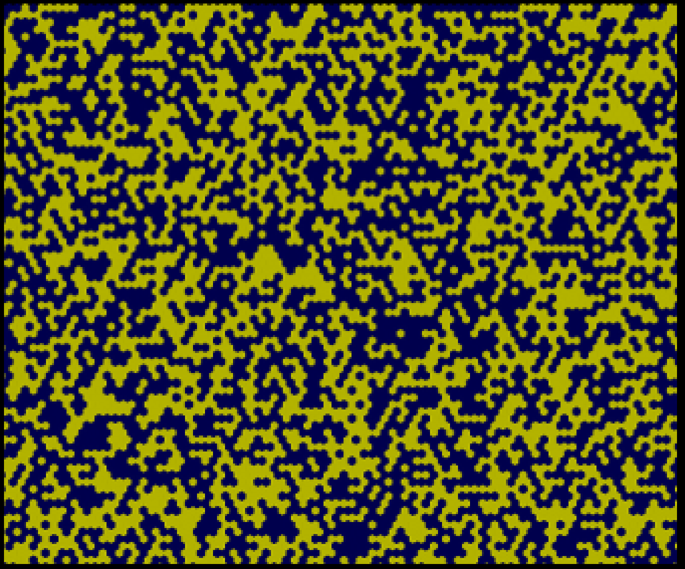

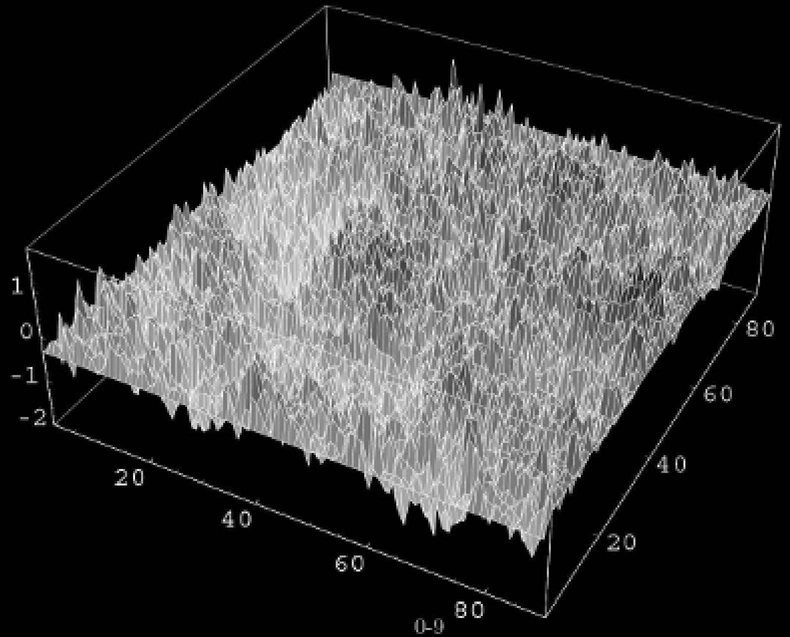

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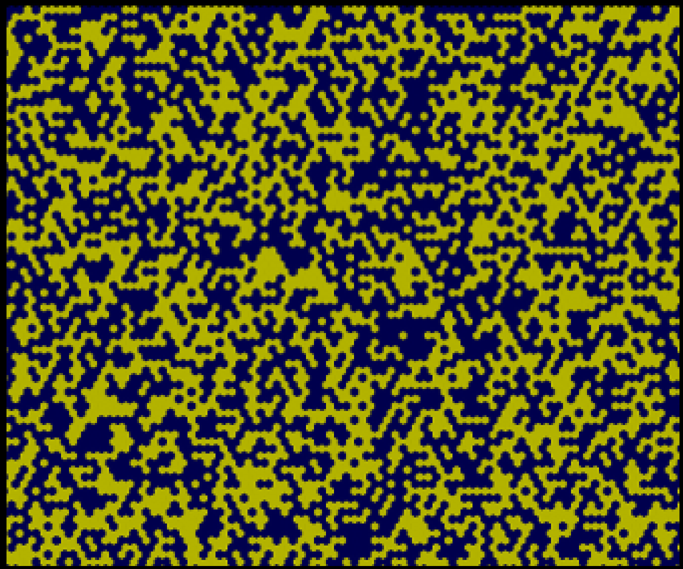

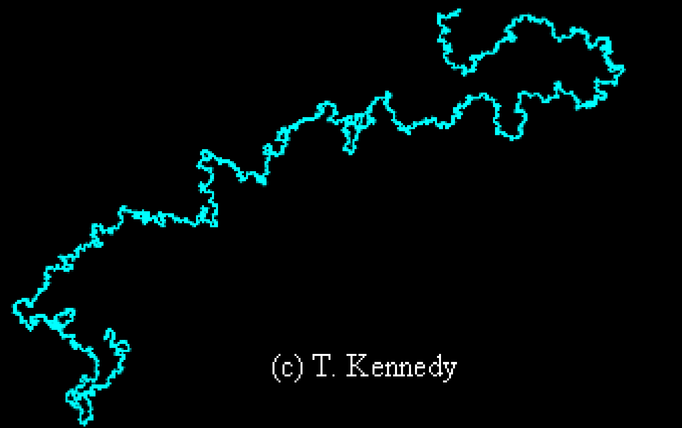

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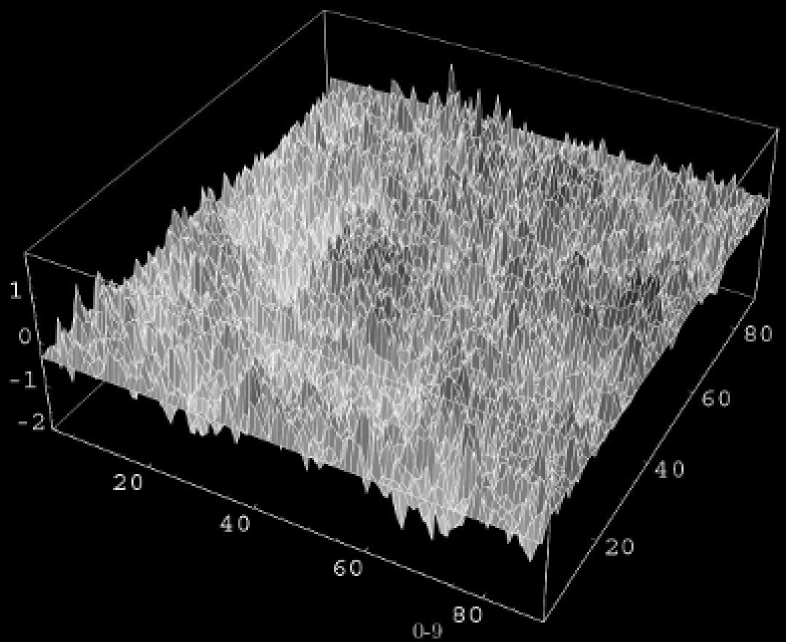

