1st PRIMA Congress

Conformal invariance and universality in the 2D Ising model

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July 6, 2009

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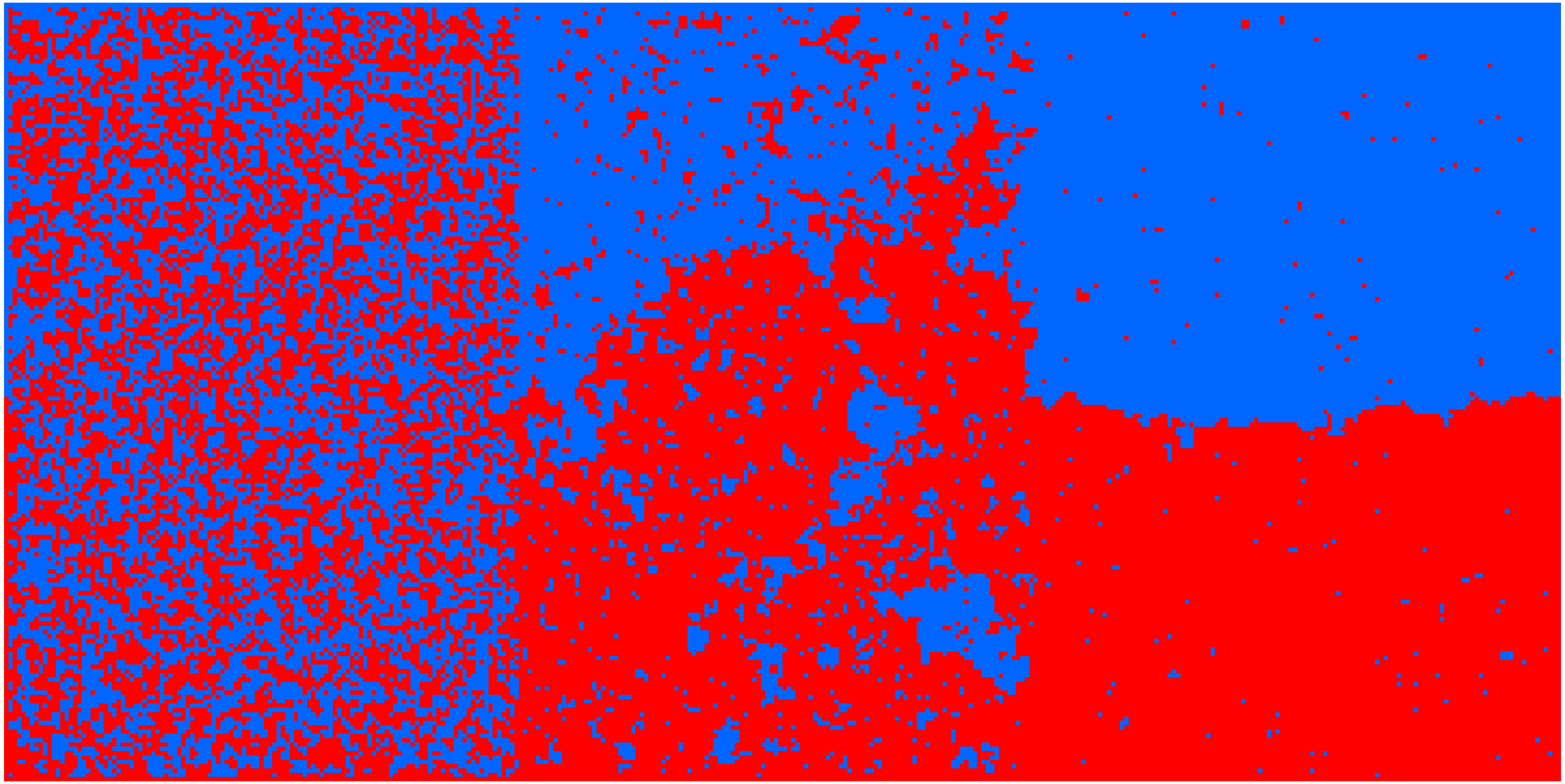
Archetypical example of a phase transition: 2D Ising model

Configurations of $+$ and $-$ spins (red and blue squares)

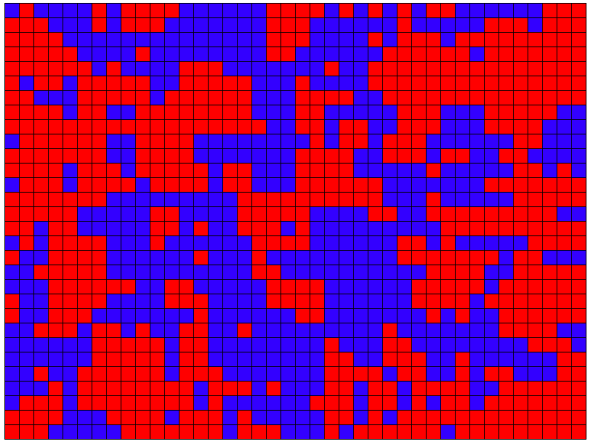
$\mathbf{Prob}(\text{config}) \asymp e^{-\ \#\ \{+-\ \text{neighbor pairs}\}\ /\ \mathrm{T}} = x^{\text{length of loops}}$

[Kramers-Wannier, 1941]: on the square lattice $x_c = 1/(1+\sqrt{2})$

$x \approx 1,\ T \approx \infty$ $\qquad$ $x = x_c$ $\qquad$ $x \approx 0,\ T \approx 0$



2D Ising model:



• Physically "realistic" model of order–disorder **phase transitions**

• **" Exactly solvable"** – many parameters computed exactly, but usually non-rigorously [Onsager, Kaufman, Yang, Kac, Ward, Potts, Montroll, Hurst, Green, Kasteleyn, Vdovichenko, Fisher, Baxter, . . .]

• Connections to **Conformal Field Theory** – allow to compute more things in a more general setting [den Nijs, Nienhuis, Belavin, Polyakov, Zamolodchikov, Cardy, Duplantier, . . .]

• Much progress in physics, but for a long time poor mathematical understanding.

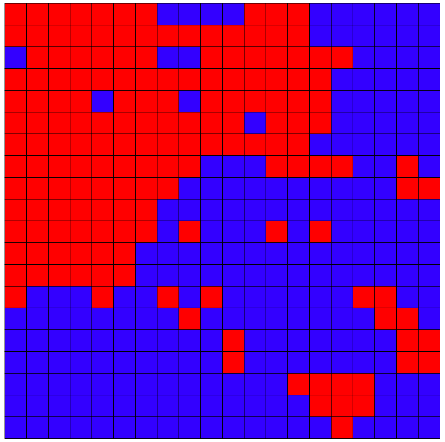
Structure of CFT arguments: at critical temperature
(A) the model has a continuum scaling limit (as mesh$\to 0$),
the limit is universal (independent of the lattice)
and conformally invariant (preserved by conformal maps)
(B) conformal invariance allows to describe the limit.

