On the Connectivities of Subcritical Random Cluster Models

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The random cluster model

$\mathcal{E}$: unordered pairs of nearest-neighbour sites of $\mathbb{Z}^d$

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$$e \in \mathcal{E} \text{ is } \begin{cases} \text{open} \\ \text{closed} \end{cases} \text{ in } \omega \in \Omega \text{ if } \begin{cases} \omega(e) = 1 \\ \omega(e) = 0 \end{cases}$$

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ω identified with the graph $(\mathbb{Z}^d, \{e : \omega(e) = 1\})$

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$x \leftrightarrow \infty$: $|\{y \,:\, y \leftrightarrow x\}| = \infty$

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Random cluster model depends on two parameters: $p \in [0, 1]$, $q \geq 1$.

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$q \geq 1$:

$$\text{“ } \mathbb{P}(\omega) = \prod_e p^{\omega(e)} (1-p)^{1-\omega(e)} \, q^{\mathfrak{N}(\omega)} \text{ ”}$$

$\mathfrak{N}(\omega)$ = Number of clusters in ω

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- $q \in \{2, 3, \ldots\}$: q-states Potts model

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For all $p < p_c^1$: uniqueness of the measure

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From now on: $p < p_c^1$

Connectivity function

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How fast?

Inverse correlation length

Inverse correlation length: for $x \in \mathbb{R}^d$,

$$\xi(x) = -\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}(0 \leftrightarrow [kx])$$

$[y] \in \mathbb{Z}^d$: componentwise integer part of $y \in \mathbb{R}^d$.

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Conjecture

For any $d \geq 1$ and $q \geq 1$,

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Known $(\forall d)$ when:

- $q = 1$ [Aizenman-Barsky '87]
- $q = 2$ [Aizenman *et al* '87]
- $q \gg 1$ [Laanait *et al* '91]

The main assumption

$\hat{p}_c(q, d)$: Supremum over all values of p such that:

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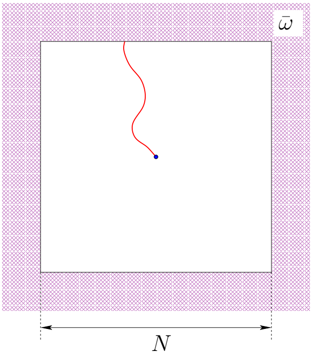
$\exists \nu_0, \nu_1 > 0$ such that, $\forall N$,

$$\sup_{\bar{\omega}} \mathbb{P}\Bigg($$



$$\Bigg| \, \bar{\omega} \Bigg) \leq \nu_0 e^{-\nu_1 N}$$

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- $d = 2$: $q \geq 1$ [Alexander '04]

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$$p < \hat{p}_{\mathrm{c}}(q,d)$$

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$$\mathbf{U}_\xi = \left\{ x \in \mathbb{R}^d \,:\, \xi(x) \le 1 \right\}$$

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$$\mathbf{K}_\xi = \left\{t \in \mathbb{R}^d \,:\, (t, \vec{n})_d \leq \xi(\vec{n}),\ \forall \vec{n} \in \mathbb{S}^{d-1}\right\}$$

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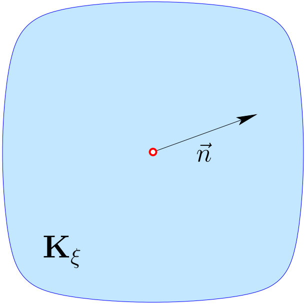
Each set encodes all information about ξ

Equidecay set and Wulff shape



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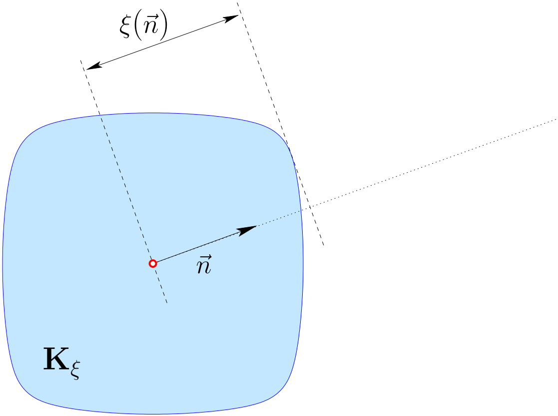

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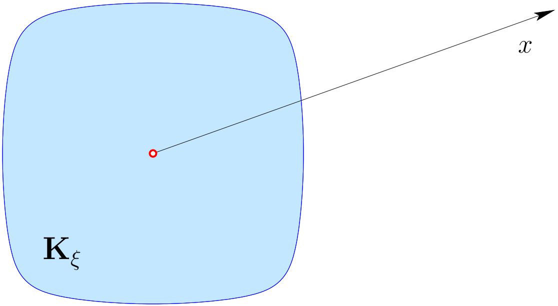

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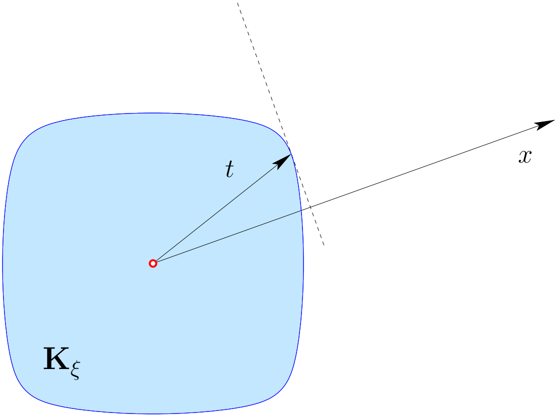

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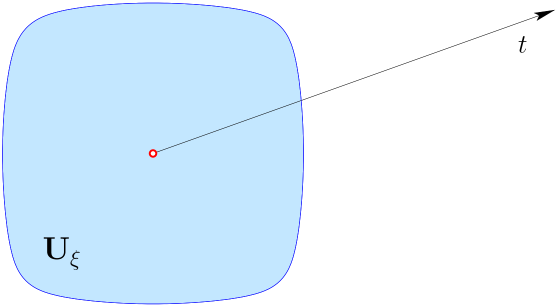

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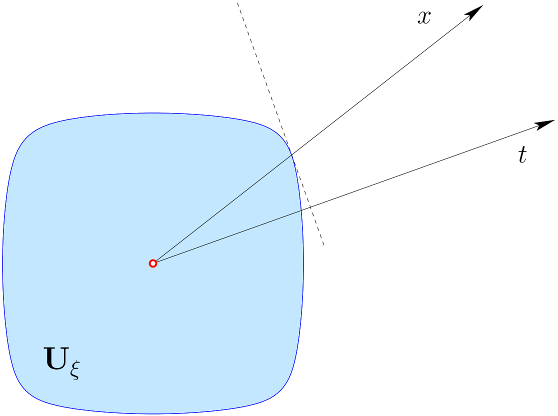

Ornstein-Zernike asymptotics

Theorem

Let $p < \hat{p}_c$. Then

$$\mathbb{P}(0 \leftrightarrow x) = \frac{\Psi(\vec{n}_x)}{|x|^{(d-1)/2}} \exp\left(-\xi(\vec{n}_x)\,|x|\right)(1 + o(1))$$

uniformly as $|x| \to \infty$. The functions Ψ and ξ are positive, analytic functions on $\mathbb{S}^{d-1}$.

Corollary: exit probability

Let $\Lambda_N = \{-N, \ldots, N\}^d$.