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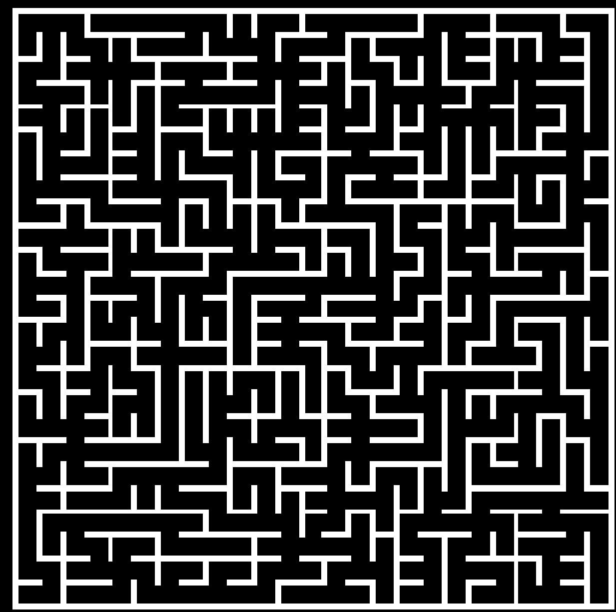

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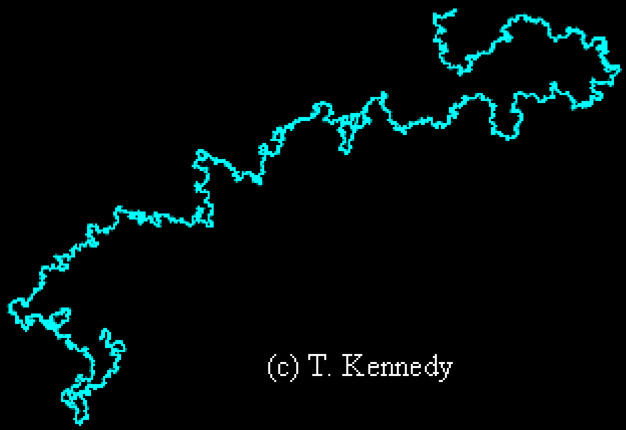

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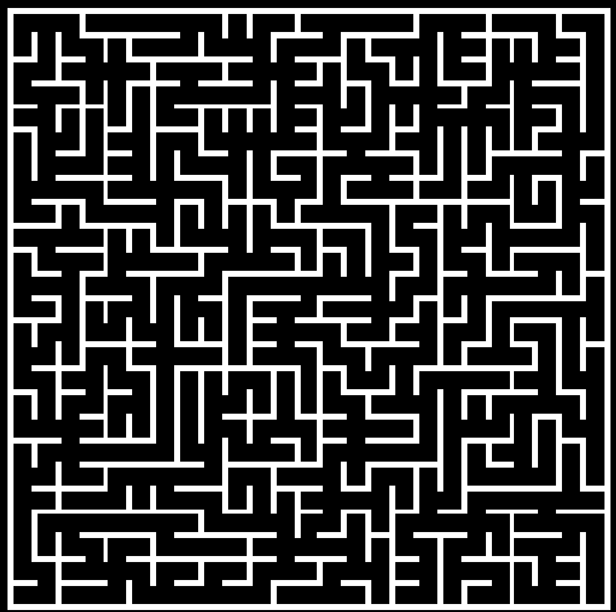

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Classification of conformally invariant systems: they form a one-parameter family:

Conformal Field Theory: physical point of view describing the random fields associated with the model	Schramm-Löwner Evolution: rigorous approach interested in the random curves arising in the system
Parametrized by central charge c	Classified by variance parameter $\kappa > 0$

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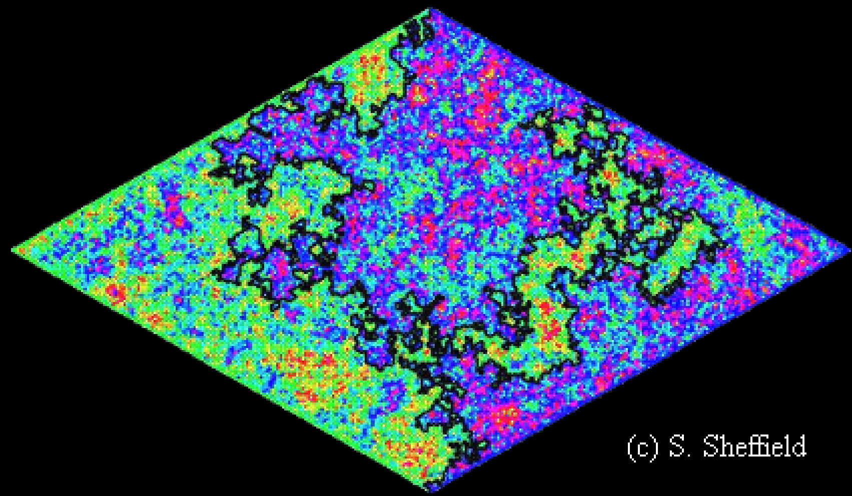

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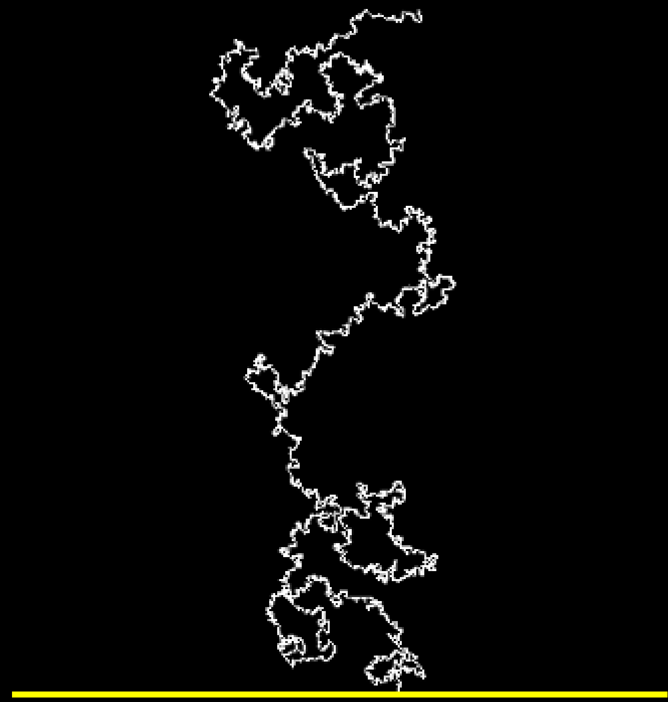
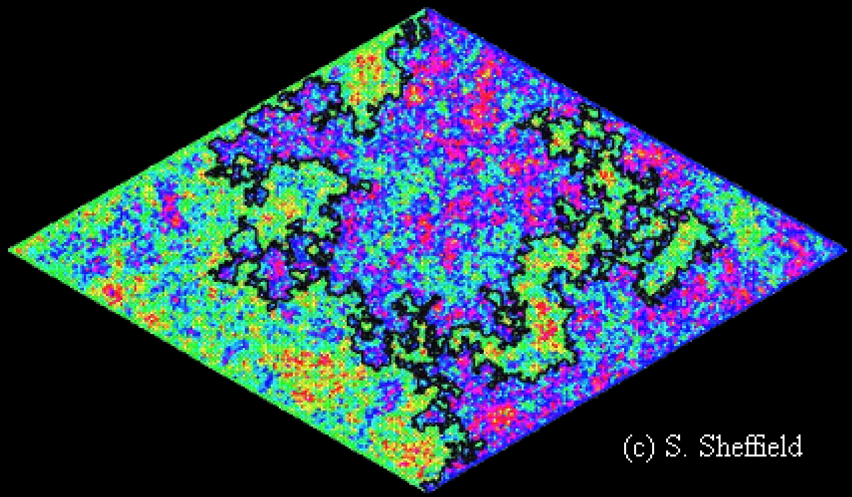