Recently mathematical progress with new, rigorous approaches. Oded Schramm described possible conformally invariant scaling limits of cluster interfaces: one-parameter family of SLE(κ) curves. Subsequently Lawler-Schramm-Werner, Rohde-Schramm, Beffara and others used SLE to prove or explain many predictions.

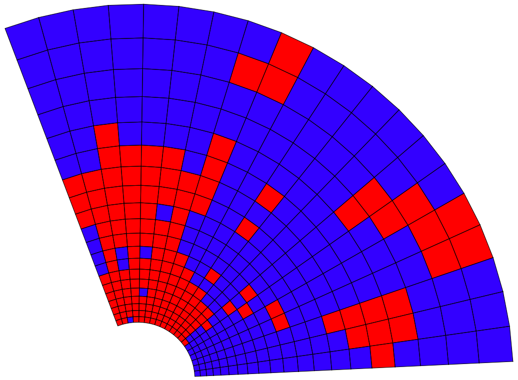
We will discuss the mathematical approaches to (A) and (B), using the Ising model as an example.

"Everybody knows that the **2D Ising model is a free fermion**"

2D Ising model at criticality is considered a classical example of conformal invariance in statistical mechanics, which is used in deriving many of its properties. However,



$\downarrow \phi$



- No mathematical proof has ever been given.

- Most of the physics arguments concern nice domains only or do not take boundary conditions into account, and thus only give evidence of the (weaker!) Möbius invariance of the scaling limit.

- Only conformal invariance of correlations is usually discussed, we discuss underlying geometric objects and distributions as well.

- We construct new objects of physical interest.

Classical example of conformal invariance:
Random Walk $\longrightarrow$ Brownian Motion

As lattice mesh goes to zero, RW $\rightarrow$ BM: probability measure on broken lines converges weakly to Wiener probability measure on continuous curves.
BM is conformally invariant [P. Lévy] and universal.

Conjecturally: in most 2-dim models at critical temperatures, universal conformally invariant SLE curves arise as scaling limits of the interfaces (cluster boundaries).

Modern example: critical percolation

to color every hexagon we toss a coin:

tails $\Rightarrow$ blue, heads $\Rightarrow$ yellow

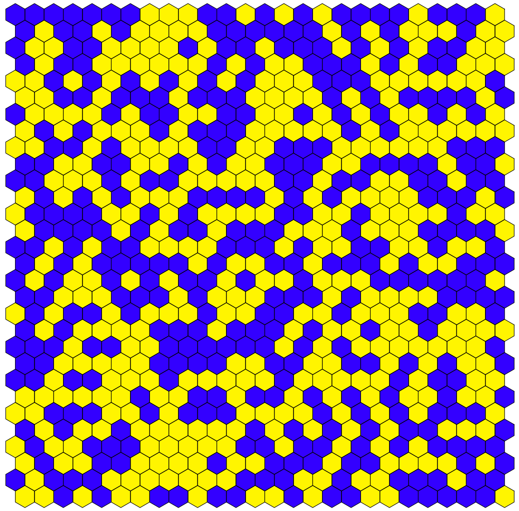
Blue hexagons are "holes" in a yellow rock.

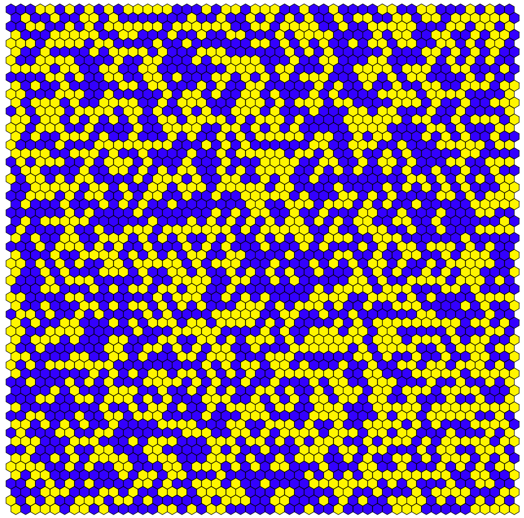
Can the water sip through? Hard to see!

The reason: clusters (connected blue holes) are complicated fractals of dimension $91/48$

(a cluster of diam D on average has $\approx D^{91/48}$ hexagons),

blue/yellow interfaces of dim $7/4$





Cardy's prediction: in the scaling limit

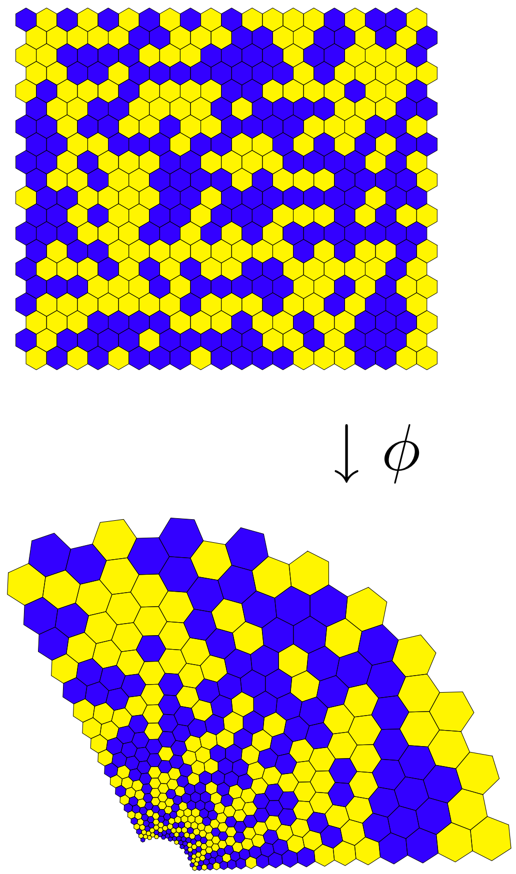
$$\mathbb{P}\,(\text{crossing}) = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}\, m^{1/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; m\right)$$

Proved on hexagonal lattice [Smirnov 2001],

cluster boundaries converge to Schramm's SLE(6) curves

Conformally invariant scaling limits of critical interfaces:

- [2001, Smirnov] critical percolation
  on hexagonal lattice
- [2003, Lawler-Schramm-Werner]
  Uniform Spanning Tree / LERW
  ([2000, Kenyon] – many observables)
- *[2003/6, Schramm-Sheffield]*
  Harmonic Explorer / Discrete GFF
- [2006, Smirnov] FK Ising model
- [2008, Smirnov] Ising model

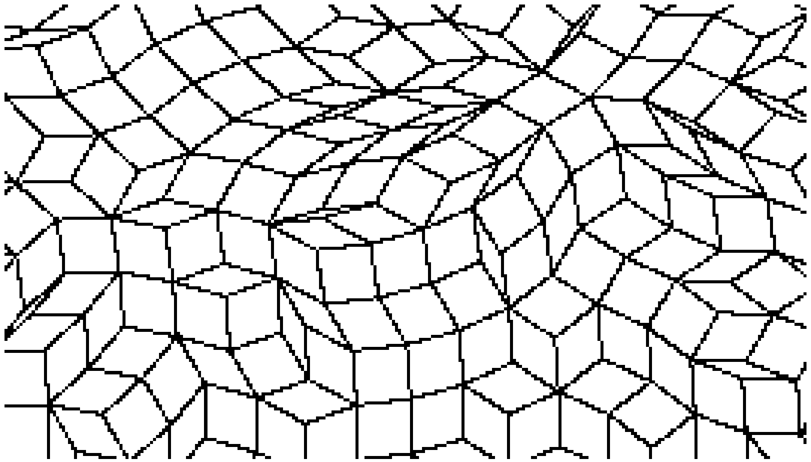


Conjectured for: self-avoiding polymers, percolation on other lattices, Potts and random cluster models, . . .

