UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES

Quasiconformal maps and harmonic measure

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In part based on joint work with

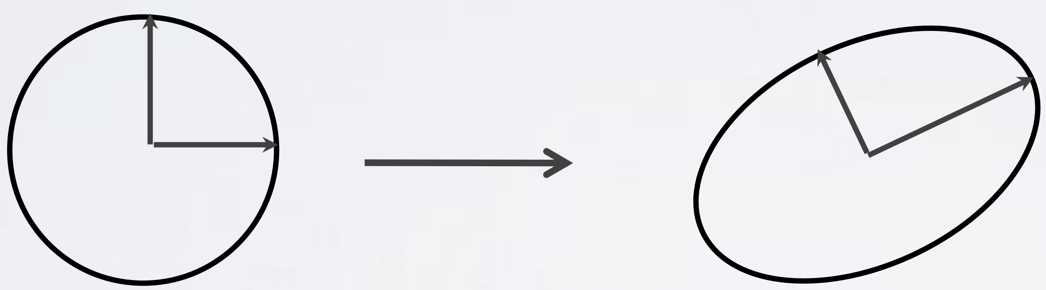
Kari Astala & **István Prause**

quasiconformal maps

$$\varphi : \Omega \to \Omega' \quad W^{1,2}_{loc}\text{-homeomorphism}$$

Def 1 $\bar{\partial}\varphi(z) = \mu(z)\partial\varphi(z) \quad \text{a.e. } z \in \Omega$ $\qquad \|\mu\|_\infty \leq k < 1$

Def 2



eccentricity ≤

$$K = \frac{1+k}{1-k}$$

measurable Riemann mapping theorem:

- (unique up to Möbius) solution exists
- depends analytically on μ

distortion of dimension

Theorem [Astala 1994] for $k-$ quasiconformal φ

$$\frac{1}{K}\left(\frac{1}{\dim E}-\frac{1}{2}\right) \le \frac{1}{\dim \varphi(E)}-\frac{1}{2} \le K\left(\frac{1}{\dim E}-\frac{1}{2}\right)$$

Rem result is sharp (easy from the proof)

In particular, $\dim E=1 \;\Rightarrow\; 1\text{-}k \le \dim \varphi(E) \le 1+k$
[Becker-Pommerenke 1987] $\dim \varphi(\mathbb{R}) \le 1+37k^2$

Conjecture [Astala] $\dim \varphi(\mathbb{R}) \le 1+k^2$

dimension of quasicircles

Thm [S] $\dim \varphi(\mathbb{R}) \leqslant 1+k^2$

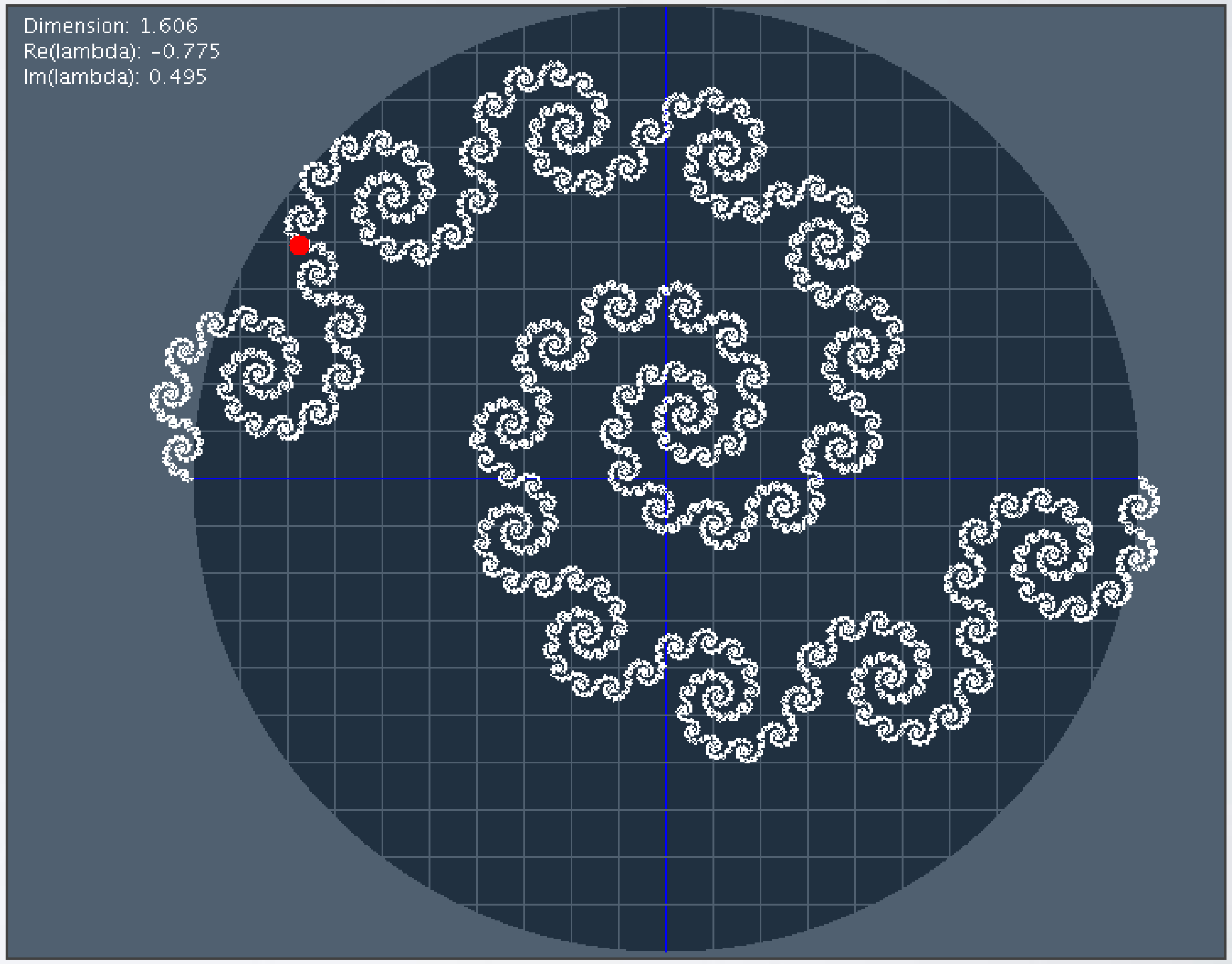
Dual statement:

φ symmetric wrt $\mathbb{R}$,

$$\left.\begin{array}{l} \operatorname{spt} \nu \subset \mathbb{R} \\ \dim \nu = 1 \end{array}\right\} \Rightarrow$$

$$\dim \varphi(\nu) \geqslant 1\text{-}k^2$$

Sharpness???



a nonrectifiable quasicircle

Proof: holomorphic motion

Any k - qc map φ_k can be embedded into a holomorphic motion of qc maps φ_λ , $\lambda \in \mathbb{D}$:

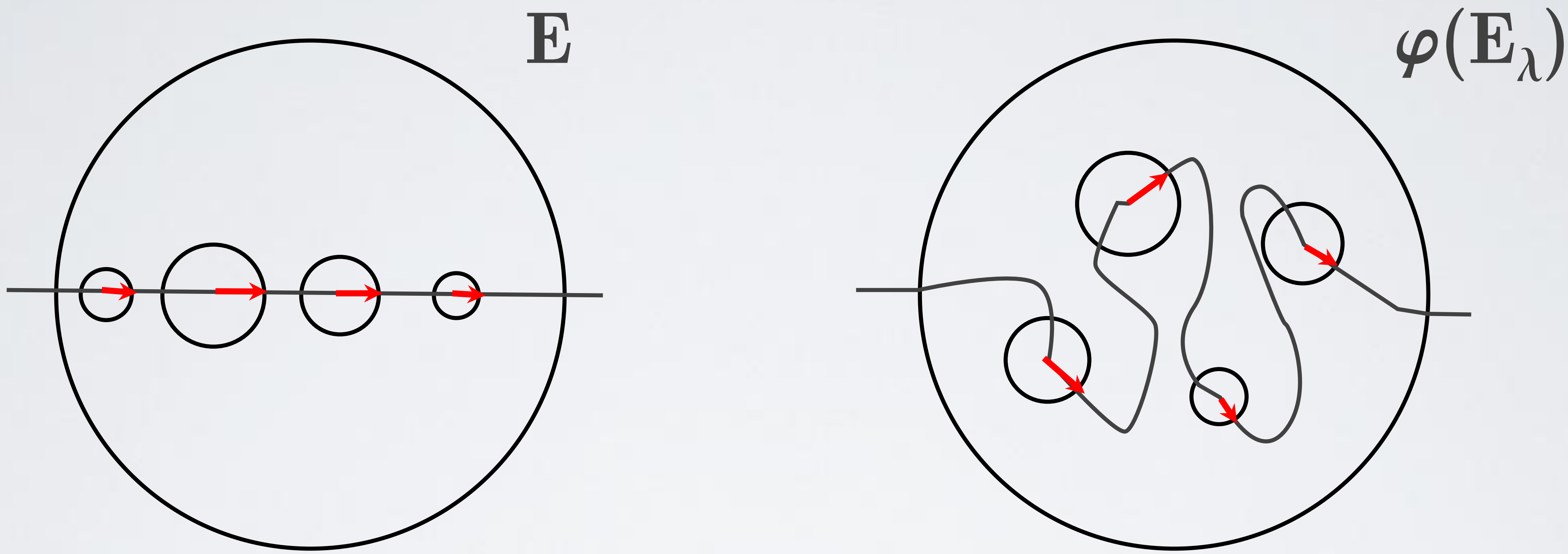
Define **Beltrami coefficient** $\mu = \mu_\varphi \,/\, \|\mu_\varphi\|$, $\|\mu\|=1$

$\lambda \in \mathbb{D}$ ⟶ $\lambda\mu$ ⟶ φ_λ which is $|\lambda|$-qc

Mañé-Sad-Sullivan, Slodkowski :

A holomorphic motion of a set can be extended to a holomorpic motion of qc maps

Proof: fractal approximation



a packing of disks evolves in the motion $\{\mathbf{B}_\lambda\}$

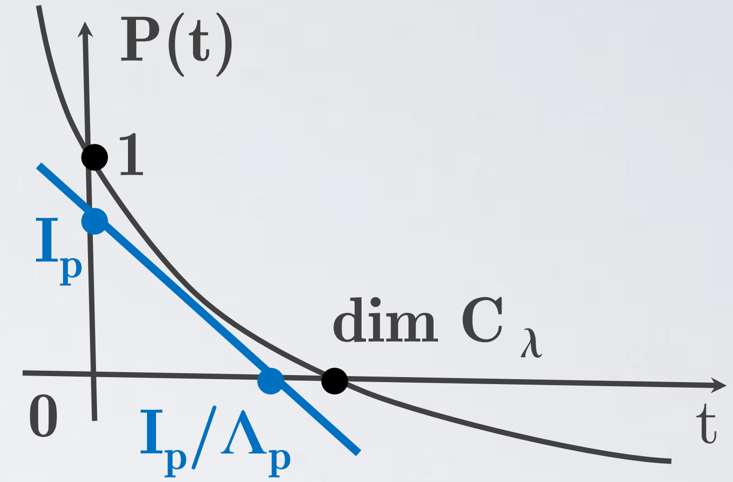
“complex radii” $\{\mathbf{r}_\lambda\}$

Cantor sets $\mathbf{C}_\lambda \approx \varphi(\mathbf{E}_\lambda)$

Proof: "thermodynamics"

Pressure [Ruelle, Bowen]

$P_\lambda(t) := \log(\Sigma\,|r_j(\lambda)|^t)$



"Entropy" $I_p := \Sigma\, p_j \log(1/p_j)$

"Lyapunov exponent" $\Lambda_p(\lambda) := \Sigma\, p_j \log(1/|r_j(\lambda)|)$

(harmonic in λ !)

Variational principle (Jensen's inequality)

$$P_\lambda(t) = \sup_{p \in \mathrm{Prob}} \Sigma\, p_j \log(|r_j(\lambda)|^t/p_j) = \sup_{p \in \mathrm{Prob}} (I_p - t\,\Lambda_p(\lambda))$$

Bowen's formula: $\dim C_\lambda = \text{root of } P_\lambda = \sup_{p \in \mathrm{Prob}} I_p / \Lambda_p(\lambda)$

Proof: Harnack's inequality

- $\dim C_0 = 1 \Longrightarrow I_p/\Lambda_p(0) \leqslant 1 \Longrightarrow \Lambda_p(0) - I_p/2 \geqslant I_p/2$
- $\dim C_\lambda \leqslant 2 \Longrightarrow I_p/\Lambda_p(\lambda) \leqslant 2 \Longrightarrow \Lambda_p(\lambda) - I_p/2 \geqslant 0$
- **Harnack** $\Longrightarrow \Lambda_p(\lambda) - \frac{I_p}{2} \geqslant \frac{1-|\lambda|}{1+|\lambda|}\frac{I_p}{2}$

  $\Longrightarrow \Lambda_p(\lambda) \geqslant \frac{1}{1+|\lambda|} I_p$

  $\Longrightarrow \dim C_\lambda = \sup_p I_p/\Lambda_p(\lambda) \leqslant 1+|\lambda|$ ■
- Quasicircle $\Longrightarrow$ **(anti)symmetric motion** $\Longrightarrow$ even Λ

  $\Longrightarrow$ "quadratic" Harnack $\Longrightarrow$ $\dim C_\lambda \leqslant 1+|\lambda|^2$ ■

Proof: symmetrization

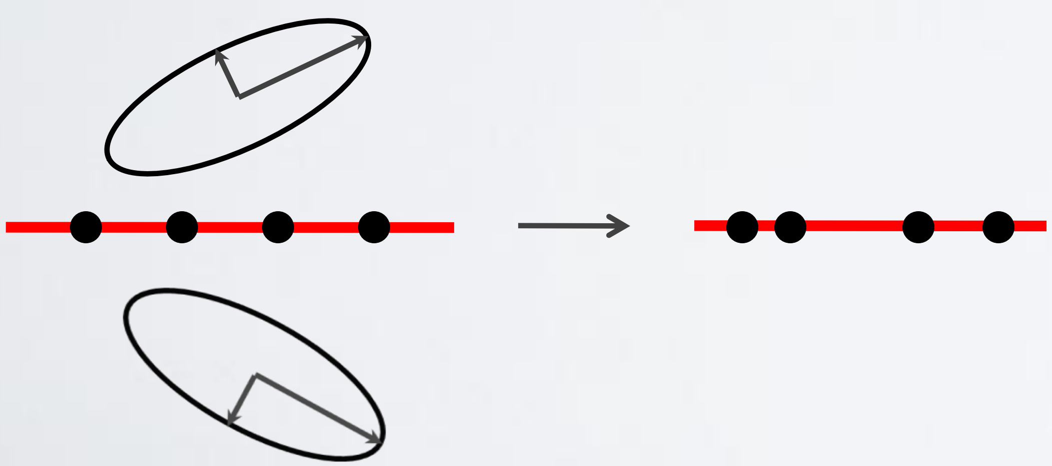
Thm [S] the following are equivalent:

a. $\Gamma = \varphi(\mathbb{R})$ and φ is k-qc

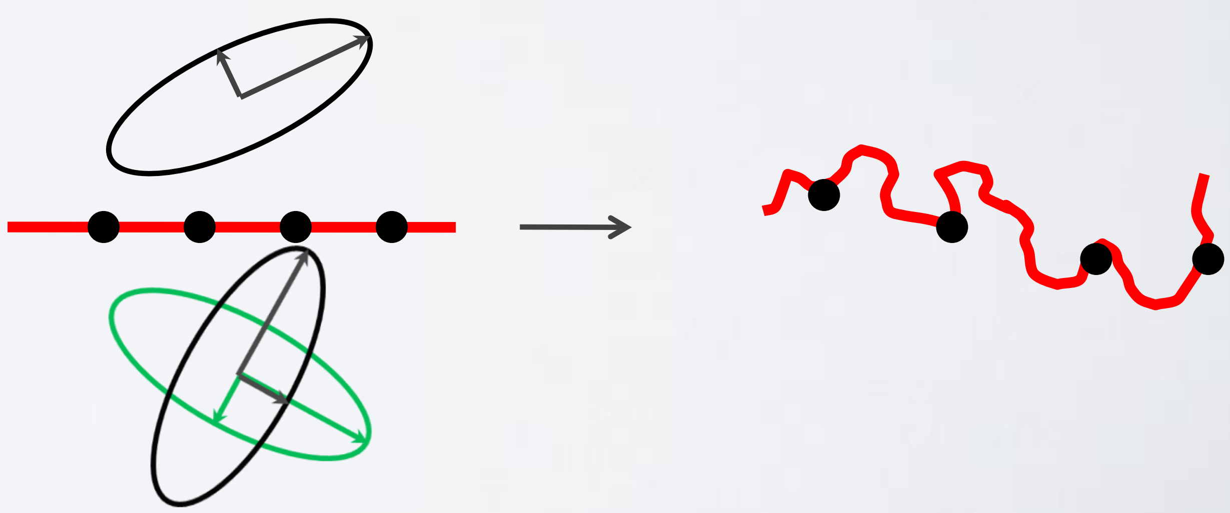
b. $\Gamma = \varphi(\mathbb{R})$ and φ is $\frac{2k}{1+k^2}$ qc in $\mathbb{C}_+$ and conformal in $\mathbb{C}_-$

c. $\Gamma = \varphi(\mathbb{R})$ and φ is k-qc and antisymmetric

symmetric:



antisymmetric:



harmonic measure ω

- **Brownian motion**

  exit probability

- **conformal map**

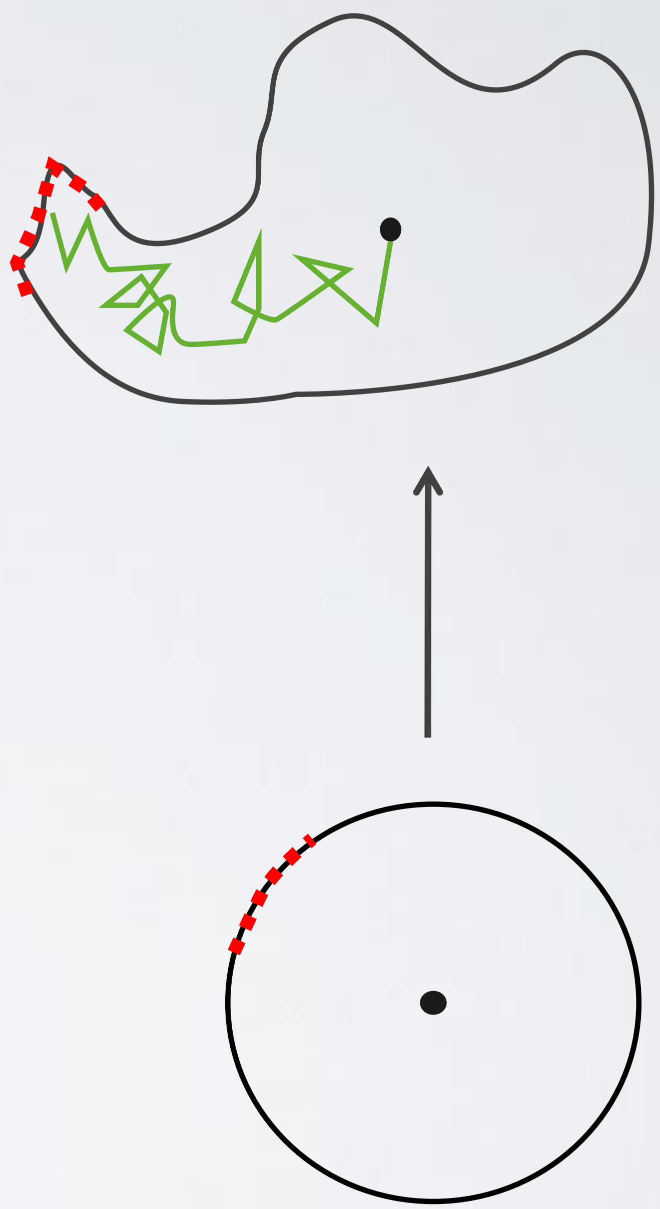
  image of the length

- **potential theory**

  equilibrium measure

- **Dirichlet problem for Δ**

$$u(z_0) = \int_{\partial\Omega} u(z)\,d\omega(z)$$

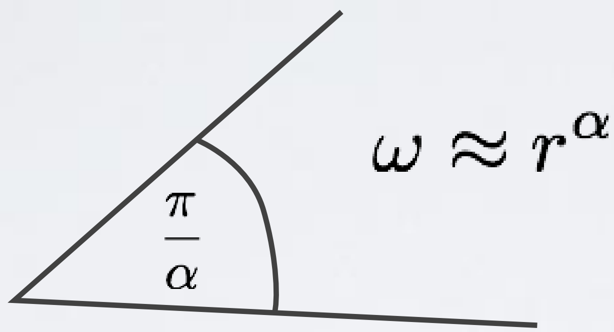


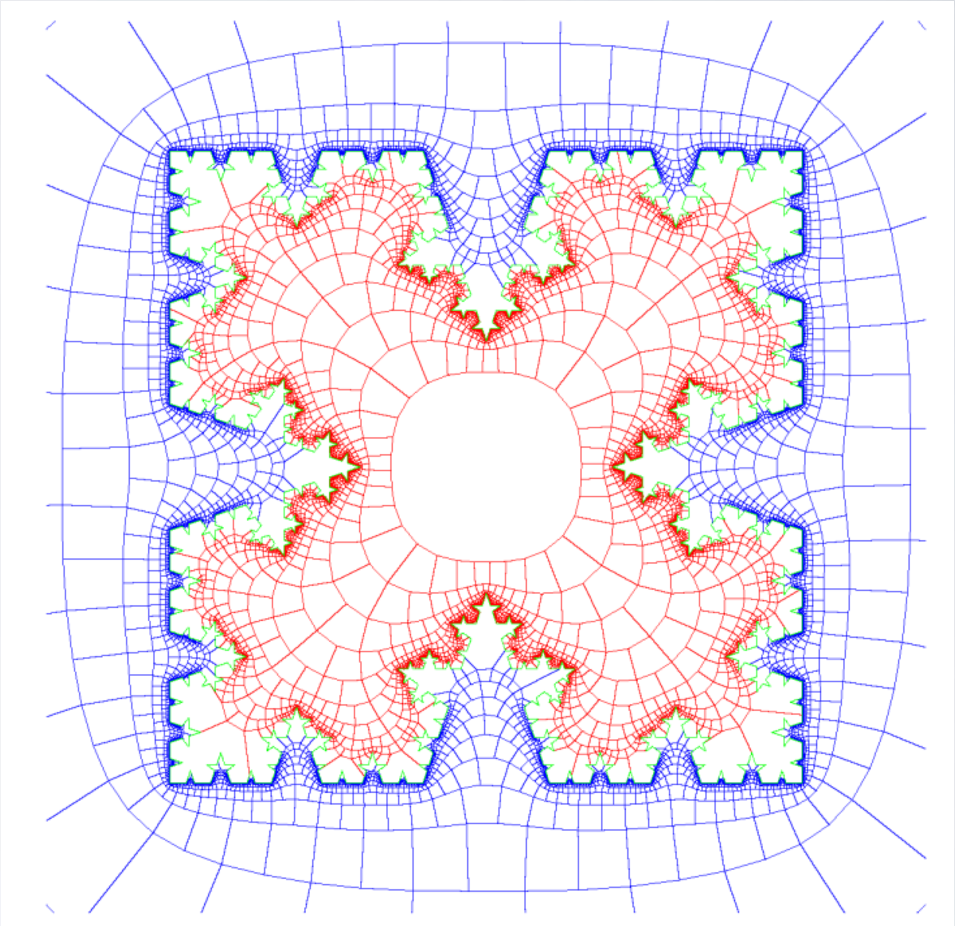
multifractality of ω

"fjords and spikes"

