Upper bound on correlations for two-dimensional O(N)-symmetric models

Yvan Velenik

based on (an old) joint work with **Dima Ioffe** and **Senya Shlosman**
and (a recent one) with **Maxime Gagnebin**

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Structure of the talk

O(N) model

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- Given $\Lambda \Subset \mathbb{Z}^d$ and $\beta \geq 0$, the **energy** of $\vec{\mathbf{S}}$ in Λ is

$$\mathcal{H}_\Lambda(\vec{\mathbf{S}}) = -\beta \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\sim j}} \vec{S}_i \cdot \vec{S}_j$$

Note that $\vec{S}_i \cdot \vec{S}_j$, and thus $\mathcal{H}_\Lambda$, is invariant under simultaneous and identical rotation of all spins $\vec{S}_i$.

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- **Boundary condition:** a configuration $\vec{\mathbf{S}}^* = (\vec{S}_i^*)_{i \in \mathbb{Z}^d}$
- **Gibbs measure in Λ with b.c. $\vec{\mathbf{S}}^*$:**

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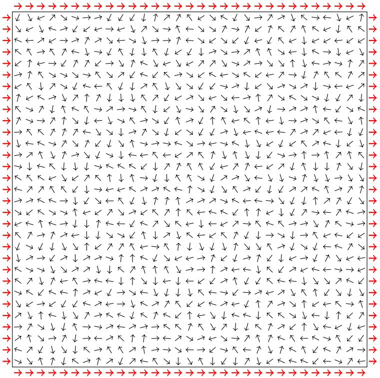
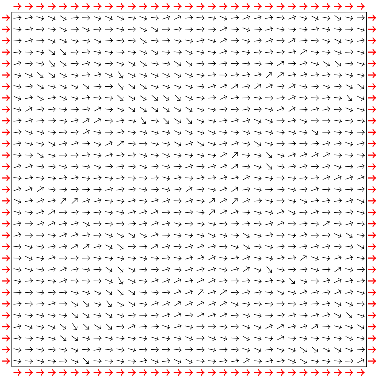
- **Examples:** Ising model ($N = 1$), XY model ($N = 2$), Heisenberg model ($N = 3$)

Typical configurations ($N = 2$)



High temperature (β small)



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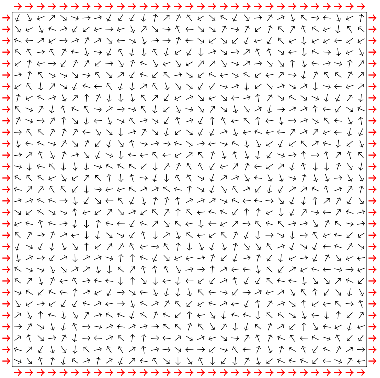

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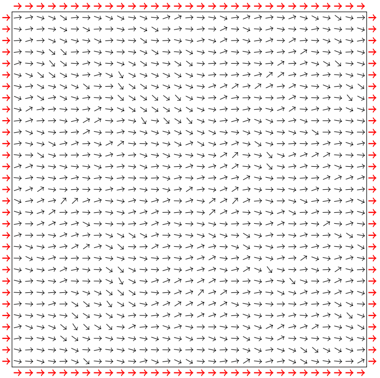

Low temperature (β large)

Basic question: does one still feel the effect of the boundary condition at the center of the box as the size of the box diverges?

O(N) model

While the answer is always **No** when β is taken sufficiently small, the behavior at large β (low temperatures) can depend on N:

	$N = 1$	$N \geq 2$
$d = 1$	**No**	**No**
$d = 2$	**Yes**	**No**
$d \geq 3$	**Yes**	**Yes**

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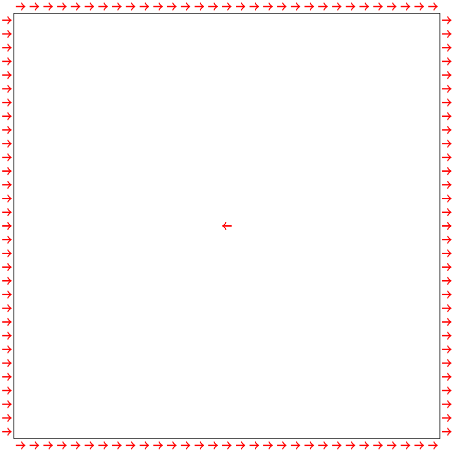
Why does the Ising model behave differently from models with larger values of N (when $d = 2$)?

