Ferrari–Spohn asymptotics for an Ising model interface

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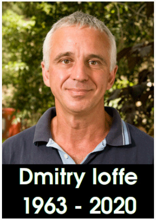
Joint work with Dmitry Ioffe, Sébastien Ott and Senya Shlosman



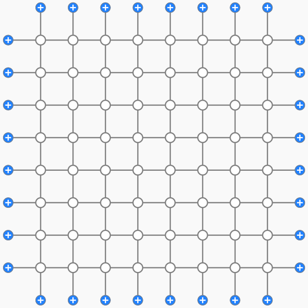
— Introduction —

- **Box:** $\mathrm{B}_N = \{-N+1, \dots, N\}^2$
- ⊕ **boundary condition:**

$$\Omega_N^{+} = \{\sigma = (\sigma_i)_{i \in \mathbb{Z}^2} \in \{\pm 1\}^{\mathbb{Z}^2} \ : \ \forall i \notin \mathrm{B}_N, \sigma_i = 1\}$$

- **Hamiltonian:** $\mathscr{H}_N(\sigma) = -\beta \sum_{\substack{\{i,j\} \cap \mathrm{B}_N \neq \varnothing \\ i \sim j}} \sigma_i \sigma_j$
- **Gibbs measure:** Probability measure on Ω_N^{+} s.t.

$$\mu_{N;\beta}^{+}(\sigma) = \frac{1}{\mathscr{Z}_{N;\beta}^{+}} \mathrm{e}^{-\mathscr{H}_N(\sigma)}$$



- **Box:** $\mathrm{B}_N = \{-N+1, \ldots, N\}^2$
- ⊕ **boundary condition:**

$$\Omega_N^{\oplus} = \{\sigma = (\sigma_i)_{i\in\mathbb{Z}^2} \in \{\pm 1\}^{\mathbb{Z}^2} \,:\, \forall i \notin \mathrm{B}_N, \sigma_i = 1\}$$

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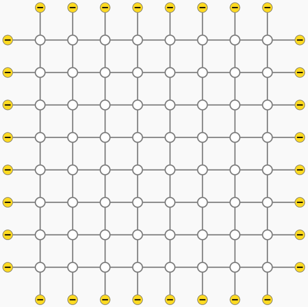
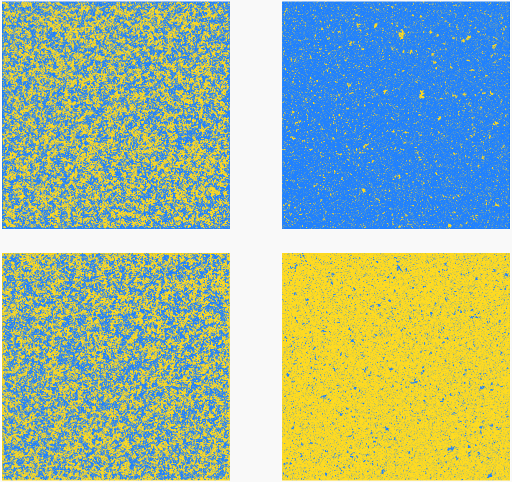


Extends trivially to other boundary conditions.
For instance, the ⊖ **boundary condition**: $\mu_{N;\beta}^{\ominus}, \ldots$

Phase transition

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. **Typical configurations** at $\beta \in [0,\infty)$ for $N > N_0(\beta)$:

$\beta < \beta_c$	$\beta > \beta_c$	



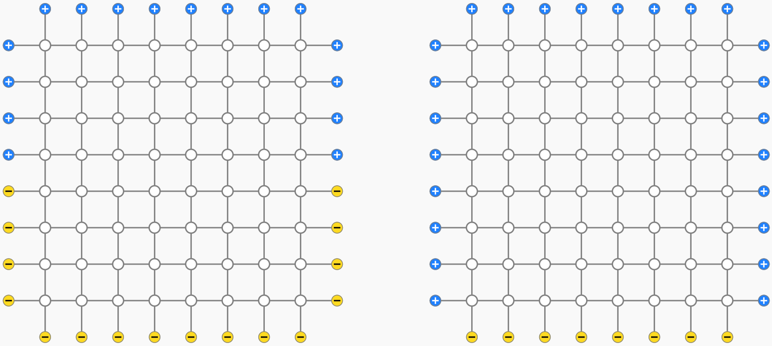
under $\mu^{\oplus}_{N;\beta}$

under $\mu^{\ominus}_{N;\beta}$

— Phase coexistence —

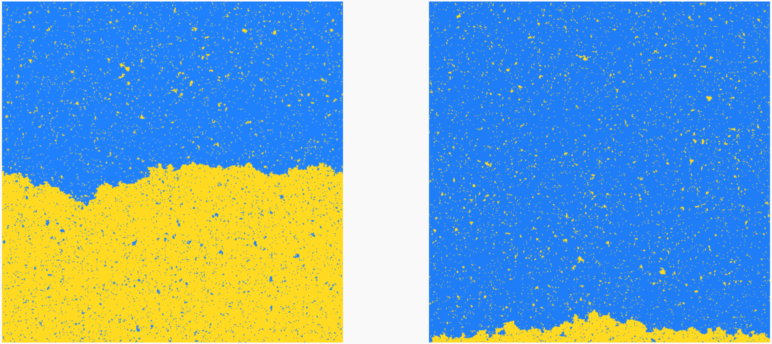
Phase coexistence: boundary conditions

To force spatial coexistence, consider 2 types of **Dobrushin boundary condition**:



Phase coexistence: scaling limit of the interface

Typical configurations induced by these boundary conditions when $\beta > \beta_c$



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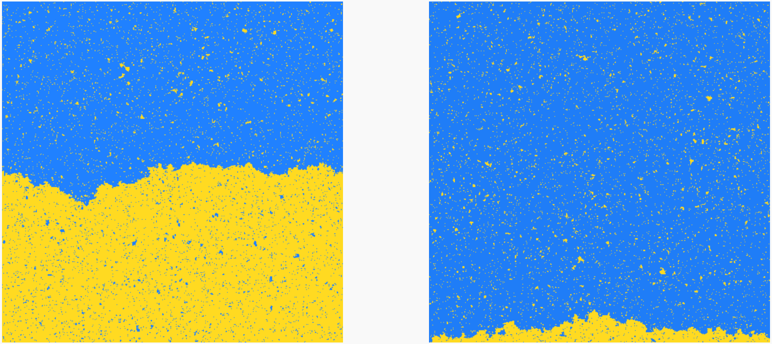


Corresponding **(diffusive) scaling limits** of the interface

Brownian bridge

[Greenberg, Ioffe 2005]

Brownian excursion

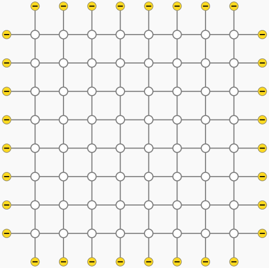
[Ioffe, Ott, V., Wachtel 2020]

(Diffusive constant $= \chi_\beta =$ the curvature of the Wulff shape in direction $\mathbf{e}_1$.)

— Metastability —

Effect of a magnetic field: metastability

- Let us consider again the ⊖ boundary condition



but let us add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in \mathrm{B}_N} \sigma_i$$

with $\boldsymbol{h} > \mathbf{0}$.

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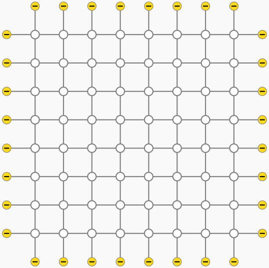

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$$-h \sum_{i \in \mathrm{B}_N} \sigma_i$$

with $\boldsymbol{h > 0}$.

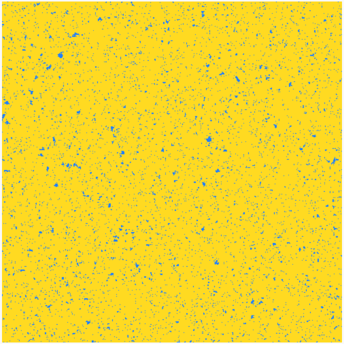
- This induces a **competition between the boundary condition and the magnetic field**:

effect of the boundary condition $\sim N$ effect of the field $\sim hN^2$

competition if $h \sim 1/N$

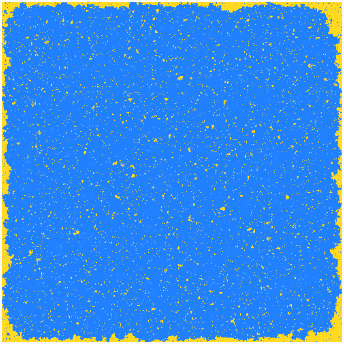
Effect of a magnetic field: metastability

- Let $h = \lambda/N$. [Schonmann and Shlosman 1996] proved: $\exists \lambda_c \in (0, \infty)$ such that



$\lambda < \lambda_c$

— phase is **metastable**



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— phase is **unstable**

- **Question:** Behavior of the layer of unstable — phase along the walls?