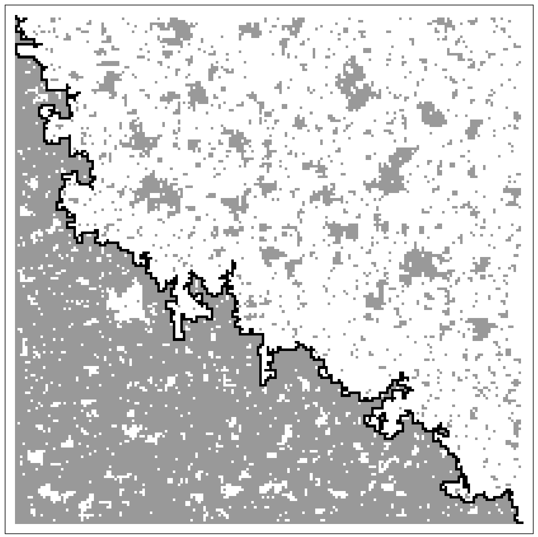
Theorem [Chelkak–Smirnov]. Ising model on *isoradial graphs* at T_c has a conformally invariant scaling limit as mesh $\epsilon \to 0$. Interfaces in spin and random cluster representations converge to Schramm's curves SLE(3) and SLE$(16/3)$.

- Square lattice case is easier **[Smirnov]**.
- At $T < T_C$ interface $\to$ an interval **[Pfister-Velenik]**.
- **Conj** At $T > T_C$ interface $\to$ SLE(6), same as percolation.

  Known only for triangular lattice and $T = \infty$ **[Smirnov]**.



An isoradial graph with its dual give a tiling by rhombi



Ising $\to$ SLE(3), $\mathrm{Dim} = 11/8$

Using Ising model as an example we will discuss how to

(A) find an observable with a conformally invariant scaling limit
(Tools: discrete complex analysis, conformal invariants)

(B) using one observable, construct (conformally invariant) scaling limits of the interfaces (Tools: Schramm–Loewner Evolution)

Related topics:

- universality — discrete complex analysis is more interesting **[Chelkak - S]**
- deriving (some) exponents directly from observables **[Hongler - S]**
- interfaces on Riemann surfaces, general boundary conditions — interesting conformal invariants, spin structures
- full scaling limit — SLE loop soups **[Kemppainen - S]**
- perturbation $p \approx p_c$ — no conformal invariance **[Makarov - S]**

(A) How to find a conformally invariant observable?

We need a *discrete conformal invariant*

Discrete harmonic or **dicrete analytic** (=preholomorphic) function solving prescribed boundary value problem

- more accessible in the discrete case than other invariants
- most other invariants can be reduced to it

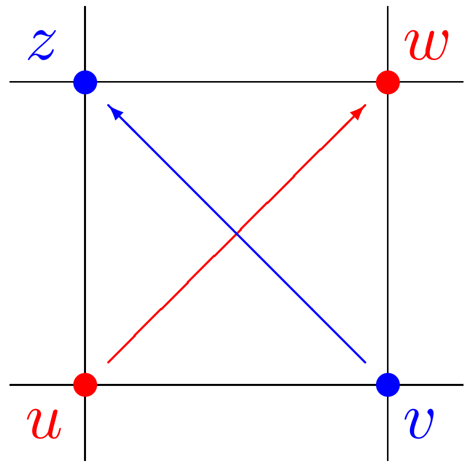
Boundary value problems

- **Dirichlet** or **Neumann**: clear discretization, scaling limit.
- **Riemann-Hilbert**: wider choice! discretization? scaling limit?

Leads to conformally covariant functions, "**spinors**:" $F(z)\,(dz)^{\alpha}(d\bar{z})^{\beta}$

Discrete analytic (preholomorphic): some discrete version of the Cauchy-Riemann equations $\partial_{i\alpha} F = i\partial_\alpha F$. On square lattice, e.g.

$$\boldsymbol{F(z) - F(v) = i\,(F(w) - F(u))}$$



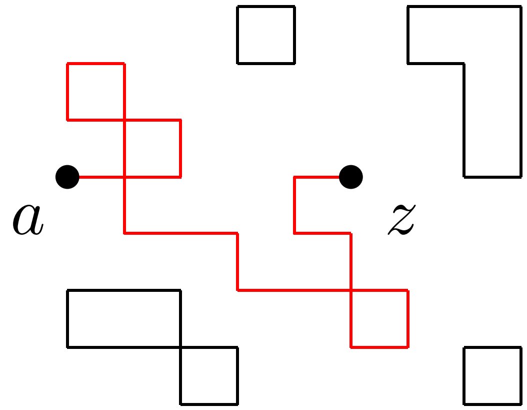
Discrete complex analysis starts like the usual one.
Easy to prove: if $F, G \in \mathrm{Hol}$, then

- $F \pm G \in \mathrm{Hol}$
- $F' \in \mathrm{Hol}$ (defined on the dual lattice)
- $\oint F = 0$
- $\int^z F$ is well-defined and $\int^z F \in \mathrm{Hol}$
- maximum principle
- $F = H + i\tilde{H} \Rightarrow H$ discrete harmonic (mean-value property)
- H discrete harmonic $\Rightarrow \exists\, \tilde{H}$ such that $H + i\tilde{H} \in \mathrm{Hol}$

Problem: $F, G \in \mathrm{Hol} \not\Rightarrow F \cdot G \in \mathrm{Hol}$. For general lattices $F' \notin \mathrm{Hol}$.

