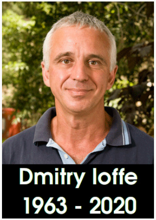
Failure of the Ornstein–Zernike asymptotics for the pair correlation function at high temperature and small density

Yvan Velenik
Université de Genève

Joint work with Yacine Aoun, Dmitry Ioffe and Sébastien Ott



— Introduction —

Ornstein–Zernike asymptotics for the pair correlation function

- In 1914 and 1916, Ornstein and Zernike developed a (heuristic) theory of correlations in fluids with quickly decaying interactions. In particular, they concluded that, at large distances, the density-density correlation satisfies

$$G(r) \sim r^{-(d-1)/2} e^{-\nu r},$$

where ν is the **inverse correlation length**.

- The OZ theory has become a major piece in the modern statistical theory of fluids and can be found in most textbooks today.
- In the 1960s, it was realized that the above prediction fails close to a critical point (where $\nu = 0$). It was however generally expected to hold away from the critical point, in particular at sufficiently high temperatures and/or sufficiently small densities.

As I'll explain, the situation is actually more subtle than previously thought...

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- The **Hamiltonian** in $\Lambda \Subset \mathbb{Z}^d$ is the function

$$\mathscr{H}_\Lambda(\sigma) = -\sum_{\{x,y\}\subset\Lambda} J_{y-x}\,\sigma_x\sigma_y - h\sum_{x\in\Lambda}\sigma_x$$

defined on configurations $\sigma = (\sigma_x)_{x\in\Lambda} \in \{\pm 1\}^\Lambda$.

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- The **Gibbs measure in** Λ at inverse temperature $\beta \geq 0$ is the probability measure

$$\mathbb{P}_{\Lambda;\beta,h}(\sigma) = \frac{e^{-\beta\mathscr{H}_\Lambda(\sigma)}}{Z_{\Lambda;\beta,h}}.$$

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- We are interested in the (infinite-volume) **Gibbs measure** uniquely defined by

$$\mathbb{P}_{\beta,h} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \mathbb{P}_{\Lambda;\beta,h} \quad (\text{when } h \neq 0), \qquad \mathbb{P}_{\beta,0} = \lim_{h\downarrow 0} \mathbb{P}_{\beta,h}.$$

2-point function and inverse correlation length

- The **2-point function** is defined, for any $x \in \mathbb{Z}^d$, by

$$G_{\beta,h}(x) = \mathrm{Cov}_{\mathbb{P}_{\beta,h}}(\sigma_0, \sigma_x).$$

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$$\nu_{\beta,h}(\vec{s}) = -\lim_{n\to\infty} \frac{1}{n} \log G_{\beta,h}([n\vec{s}]),$$

where $[x] \in \mathbb{Z}^d$ is the coordinatewise integer part of $x \in \mathbb{R}^d$.

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- There exists $\beta_c = \beta_c(d) \in (0, +\infty]$ such that

$$(\beta, h) \neq (\beta_c, 0) \implies \min_{\vec{s}} \nu_{\beta,h}(\vec{s}) > 0$$

$$\forall \vec{s} \in \mathbb{S}^{d-1}, \nu_{\beta_c,0}(\vec{s}) = 0$$

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- When $(\beta, h) \neq (\beta_c, 0)$, $\nu_{\beta,h}$ can be extended to a norm on $\mathbb{R}^d$:

$$\nu_{\beta,h}(x) = \|x\| \cdot \nu_{\beta,h}(\hat{x}),$$

where $\hat{x} = x / \|x\| \in \mathbb{S}^{d-1}$.

— Rigorous results —

1. Finite-range interactions

$$\exists R < \infty, \quad \|x\| > R \implies J_x = 0.$$

Asymptotic behavior of the 2-point function

Ornstein–Zernike asymptotic behavior holds when $\beta < \beta_c$ or $h \neq 0$:

Theorem

Assume that $\beta < \beta_c$ or $h \neq 0$. Then, as $n \to \infty$,

$$G_{\beta,h}([n\vec{s}]) = \frac{\Psi_{\beta,h}(\vec{s})}{n^{(d-1)/2}} e^{-\nu_{\beta,h}(\vec{s})n} (1 + o(1))$$

Moreover, the functions $\Psi_{\beta,h}$ and $\nu_{\beta,h}$ are analytic in $\vec{s}$.