$\mathcal{F}_\alpha$ scaling: $\omega B(z,r) \approx r^\alpha$

geometric Meaning :





Courtesy of D. Marshall

Beurling's theorem: $\alpha \geq 1/2$

spectrum: $f(\alpha) = \dim \mathcal{F}_\alpha$

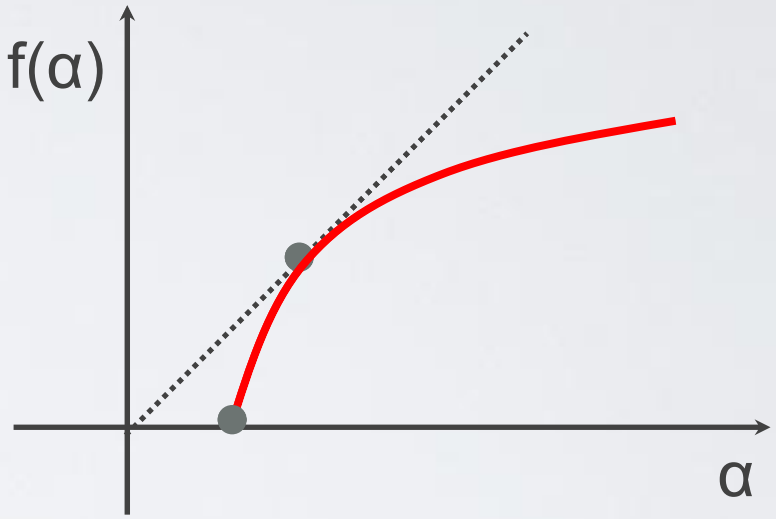
Makarov's theorem: Borel dim $\omega = 1, \; f(1) = 1$

Many open problems reduce to estimating the

universal spectrum

$$f(\alpha) = \sup_{\Omega} f_{\Omega}(\alpha)$$

over all simply connected domains



Conjecture : $f(\alpha) \stackrel{?}{=} 2 - \frac{1}{\alpha}$

[Brennan-Carleson-Jones-Krätzer-Makarov]

Legendre transform & pressure

Restrict pressure to **conformal maps** $\varphi : \mathbb{C}_+ \to \Omega$

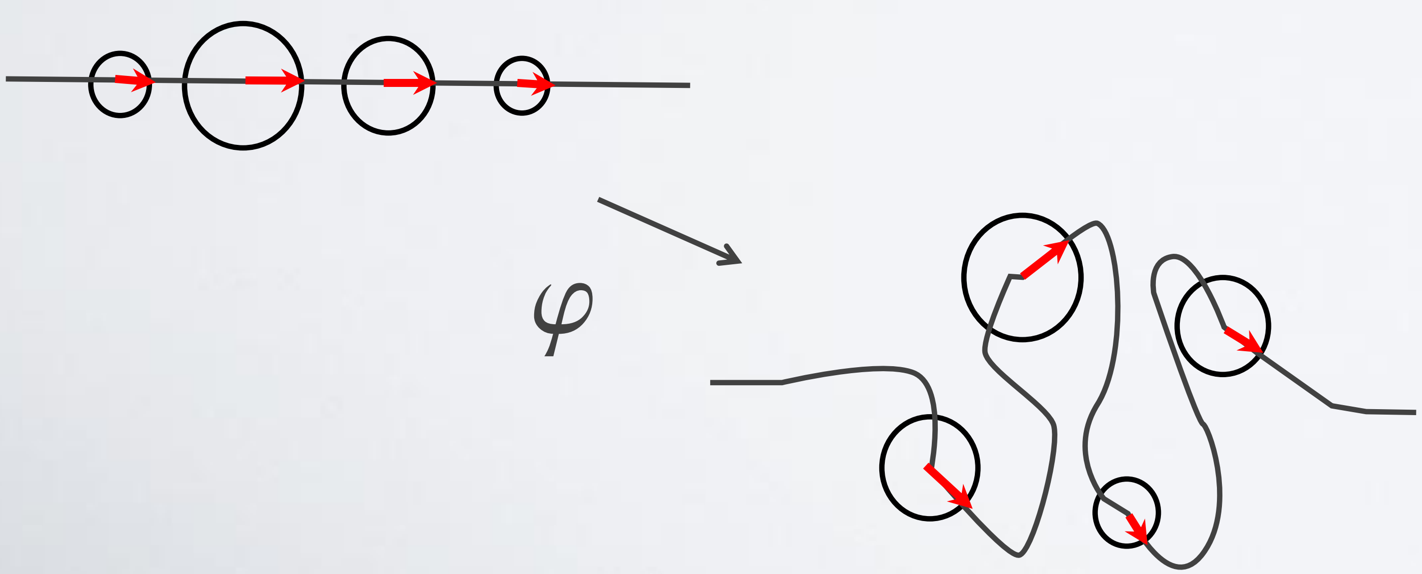
$$\pi_\Omega(t) := \log(\Sigma\, |r_j(\lambda)|^t)$$

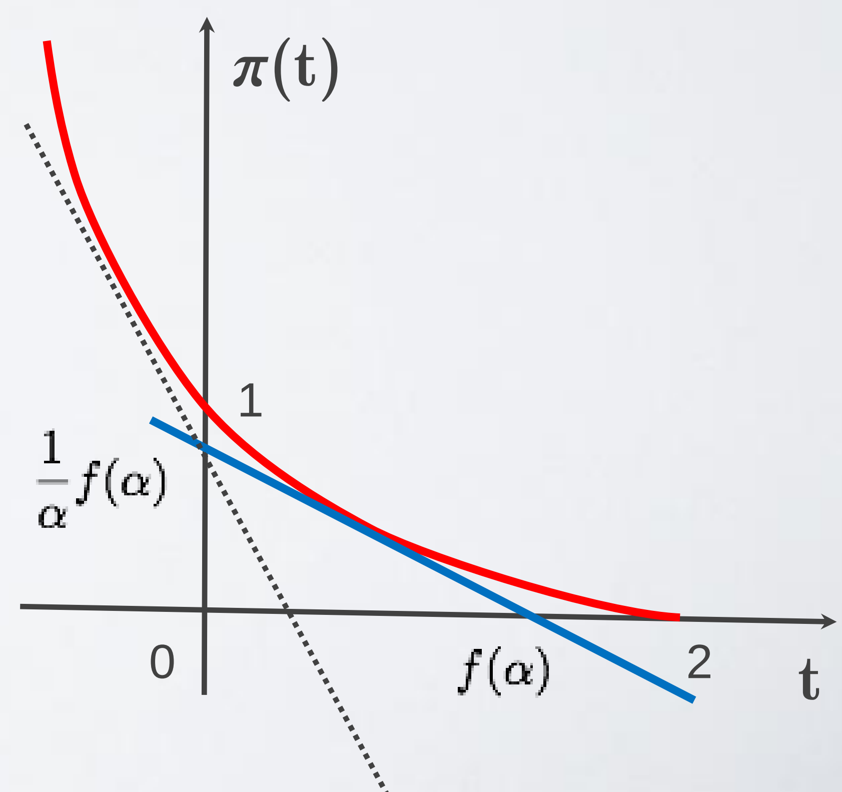
Universal pressure $\pi(t) := \sup_\Omega \pi_\Omega(t)$