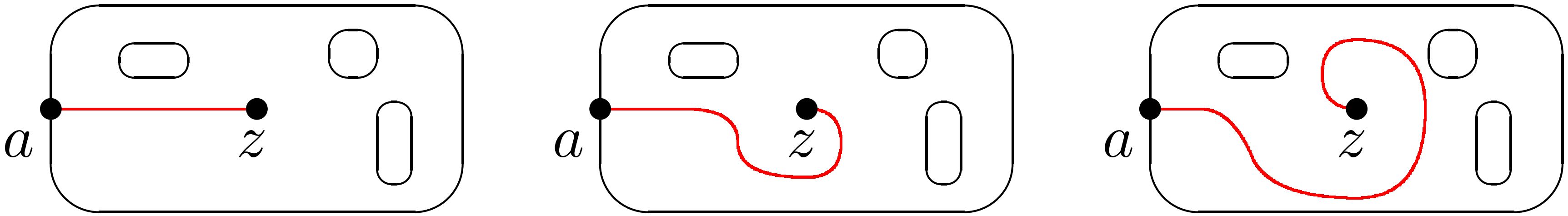
Ising preholomorphic observable: $F(z) := \sum_\omega x^{\#\text{edges}}\mathcal{W}$

- represent a configuration by a collection of interfaces between $+$ and $-$ spins.
- consider configurations ω which have loops plus an interface between a and z.
- introduce **Fermionic** complex weight:



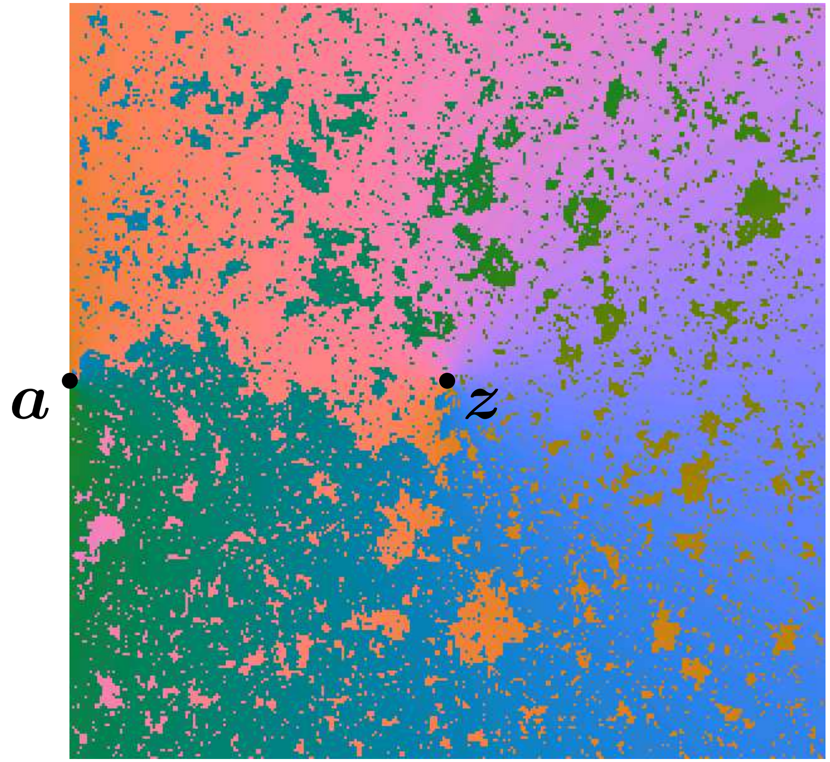
$$\mathcal{W} := \exp\left(-i\,\tfrac{1}{2}\,\textbf{winding}(\gamma, a \to z)\right)$$
$$= \lambda^{\#\text{ signed turns of }\gamma}, \quad \lambda := e^{-\pi i/4}$$

weight $\mathcal{W}$ $\quad 1 \quad -i \quad -1$



Rem Removing complex weight $\mathcal{W}$ one obtains correlation of spins at a and z on the dual lattice at the dual temperature $\tilde{x}$

Rem One can obtain such configurations by creating a disorder operator, i.e. a monodromy at z: when one goes around, $+$ spins become $-$and vice versa.



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Rem $F(z)\sqrt{dz}$ is a **fermion**

Theorem. For Ising model at T_c, F is a preholomorphic solution of a Riemann boundary value problem. When mesh $\epsilon \to 0$,

$$F(z)\,/\,\sqrt{\epsilon} \rightrightarrows \sqrt{P'(z)} \ \text{ inside } \Omega,$$

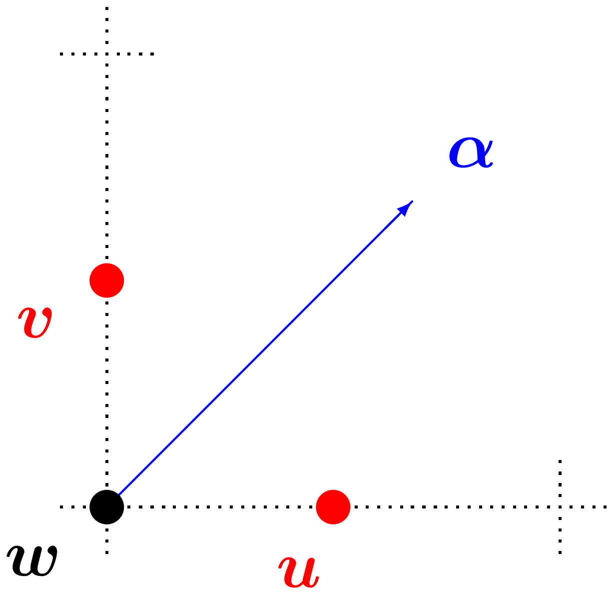
where P is the complex Poisson kernel at a: a conformal map $\Omega \to \mathbb{C}_+$ such that $a \mapsto \infty$.

Rem Both sides should be normalised in the same chart

Rem Off criticality **massive holomorphic:** $\bar{\partial}\, F = im(x - x_c)\bar{F}$

Proof: discrete CR relation

Consider function F on the centers of edges.
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the centers of two neighboring edges from the vertex $\boldsymbol{w}$.
Let $\boldsymbol{\alpha}$ be the unit bisector of the angle $\boldsymbol{uwv}$.



"Strong" Cauchy-Riemann relation:

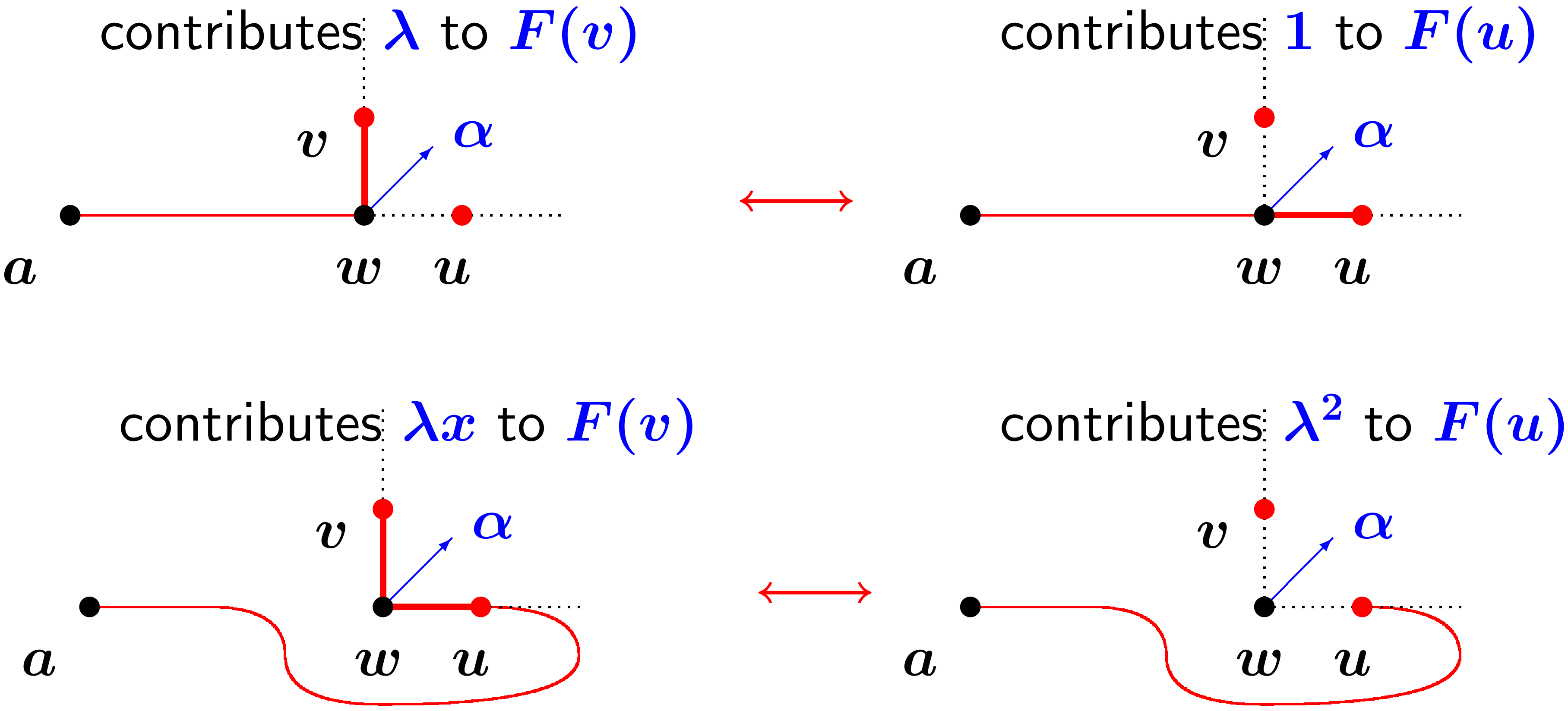
$$\mathrm{Proj}(\boldsymbol{F(v)}, \boldsymbol{1/\sqrt{\alpha}}) = \mathrm{Proj}(\boldsymbol{F(u)}, \boldsymbol{1/\sqrt{\alpha}}), \text{ or equivalently}$$

$$\boldsymbol{F(v) + \bar{\alpha}\,\overline{F(v)} = F(u) + \bar{\alpha}\,\overline{F(u)}}$$

- Implies the classical one for the square lattice
- Same formula works on any rhombic lattice
- Proved by constructing a bijection between configurations included into $F(v)$ and $F(u)$

Proof: discrete CR by local rearrangement

Let $\lambda = \exp(-\pi i/4)$ be the complex weight of a $\pi/2$ turn.
Erasing/drawing half-edges wu and wv gives a **bijection**:



There are more pairs, but relative contributions are always easy

Proof: discrete CR by local rearrangement

It remains to check that discrete CR is satisfied by every pair.
In our picture $\boldsymbol{\alpha = \exp(\pi i/4) = \bar{\lambda}}$, so the discrete CR takes form

$$\boldsymbol{F(v) + \lambda\,\overline{F(v)} = F(u) + \lambda\,\overline{F(u)}}$$

Plugging in 2 configurations above, we must check that:

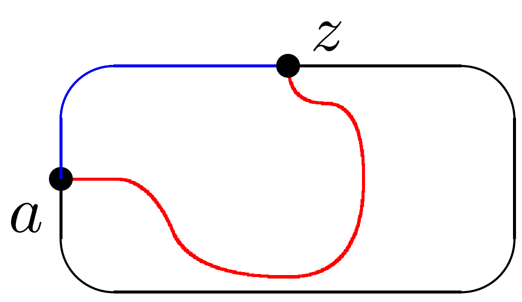
$$\begin{aligned} \boldsymbol{\lambda + \lambda\bar{\lambda} = 1 + \lambda\bar{1}} \quad &\Leftrightarrow \quad \boldsymbol{\lambda + 1 = 1 + \lambda} \\ \boldsymbol{\lambda x + \lambda\overline{\lambda x} = \lambda^2 + \lambda\bar{\lambda}^2} \quad &\Leftrightarrow \quad \boldsymbol{x = \lambda + \bar{\lambda} - 1} \end{aligned}$$

Other pairs lead to the same 2 possibilities.
The first identity always holds, the second one holds for $x = x_c$:
indeed, on the square lattice $\boldsymbol{\lambda = \exp(-\pi i/4)}$ and $\boldsymbol{x_c = \sqrt{2} - 1}$.

Rem For $x \neq x_c$ one gets **massive CR:** $\boldsymbol{\bar{\partial}\,F = im(x - x_c)\bar{F}}$
$\Rightarrow$ new derivation of criticality at x_c

Proof: Riemann-Hilbert boundary value problem

When z is on the boundary, winding of the interface $a \to z$ is uniquely determined, and coincides with the winding of $\partial\Omega$, $a \to z$. So we know $\mathrm{Arg}(F)$ on $\partial\Omega$.



F solves the discrete version of the covariant Riemann BVP

$$\mathrm{Im}\left(\boldsymbol{F(z)} \cdot (\text{tangent to } \boldsymbol{\partial\Omega})^{\mathbf{1/2}}\right) = \mathbf{0} \text{ with } \sigma = 1/2.$$

$$F \parallel \tau^{-1/2} \;\Rightarrow\; F^2 \parallel \tau^{-1} \;\Rightarrow\; F^2 dz \parallel 1 \text{ on } \partial\Omega$$

Continuum case: $F = (P')^{1/2}$, where $P : \Omega \to \mathbb{C}_+$, $a \mapsto \infty$.

Proof: convergence Consider $\int_{z_0}^{z} F^2(u)du$ – solves Dirichlet BVP.

Big problem: in the discrete case F^2 is no longer analytic!!!

Proof of convergence: set $H := \frac{1}{2\epsilon}\mathrm{Im}\int^z F(z)^2 dz$

- well-defined
- approximately discrete harmonic: $\Delta H = \pm\left|\partial F\right|^2$
- $H = 0$ on the boundary, blows up at a

$\Rightarrow \boldsymbol{H \rightrightarrows \mathrm{Im}\, P}$ where $\boldsymbol{P}$ is the complex Poisson kernel at $\boldsymbol{a}$