Let us consider the **minimal energetic cost** of flipping a spin in the middle of a square box of “radius” n...

$N = 1$ (Ising model), $d = 2$



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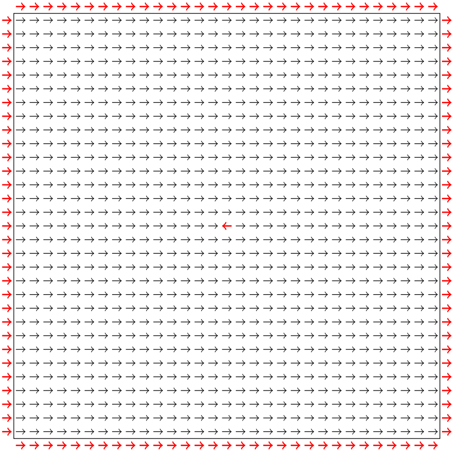
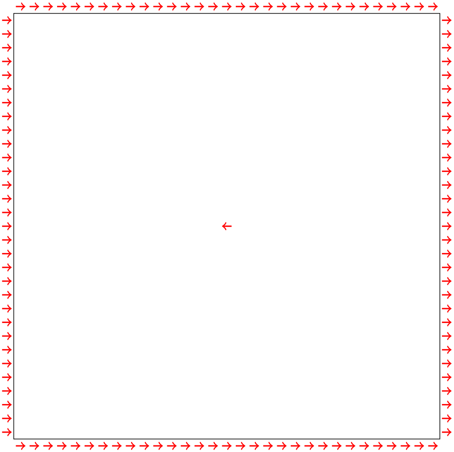


Min. energy cost: 8β

$N = 2$ (XY model), $d = 2$



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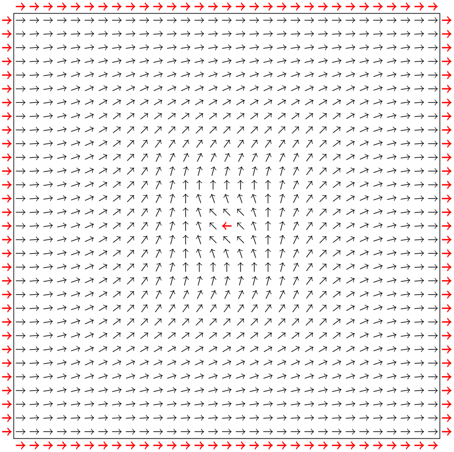

Min. energy cost: $O(\beta / \log n)$

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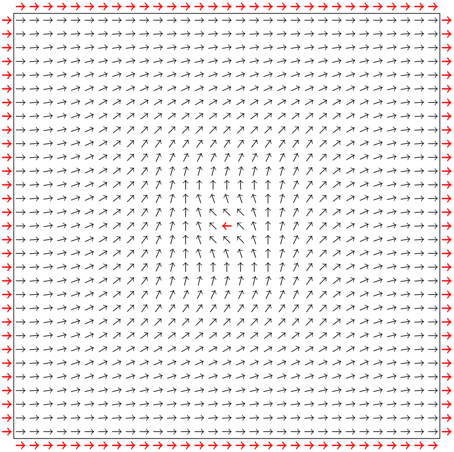

Min. energy cost: $O(\beta / \log n)$

This corresponds to the following configuration:

If $\|i\|_\infty = r$ then

$\vec{S}_i = (\cos\alpha_r, \sin\alpha_r)$ with

$$\alpha_r = \left(1 - \frac{\log(1+r)}{\log(1+n)}\right)$$

Let us now consider a more general class of models. These are constructed exactly as before, but with a Hamiltonian

$$\mathcal{H}_\Lambda(\vec{\mathbf{S}}) = \sum_{\{i,j\}\cap\Lambda\neq\emptyset} p(i,j)\, W(S_i \cdot S_j)$$

where $p(i,j) = p(j-i) \geq 0$ and $\sum_i p(i) = 1$ (that is, these are the transition probabilities of a random walk on $\mathbb{Z}^d$)

Mermin-Wagner theorem: statement

The following classical result is due to works of Mermin, Wagner, Dobrushin, Shlosman, Pfister, and many others:

Theorem

Consider $\mathrm{O}(N)$*-symmetric models on* $\mathbb{Z}^d$*, with* $p(\cdot)$ *such that the associated random walk is recurrent, and* W *twice continuously differentiable. Let* $\Lambda_n = \{-n, \ldots, n\}^d$ *and let* $\vec{\mathbf{S}}^*$ *be some boundary condition. Then every cluster point* μ *of the sequence* $(\mu_{\Lambda_n}^{\vec{\mathbf{S}}^*})_{n \geq 1}$ *is rotation invariant.*

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The recurrence condition is known to be optimal: there is spontaneous magnetization in the $\mathrm{O}(N)$ models as soon as the random walk is transient [Bonato, Perez, Klein, JSP 1982].