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The above result has a **long history**. Some milestones are

- ▷ *Wu 1966, Wu–McCoy–Tracy–Barouch 1976:* exact computation in $d = 2$ when $h = 0$
- ▷ *Abraham–Kunz 1977, Paes-Leme 1978:* any dimension, $h = 0$ and $\beta \ll 1$
- ▷ *Campanino–Ioffe–V. 2003:* any dimension, $h = 0$ and $\beta < \beta_c$
- ▷ *Campanino–Ioffe–V. 2008:* extension to Potts models
- ▷ *Ott 2020:* any dimension, $h \neq 0$ and β arbitrary

Asymptotic behavior of the 2-point function

Knowledge much less complete when $h = 0$ and $\beta > \beta_c$, but **OZ can be violated**:

Theorem

Assume that $h = 0$ and $\beta > \beta_c$. Then, as $n \to \infty$,

$$G_{\beta,h}([n\vec{s}]) = \begin{cases} \frac{\Psi_{\beta,0}(\vec{s})}{n^{2}} e^{-\nu_{\beta,0}(\vec{s})n}(1 + o(1)) & \text{in the planar case} \\ \frac{\Psi_{\beta,0}(\vec{s})}{n^{(d-1)/2}} e^{-\nu_{\beta,0}(\vec{s})n}(1 + o(1)) & \text{when } d \geq 3, \beta \gg 1 \end{cases}$$

- ▷ *Wu–McCoy–Tracy–Barouch 1976:* exact computations in the planar case
- ▷ *Bricmont–Fröhlich 1985:* $d \geq 3$, $\beta \gg 1$

Remarks: 1. OZ decay expected to persist for all $\beta > \beta_c$ when $d \geq 3$.

2. OZ expected to hold when $d = 2$ and the graph is not planar.

— Rigorous results —

2. Infinite-range interactions

[From now on, for simplicity, we assume that $h = 0$ and omit it from the notations. However, many results extend to all $h \neq 0$.]

Subexponential decay of interactions

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- In fact, sharp asymptotics have been obtained:

Theorem [Newman–Spohn 1998]

Assume that $\beta < \beta_c$. Then, as $\|x\| \to \infty$,

$$G_\beta(x) = \beta \chi_\beta^2 J_x (1 + o(1)),$$

where $\chi_\beta = \sum_x G_\beta(x)$ is the magnetic susceptibility.

- This result was extended to Potts models by Aoun (2020).

Superexponential decay of interactions

- Assume that interactions decay **strictly faster than exponentially**:

$$\forall c \in \mathbb{R}, \quad \lim_{\|x\|\to\infty} e^{c\|x\|} J_x = 0.$$

In this case, the results obtained in the finite-range case when $\beta < \beta_c$ extend verbatim:

Work in progress [Aoun–Ott–V.]

Assume that $\beta < \beta_c$. Then, as $n \to \infty$,

$$G_\beta([n\vec{s}]) = \frac{\Psi_\beta(\vec{s})}{n^{(d-1)/2}} e^{-\nu_\beta(\vec{s})n} (1 + o(1))$$

Moreover, the functions Ψ_β and ν_β are analytic in $\vec{s}$.

In particular, **OZ asymptotic behavior holds in all dimensions**.

Exponential decay of interactions

Let us thus consider the “critical” case of exponentially decaying interactions:

$$J_x = \psi(x) e^{-|||x|||},$$

where

- ▷ $|||\cdot|||$ denotes an arbitrary norm on $\mathbb{R}^d$
- ▷ ψ is subexponential

and we assume in this talk, for simplicity, that

$$\forall x \in \mathbb{R}^d, \quad \psi(x) = \psi(|||x|||) > 0.$$

Qualitative behavior of the inverse correlation length

$\beta \uparrow \beta_c$

It is a general result that $\lim_{\beta \uparrow \beta_c} \nu_\beta(\vec{s}) = 0$ for all $\vec{s} \in \mathbb{S}^{d-1}$.