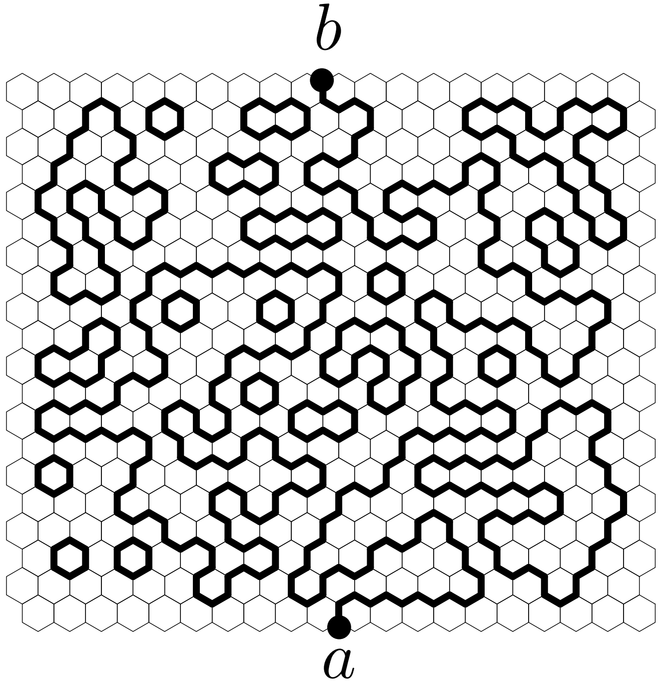
$\Rightarrow \boldsymbol{\nabla H \rightrightarrows P'} \quad \Rightarrow \quad \boldsymbol{\frac{1}{\sqrt{\epsilon}} F \rightrightarrows \sqrt{P'}}$ $\square$

Problems: we must do all sorts of estimates (Harnack inequality, normal familes, harmonic measure estimates, . . .) for *approximately* discrete harmonic or holomorphic functions in the absence of the usual tools. For more general graphs even worse, moreover there are no known Ising estimates to use **[Chelkak - S]**.

Question: what is the most general discrete setup when one can get the usual complex analysis estimates? (without using multiplication)

Possible generalization:

$\boldsymbol{O(n)}$ **loop gas.** Configurations of disjoint simple loops on hexagonal lattice. Loop-weight $n \in [0, 2]$, edge-weight $x > 0$.



$$\boldsymbol{Z = \sum_{\text{configs}} n^{\#\ \text{loops}}\, x^{\#\ \text{edges}}}$$

Dobrushin boundary conditions:

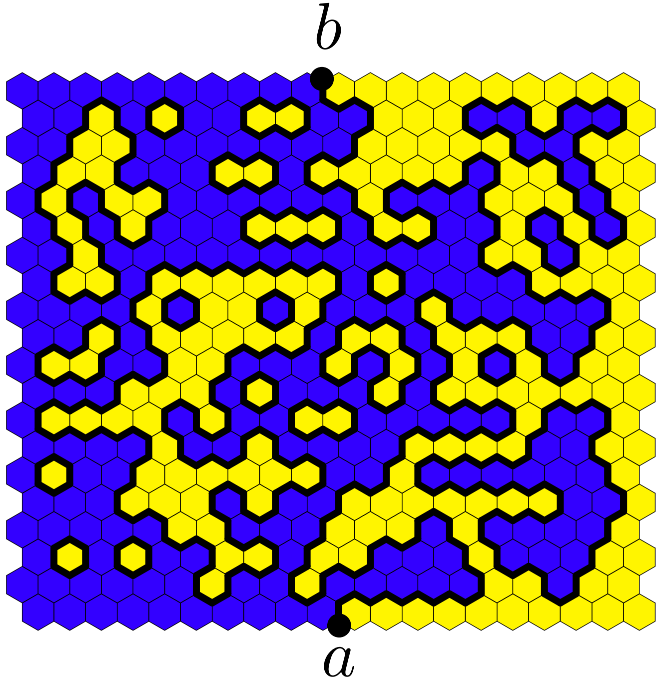
besides loops, an interface $\gamma : a \leftrightarrow b$.

Conjecture [Kager-Nienhuis,...]. $\exists$ *conformally invariant scaling limits for* $x = x_c(n) := 1/\sqrt{2 + \sqrt{2 - n}}$ *and* $x \in (x_c(n), +\infty)$.

Two different limits correspond to dilute / dense phases

(limiting loops are simple / non-simple)

Hexagons of two colors (Ising spins ± 1), which change whenever a loop is crossed.



For $\boldsymbol{n = 1}$ the partition function becomes

$$\boldsymbol{Z = \sum x^{\#\ \text{edges}}}$$
$$\boldsymbol{= \sum x^{\#\ \text{pairs of neighbors of opposite spins}}}$$

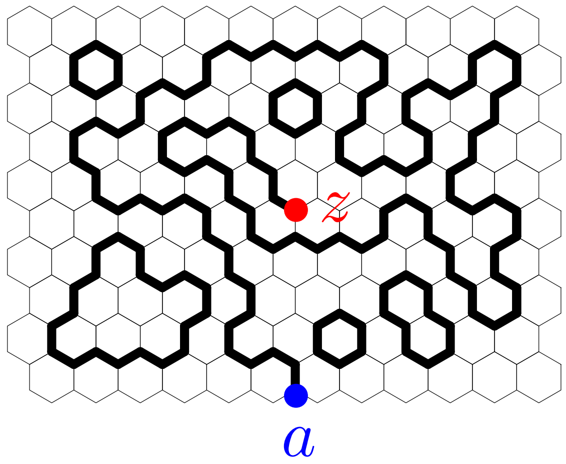
$\boldsymbol{n = 1,\ x = 1/\sqrt{3}}$: **Ising model at T_c**

Note: critical value of x is known [Wannier]

$\boldsymbol{n = 1,\ x = 1}$: **critical percolation** (on hexagons $=$ sites of the dual triangular lattice) All configs are equally likely ($p_c = 1/2$ [Kesten, Wierman]).

$\boldsymbol{n = 0,\ x = 1/\sqrt{2+\sqrt{2}}}$: a version of **self-avoiding random walk** (no loops, only a simple curve from a to b with weight x^{length}, cf. prediction [Nienhuis] that number of length ℓ simple curves is $\approx \sqrt{2+\sqrt{2}}^{\,\ell}\,\ell^{11/32}$)

Preholomorphic parafermion for the $O(n)$ model



Set $\boldsymbol{F(z) := \sum_{\omega} n^{\#loops} x^{\#\text{edges}} \mathcal{W}}$
Interface runs from a to z. $2\cos(2\pi k) := n$.
Replace power $1/2$ in Ising complex weight
by spin $\sigma = 1/4 + 3k/2$ for $x = x_c$,
$\sigma = 1/4 - 3k/2$ for $x > x_c$.

Conjecture. *For the $O(n)$ model at x_c and $x > x_c$*

$$\boldsymbol{\epsilon^{-\sigma} F(z) \rightrightarrows (P'(z))^{\sigma}} \quad \text{inside } \Omega$$

as lattice mesh $\epsilon \to 0$. Here P is the complex Poisson kernel at a.