Theorem

Let $p < \hat{p}_c$. There exists a constant $\psi(p, q, d)$, such that

$$\mathbb{P}(0 \leftrightarrow \partial\Lambda_N) = \psi\, e^{-N\xi(\mathsf{e}_1)}(1 + o(1))$$

Strict convexity of ξ

Theorem

Let $p < \hat{p}_c$. Then $\mathbf{K}_\xi$ and $\mathbf{U}_\xi$ have analytic, strictly convex boundaries, with uniformly positive Gaussian curvature.

Forward cone

$$Y_\delta(t) = \left\{x \in \mathbb{R}^d \,:\, (x,t)_d \geq (1-\delta)\xi(x)\right\} \quad (\delta \in (0,1))$$

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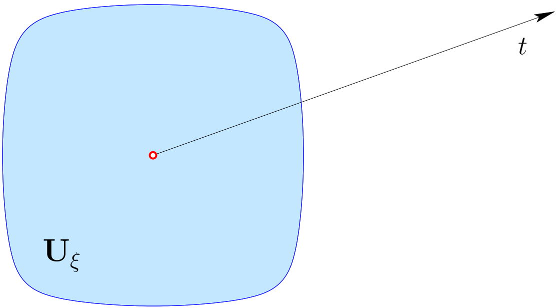

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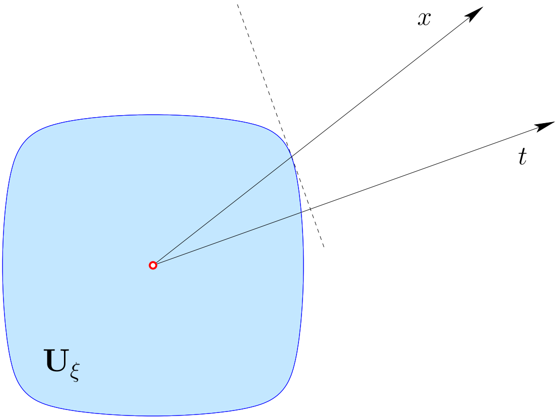

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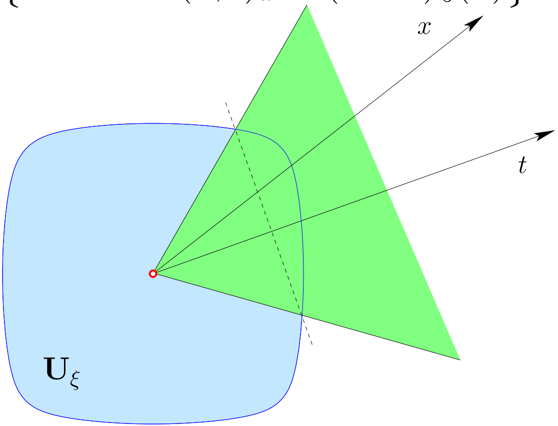

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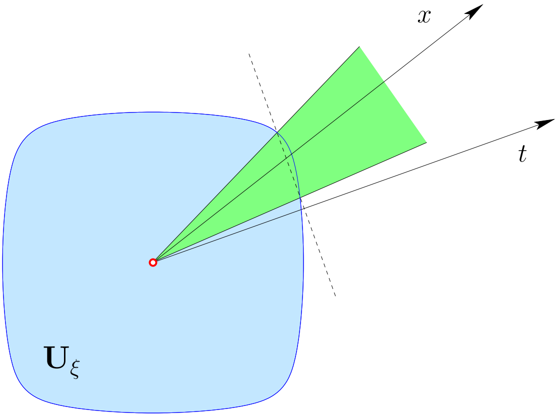
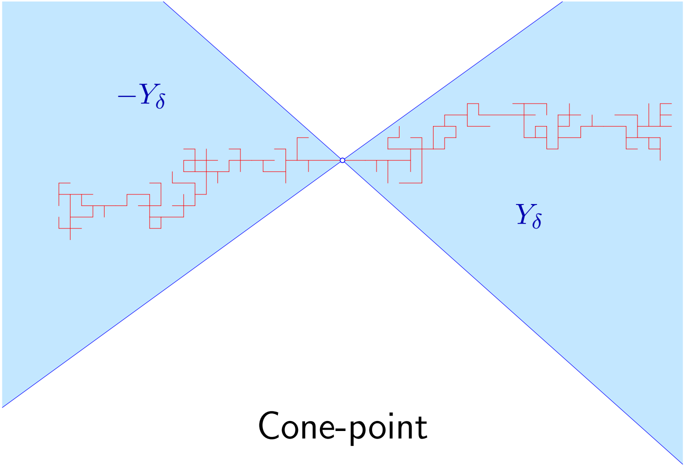


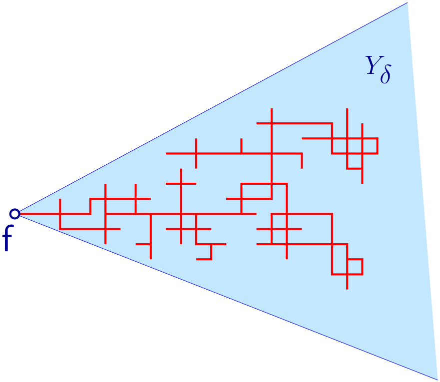
Irreducible pieces



Cone-point

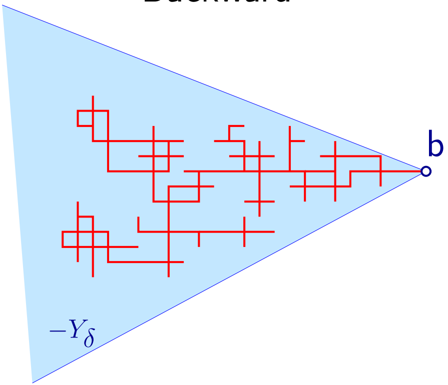
Irreducible pieces

Forward

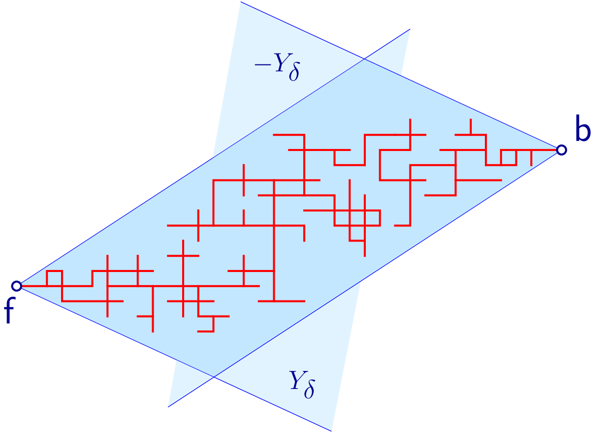


Irreducible pieces

Backward



Irreducible pieces



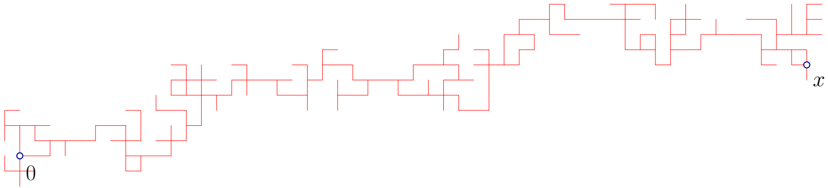
Irreducible pieces

Theorem: Separation of masses

$$\sum_{\substack{\gamma:0\leftrightarrow x \\ \text{(f/b)-irreducible}}} \mathbb{P}(\gamma) \leq e^{-\xi(x)-\kappa_1|x|}$$

uniformly in x and t dual to x.