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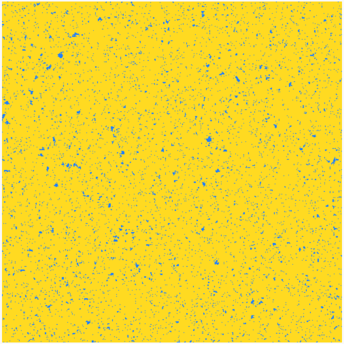

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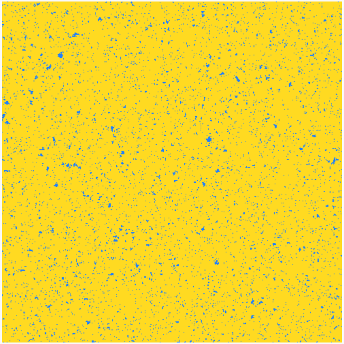

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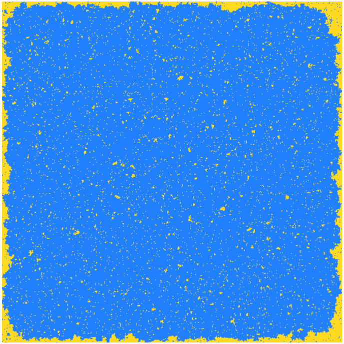

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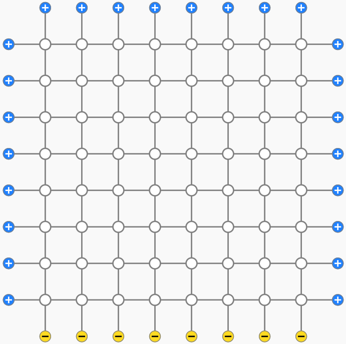
— Behavior of an unstable layer —

Effect of a magnetic field: simpler geometry

We consider again the boundary condition



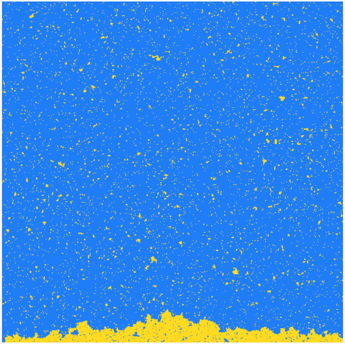
but add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in \mathrm{B}_N} \sigma_i$$

with $\boldsymbol{h} > \mathbf{0}$.

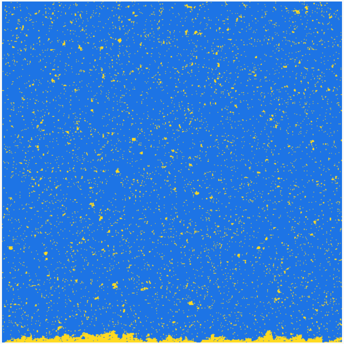
Effect of a magnetic field: critical prewetting

Let $\beta > \beta_c$. Since $h > 0$, the layer of $-$ phase becomes **unstable**:



$h = 0$

average width $= O(N^{1/2})$

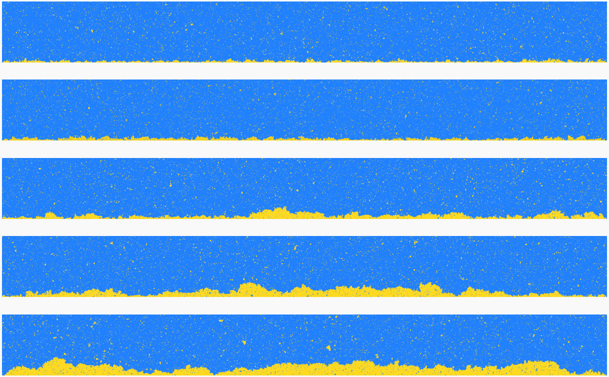


$h > 0$

average width $= O(1)$

Effect of a magnetic field: critical prewetting

The width of the layer increases as h decreases:



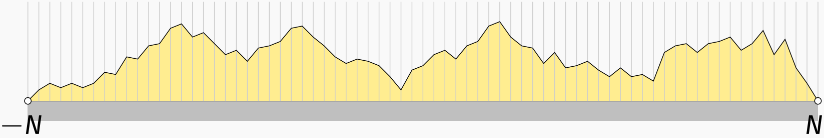
To get a meaningful scaling limit and mimic the previous situation, we choose $h = h(N)$ to be of the form

$$h = \frac{\lambda}{N}$$

for some $\lambda > 0$.