(c) V. Beffara



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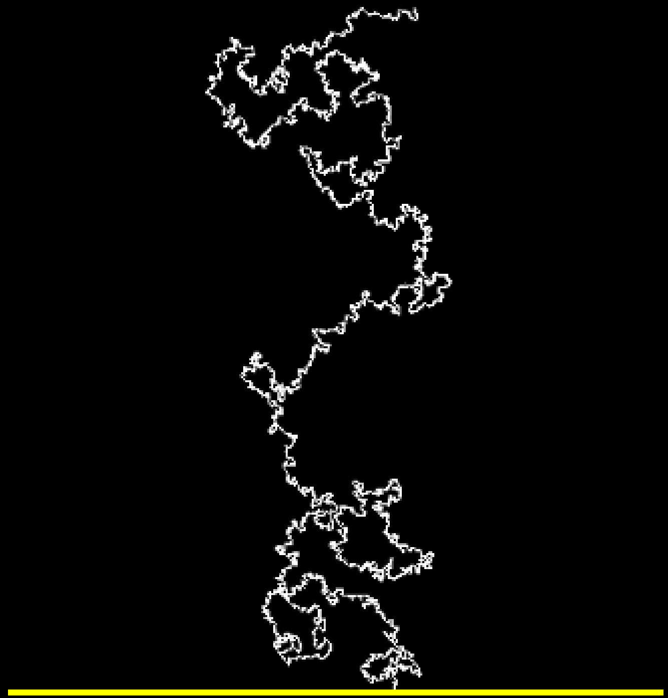

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The Ising model was introduced in 1920 by Lenz as a model for ferromagnetism

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The Ising model on a graph G is a random assignment of up/down (or ± 1) spins to the vertices $V(G)$ that interact via the edges $E(G)$

The probability of a spin configuration $\sigma \in \{\pm 1\}^{V(G)}$ is given by

$$\mathbb{P}[\sigma] = \frac{1}{Z_\beta} \exp\left(-\beta H(\sigma)\right)$$

where

- $\beta \propto \frac{1}{T}$ is the inverse temperature
- $H(\sigma) = -\sum_{\langle i,j \rangle \in E(G)} \sigma_i \sigma_j$ is the energy
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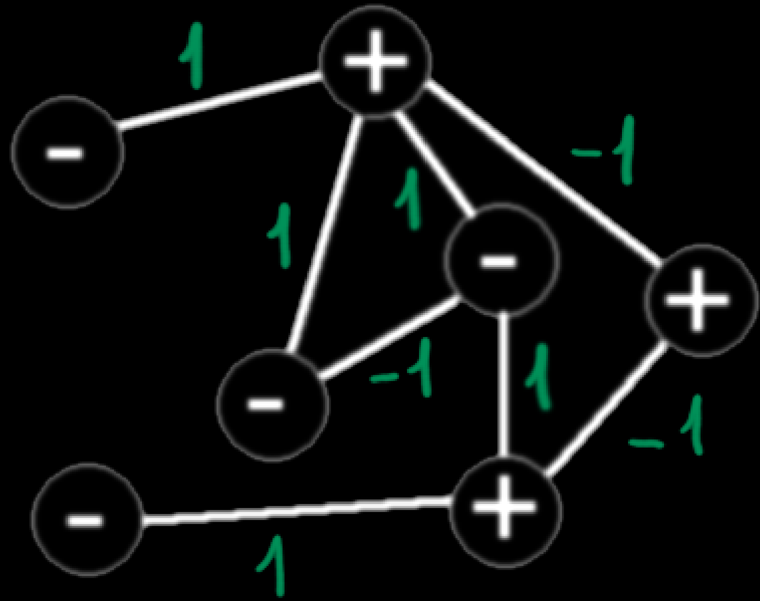
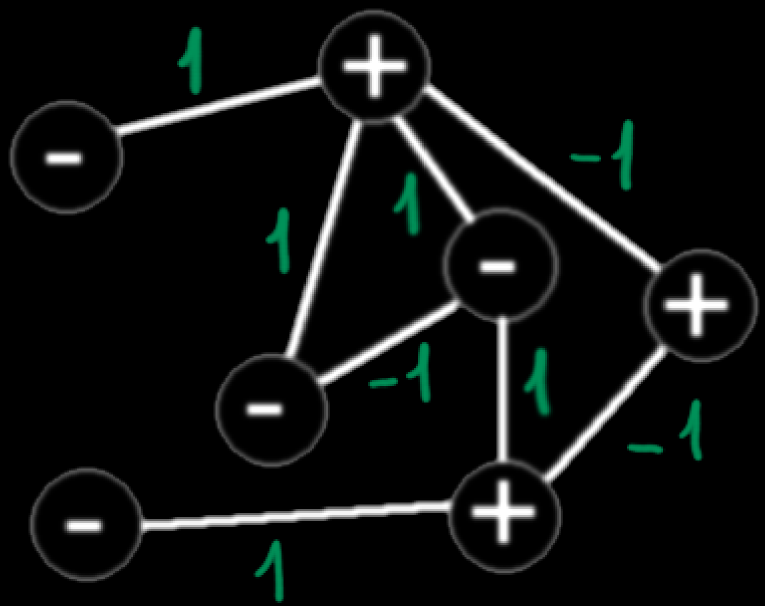


$H = 5 - 3 = 2$



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- Study of various phase transitions
- Image processing
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in 2D or 3D

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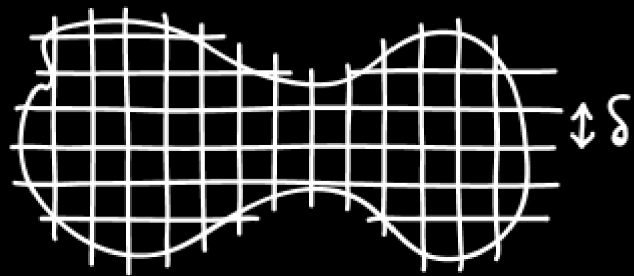
The Ising model on the square grid

Let Ω be a smooth domain

For each $\delta>0$, consider the Ising model on the largest connected component of $\Omega_\delta := \Omega \cap \delta\mathbb{Z}^2$, where $\delta\mathbb{Z}^2$ is the square lattice of mesh size δ



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- plus: condition the boundary spins to be +1
- minus: condition them to be −1
- free: no conditioning
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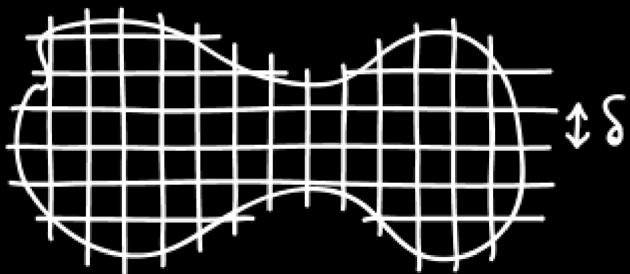

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What happens as $\delta \to 0$?