Thm [Makarov 1998] Legendre transforms:

$f(\alpha) = \inf_t \{\alpha\pi(t)+t\}$ $\pi(t) = \sup_\alpha\{(f(\alpha)-t)/\alpha\}$

Conjecture: $\pi(t) = (2-t)^2/4$





finding the universal spectrum

- no real intuition
- some numerical evidence
- only weak estimates

Example: $\pi(1)$ gives optimal

- coefficient decay rate for bounded conformal maps
- growth rate for the length of Green's lines

Conjecturally $\pi(1) = 0.25$, best known estimates:

$$\mathbf{0.23} \leqslant \pi(1) \leqslant \mathbf{0.46}$$

[Beliaev, Smirnov] [Hedenmalm, Shimorin]

fine structure of harmonic measure via the holomorphic motions

I. qc deformations of conformal structure and harmonic measure
II. motions in bi-disk
III. welding conformal structures and Laplacian on 3-manifolds

joint work with
Kari Astala and István Prause

I. deforming conf structure

Recall: $\text{spt}\ \nu \subset \mathbb{R}\ \&\ \dim \nu = 1 \implies \dim \varphi(\nu) \geqslant 1\text{-}k^2$

Thm assume that the statement above is sharp:

$$\left.\begin{array}{l}\text{spt}\ \sigma \subset \mathbb{R} \\ \dim \sigma = 1\text{-}k^2\end{array}\right\} \implies \exists\ k\text{-qc}\ \varphi\ \text{s.t.}\ \varphi(dx) = \sigma$$

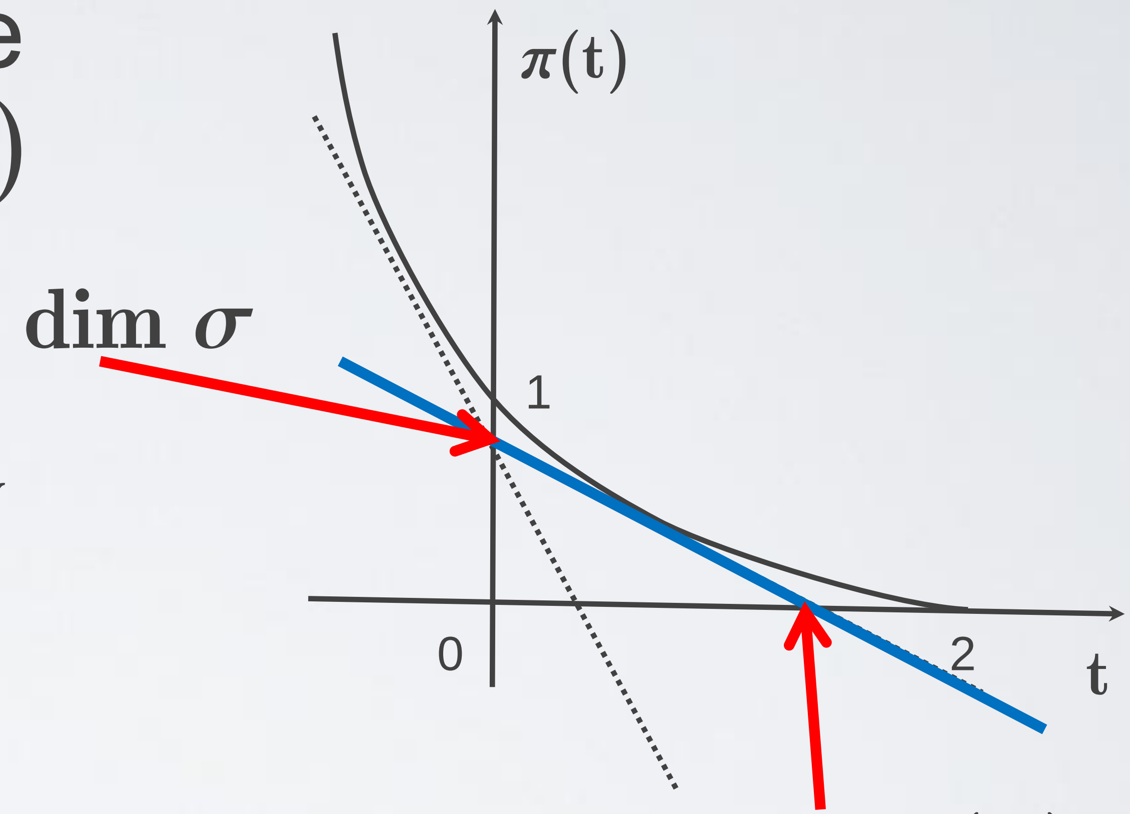
then the universal spectrum conjecture holds

Rem in general no sharpness (e.g. any porous σ), but we need it only for relevant “Gibbs” measures

Question: how to deform? (use φ?)

I. proof: deforming to ω

For "Gibbs" measures the blue line is tangent to $\pi(\mathrm{t})$



Set $1\text{-}k^2 := \dim \sigma$ and take holomorphic motion ψ such that $\psi_k(\mathrm{dx}) = \sigma$

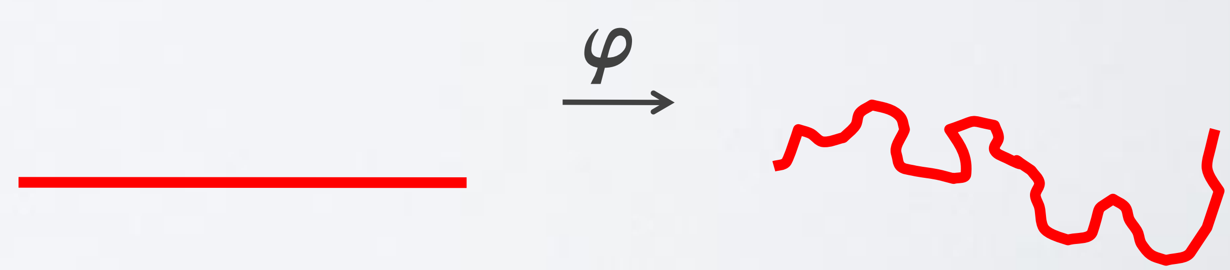
By Makarov's theorem

$$\dim \varphi(\psi_k^{-1}(\sigma)) = \dim \varphi(\mathrm{dx}) = 1$$

By Astala's theorem

$$\dim \varphi(\sigma) \leqslant 1+k$$

$$\Longrightarrow \pi(\mathrm{t}) \leqslant (2\text{-}\mathrm{t})^2 / 4$$



measure σ

measure $\varphi(\sigma)$

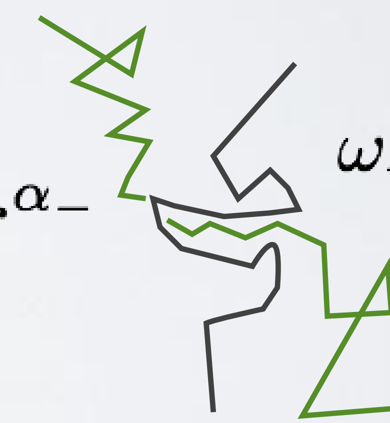
II. two-sided spectrum

rotation [Binder]

$$f(\alpha,\gamma) = \dim \mathcal{F}_{\alpha,\gamma} \qquad \omega \approx r^{\alpha} \ \& \ \gamma\text{-spiraling}$$