Earlier rigorous results

- This type of problem was first studied for **effective models** in
  - [Abraham, Smith 1986] specific integrable model: width $\sim N^{1/3}$, corr. length $\sim N^{2/3}$
  - [Hryniv, V. 2004] general class: width $\sim N^{1/3}$, correlation length $\sim N^{2/3}$
  - [Ioffe, Shlosman, V. 2015] general class: weak convergence to Ferrari–Spohn diffusion



$$\text{Prob(path)} \propto e^{-\frac{\lambda}{N}\,\text{Area}}\ \text{Prob}_{\text{RW}}(\text{path})$$

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- Results for the **2d Ising model** were obtained in
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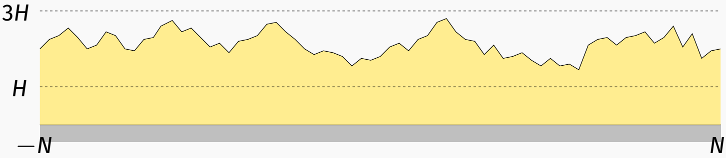
Goal of this work: complete the analysis by proving weak convergence to a Ferrari–Spohn diffusion for the 2d Ising interface

Parenthesis: why $N^{1/3}$?

► In effective models, it is easy to understand heuristically the origin of the $N^{1/3}$ scaling:

▷ consider a path staying in the tube $[-N, N] \times [H, 3H]$ for some fixed $H > 0$.



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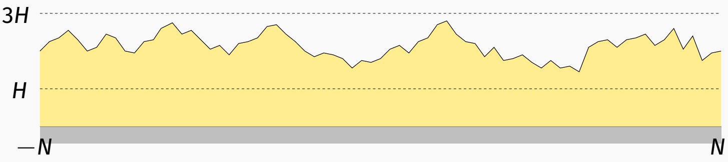

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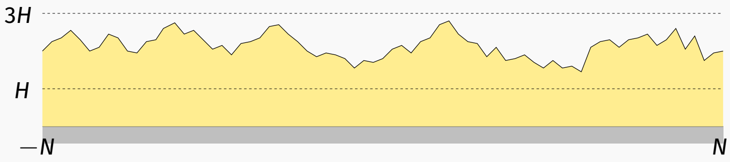

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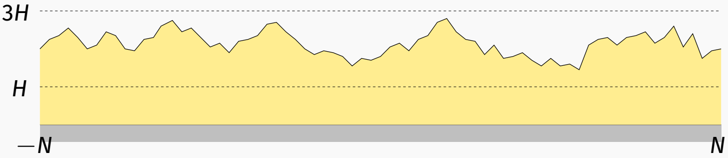

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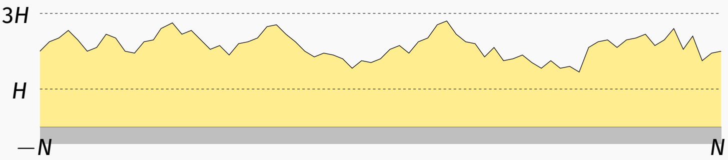

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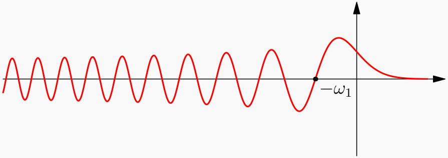
- This argument can be converted into a rigorous proof (for effective models).

- Let us introduce
  - the **spontaneous magnetization**: m_β^*
  - the **curvature of the Wulff shape** (in direction e_1): χ_β
  - the **Airy function** Ai and its first zero $-\omega_1$



- Set $\varphi_0(r) = \mathrm{Ai}\big((4\lambda m_\beta^* \sqrt{\chi_\beta})^{1/3}\, r - \omega_1\big)$.

- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0,\infty)$ with generator

$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$

and Dirichlet boundary condition at 0.