It depends on β: there is a phase transition at $\beta_c := \frac{1}{2}\ln(\sqrt{2}+1)$:

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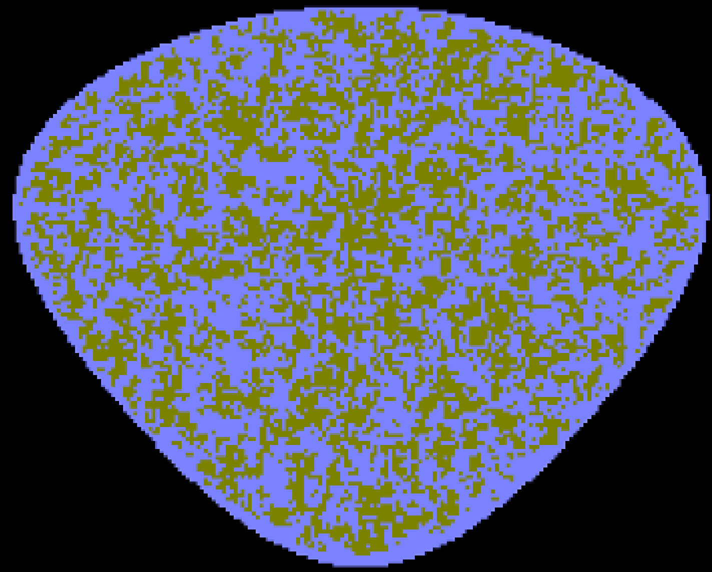

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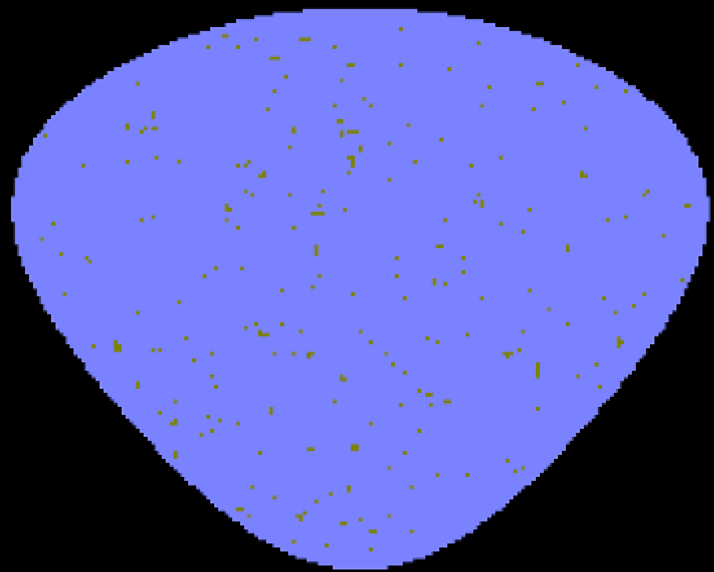

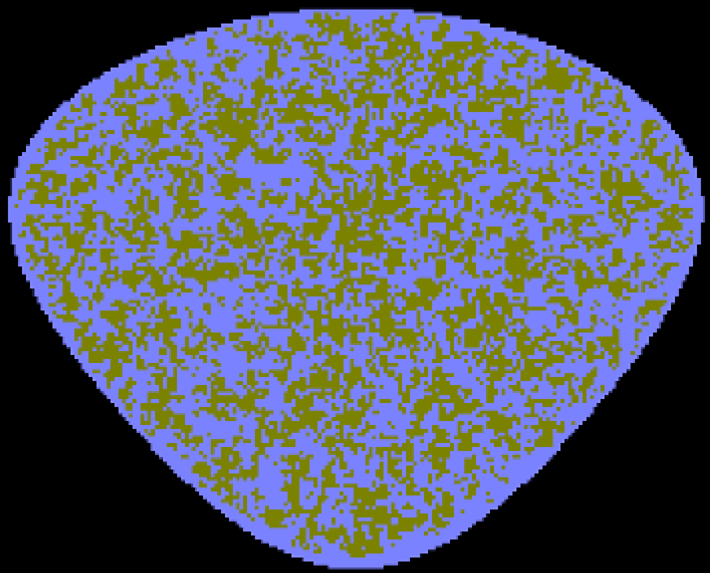

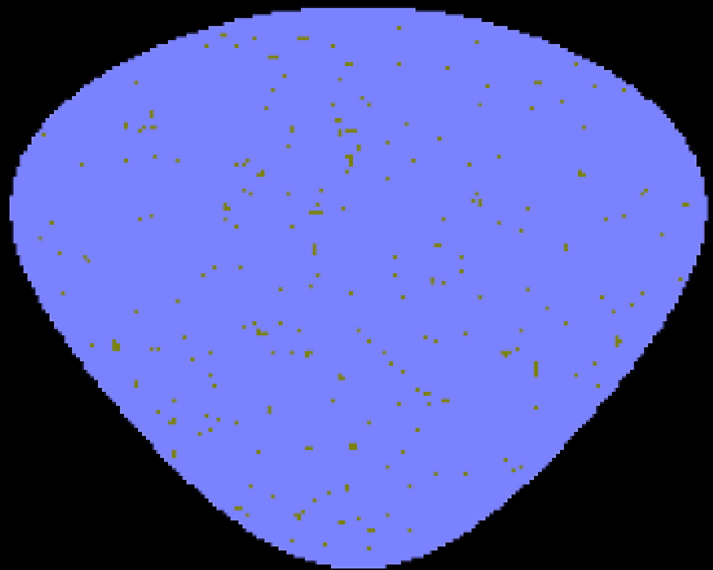

What happens at the critical point?

- Universality: the scaling limit at criticality should be independent of the lattice and of other details

- Conformal invariance: SLE approach
  Describe the interfaces between + and − spin clusters

SLE_3
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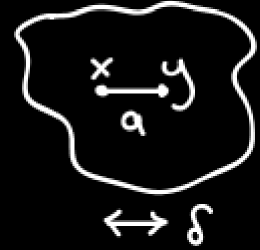
Main theorems about the energy density

Let Ω be a smooth simply connected bounded domain

For each $\delta > 0$, identify the points of Ω with the closest edges of Ω_δ



Lemma (local effects) As $\delta \to 0$, the average contribution of an edge to H tends to $-\frac{\sqrt{2}}{2}$

For an edge $a = \langle x, y\rangle$, set $\epsilon_\delta(a) := \frac{\sqrt{2}}{2} - \sigma_x \sigma_y$

Denote by ℓ_Ω the hyperbolic metric element of Ω, defined by $\ell_{D(0,1)}(a) = \frac{1}{1-|a|^2}$ and by $\ell_{\varphi(\Omega)}(\varphi(z)) = |\varphi'(z)| \cdot \ell_\Omega(z)$ $\forall \varphi$ conformal

Theorem 1: one-point (H.-Smirnov)

With + boundary condition, uniformly on the compact subsets,

$$\frac{1}{\delta} \mathbb{E}[\epsilon_\delta(a)] \xrightarrow[\delta \to 0]{} -\frac{1}{\pi} \ell_\Omega(a).$$

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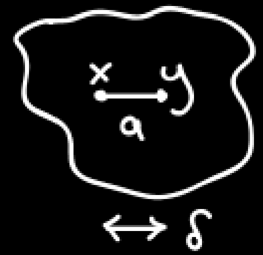

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For any simply connected domain D and any $a_1, \dots, a_n \in D$, there exists a correlation function $\langle \epsilon(a_1) \cdot \ldots \cdot \epsilon(a_n) \rangle_D$ with:

$$\langle \epsilon(\varphi(a_1)) \cdot \ldots \cdot \epsilon(\varphi(a_n)) \rangle_{\varphi(D)}$$

$$= \prod_{i=1}^{n} |\varphi'(a_i)| \, \langle \epsilon(a_1) \cdot \ldots \cdot \epsilon(a_n) \rangle_D$$

for any conformal map φ, and such that the following holds:

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Explicit formulae: Pfaffians