Effective 1D structure of long clusters



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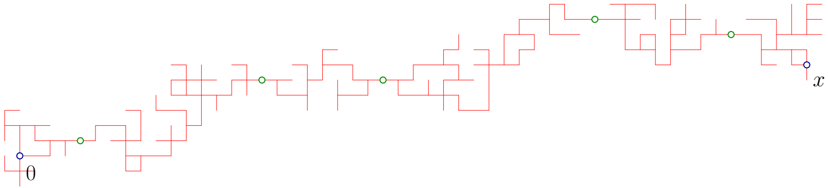

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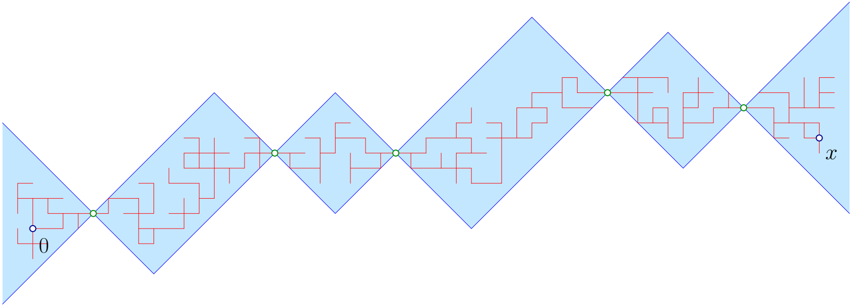

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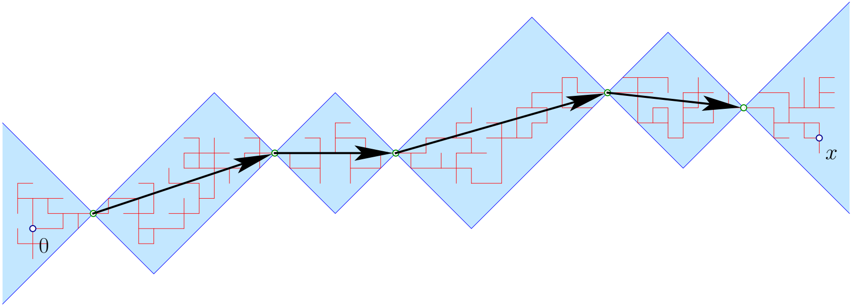

Invariance principle for long clusters

Let $x \in \mathbb{S}^{d-1}$ and let $t \in \partial \mathbf{K}_\xi$ be the dual point

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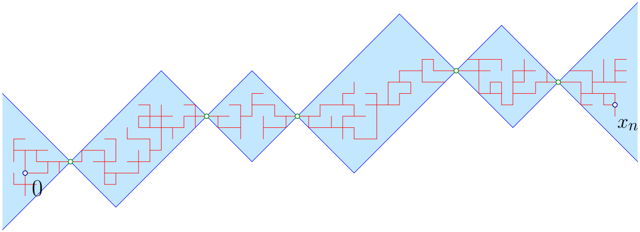
Sequence of vertices: $x_n = [nx]$

Corresponding sequence of conditional measures:

$$\mathbb{P}_{x,n}(\cdot) = \mathbb{P}\left(\cdot \,\middle|\, 0 \leftrightarrow x_n\right)$$

Invariance principle for long clusters



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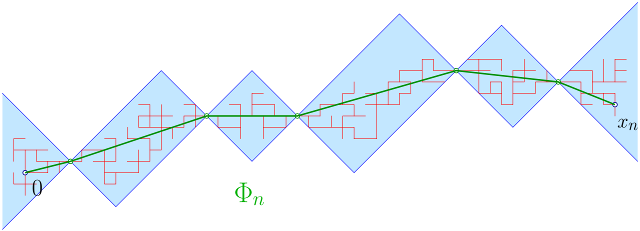

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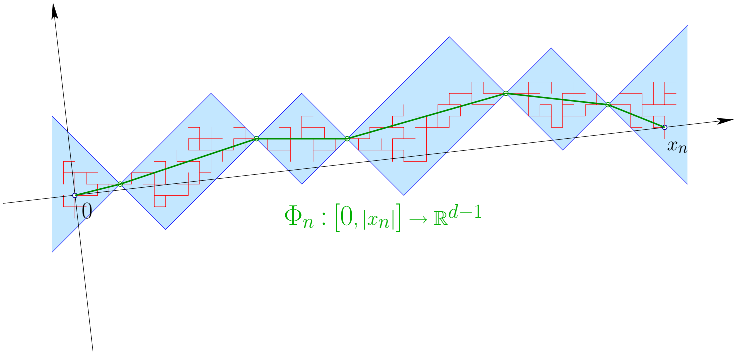

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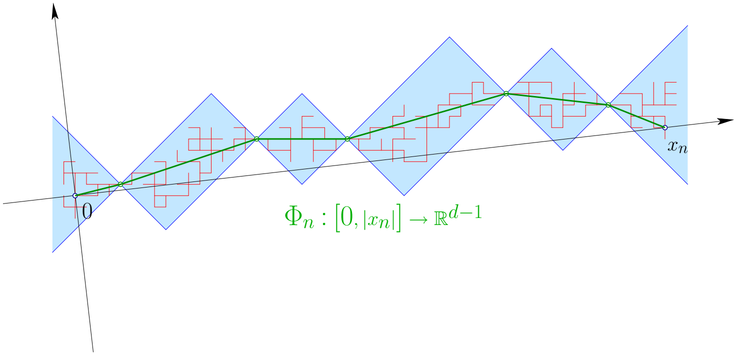
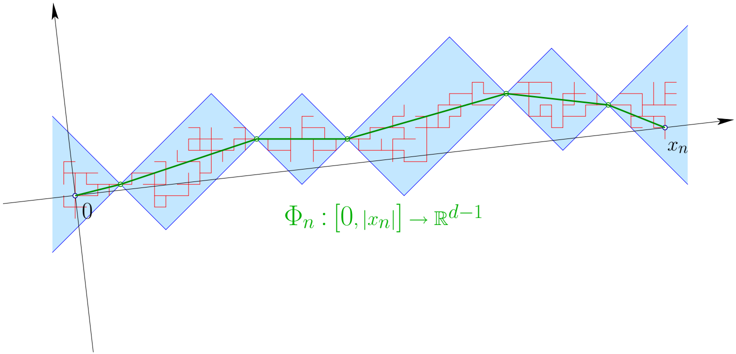


$$\lim_{n\to\infty} \mathbb{P}_{x,n}\left(\mathrm{d_H}(\mathbf{C}_{0,x_n}, \Phi_n) > (\log n)^2\right) = 0$$

Invariance principle for long clusters



$$\phi_n(r) = \frac{1}{\sqrt{n}}\Phi_n(nr)$$

Invariance principle for long clusters

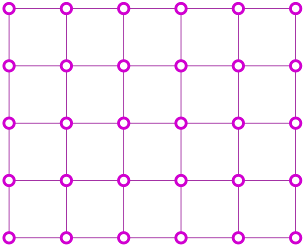
Theorem

Let $p < \hat{p}_c$. Then $\{\phi_n(\cdot)\}$ weakly converges under $\{\mathbb{P}_{x,n}\}$ to the distribution of

$$\left(\sqrt{\chi_1}B_1(\cdot), \ldots, \sqrt{\chi_{d-1}}B_{d-1}(\cdot)\right),$$

where $B_1, \ldots, B_{d-1}$ are independent standard Brownian bridges on $[0,1]$, and $\{\chi_i(t)\}$ are the principal curvatures of $\partial\mathbf{K}_\xi$ at t.