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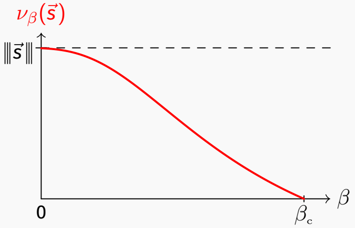
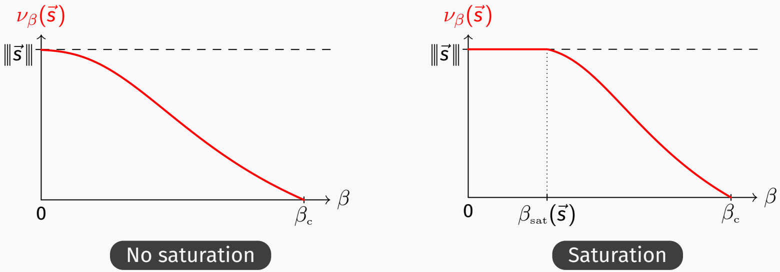
- This leads to **two scenarios**:



Saturation



- We define $\beta_{sat}(\vec{s}) = \sup\{\beta \geq 0 \,:\, \nu_\beta(\vec{s}) = \|\vec{s}\|\}$ and say that there is **saturation** in direction $\vec{s}$ at inverse temperature β if $\beta < \beta_{sat}(\vec{s})$.
- Observe that $\beta_{sat}(\vec{s}) > 0$ implies that $\beta \mapsto \nu_\beta(\vec{s})$ **is not analytic** on $[0, \beta_c)$.

Criterion for the existence of a saturation regime

- Let us introduce the generating functions (for $t \in \mathbb{R}^d$)

$$\mathbb{G}_\beta(t) = \sum_{x \in \mathbb{Z}^d} e^{t \cdot x} \, G_\beta(x), \qquad \mathbb{J}(t) = \sum_{x \in \mathbb{Z}^d} e^{t \cdot x} J_x.$$

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$$\mathscr{U} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}, \quad \mathscr{W} = \{t \in \mathbb{R}^d : \forall x \in \mathbb{R}^d, \; t \cdot x \leq \|x\|\}.$$

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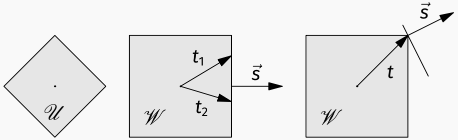
Easy fact: $\mathscr{W}$ is the closure of the domain of convergence of $\mathbb{J}$.

- $t \in \partial\mathscr{W}$ is **dual** to $\vec{s} \in \mathbb{S}^{d-1}$ if

$$t \cdot \vec{s} = \|\vec{s}\|$$



Criterion for the existence of a saturation regime

Theorem [Aoun–Ioffe–Ott-V. 2021]

Let $\vec{s} \in \mathbb{S}^{d-1}$ and $\mathscr{T}_{\vec{s}} = \{t \in \partial\mathscr{W} \ : \ t \text{ is dual to } \vec{s}\}$. Then

$$\beta_{\text{sat}}(\vec{s}) > 0 \iff \inf_{t \in \mathscr{T}_{\vec{s}}} \mathbb{J}(t) < \infty.$$

► Shows that the correlation length does not always depend analytically on β in the whole high-temperature region (even in dimension 1!).

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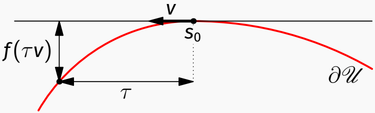
- Extends to many models: Potts, XY, (FK)-percolation, GFF with mass as parameter, Ising with magnetic field as parameter, etc.