The Pfaffian $\mathrm{Pf}(A)$ of a $2n \times 2n$ antisymmetric matrix A is defined as

$$\frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)\sigma(2)} \cdot \ldots \cdot a_{\sigma(2n-1)\sigma(2n)}$$

Also: $(\mathrm{Pf}(A))^2 = \det(A)$

We can show an improved version of a CFT prediction (Burkhardt & Guim, 1993): on the half-plane, we have

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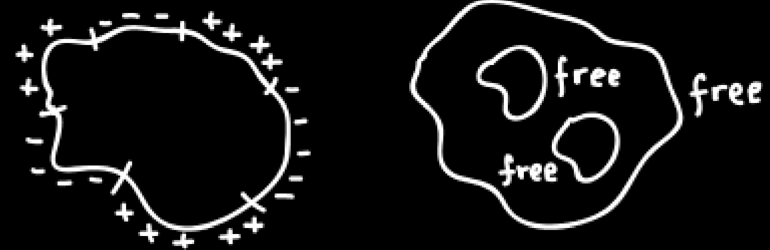
Theorem 2 admits generalizations with free and mixed boundary conditions and with multiply connected domains



Informally, it means that the energy field (whatever it is) behaves in law like a covariant 1-tensor:

boundary conditions

$$\epsilon^{\varphi(b)}_{\varphi(\Omega)}(\varphi(z)) = |\varphi'(z)| \cdot \epsilon^{b}_{\Omega}(z)$$

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We will first give the proof for the one-point function and then sketch the one for the general case

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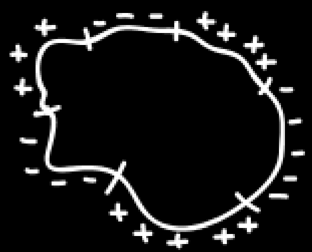

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Proof of Theorem 1

Overall strategy:

(A) Graphical representation of the energy as a statistics over a family of weighted contours

(B) Discrete holomorphic deformation of the contour statistics: introduction of a fermionic two-point observable

(C) Convergence of the observable in the scaling limit

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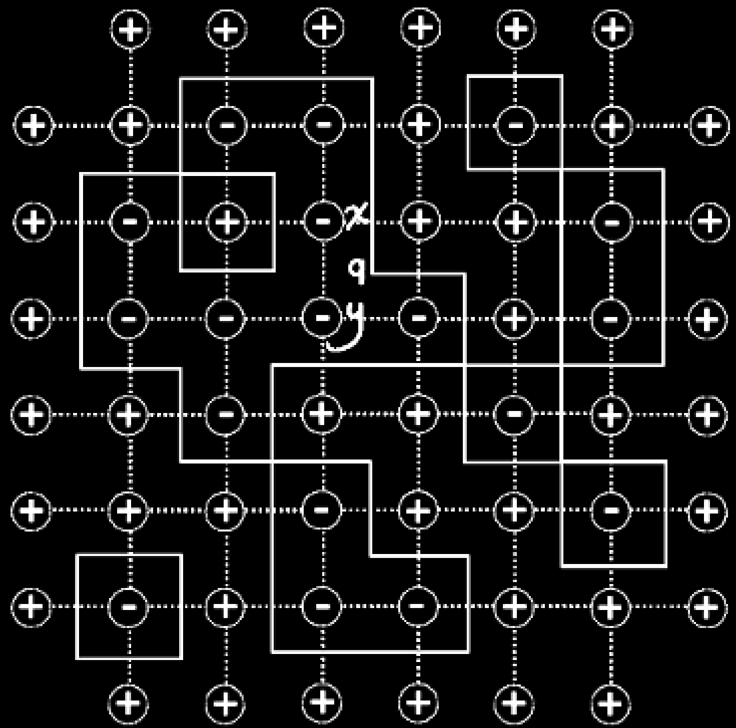

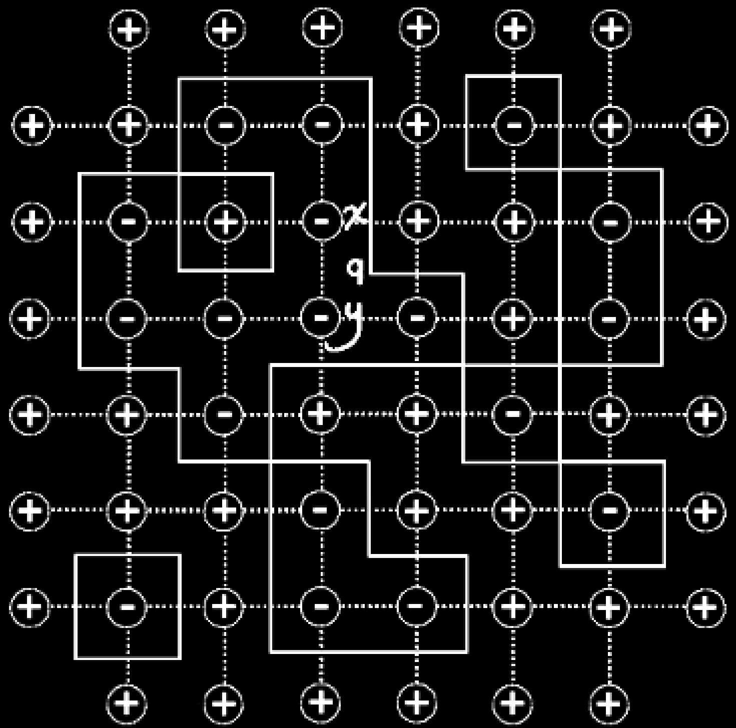

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$\mathcal{C}_\delta := \{\gamma \subseteq E(\Omega_\delta^*) :$ set of closed loops$\}$

$$\mathbb{P}[\gamma] = \alpha^{\#\text{edges}(\gamma)} / Z_\delta$$

$\alpha = \alpha_c = \sqrt{2} - 1$

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(B) Contour deformation

Let z be the midpoint of an edge

Set $\mathcal{C}_\delta^a(z) :=$

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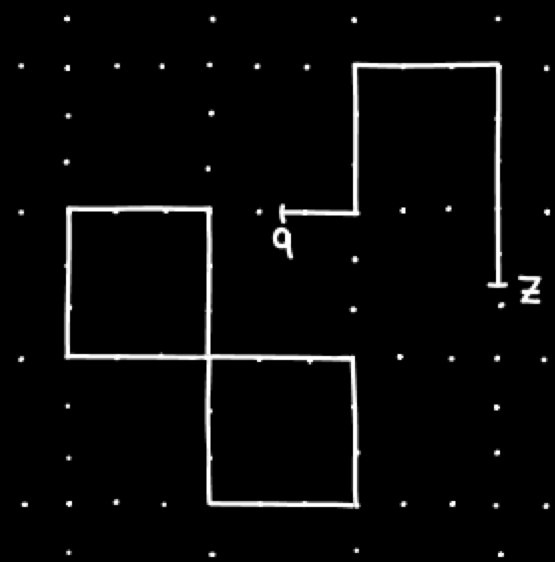