two-sided spectrum

$$f(\alpha_-,\alpha_+,\gamma) = \dim \mathcal{F}_{\alpha_-,\alpha_+,\gamma}$$

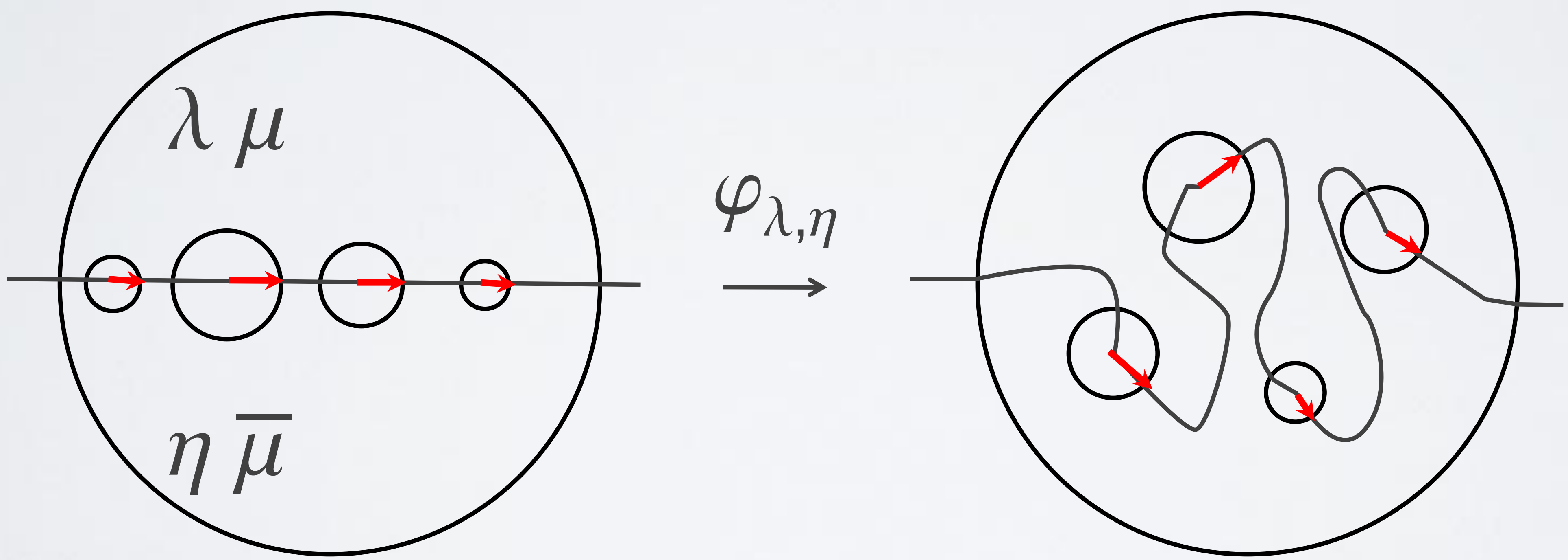


Beurling's estimate $$\frac{1}{\alpha_-} + \frac{1}{\alpha_+} \leq \frac{2}{1+\gamma^2}$$

II. bidisk motion

Take Beltrami μ in $\mathbb{C}_+$ of norm 1, symmetrize it

$$\mu_{\lambda,\eta} = \begin{cases} \lambda\, \mu(z) \text{ in } \mathbb{C}_+ \\ \eta\, \overline{\mu(\overline{z})} \text{ in } \mathbb{C}_- \end{cases} \longrightarrow \quad \varphi_{\lambda,\eta}(z) \quad (\lambda,\eta) \in \mathbb{D}^2$$



symmetric for $\lambda=\overline{\eta}$, antisymmetric for $\lambda=-\overline{\eta}$

II. thermodynamics

$$P_{\lambda,\eta}(t) = \log\left(\sum |r(B_{\lambda,\eta})|^t\right) = \sup_p(\mathrm{I} - t\,\mathrm{Re}\,\Lambda_{\lambda,\eta})$$

$$\mathrm{I} = \sum p_i \log\frac{1}{p_i}$$

entropy

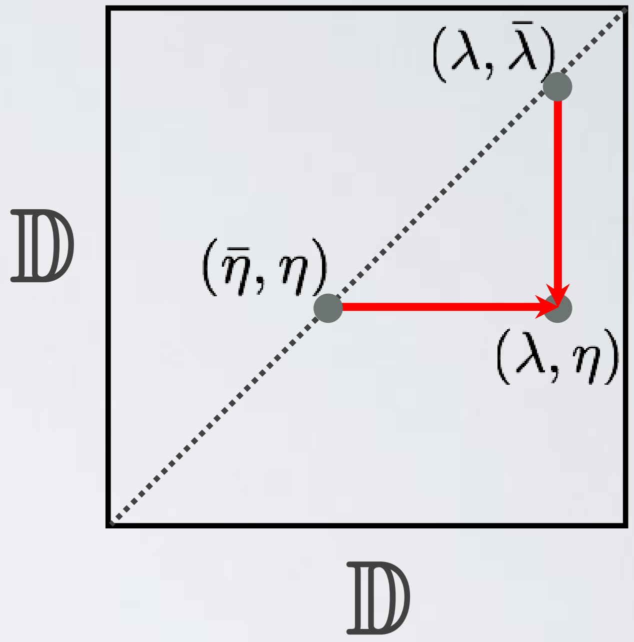
$$\Lambda_{\lambda,\eta} = \sum p_i \log\frac{1}{r_i(\lambda,\eta)}$$

(complex) Lyapunov exponent

$$\dim(C_{\lambda,\eta}) = \sup_p \dim p = \sup_p \frac{\mathrm{I}}{\mathrm{Re}\,\Lambda_{\lambda,\eta}}$$

II. "easy" estimates

- reflection symmetry $\varphi_{\lambda,\eta}(z) = \overline{\varphi_{\bar{\eta},\bar{\lambda}}(\bar{z})}$
- diagonal $(\lambda, \bar{\lambda})$
- projections $(\lambda, \eta)_+ = (\lambda, \bar{\lambda}),\ (\lambda, \eta)_- = (\bar{\eta}, \eta)$

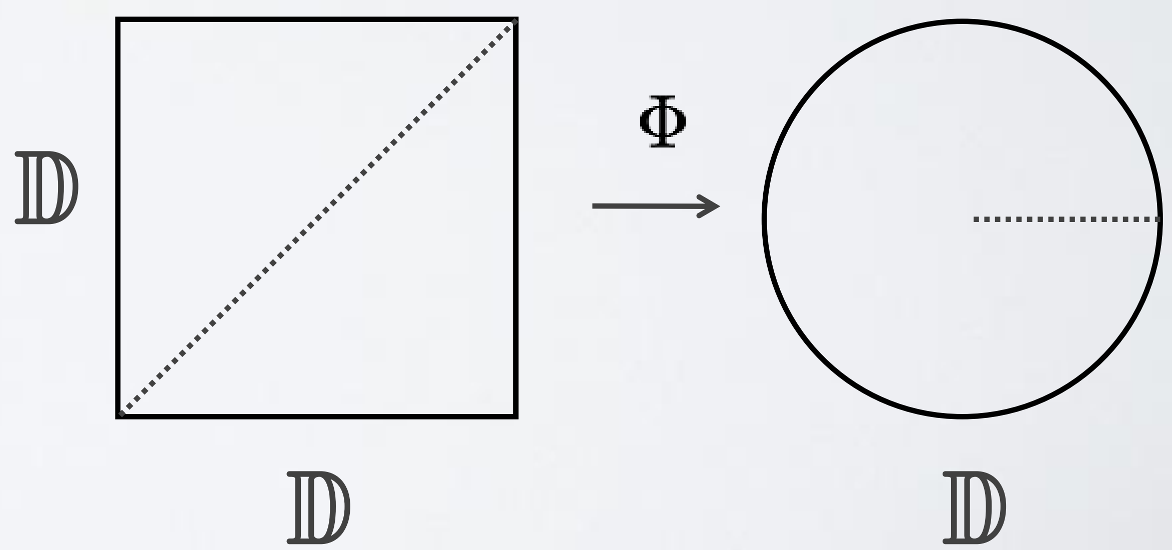


$$\Phi(\lambda, \eta) = 1 - \frac{\mathrm{I}}{\Lambda_{\lambda,\eta}}$$

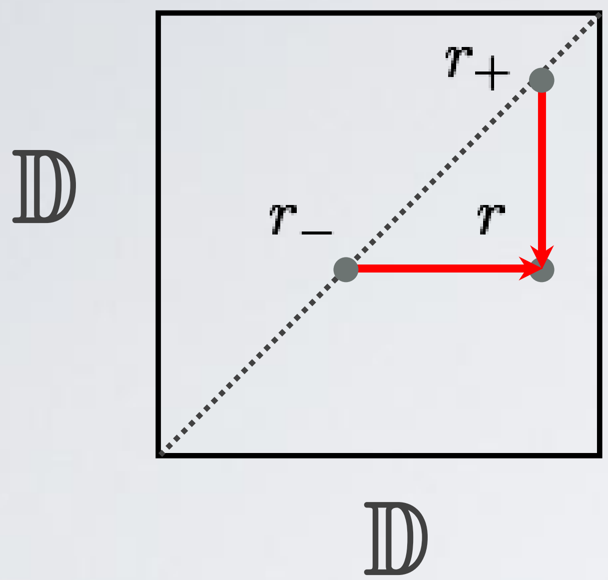
$\Phi : \mathbb{D}^2 \to \mathbb{D}$ $\quad \dim C_{\lambda,\eta} \leq 2$