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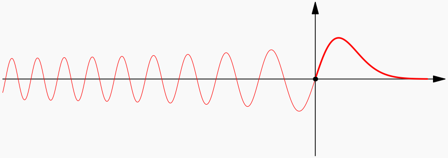

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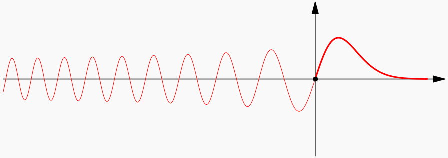

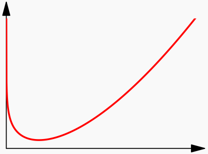

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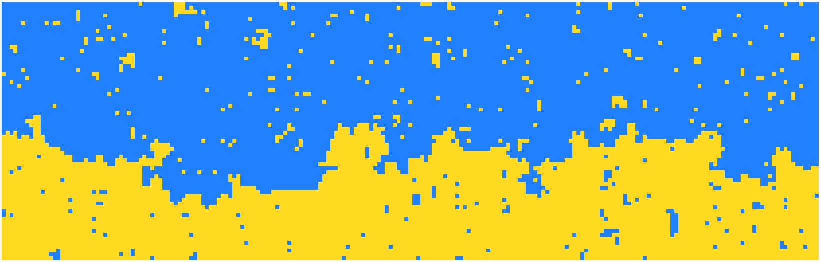
Structure of the interface

- We want to prove weak convergence of the interface towards the FS diffusion, but the interface is not the graph of a function:

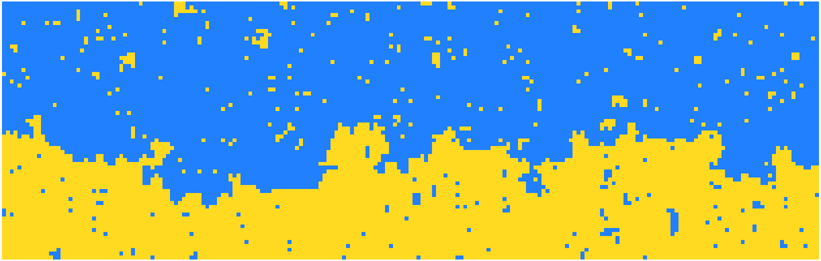


[zoom on a piece of interface]

- We thus need to explain what we mean by the above-mentioned convergence.

Structure of the interface

- We consider the upper and lower envelopes, whose linear interpolations are graphs of functions from $\mathbb{R}$ to $\mathbb{R}$.



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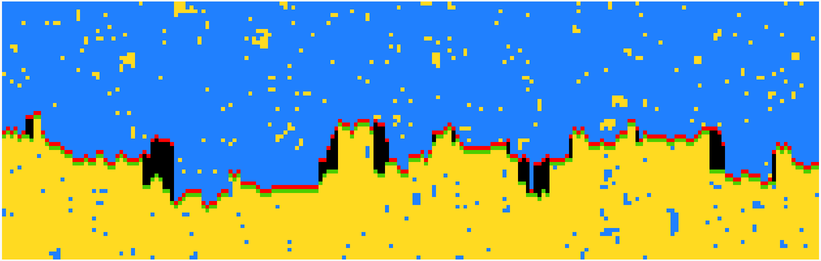

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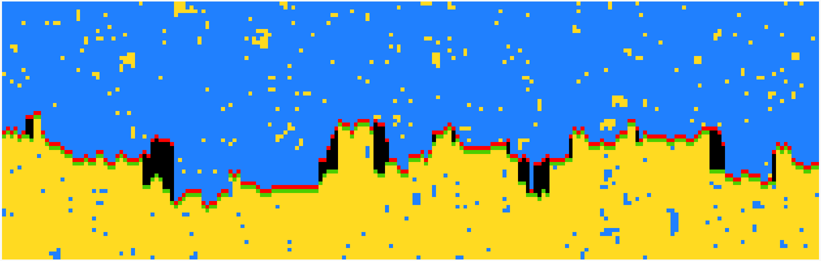


- It can be shown that there exists $K = K(\beta)$ such that the probability that these two envelopes differ by less than $K \log N$ everywhere tends to 1 as $N \to \infty$.
- Since the relevant vertical scale for our scaling will be $N^{1/3}$, one can use any of these envelopes for the weak convergence.

Main result

Theorem (Informal statement [Ioffe, Ott, Shlosman, V. 2020]**)**

Let $\hat{\gamma}^+ : \mathbb{R} \to \mathbb{R}$ be the function obtained from the (linearly interpolated) upper envelope by

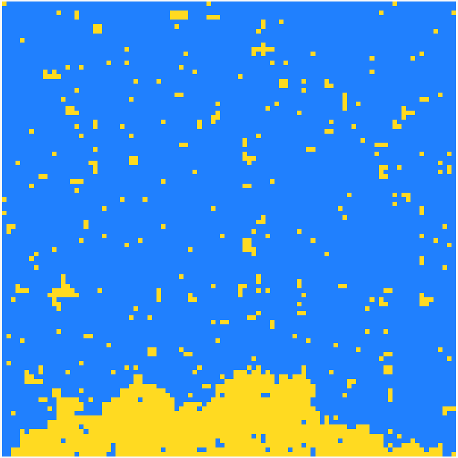
- *scaling it horizontally by $N^{-2/3}$*
- *scaling it vertically by $\chi_\beta^{-1/2} N^{-1/3}$*

Then, as $N \to \infty$, the distribution of $\hat{\gamma}^+$ converges weakly to that of the Ferrari–Spohn diffusion introduced in a previous slide.