Duality



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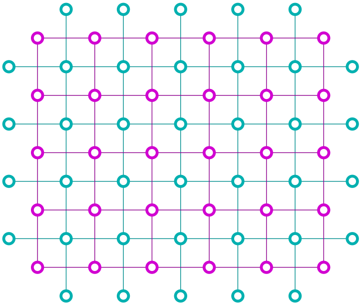

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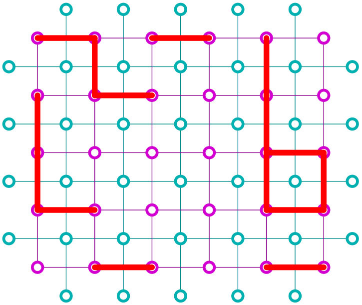

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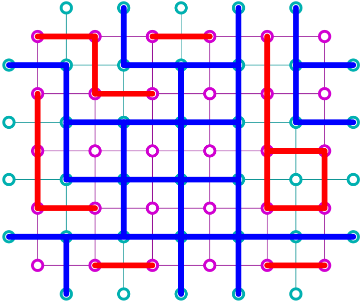

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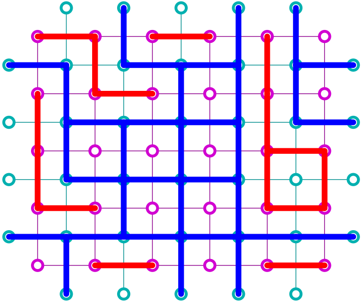
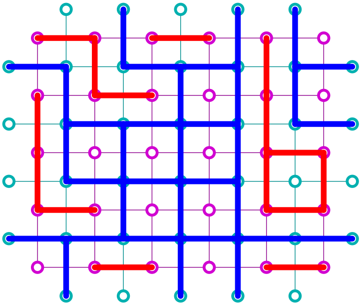


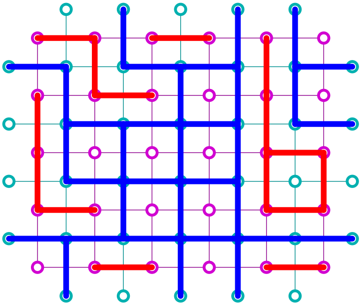
$$(p, q) \longleftrightarrow (p^\star, q)$$

Duality



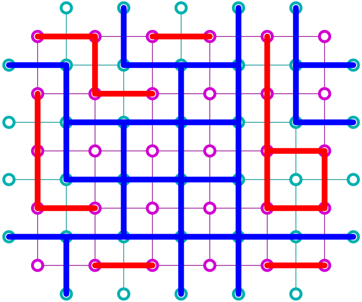
$$\frac{p^\star}{1-p^\star} = q\,\frac{1-p}{p}$$

Duality



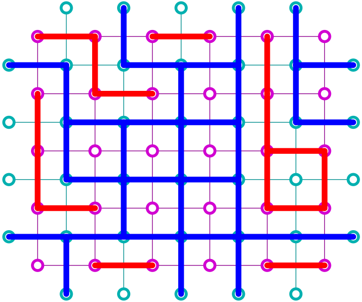
Subcritical ⟷ Supercritical

Duality



$\xi \longleftrightarrow$ surface tension

Duality



$\xi > 0 \longleftrightarrow$ positive surface tension

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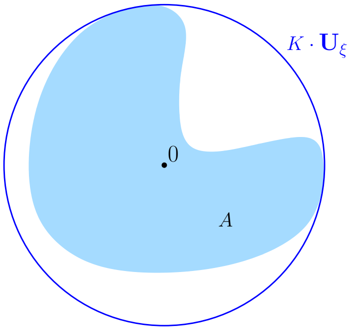
Preceding results can be reformulated as: At any temperature at which the surface tension is positive,

- No roughening transition: The equilibrium crystal shape $\mathbf{K}_\xi$ has an analytic, strictly convex boundary, with strictly positive curvature.
- Diffusive scaling of interfaces: Interface weakly converges to a (suitably scaled) Brownian bridge.

Mixing for connectivities



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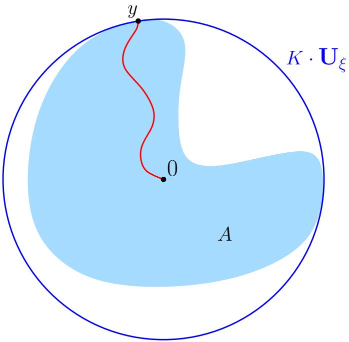
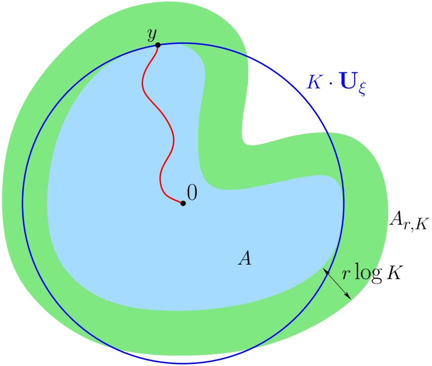


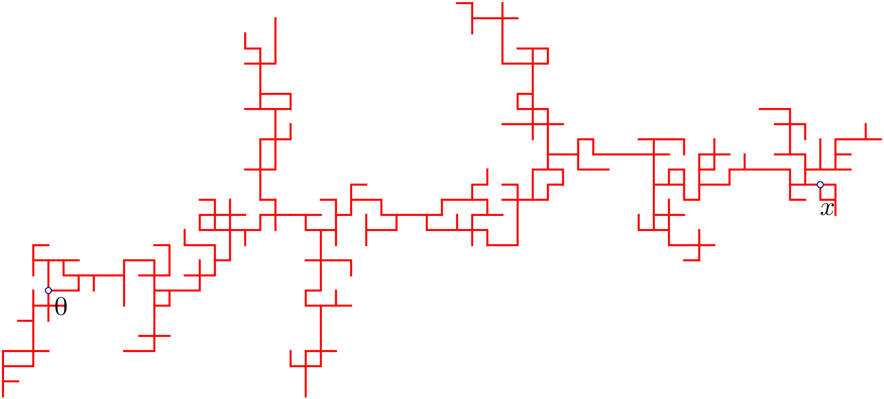
$$\mathbb{P}(0 \overset{A}{\leftrightarrow} y \qquad\qquad\qquad\qquad)$$

Mixing for connectivities



$$\sup_{\bar\omega} \mathbb{P}(0 \overset{A}{\leftrightarrow} y \,|\, \omega \equiv \bar\omega \text{ off } A_{r,K}) \le e^{-K}(1 + o_K(1))$$

Coarse-graining



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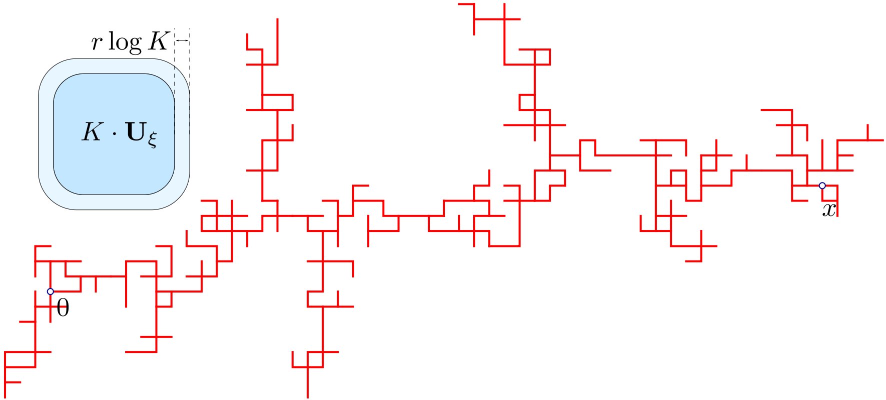