$\Phi(\lambda, \bar{\lambda}) \geq 0$ $\quad \dim C_{\lambda,\bar{\lambda}} \leq 1$

$\Phi(\lambda, \eta) = \overline{\Phi(\bar{\eta}, \bar{\lambda})}$

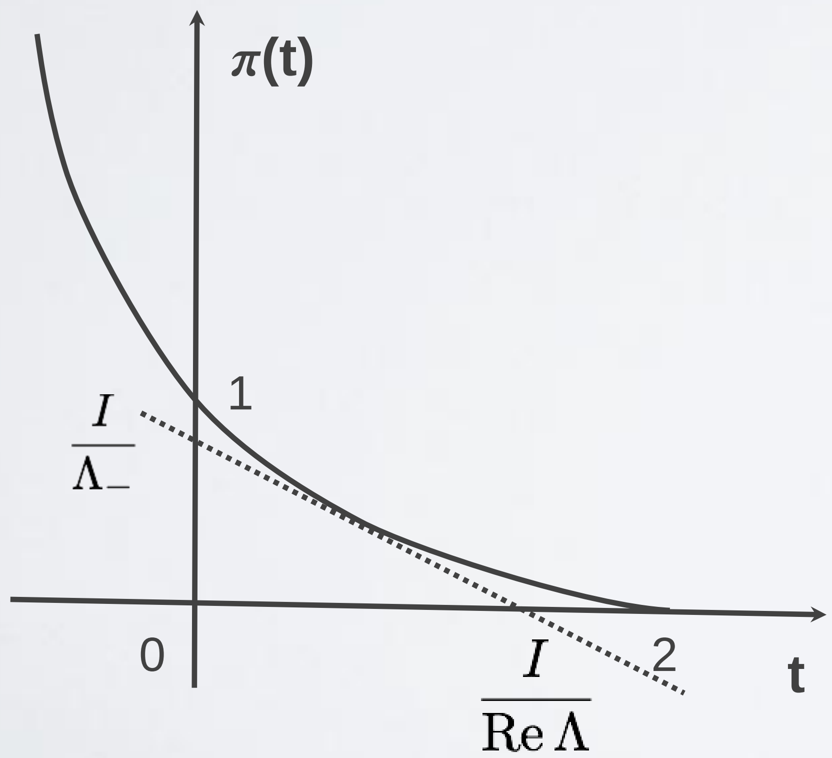


II. scaling relations



$$\omega_\pm B_{\lambda,\eta} \approx r_\pm$$

$$\log \frac{1}{r} = \frac{1+i\gamma}{\alpha_\pm} \log \frac{1}{r_\pm}$$



$$\frac{\mathrm{I}}{\Lambda} = \frac{f}{1+i\gamma}, \quad \frac{\mathrm{I}}{\Lambda_-} = \frac{f}{\alpha_-}, \quad \frac{\mathrm{I}}{\Lambda_+} = \frac{f}{\alpha_+}$$

II. Beurling and Brennan

Beurling $\Rightarrow \frac{1}{\alpha_-} + \frac{1}{\alpha_+} \leq \frac{2}{1+\gamma^2} \Rightarrow \quad 2\,\mathrm{Re}\,\Phi(\lambda,\eta) \leq \Phi(\bar{\eta},\eta) + \Phi(\lambda,\bar{\lambda})$

Corollary: $\lambda \mapsto \Phi(\lambda,\bar{\lambda})$ is subharmonic

Brennan's conjecture: $F : \Omega \to \mathbb{D}, \quad F' \in L^{4-\epsilon}$

Equivalent question: $f(\alpha) \leq 4(\alpha - \frac{1}{2})$ **?**

Two-sided: $2\,\mathrm{Re}\,\frac{1-\Phi}{1+\Phi} \geq \frac{1-\Phi_-}{1+\Phi_-} + \frac{1-\Phi_+}{1+\Phi_+}$ **?**

II. two-sided spectrum

Conjecture: $|\Phi|^2 \leq \Phi_-\Phi_+$ or $\begin{pmatrix} \Phi(\lambda,\bar{\lambda}) & \Phi(\lambda,\bar{\eta}) \\ \Phi(\eta,\bar{\lambda}) & \Phi(\eta,\bar{\eta}) \end{pmatrix} \geq 0$

Rem it is equivalent to

$$f(\alpha_-,\alpha_+,\gamma) \leq \frac{2-(1+\gamma^2)\left(\frac{1}{\alpha_-}+\frac{1}{\alpha_+}\right)}{1-\frac{1+\gamma^2}{\alpha_-\alpha_+}}$$