— Sketch of proof —

Sketch of proof

- Any realization of the interface γ splits the box B_N into two sets:
  - $B_N^+[\gamma]$ **above** γ
  - $B_N^-[\gamma]$ **below** γ



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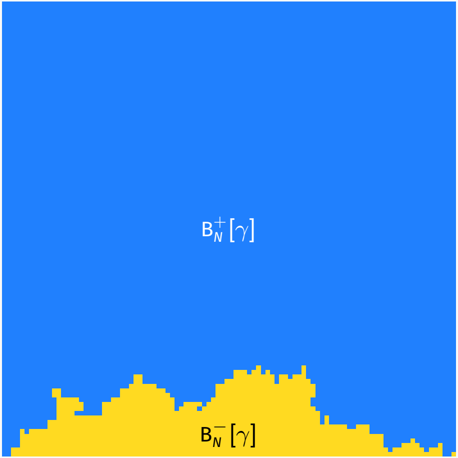
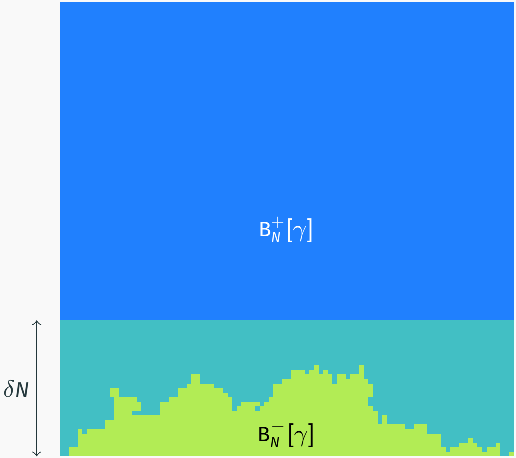


Sketch of proof — Step 1: (very) weak localization of the interface

- For any fixed $\delta > 0$, with high probability, γ remains at a distance at most δN from the bottom wall.



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▷ Obvious inside $B_N^+[\gamma]$: follows from FKG, since already true without magnetic field...

▷ Not so clear inside $B_N^-[\gamma]$: the $-$ phase is not stable $\rightsquigarrow$ may be favorable to create giant droplets of $+$ phase!

▷ However, the critical droplet of $+$ phase is a “square” of sidelength D such that $2\beta \cdot 4D \lesssim 2\frac{\lambda}{N} \cdot D^2$, that is, $D \gtrsim \frac{4\beta}{\lambda} N$.
$\rightsquigarrow$ choosing $\delta \ll \beta/\lambda$, we see that there is not enough room in $B_N^-[\gamma]$ to accommodate a critical droplet and **the layer of $-$ phase is metastable!**

Sketch of proof — Step 3: effective weight due to the magnetic field

- Since all contours are small, we can prove that, conditionally on the realization of γ, the magnetization concentrates (using results from [Ioffe, Schonmann 1998]):

$$\sum_{i \in \mathrm{B}_N} \sigma_i \approx m^*_\beta |\mathrm{B}^+_N[\gamma]| - m^*_\beta |\mathrm{B}^-_N[\gamma]| = m^*_\beta |\mathrm{B}_N| - 2m^*_\beta |\mathrm{B}^-_N[\gamma]|$$

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- From this, we deduce an **effective probability** for the contour γ in terms of the probability when $h = 0$: roughly speaking,

$$\mathrm{Prob}_{\beta,h=\lambda/N}(\gamma) \propto \exp\Big[-\frac{2\lambda m^*_\beta}{N} |\mathrm{B}^-_N[\gamma]|\Big] \ \mathrm{Prob}_{\beta,h=0}(\gamma)$$

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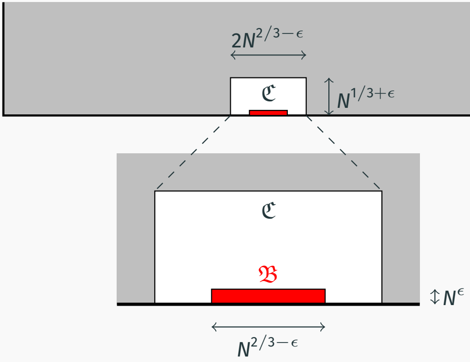
- Advantage: properties of γ well understood when $h = 0$ using the Ornstein–Zernike theory, which yields a **coupling to a random walk**.