$W(\gamma) = -\frac{\pi}{2}$

#edges: 12

(B) Contour deformation

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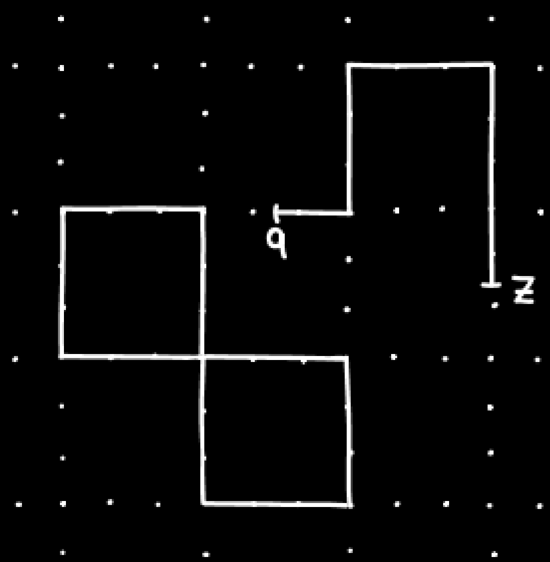

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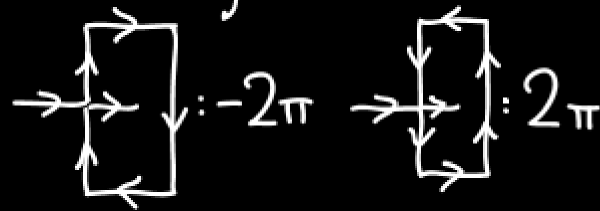
(B) Holomorphic deformation: fermionic observable

W: winding number (i.e total rotation) of the path from a to z

well-defined mod 4π:



$$h_\delta^a(z) := \frac{\sum_{\gamma \in \mathcal{C}_\delta^a(z)} \alpha^{\#\text{edges}(\gamma)} e^{-i\frac{W(\gamma)}{2}}}{Z_\delta}$$

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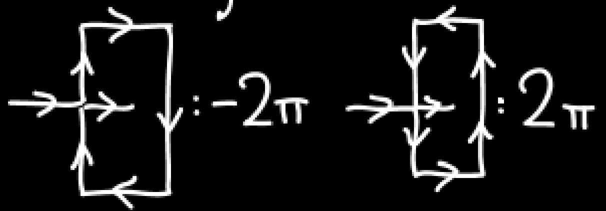

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(B) Properties of $z \mapsto h_\delta^a(z)$

(1) Discrete holomorphic on $\Omega_\delta \setminus \{a\}$

Discrete Cauchy-Riemann equations

$$(\bar{\partial}_\delta h_\delta^a = 0)$$

(2) Discrete singularity at a

$$(\bar{\partial}_\delta h_\delta^a)(a) = \frac{1}{2}$$

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$$h_a^\delta \big|_{\partial\Omega_\delta} \parallel \frac{1}{\sqrt{n}}$$

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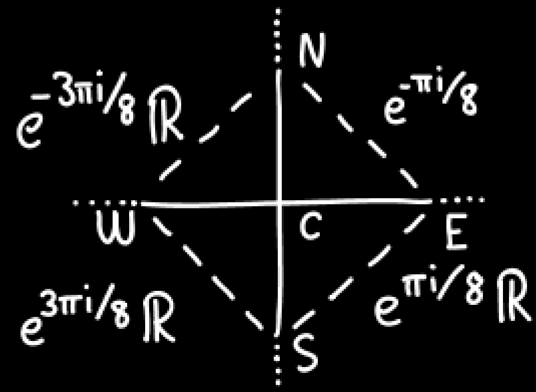
(B1) Discrete s-holomorphicity



$$\begin{cases} \mathrm{Proj}\left(h_\delta^a(E) - h_a^\delta(N); e^{-\pi i/8}\mathbb{R}\right) = 0 \\ \mathrm{Proj}\left(h_\delta^a(N) - h_\delta^a(W); e^{-3\pi i/8}\mathbb{R}\right) = 0 \\ \mathrm{Proj}\left(h_\delta^a(W) - h_\delta^a(S); e^{3\pi i/8}\mathbb{R}\right) = 0 \\ \mathrm{Proj}\left(h_\delta^a(S) - h_\delta^a(E); e^{\pi i/8}\mathbb{R}\right) = 0 \end{cases}$$

$$\rightarrow \underbrace{h_\delta^a(E) - h_\delta^a(W) + i\left(h_\delta^a(N) - h_\delta^a(S)\right)}_{2\left(\bar\partial_\delta h_\delta^a\right)(C)} = 0$$

Stronger than discrete Cauchy-Riemann

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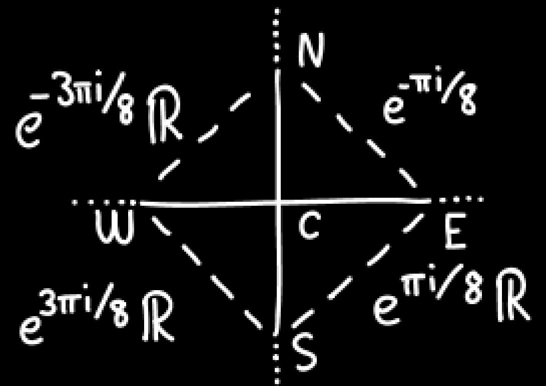

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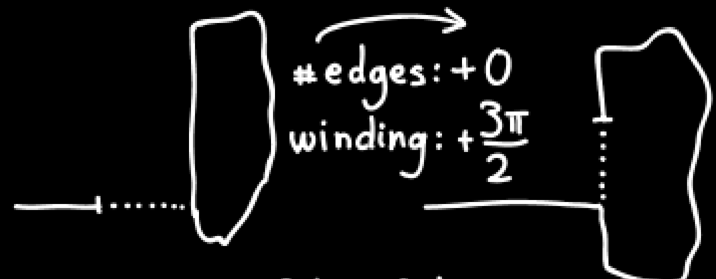
Stronger than discrete Cauchy-Riemann

(B1) Proof of s-holomorphicity of h_δ^a on $\Omega_\delta \setminus \{a\}$

Idea: exhibit bijections between the set of contours contributing to $h_\delta^a(N)$ and $h_\delta^a(W)$ preserving the projections of the weights $\alpha^{\#\text{edges}} e^{-\frac{iw}{2}}$ on $e^{-3\pi i/8}\mathbb{R}$

$X\in\mathbb{R}$: $\mathrm{Proj}\left(X-\alpha e^{-\frac{\pi i}{4}}X; e^{-\frac{3\pi i}{8}}\mathbb{R}\right)=0$



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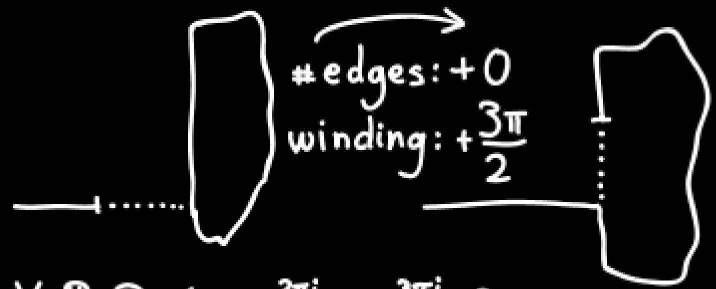

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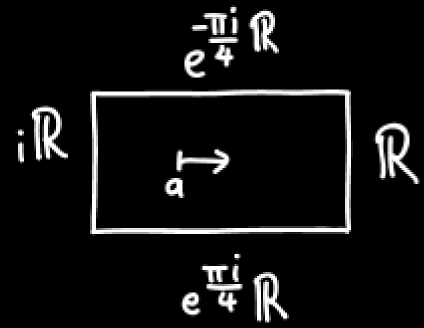
(B2) Discrete singularity at a