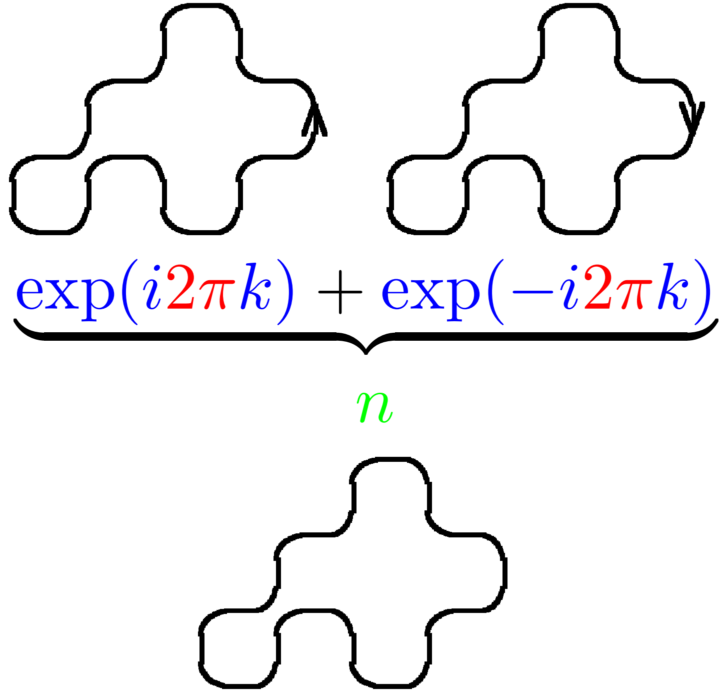
Same proof almost works, but one lemma is still missing. . .
Explains Nienhuis predictions of critical temperature x_c!

Why complex weights? [cf. Baxter]

Set $\mathbf{2\cos(2\pi k) = n}$. Orient loops $\Leftrightarrow$ height function changing by ± 1 whenever crossing a loop *(think of a geographic map with contour lines)*

New $\mathbb{C}$ partition function (local!):

$$\boldsymbol{Z^{\mathbb{C}} = \sum \prod_{\text{sites}} x^{\#\text{edges}} e^{(i\ \text{winding}\cdot k)}}$$



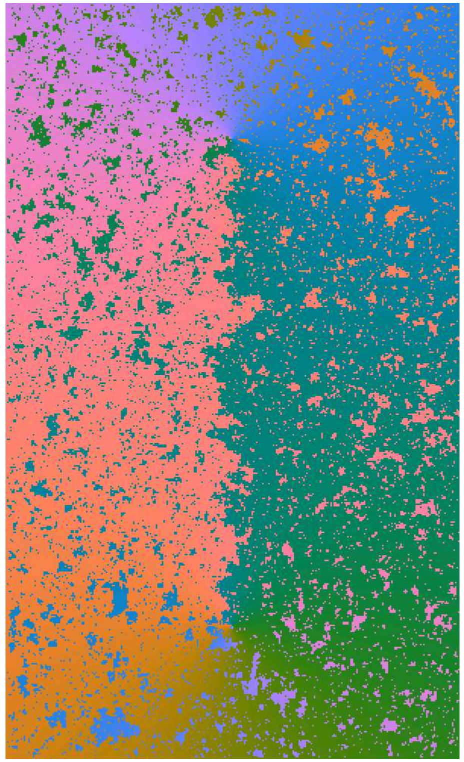
Forgetting orientation projects onto the original model: $\mathrm{Proj}\left(Z^{\mathbb{C}}\right) = Z$

Oriented interface $a \to z \Leftrightarrow +1$ monodromy at z

Can rewrite our observable as $\boldsymbol{F(z) = Z^{\mathbb{C}}_{+1\ \text{monodromy at z}}}$

Note: being attached to $\partial\Omega$, γ is weighted differently from loops

What can we deduce from one observable?
Interfaces converge to Schramm's SLE curves and loop soups. Then one can use the machinery of Itô calculus to calculate almost anything.



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But even beforehand one can say many things. Putting both points a and b inside, one obtains a discrete version of the Green's function with Riemann boundary values.
One of corollaries:

Theorem [Hongler - Smirnov]. At T_c the correlation of two neighboring spins σ_1, σ_2 near a vertex $z \in \Omega$ satisfies

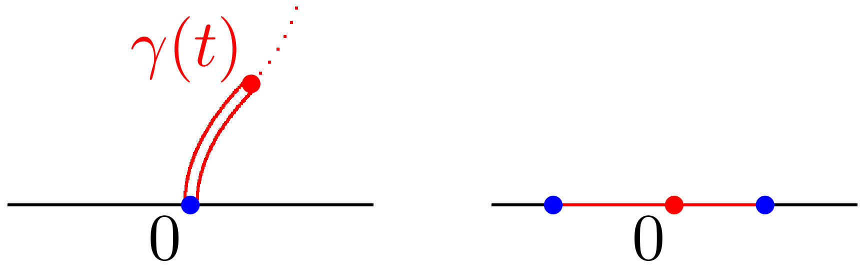
$$\mathbb{E}\,\sigma_1\sigma_2 \;=\; \frac{1}{\sqrt{2}} \pm \frac{1}{2\pi}\,\rho_\Omega(z)\,\epsilon + O(\epsilon^2),$$

here ρ is the element of the hyperbolic metric, and the sign $\pm$ depends on the boundary conditions ("+" or free).

(B) Schramm-Loewner Evolution.

LE is a slit $\gamma(t)$ obtained by solving an ODE for the Riemann map G_t:

$$\mathbb{C}_+ \setminus \gamma[0,t] \xrightarrow{G_t} \mathbb{C}_+$$



$$\partial_t \left(G_t(z) - w(t) \right) \;=\; 2/G_t(z)$$

$$G_t(z) = z - w(t) + 2t/z + \mathcal{O}\left(1/z^2\right)$$ — normalization at ∞.

SLE(κ) is a random curve obtained by taking $w(t) := \sqrt{\kappa} B_t$.

Schramm's Principle: *if an interface has a conformally invariant scaling limit, it is SLE(κ) for some $\kappa \in [0, \infty)$.*

Proof: Conformal invariance with Markov property (interface does not distinguish its past from the domain boundary) translates into $w(t)$ having i.i.d. increments.$\square$

To use the Principle one still has

(i) to show existence of the **scaling limit**

(ii) to prove its **conformal invariance**

(iii) calculate some observable to determine κ

For **(i)** in principle one needs $infinitely\ many\ observables$.
For percolation constructed from one observable using $locality$.

Fortunately SLE can be used to do **(iii)**$\Rightarrow$**(i-ii)**, see [Lawler-Schramm-Werner, Smirnov] for UST/LERW and percolation with invariant observables. A generalization of Schramm's Principle:

If a "martingale" observable has a conformally covariant limit, then the interface converges to SLE(κ) *with particular* $\kappa \in [0, \infty)$*, and the full collection of interfaces – to the corresponding loop ensemble.*

Proof: convergence of interfaces. Assume $\exists$ observable with a conformally invariant limit $\Rightarrow$ [Kemppainen-Smirnov] $\Rightarrow$ a priori estimates $\Rightarrow \{\gamma\}_{\text{mesh}}$ is precompact in a nice space.

Enough to show: limit of any converging subsequence $=$ SLE.

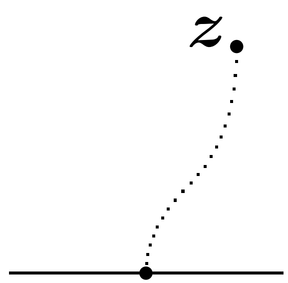
Pick a subsequential limit, map to $\mathbb{C}_+$, describe by Loewner Evolution with unknown random driving force $w(t)$.