Unfortunately, at this stage this random walk is **spatially inhomogeneous**, the increments depending in a complicated way on the distance to the bottom wall...

This problem is dealt with in the next steps.

Sketch of proof — Step 4: entropic repulsion

- First, given the following setting: for any fixed (small) $\epsilon > 0$,

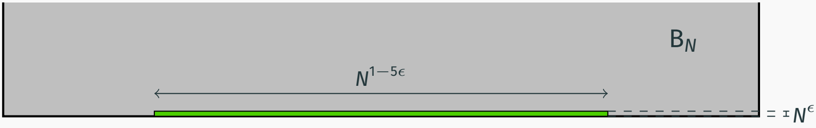


we show that, with high probability, γ does not intersect $\mathfrak{B}$, using the following facts:

- we can restrict to the same event in the box $\mathfrak{C}$ (by FKG)
- in the box $\mathfrak{C}$, the magnetic field is irrelevant ($\frac{\lambda}{N}|\mathfrak{C}| = 2\lambda$ is of order 1)
- this allows us to use weak convergence of the interface to Brownian excursion proved in [Ioffe, Ott, V., Wachtel 2020]

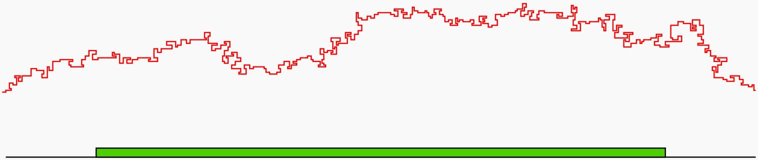
Sketch of proof — Step 4: entropic repulsion

► A union bound then allows one to conclude that, with high probability, γ stays above the following green rectangle:



Sketch of proof — Step 5: Effective model at $h = 0$

- When $h = 0$, using the Ornstein–Zernike techniques, as developed in [Campanino, Ioffe, V. 2003] and [Ott, V. 2018], we can couple the interface γ with a directed random walk on $\mathbb{Z}^2$.



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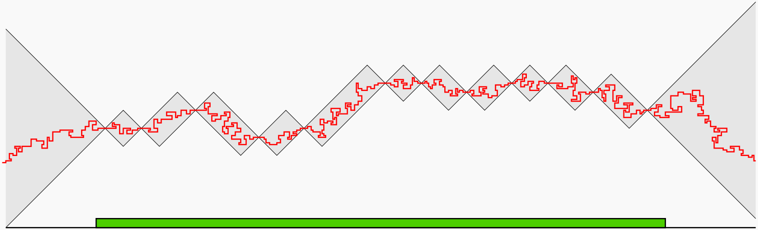

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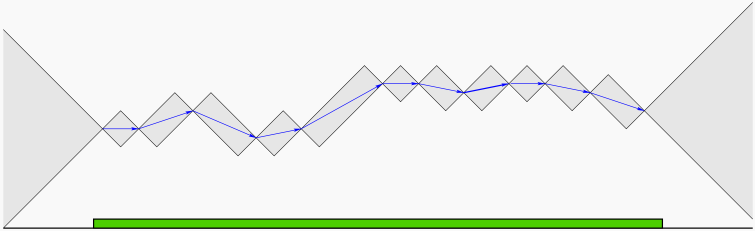

- By the previous step, above the green rectangle, the distance between γ and the bottom wall is at least N^{ϵ}. It follows that the finite-volume weights are well approximated by infinite-volume weights. Therefore, the resulting effective random walk can be taken spatially homogeneous.