- Application of this criterion yields, for instance, that
  - If $\sum_{\ell \in \mathbb{N}} \Psi(\ell) = \infty$, then $\beta_{\text{sat}}(\vec{s}) = 0$ for all $\vec{s} \in \mathbb{S}^{d-1}$.
  - If $\sum_{\ell \in \mathbb{N}} \ell^{d-1} \Psi(\ell) < \infty$, then $\beta_{\text{sat}}(\vec{s}) > 0$ for all $\vec{s} \in \mathbb{S}^{d-1}$.

It would be useful to have a more explicit criterion in general.

A more explicit form of the criterion

- Local parametrization of $\partial\mathscr{U}$ at $s_0 = \vec{s}/\|\vec{s}\|$:



- Assume there exist $C, c > 0$ and a nonnegative nondecreasing function g such that

$$C\,g(\tau) \geq f(\tau v) \geq c\,g(\tau)$$

for all small $\tau \geq 0$ and all vectors v in a supporting hyperplane to $\partial\mathscr{U}$ at s_0.

Then the following more explicit version of the criterion holds:

$$\beta_{\text{sat}}(\vec{s}) > 0 \iff \sum_{\ell \geq 1} \psi(\ell)\big(\ell g^{-1}(1/\ell)\big)^{d-1} < \infty.$$

An example

- Let us illustrate the previous criterion for the model on $\mathbb{Z}^2$ with

$$J_x = \|x\|_p^{\alpha} \, \mathrm{e}^{-\|x\|_p},$$

where $\|\cdot\|_p$ is the p-norm, $\alpha \in \mathbb{R}$ and we assume $p \in (2, \infty)$.

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- Let $\vec{s} \in \mathbb{S}^1$ and $s_0 = (x_0, y_0) = \vec{s} / \|\vec{s}\|_p$.

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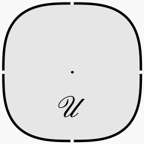
- Let $\vec{s} \in \mathbb{S}^1$ and $s_0 = (x_0, y_0) = \vec{s} / \|\vec{s}\|_p$.
- When both x_0 and y_0 are nonzero,



$$f(\tau v) = \frac{p-1}{2} \frac{x_0^{p-2} y_0^{p-2}}{(x_0^{2p-2} + y_0^{2p-2})^{3/2}} \tau^2 + o(\tau^2).$$

We can thus choose $g(\tau) = \tau^2$. It follows that

$$\beta_{\text{sat}}(\vec{s}) > 0 \iff \sum_{\ell \geq 1} \ell^{\alpha} (\ell \sqrt{1/\ell}) < \infty \iff \alpha < -\frac{3}{2}.$$

An example

- In the remaining cases, that is, when $\vec{s} \in \{\pm(1,0), \pm(0,1)\}$,

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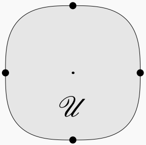

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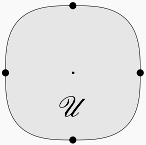

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$$\beta_{\text{sat}}(\vec{s}) > 0 \iff \sum_{\ell \geq 1} \ell^\alpha (\ell \sqrt[p]{1/\ell}) < \infty \iff \alpha < \frac{1}{p} - 2.$$

▶ This shows that the positivity of $\beta_{\text{sat}}(\vec{s})$ (and a fortiori its actual value) depends on the norm. It also **depends in general on the direction**: when $-\frac{3}{2} > \alpha \geq \frac{1}{p} - 2$,

- ▷ $\beta_{\text{sat}}(\vec{s}) = 0$ for $\vec{s} \in \{\pm(1,0), \pm(0,1)\}$
- ▷ $\beta_{\text{sat}}(\vec{s}) > 0$ in all the other directions.