(Follows [Pfister, CMP, 1981] with some simplifications.
For details, see Chapter 8 of the book by S. Friedli and YV, at
`http://www.unige.ch/math/folks/velenik/smbook`)

Suffices to prove:

$$\lim_{n\to\infty} \left| \mu_{\Lambda_n}^{\vec{\mathbf{S}}^*}(f) - \mu_{\Lambda_n}^{\vec{\mathbf{S}}^*}(f \circ \mathsf{R}) \right| = 0,$$

for all local functions f, all b.c. $\vec{\mathbf{S}}^*$ and all (simultaneous and identical) rotations R of all the spins.

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To simplify notations:

- Only $N = 2$. In this case, configurations are parametrized by angles $\boldsymbol{\theta} = (\theta_i)_{i\in\mathbb{Z}^d}$, $\theta_i \in (-\pi, \pi]$, such that $\vec{S}_i = (\cos\theta_i, \sin\theta_i)$.
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Mermin-Wagner theorem: proof

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To prove:

$$\lim_{n\to\infty} \left|\mu^{\boldsymbol{\theta}^*}_{\Lambda_n}(f(\theta_0)) - \mu^{\boldsymbol{\theta}^*}_{\Lambda_n}(f(\theta_0 + \alpha))\right| = 0,$$

for all function f, all b.c. $\boldsymbol{\theta}^*$ and all $\alpha \in (-\pi, \pi]$.

Mermin-Wagner theorem: proof

Let $\mathbf{\Psi} = (\Psi_i)_{i\in\mathbb{Z}^d}$, $\Psi_i \in (-\pi, \pi]$, s.t. $\Psi_0 = \alpha$ and $\Psi_i = 0$ for all $i \notin \Lambda_n$.

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Denoting by $\mathrm{BC} = \mathrm{BC}(\boldsymbol{\theta}^*, n) = \{\theta_i = \theta_i^* \text{ for all } i \notin \Lambda_n\}$, define

$$\mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\mathbf{\Psi}}(\mathrm{d}\boldsymbol{\theta}) = 1_{\mathrm{BC}}(\boldsymbol{\theta})\,\frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*,\mathbf{\Psi}}}\,e^{-\mathcal{H}_{\Lambda_n}(\boldsymbol{\theta}-\mathbf{\Psi})}\,.$$

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Easy to check (Jacobian= 1): $\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}} = \mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}$. In particular,

$$\begin{aligned}
\mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}}(f(\theta_0)) &= \frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}}\int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n}\, e^{-\mathcal{H}_{\Lambda_n}(\overbrace{\boldsymbol{\theta}-\boldsymbol{\Psi}}^{=\tilde{\boldsymbol{\theta}}})} f(\theta_0) \\
&= \frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}}\int \mathrm{d}\tilde{\boldsymbol{\theta}}_{\Lambda_n}\, e^{-\mathcal{H}_{\Lambda_n}(\tilde{\boldsymbol{\theta}})} f(\tilde{\theta}_0 + \underbrace{\Psi_0}_{=\alpha}) \\
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\end{aligned}$$

Therefore,

$$\left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}(f(\theta_0)) - \mu_{\Lambda_n}^{\boldsymbol{\theta}^*}(f(\theta_0+\alpha))\right| = \left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}(f(\theta_0)) - \mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}}(f(\theta_0))\right|.$$

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Denote by $H(\mu|\nu) = \mu\big(\log(\mathrm{d}\mu/\mathrm{d}\nu)\big)$ the **relative entropy** of μ with respect to $\nu \gg \mu$.