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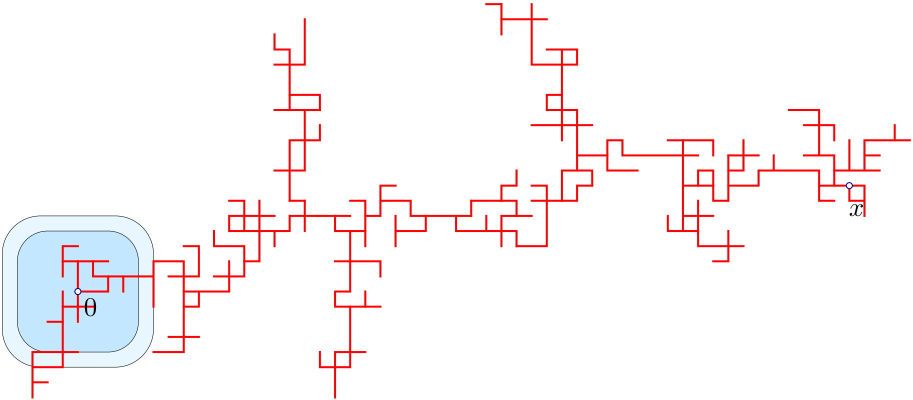

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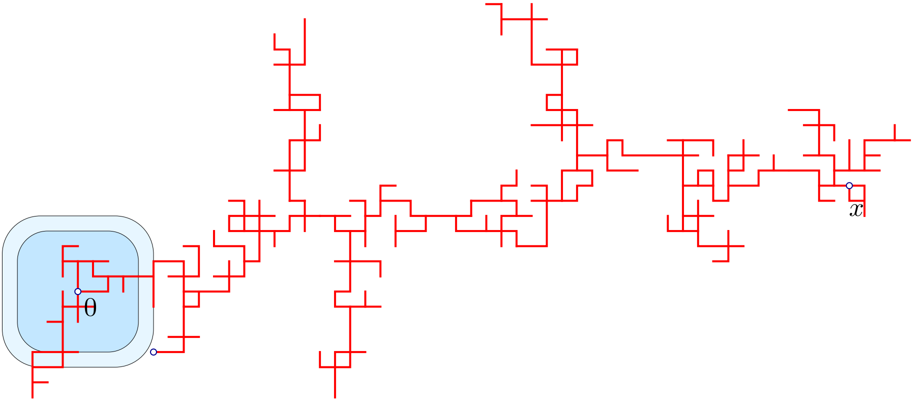

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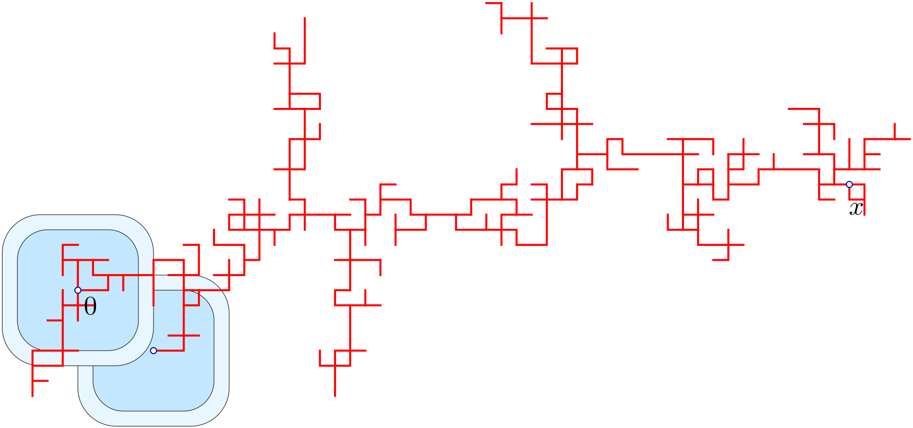

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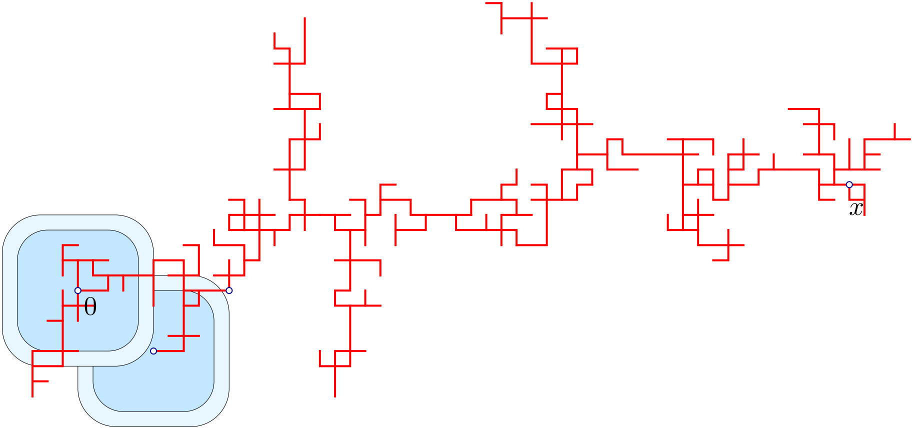

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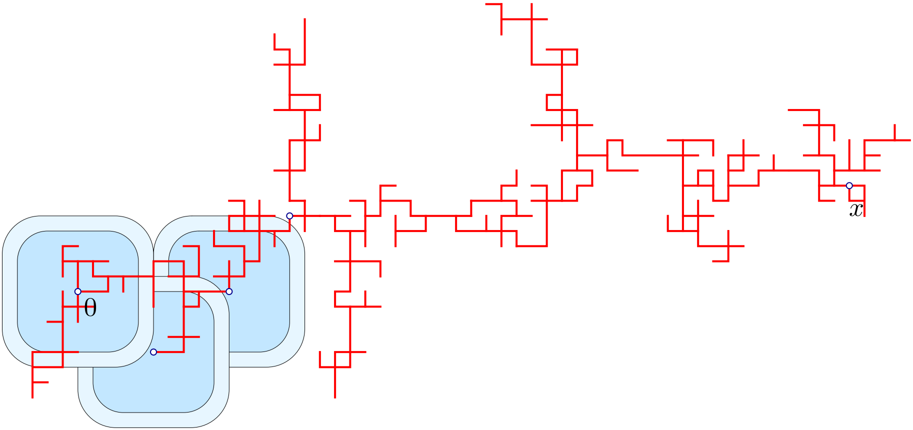

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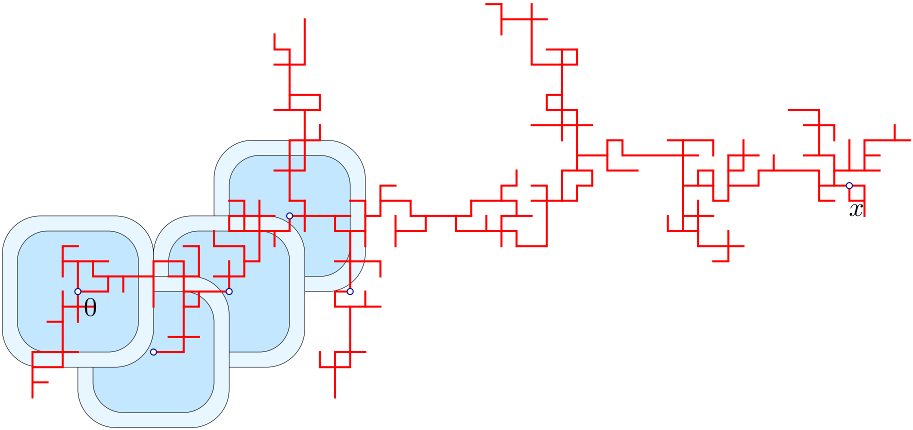