Consequences for OZ behavior

▶ When $\beta_{\text{sat}}(\vec{s}) > 0$, **OZ asymptotics are known not to hold at sufficiently high temperatures** for various classes of interactions. For instance:

- ▷ $\psi(x) = C\,\mathrm{e}^{-c\|x\|^{\alpha}}$ with $0 < \alpha < 1$
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Theorem [Aoun–Ioffe–Ott–V. 2021]

Assume that $\beta_{\text{sat}}(\vec{s}) > 0$ and that ψ is as above. Then, for all β small enough, there exist $c_+ > c_- > 0$ such that, for all n,

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Consequences for OZ behavior

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- Current **work in progress**:
  - more general classes of prefactors
  - extension to all $\beta \in (0, \beta_{\text{sat}}(\vec{s}))$
  - sharp asymptotics
  - proof that OZ holds for all $\beta \in (\beta_{\text{sat}}(\vec{s}), \beta_c)$
  - proof that OZ usually holds at $\beta_{\text{sat}}(\vec{s})$

— Comments on the proof —

Proof of the criterion when $d = 1$

1. Preliminaries.

- When $\beta < \beta_c$,

$$G_\beta(x) = \sum_{\gamma\colon 0 \to x} q_\beta(\gamma), \tag{1}$$

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- One can show that there exists $C_\beta > 0$ such that, if $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ is self-avoiding,

$$\prod_{i=1}^{n} \beta J_{\gamma_i - \gamma_{i-1}} \geq \mathrm{q}_\beta(\gamma) \geq \prod_{i=1}^{n} C_\beta J_{\gamma_i - \gamma_{i-1}}. \tag{2}$$

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- Recall that $\mathbb{G}_\beta(t) = \sum_{x\in\mathbb{Z}} e^{tx} G_\beta(x)$ and $\mathbb{J}(t) = \sum_{x\in\mathbb{Z}} e^{tx} J_x$.

Observe that

- the radii of convergence of $\mathbb{G}_\beta$ and $\mathbb{J}$ are given by $\nu_\beta(1)$ and $|||1|||$, respectively;
- (1) and (2) imply that $\nu_\beta(1) \leq |||1|||$ (use $\gamma = \{0, x\}$ for a lower bound on $\mathbb{G}_\beta$).

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&= \sum_{x\in\mathbb{Z}} e^{tx} \sum_{n\geq 1} \sum_{\substack{y_1,\dots,y_n \\ \sum_k y_k = x}} q_\beta\big((0, y_1, y_1+y_2, \dots, y_1+\cdots+y_n)\big) \\
&= \sum_{x\in\mathbb{Z}} \sum_{n\geq 1} \sum_{\substack{y_1,\dots,y_n \\ \sum_k y_k = x}} e^{t(y_1+\cdots+y_n)} q_\beta\big((0, y_1, y_1+y_2, \dots, y_1+\cdots+y_n)\big) \\
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- Let $\vec{s} = 1$. Then $\mathscr{T}_{\vec{s}} = \{|\!|\!|1|\!|\!|\}$.

The criterion thus reduces to: $\beta_{\text{sat}}(1) > 0 \iff \mathbb{J}(|\!|\!|1|\!|\!|) < \infty$

2. Proof that $\mathbb{J}(\|1\|) < \infty \implies \beta_{\text{sat}}(1) > 0$.

- By (3) and the upper bound in (2),

$$\mathbb{G}_\beta(t) \le \sum_{n \ge 1} \sum_{y_1, \dots, y_n} \prod_{k=1}^{n} e^{t y_k} \beta J_{y_k} = \sum_{n \ge 1} (\beta \mathbb{J}(t))^n.$$

Therefore,

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Therefore,

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- In particular, with $t = \|1\|$,

$$\mathbb{J}(\|1\|) < \infty \implies \mathbb{G}_\beta(\|1\|) < \infty \implies \|1\| \le \nu_\beta(1) \implies \|1\| = \nu_\beta(1),$$

for all $\beta < 1/\mathbb{J}(\|1\|)$. This show that $\beta_{\text{sat}}(1) > 0$.