Idea: the bijections do not work near a. They give a discrete holomorphicity defect:

$$(\partial_\delta h_\delta^a)\left(a - \frac{\delta}{2}\right) = \frac{1}{2}$$

(B3) Boundary conditions

Idea: the winding of a path in a configuration in $\mathcal{C}_\delta^a(z)$ is fixed mod 2π when $z \in \partial\Omega_\delta$. Hence, the phase is fixed:



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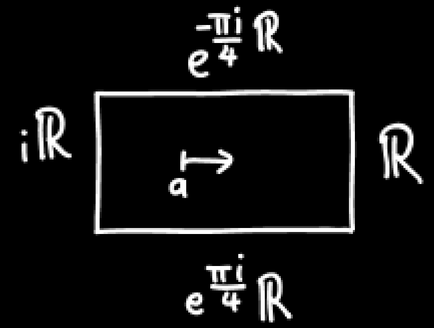

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Derivation of the energy density

Idea: introduce a $\bar\partial_\delta$-Green's function $\tilde h_\delta^a$ (defined on the full plane) that kills the singularity: we have

$$\left.\begin{aligned}&\bar\partial_\delta\tilde h_\delta^a\left(a-\tfrac{\delta}{2}\right)=\frac{1}{2}\\&\tilde h_\delta^a(z)\xrightarrow[z\to\infty]{}0\end{aligned}\right\}\text{ determine }\tilde h_\delta^a\text{ uniquely}$$

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(B1) and (B2) give that $v_\delta^a:=h_\delta^a-\tilde h_\delta^a$ is s-holomorphic on Ω_δ and that

$$2\cdot v_\delta^a(a)=2\,\mathbb{P}[\sigma_x=\sigma_y]-\frac{\sqrt{2}+2}{2}=\epsilon_\delta(a)$$

Hence $\tilde h_\delta^a$ also eliminates the local effects

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(C) Convergence results

$$\frac{1}{\delta} h_a^\delta(z) \to \frac{\psi_a(z)+1}{2\pi\psi_a(z)} \sqrt{\psi_a'(a)\,\psi_a'(z)}$$

with $\psi_a : \Omega \to D(0,1)$, $a \mapsto 0$, $\psi_a'(a) > 0$

$$\frac{1}{\delta} \tilde{h}_\delta^a(z) \xrightarrow{\delta\to 0} \frac{1}{2\pi(z-a)} \quad \text{(Kenyon)}$$

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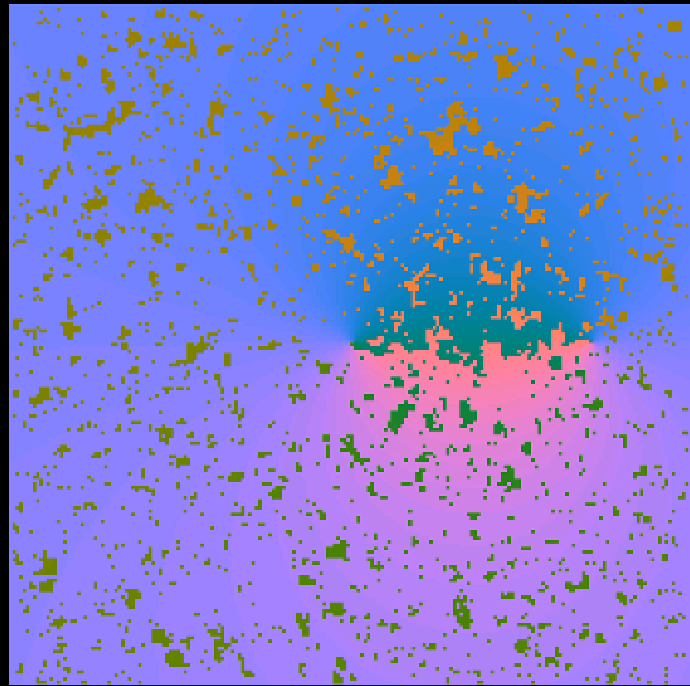
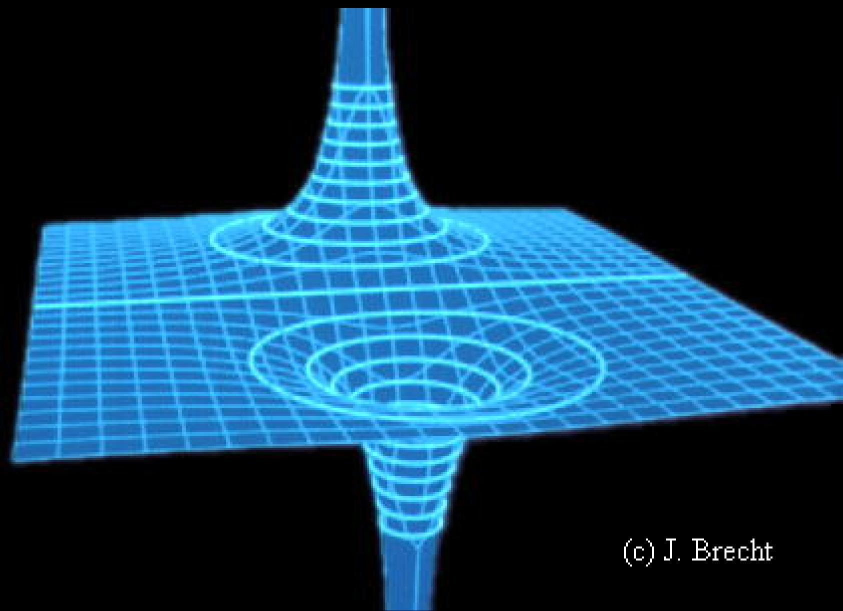
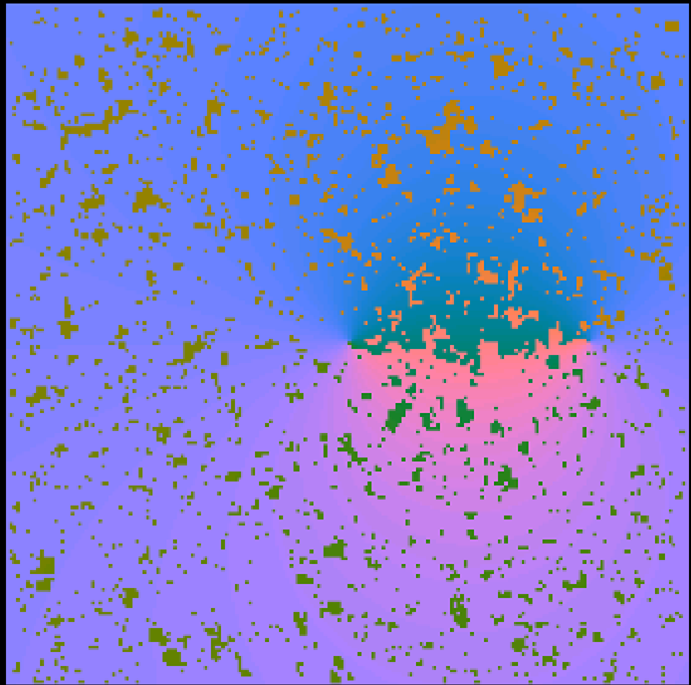
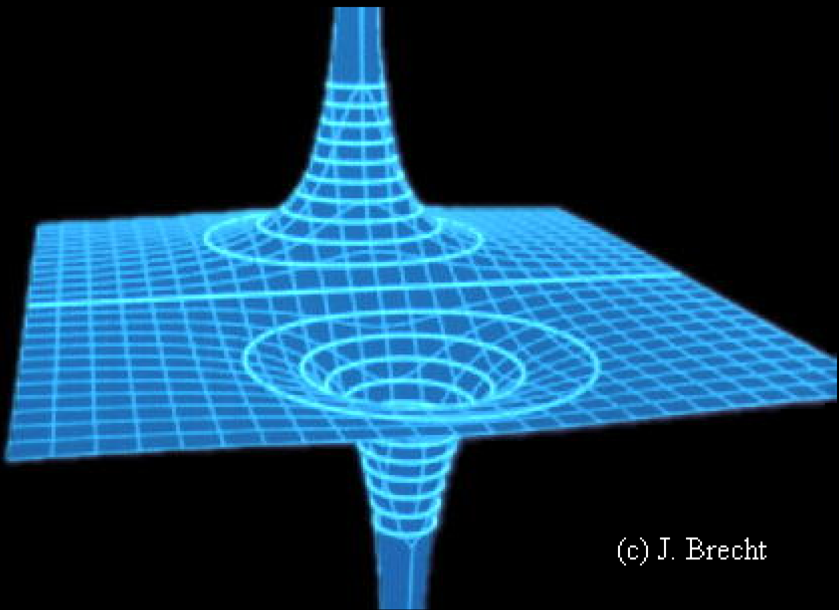