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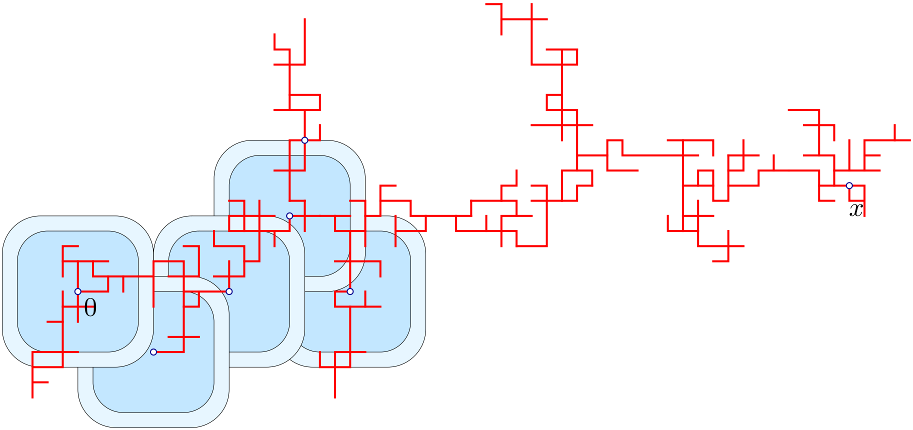

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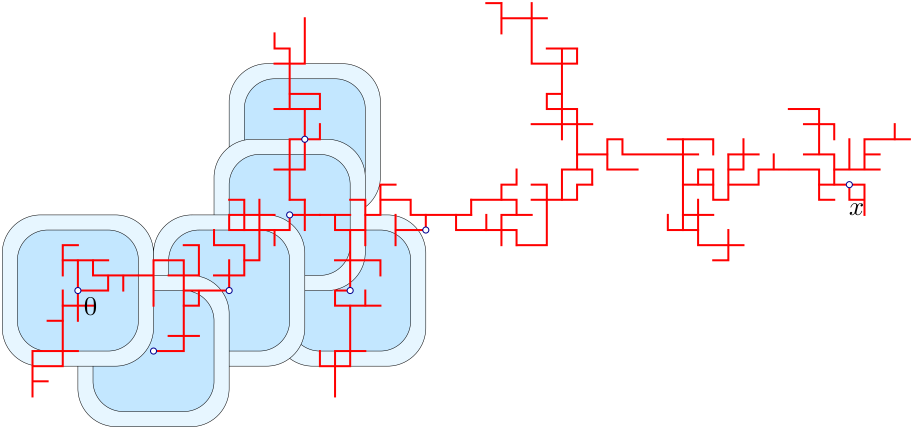

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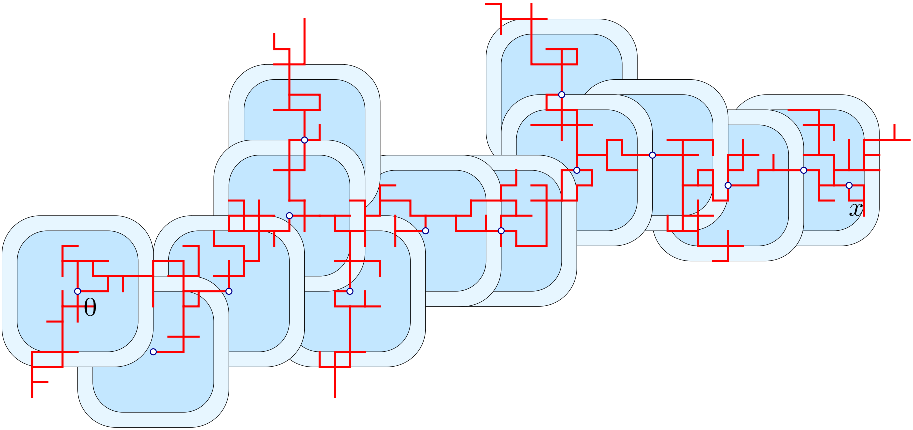

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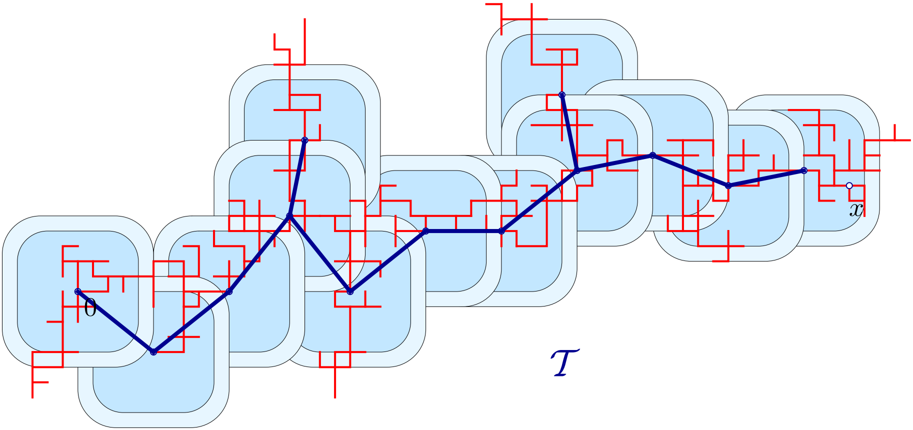

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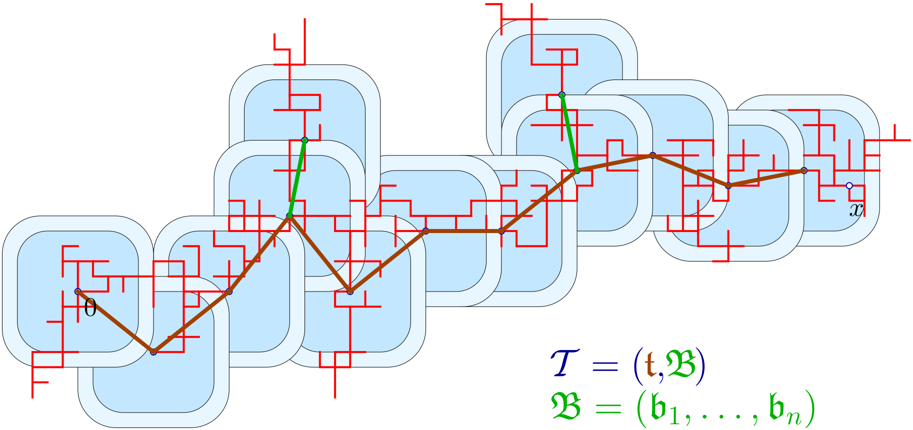
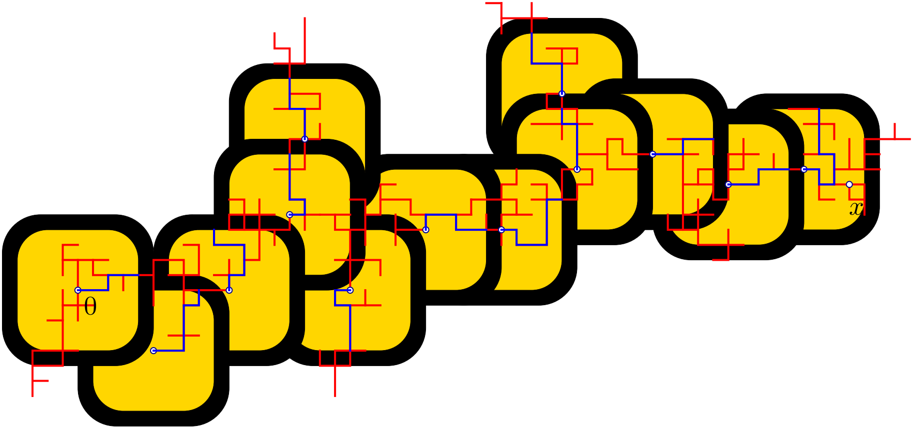


$$\mathcal{T} = (\mathfrak{t}, \mathfrak{B})$$
$$\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$$