Proof of the criterion when $d = 1$

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\mathbb{G}_\beta(t) &\geq \sum_{n\geq 1} \sum_{y_1 \geq 1} \cdots \sum_{y_n \geq 1} \prod_{k=1}^{n} C_\beta J_{y_k} e^{t y_k} \\
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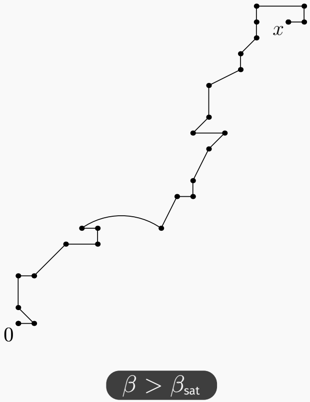
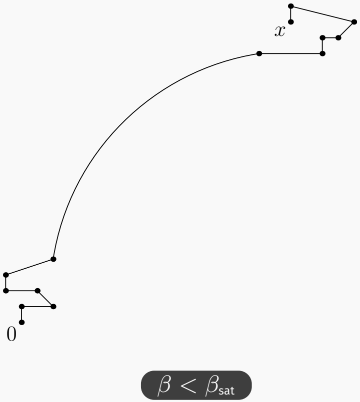
- This implies that $(1-\epsilon)\|1\| \geq \nu_\beta(1)$ and thus $\|1\| > \nu_\beta(1)$.
- We conclude that $\beta_{\text{sat}}(1) \leq \beta$. Since $\beta > 0$ was arbitrary, $\beta_{\text{sat}}(1) = 0$.

— One final remark —

An associated condensation phenomenon

► Sketches of typical paths γ contributing to $\mathsf{G}_\beta(x) = \sum_{\gamma:\, 0 \to x} \mathsf{q}_\beta(\gamma)$:





► Reminiscent of the condensation phenomenon for large deviations of the sum of independent random variables, depending on the fatness of their tail.

Thank you for your attention!

- D. B. Abraham and H. Kunz.
  Ornstein-Zernike theory of classical fluids at low density.
  Phys. Rev. Lett., 39(16):1011–1014, 1977.
- Y. Aoun.
  Sharp asymptotics of correlation functions in the subcritical long-range random-cluster and Potts models.
  Electron. Commun. Probab., 26:1 – 9, 2021.
- Y. Aoun, D. Ioffe, S. Ott, and Y. Velenik.
  Failure of Ornstein–Zernike asymptotics for the pair correlation function at high temperature and small density.
  Phys. Rev. E, 103:L050104, 2021.
- Y. Aoun, Dmitry Ioffe, Sébastien Ott, and Yvan Velenik.
  Non-analyticity of the Correlation Length in Systems with Exponentially Decaying Interactions.
  Commun. Math. Phys., 2021.

- J. Bricmont and J. Fröhlich.
  Statistical mechanical methods in particle structure analysis of lattice field theories. II. Scalar and surface models.
  Comm. Math. Phys., 98(4):553–578, 1985.
- M. Campanino, D. Ioffe, and Y. Velenik.
  Ornstein-Zernike theory for finite range Ising models above T_c.
  Probab. Theory Related Fields, 125(3):305–349, 2003.
- M. Campanino, D. Ioffe, and Y. Velenik.
  Fluctuation theory of connectivities for subcritical random cluster models.
  Ann. Probab., 36(4):1287–1321, 2008.
- C. L. Newman and H. Spohn.
  The Shiba relation for the spin-boson model and asymptotic decay in ferromagnetic Ising models, 1998.
  Unpublished.

- L. S. Ornstein and F. Zernike.
  Accidental deviations of density and opalescence at the critical point of a single substance.
  KNAW, Proceedings, 17 II:793–806, 1914.
- S. Ott.
  Sharp asymptotics for the truncated two-point function of the Ising model with a positive field.
  Commun. Math. Phys., 374(3):1361–1387, 2020.
- P. J. Paes-Leme.
  Ornstein-Zernike and analyticity properties for classical lattice spin systems.
  Ann. Physics, 115(2):367–387, 1978.
- B. Widom.
  On the radial distribution function in fluids.
  J. Chem. Phys., 41:74–77, 1964.

- T. T. Wu.
  Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. I.
  Phys. Rev., 149:380–401, 1966.
- T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch.
  Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region.
  Phys. Rev. B, 13:316–374, 1976.
- F. Zernike.
  The clustering-tendency of the molecules in the critical state and the extinction of light caused thereby.
  KNAW, Proceedings, 18 II:1520–1527, 1916.