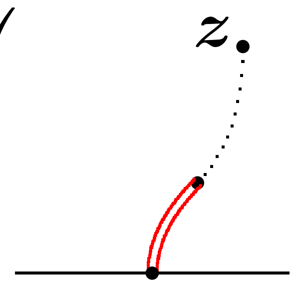
From the martingale property $\boldsymbol{F(z,\Omega) = \mathbb{E}_{\gamma'} F(z,\Omega\setminus\gamma')}$ of the observable extract expectation of increments of $w(t)$ and $w(t)^2$, conclude that $w(t)$ and $w(t)^2 - 3t$ are martingales.

By Lévy characterization theorem $\mathbf{w(t) = \sqrt{3}B_t}$.

So interface converges to SLE(3). $\square$

Suppose that $(P'(z))^\sigma$ is an observable ($\sigma = \frac{1}{2}$ for Ising), then

F 

$= \mathbb{E}_{\gamma[0,t]}\ F$ 

$$\| \qquad\qquad\qquad\qquad \|$$

$$(-(1/z)')^\sigma \qquad\qquad \mathbb{E}_{G_t}\,(-(1/G_t)')^\sigma$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{1}{z^{2\sigma}} \qquad\qquad \text{use expansion of } G_t \text{ at } \infty$$

$$\|$$

$$\mathbb{E}_{G_t}\frac{1}{z^{2\sigma}}\left(1+\frac{2\sigma}{z}w(t)+\frac{\sigma(2\sigma+1)}{z^2}\left(w(t)^2-\frac{6t}{2\sigma+1}\right)+\mathcal{O}\left(\frac{1}{z^3}\right)\right)$$

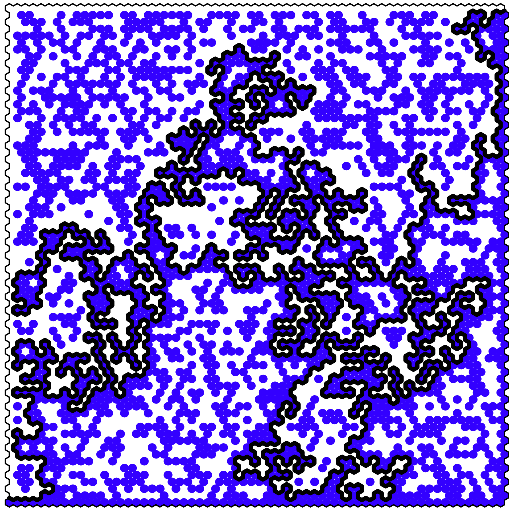
Rem A posteriori the method calculates all martingale observables for SLE!

What do we know about other models?

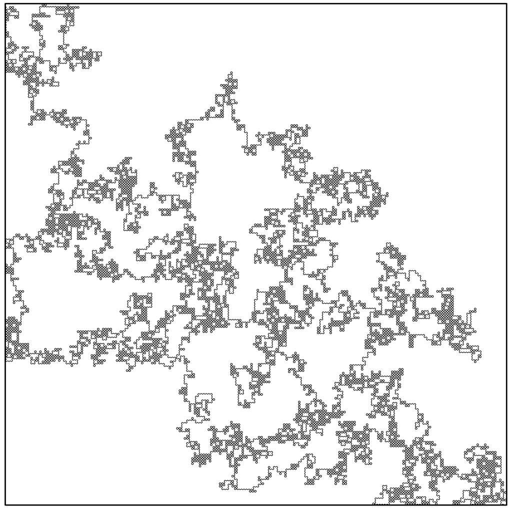
Interface converges to conformally invariant SLE(κ) curve for

c	κ	n	$O(n)$ loop gas dense/dilute	FK loops, $n = \sqrt{q}$
-2	8	0	$\cdots$	**uniform spanning tree, lerw [Lawler-Schramm-Werner 2003]**
0	6	1	**site percolation on the triangular lattice [S 2001]**	bond percolation on the square lattice
$\frac{1}{2}$	$\frac{16}{3}$	$\sqrt{2}$	$\cdots$	**FK Ising [S 2006]**
1	4	2	$\cdots$	FK 4-Potts
$\frac{1}{2}$	3	1	**Ising [S 2008]**	$\cos\left(\frac{4\pi}{\kappa}\right) = -\frac{n}{2}$
0	$\frac{8}{3}$	0	Self Avoiding Random Walk	

Also: Discrete Gaussian Free Field, $\kappa = 4$ [Schramm-Sheffield, 2006] percolation $\kappa = 6$

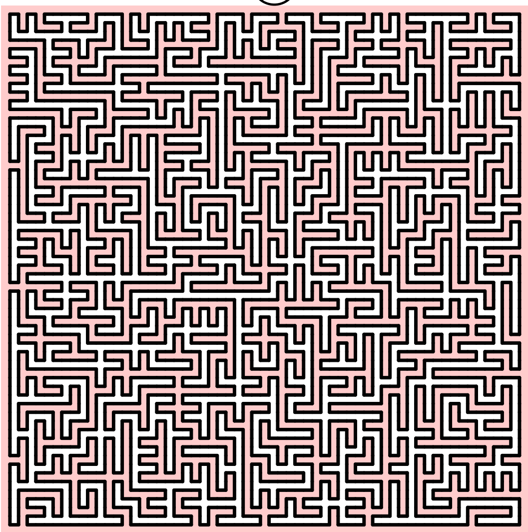


FK Ising $\kappa = 16/3$

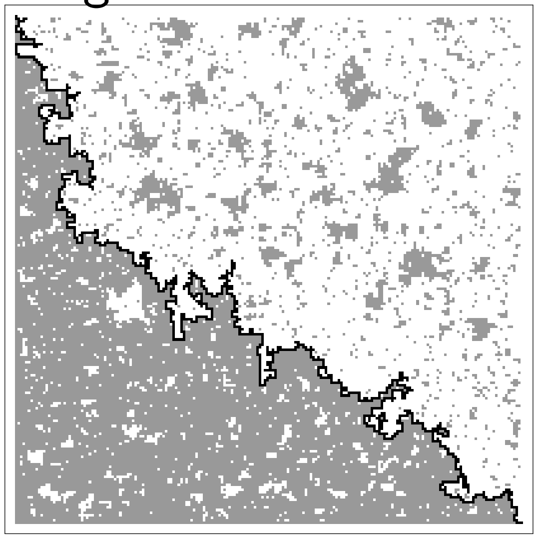


Square bond percolation?

ust $\kappa = 8$ ©O. Schramm



Ising $\kappa = 3$



Self-avoiding random walk?

CONCLUSION In several cases proof of conformally invariant scaling limits

- Some universality
- Fair understanding in other cases
- Many things new for physicists
- Heavy use of complex analysis

Can we say something about

- Other models?
- Renormalization?
- Connection to Yang-Baxter?

To answer we must learn more about

- Discrete complex analysis
- Conformal geometry
- Integrable systems

Ising $\to$ SLE(3), Dimension $= 11/8$