Coarse-graining



Coarse-graining

Mixing estimate

$$\Downarrow$$

$$\mathbb{P}(\mathcal{T}) \leq \exp\{-K(1 - o_K(1))N(\mathcal{T})\}$$

$N(\mathcal{T})$: number of vertices in $\mathcal{T}$

Rough bounds

By energy/entropy arguments:

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- Typical trees have a small trunk

$$\exists c_1, c_2 : \ \mathbb{P}\Big(N(\mathfrak{t}) > c_1 \frac{|x|}{K} \ \Big| \ 0 \leftrightarrow x\Big) \leq e^{-c_2|x|}$$

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- Typical trees have few and small branches

$$\forall \kappa > 0 : \; \mathbb{P}\Big(N(\mathfrak{B}) > \kappa \frac{|x|}{K} \;\Big|\; 0 \leftrightarrow x\Big) \leq e^{-\frac{1}{2}\kappa|x|}$$

Surcharge function

t dual to x

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The surcharge function

$$\mathfrak{s}_t(y) = \xi(y) - (t, y)_d$$

measures the typicality of an increment

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measures the typicality of an increment

For a trunk $\mathfrak{t} = (\mathfrak{t}_0, \ldots, \mathfrak{t}_N)$, we set

$$\mathfrak{s}_t(\mathfrak{t}) = \sum_{l=1}^{N} \mathfrak{s}_t(\mathfrak{t}_l - \mathfrak{t}_{l-1})$$

Surcharge function

Surcharge inequality

Let $\epsilon > 0$. There exists $K_0(\epsilon)$ such that

$$\mathbb{P}\big(\mathfrak{s}_t(\mathfrak{t}) > 2\epsilon|x| \;\big|\; 0 \leftrightarrow x\big) \leq \mathrm{e}^{-\epsilon|x|}$$

uniformly in $x \in \mathbb{Z}^d$, $t \in \partial\mathbf{K}_\xi$ dual to x, and $K > K_0$.

Forward cone

$$Y_\delta(t) = \left\{ y \in \mathbb{R}^d \,:\, (y,t)_d > (1-\delta)\xi(y) \right\}$$
$$= \left\{ y \in \mathbb{R}^d \,:\, \mathfrak{s}_t(y) < \delta\xi(y) \right\}$$

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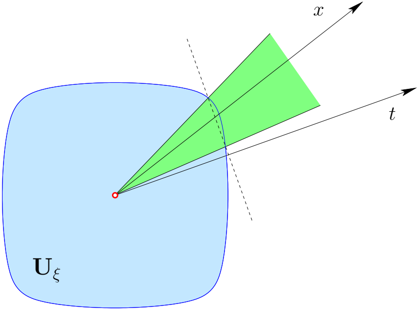
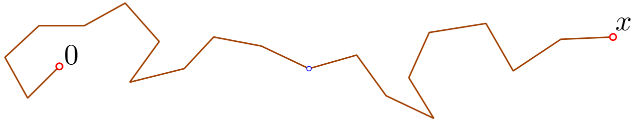


Cone points of trunks



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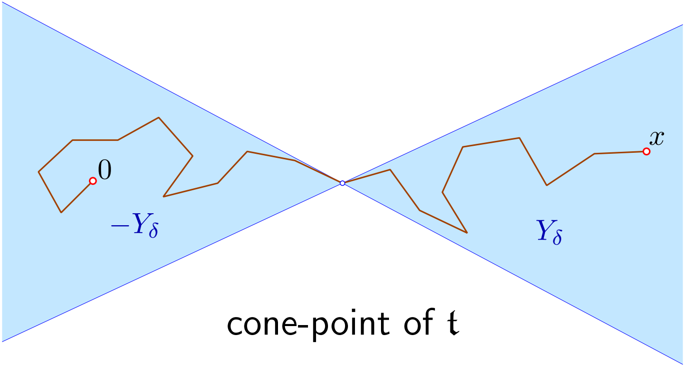


cone-point of $\mathfrak{t}$

Cone points of trunks

$$\#^{\text{n.c.p.}}(\mathfrak{t}) = \#\{\text{non-cone-points of } \mathfrak{t}\}$$

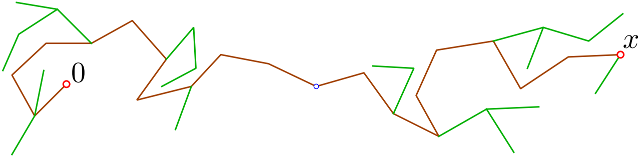
Lemma

$$\mathfrak{s}_t(\mathfrak{t}) \geq c_4\, \delta\, K\, \#^{\text{n.c.p.}}(\mathfrak{t})$$

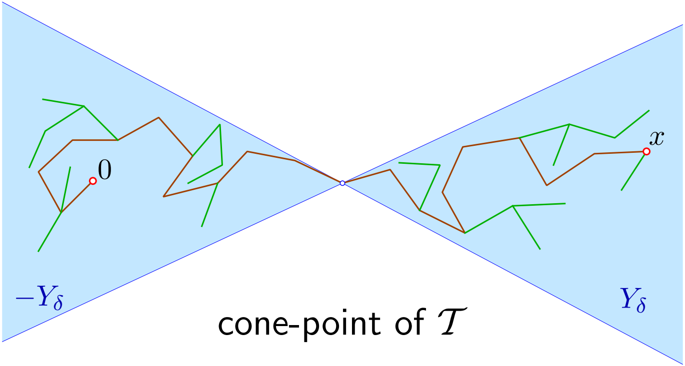
Consequently,

$$\mathbb{P}\Big(\#^{\text{n.c.p.}}(\mathfrak{t}) \geq \epsilon N(\mathfrak{t}) \;\Big|\; 0 \leftrightarrow x\Big) \leq \mathrm{e}^{-c_5 \epsilon |x|}$$

Cone points of trees



Cone points of trees



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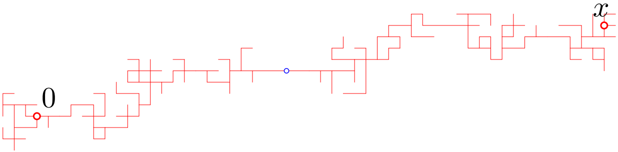
- The total size of the branches is small

Lemma

There exist $\nu > 0$ and c such that

$$\mathbb{P}\big(\#\{\text{cone-points of } \mathcal{T}\} < \nu \frac{|x|}{K} \mid 0 \leftrightarrow x\big) \leq \mathrm{e}^{-c|x|}$$