(c) J. Brecht





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(1) Precompactness

Difficulty: control a function only knowing its argument on $\partial\Omega$

Idea: develop tools to control v_δ^a (we know that $\tilde{h}_\delta^a$ converges)

(2) Identification of the limit

Idea: use Smirnov's integral trick: $h_\delta^a \parallel \frac{1}{\sqrt{n}}$ can be turned into a Dirichlet boundary condition by defining a discrete version of $\mathrm{Re}\left(\int (h_\delta^a)^2\right)$, constant on $\partial\Omega$

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Proof of precompactness

Idea: formulate v_δ^a as the solution to a discrete Riemann-Hilbert boundary value problem

$(RH_\delta, \tilde{h}_\delta^a)$:

- v_δ^a s-holomorphic
- $v_\delta^a + \tilde{h}_\delta^a \parallel \frac{1}{\sqrt{n}}$ on $\partial\Omega_\delta$

Obtain an inequality:

$$\|v_\delta^a\|_{1,\partial\Omega_\delta} \leq 2 \|\tilde{h}_\delta^a\|_{1,\partial\Omega_\delta}$$

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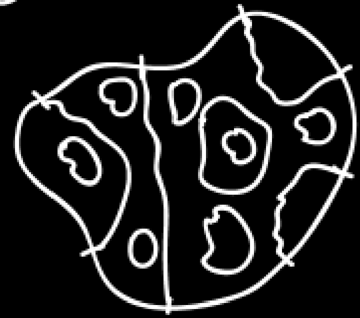
Sketch of proof of Theorem 2

Let $a_1, \dots, a_n \in \Omega$ be the points where we observe the field and let $b_1, \dots, b_{2k}$ be boundary changing operators

Strategy:

(A) Represent the correlation $\mathbb{E}[\epsilon_\delta(a_1) \cdot \ldots \cdot \epsilon_\delta(a_n)]$ in terms of probabilities of presence/absence of the edges $a_1, \dots, a_n$ in the contours



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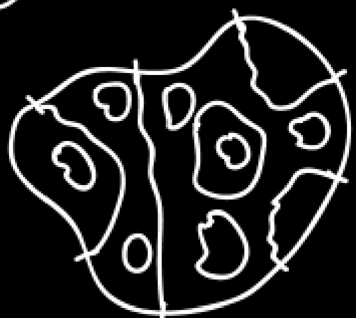

(B) Introduce a 2m-point fermionic observable, with $2m = 2n + 2k$ which is an antisymmetric function of 2m decorated midpoints of edges $z_1^{o_1}, \dots, z_{2m}^{o_{2m}}$, where $o_1, \dots, o_{2m}$ are edge orientations with a specified square root

$$f(z_1^{o_1}, \dots, z_{2m}^{o_{2m}}) = \sum_{\omega \in \mathcal{C}(z_1^{o_1}, \dots, z_{2m}^{o_{2m}})} \underbrace{\alpha}_{\sqrt{2}-1}^{\#\text{edges}(\omega)} \phi(\omega)$$

$\omega \in \mathcal{C}(\dots)$: loops + m paths linking $z_1, \dots, z_{2m}$

$\phi(\omega)$: phase involving windings, pairing signature

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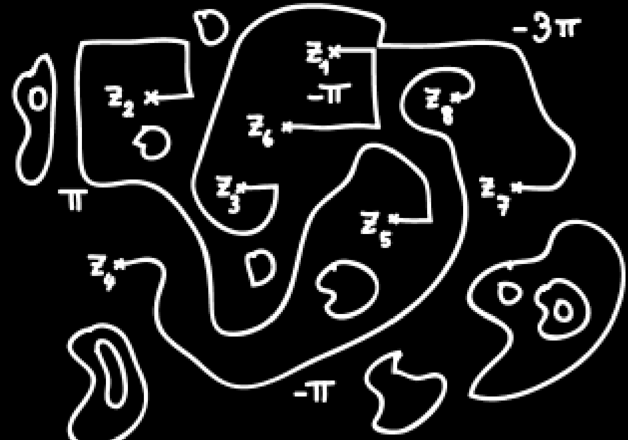
phase involving windings, pairing signature

(B) Definition of $\phi(\omega)$:

- If all edges are horizontal and oriented from left to right,

$$\phi(\omega) = e^{-\frac{i}{2} \sum_{i<j : z_i \leadsto z_j} W(\omega : z_i \leadsto z_j)} (-1)^{n_c}$$

n_c: number of crossings of $\{\{i,j\} : z_i \leftrightsquigarrow z_j\}$



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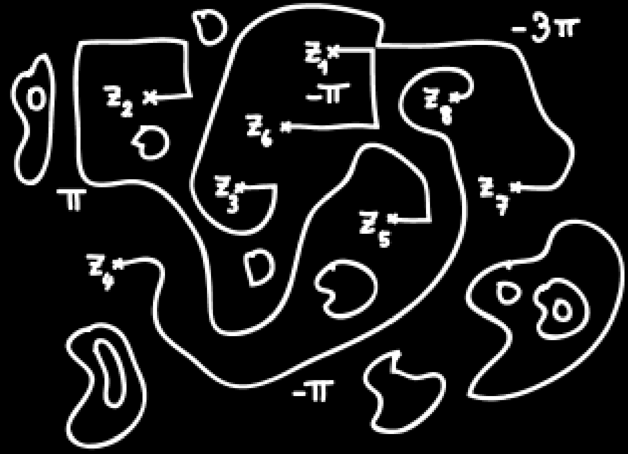

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(C) Use discrete complex analysis techniques to obtain

$$f(z_1^{o_1}, \dots, z_{2m}^{o_{2m}}) = \mathrm{Pfaff}\left[\left(f(z_i^{o_i};\, z_j^{o_j})\right)_{i,j}\right],$$

where $f(z_i^{o_i}; z_j^{o_j})$ is essentially the two-point observable of the proof of Theorem 1

Convergence follows readily

(D) Fuse pairwise $2n$ of the points $z_1, \dots, z_{2n}$ at the locations of the points $a_1, \dots, a_n$ with a suitable renormalization and move the $2k$ remaining z_i's to the points $b_1, \dots, b_{2k}$