Sketch of proof — Step 6: Full effective model

- Remember that

$$\text{Prob}_{\beta,h=\lambda/N}(\gamma) \propto \exp\left[-\frac{2\lambda m_\beta^*}{N}|\mathsf{B}_N^-[\gamma]|\right] \text{Prob}_{\beta,h=0}(\gamma)$$

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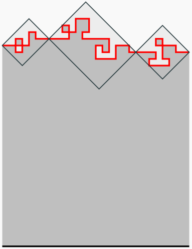
- This leads, *in the presence of the magnetic field λ/N*, to a coupling between γ and an effective RW model subject to an area-tilt: roughly speaking,

$$\mathrm{Prob}^{\mathrm{RW}}_{\beta,h=\lambda/N}(X) \propto \exp\Big[-\frac{2\lambda m_\beta^*}{N}\,\mathrm{Area}(X)\Big]\ \mathrm{Prob}^{\mathrm{RW}}_{\beta,h=0}(X)$$



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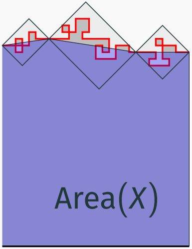

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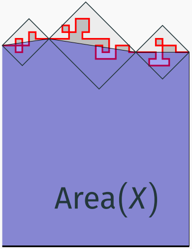

- This reduces our task to proving the desired weak convergence for this effective model.

Sketch of proof — Step 7: proof of convergence for the effective model

- This part is done in a way very similar to the analysis in [Ioffe, Shlosman, V. 2015] and [Ioffe, V., Wachtel 2018]:
  - Express the relevant partition functions in terms of powers of a suitable transfer operator.
  - Compute the scaling limit of these quantities in terms of the scaling limit of the generator of the induced semigroup, which can be computed explicitly.
  - Deduce convergence of finite-dimensional distributions.
  - Complete the analysis with a proof of tightness (rough probabilistic estimates).
- The main difference is that in our earlier work, the path was the space-time trajectory of a 1d random walk rather than the spatial trajectory of a directed 2d random walk. Mainly, this results in a *random* number of steps in the present situation, which adds technicalities but does not affect the general scheme.

Some open problems

- Consider $h = N^{-\alpha}$ for other values of α, in particular $\alpha = 0$ (i.e., first the limit $N \to \infty$, then the limit $h \downarrow 0$).

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- Extend the analysis to the case of ⊖ boundary condition when $\lambda > \lambda_c$ (Schonmann–Shlosman geometry).
- In the case of ⊖ boundary condition when $\lambda > \lambda_c$, determine the limiting process at the junction between one arc of the droplet of ⊕ phase and the layer along the boundary. Fluctuations of all orders from $N^{1/2}$ to $N^{1/3}$ are expected to occur.

Thank you for your attention!

Papers mentioned in the talk i

- D. B. Abraham and E. R. Smith.
  An exactly solved model with a wetting transition.
  J. Statist. Phys., 43(3-4):621–643, 1986.
- M. Campanino, D. Ioffe, and Y. Velenik.
  Ornstein-Zernike theory for finite range Ising models above T_c.
  Probab. Theory Related Fields, 125(3):305–349, 2003.
- S. Ganguly and R. Gheissari.
  Local and global geometry of the 2D Ising interface in critical prewetting.
  Ann. Probab., 49(4):2076–2140, 2021.
- Lev Greenberg and D. Ioffe.
  On an invariance principle for phase separation lines.
  Ann. Inst. H. Poincaré Probab. Statist., 41(5):871–885, 2005.
- O. Hryniv and Y. Velenik.
  Universality of critical behaviour in a class of recurrent random walks.
  Probab. Theory Related Fields, 130(2):222–258, 2004.
- D. Ioffe, S. Ott, S. Shlosman, and Y. Velenik.
  Critical prewetting in the 2d Ising model.
  Preprint, arXiv:2011.11997, 2020.
- D. Ioffe, S. Ott, Y. Velenik, and V. Wachtel.
  Invariance principle for a Potts interface along a wall.
  J. Stat. Phys., 180(1-6):832–861, 2020.

Papers mentioned in the talk ii

- D. Ioffe and R. H. Schonmann.
  Dobrushin-Kotecký-Shlosman theorem up to the critical temperature.
  Comm. Math. Phys., 199(1):117–167, 1998.
- D. Ioffe, S. Shlosman, and Y. Velenik.
  An invariance principle to Ferrari-Spohn diffusions.
  Comm. Math. Phys., 336(2):905–932, 2015.
- D. Ioffe, Y. Velenik, and V. Wachtel.
  Dyson Ferrari-Spohn diffusions and ordered walks under area tilts.
  Probab. Theory Related Fields, 170(1-2):11–47, 2018.
- S. Ott and Y. Velenik.
  Potts models with a defect line.
  Comm. Math. Phys., 362(1):55–106, 2018.
- R. H. Schonmann and S. B. Shlosman.
  Constrained variational problem with applications to the Ising model.
  J. Statist. Phys., 83(5-6):867–905, 1996.
- Y. Velenik.
  Entropic repulsion of an interface in an external field.
  Probab. Theory Related Fields, 129(1):83–112, 2004.