Cone points of clusters



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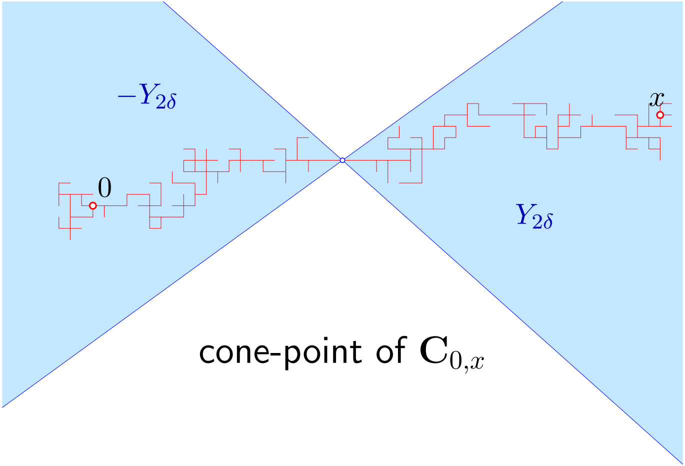
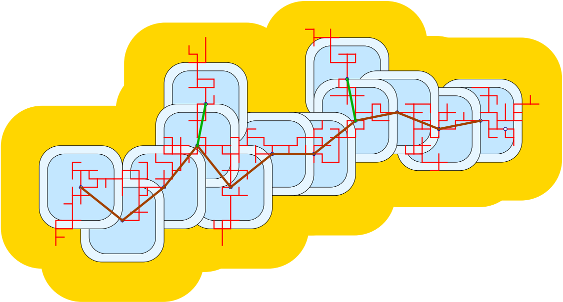


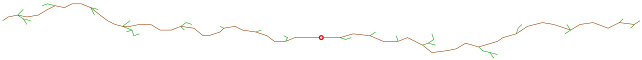
cone-point of $\mathbf{C}_{0,x}$

Cone points of clusters



Clusters remain close to their approximating tree

Cone points of clusters



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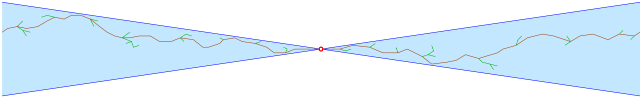

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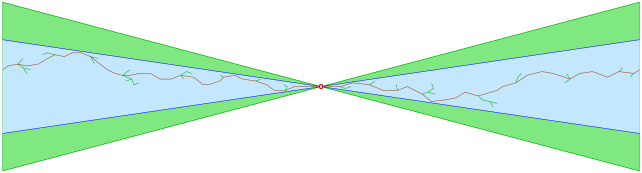

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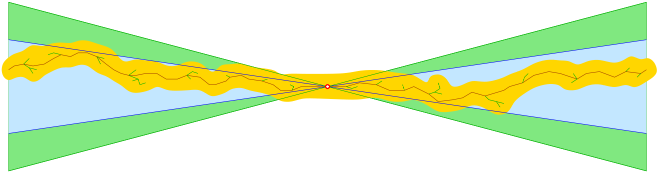

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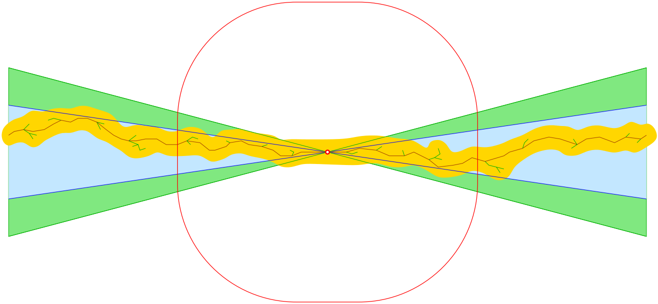

Cone points of clusters

Inside this finite ball, there is a strictly positive probability that the cluster remains inside the cone, uniformly in what happens elsewhere

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$$\Downarrow$$

Up to exponentially small error, a positive density of the cone-points of $\mathcal{T}$ are also cone-points of $\mathbf{C}_{0,x}$

Cone points of clusters

$\#^{\text{cone}}_{t,\delta}(\mathbf{C}_{0,x})$: number of cone-points of $\mathbf{C}_{0,x}$

Theorem

There exist $\delta \in (0, \frac{1}{2})$, ν and c such that

$$\mathbb{P}\Big(\#^{\text{cone}}_{t,\delta}(\mathbf{C}_{0,x}) \leq \nu|x| \;\Big|\; 0 \leftrightarrow x\Big) \leq \mathrm{e}^{-c|x|}$$

uniformly in x and dual t.

Decomposition into irreducible pieces

We can thus decompose the cluster $\mathbf{C}_{0,x}$ into irreducible pieces:

$$\mathbf{C}_{0,x} = \gamma^{\mathsf{b}} \amalg \gamma_1 \amalg \ldots \amalg \gamma_n \amalg \gamma^{\mathsf{f}}$$

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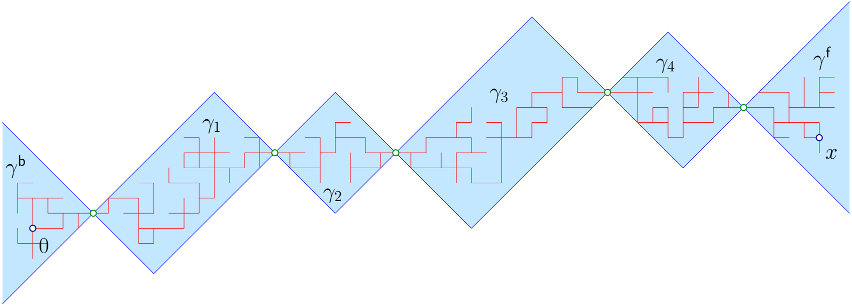

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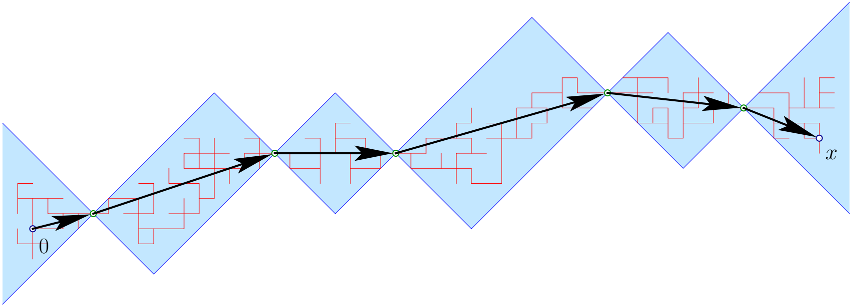
...and the Mass Separation Theorem holds, since irreducible pieces have only 1 or 2 cone-points.

One-dimensional effective process

We thus obtain an effective one-dimensional random path.



One-dimensional effective process

In the case $q = 1$, the increments are i.i.d. (except for the first and last ones, but they are small). So in that case, the proof of the Ornstein-Zernike asymptotics boils down to the classical local limit theorem for i.i.d. random variables with exponential moments.

One-dimensional effective process

For all other values of q, the distribution of an increment depends on all the others. In order to make progress, one has to understand how strong this dependency really is.

Ratio mixing

Conditional weights:

$$\mathbb{P}(\gamma_k \mid \gamma^{\mathsf{b}}, \gamma_1, \ldots, \gamma_{k-1}) = \frac{\mathbb{P}(\gamma^{\mathsf{b}} \amalg \gamma_1 \amalg \ldots \amalg \gamma_{k-1} \amalg \gamma_k)}{\mathbb{P}(\gamma^{\mathsf{b}} \amalg \gamma_1 \amalg \ldots \amalg \gamma_{k-1})}$$

Ratio mixing

One can prove that

$$\left| \frac{\mathbb{P}(\gamma_k \mid \gamma^{\mathsf{b}}, \gamma_1, \ldots, \gamma_j, \gamma_{j+1}, \ldots, \gamma_{k-1})}{\mathbb{P}(\gamma_k \mid \bar{\gamma}^{\mathsf{b}}, \bar{\gamma}_1, \ldots, \bar{\gamma}_j, \gamma_{j+1}, \ldots, \gamma_{k-1})} - 1 \right| \leq c_1 e^{-c_2(k-j)}$$

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This allows one to reformulate the problem as a local limit theorem for a suitable Ruelle's full shift operator on the countable alphabet of irreducible pieces, a problem which we already treated in our previous work on Ornstein-Zernike for Ising models.