$$\Big\downarrow \quad \alpha_+ \to \infty, \quad \gamma = 0$$

$$f(\alpha_-) \leq 2 - \frac{1}{\alpha_-}$$

II. the question

We know that

$$\Phi : \mathbb{D}^2 \to \mathbb{D}$$

$\Phi(\lambda, \bar{\lambda}) \geq 0$ and subharmonic

$$\Phi(\lambda, \eta) = \overline{\Phi(\bar{\eta}, \bar{\lambda})}$$

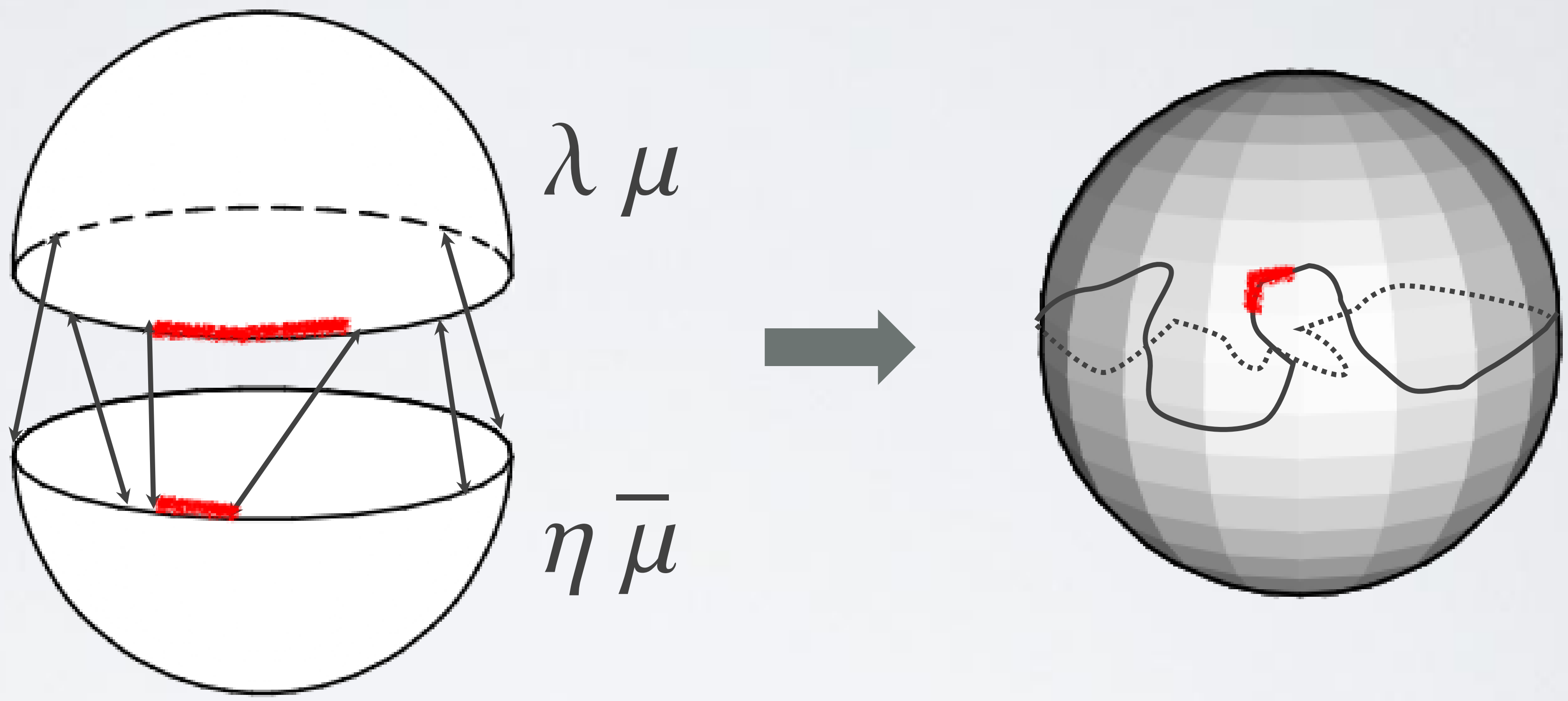
plus more…

What do we need to deduce the conjecture?

$$\begin{pmatrix} \Phi(\lambda, \bar{\lambda}) & \Phi(\lambda, \bar{\eta}) \\ \Phi(\eta, \bar{\lambda}) & \Phi(\eta, \bar{\eta}) \end{pmatrix} \geq 0$$

III. conformal welding

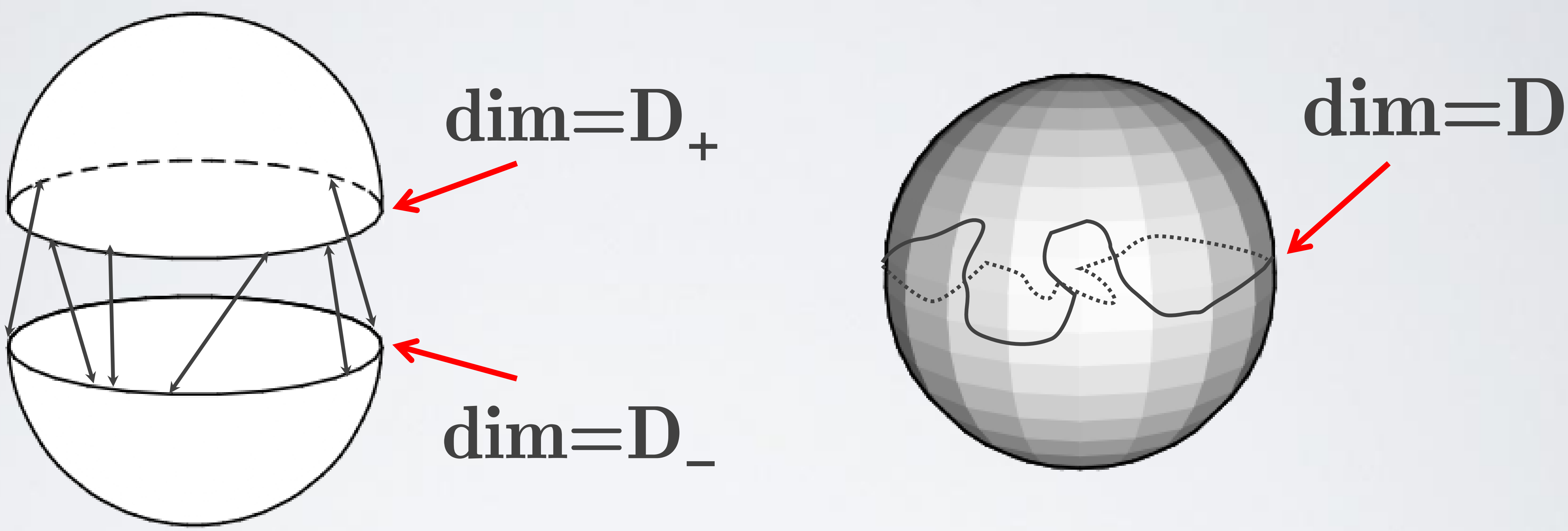
two perturbations of conformal structure



quasisymmetric welding ⟷ quasicircle

III. welding and dimensions

Take three images of the linear measure dx :



Then the conjectures before are equivalent to

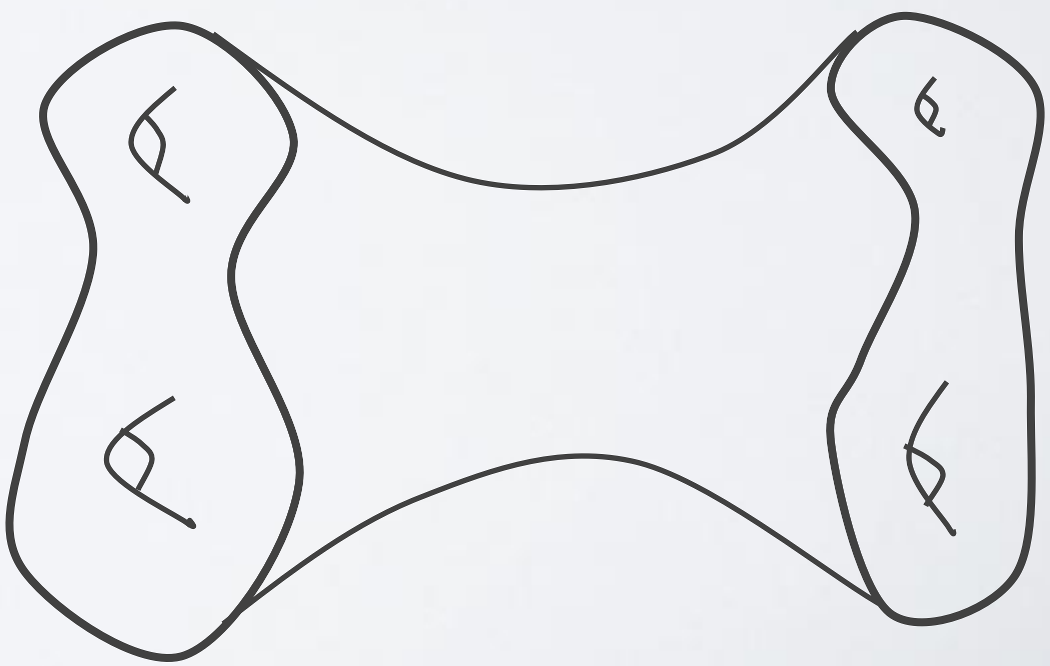
$$(1\text{-}D)^2 \le (1\text{-}D_-)\,(1\text{-}D_+)$$

III. Questions about $(1\text{-}D)^2 \leq (1\text{-}D_-)\ (1\text{-}D_+)$

Rem1 The inequality holds if $\mathbf{D_- = 1.}$

Q1 Can one interpolate to prove it in general?

Rem2 For quasicirles arising in quasi-Fuchsian groups the base eigenvalue λ_0 of the Laplacian on the associated 3-manifold has $1\text{-}\lambda_0=(1\text{-}D)^2$ for Patterson-Sullivan measure



Q2 Can one use 3D geometry ?