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Therefore,

$$\begin{aligned} H(\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}}) &= \mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\big(\mathcal{H}_{\Lambda_n}(\boldsymbol{\theta}-\boldsymbol{\Psi}) - \mathcal{H}_{\Lambda_n}(\boldsymbol{\theta})\big) \\ &= \sum_{\{i,j\}\cap\Lambda_n\neq\emptyset} p(i,j)\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\big(V(\theta_i - \Psi_i - \theta_j + \Psi_j) - V(\theta_i - \theta_j)\big)\,, \end{aligned}$$

where $V(x) = W(\cos(x))$.

Mermin-Wagner theorem: proof

Therefore,

$$\begin{aligned} H(\mu^{\boldsymbol{\theta}^*}_{\Lambda_n} | \mu^{\boldsymbol{\theta}^*, \boldsymbol{\Psi}}_{\Lambda_n}) &= \mu^{\boldsymbol{\theta}^*}_{\Lambda_n}\big(\mathcal{H}_{\Lambda_n}(\boldsymbol{\theta} - \boldsymbol{\Psi}) - \mathcal{H}_{\Lambda_n}(\boldsymbol{\theta})\big) \\ &= \sum_{\{i,j\}\cap\Lambda_n \neq \emptyset} p(i,j)\mu^{\boldsymbol{\theta}^*}_{\Lambda_n}\big(V(\theta_i - \Psi_i - \theta_j + \Psi_j) - V(\theta_i - \theta_j)\big), \end{aligned}$$

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where $V(x) = W(\cos(x))$. Since V is twice continuously differentiable,

$$V(\theta_i - \Psi_i - \theta_j + \Psi_j) = V(\theta_i - \theta_j) + V'(\theta_i - \theta_j)\,(\Psi_j - \Psi_i) + \tfrac{1}{2}\underbrace{V''(t_{ij})}_{\leq C}(\Psi_j - \Psi_i)^2\,.$$

Mermin-Wagner theorem: proof

Therefore,

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Trick: Since $H(\mu|\nu)\geq 0$ for all μ,ν, we can write

$$H(\mu_{\Lambda_n}^{\boldsymbol{\theta}^*} | \mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}}) \leq H(\mu_{\Lambda_n}^{\boldsymbol{\theta}^*} | \mu_{\Lambda_n}^{\boldsymbol{\theta}^*,\boldsymbol{\Psi}}) + H(\mu_{\Lambda_n}^{\boldsymbol{\theta}^*} | \mu_{\Lambda_n}^{\boldsymbol{\theta}^*,-\boldsymbol{\Psi}})$$
$$\leq C \sum_{\{i,j\}\cap\Lambda_n\neq\emptyset} p(i,j)(\Psi_j-\Psi_i)^2\,.$$

Mermin-Wagner theorem: proof

There only remains to prove that one can choose $\boldsymbol{\Psi}$ satisfying $\Psi_0 = \alpha$, $\Psi \equiv 0$ off Λ_n, and such that

$$\mathcal{E}(\boldsymbol{\Psi}) = \tfrac{1}{2} \sum_{\{i,j\} \cap \Lambda_n \neq \emptyset} p(i,j)(\Psi_j - \Psi_i)^2 \to 0, \quad \text{as } n \to \infty.$$

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Easy to prove: the optimal such $\mathbf{\Psi}$ is the (unique) solution to the Dirichlet problem

$$\begin{cases} (\Delta \mathbf{\Psi})_i = 0 & \forall i \in \Lambda_n \setminus \{0\} \\ \Psi_0 = \alpha & \\ \Psi_i = 0 & \forall i \notin \Lambda_n \end{cases}$$

Mermin-Wagner theorem: proof

There only remains to prove that one can choose $\boldsymbol{\Psi}$ satisfying $\Psi_0 = \alpha$, $\Psi \equiv 0$ off Λ_n, and such that

$$\mathcal{E}(\boldsymbol{\Psi}) = \tfrac{1}{2} \sum_{\{i,j\} \cap \Lambda_n \neq \emptyset} p(i,j)(\Psi_j - \Psi_i)^2 \to 0, \quad \text{as } n \to \infty.$$

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This solution is given by: for all $i \in \Lambda_n \setminus \{0\}$,

$$\Psi_i = \alpha \, \mathbb{P}_i(X \text{ visits 0 before leaving } \Lambda_n)\,,$$

and, for this $\boldsymbol{\Psi}$,

$$\mathcal{E}(\boldsymbol{\Psi}) = 2\alpha \, \mathbb{P}_0(X \text{ exits } \Lambda_n \text{ before returning to } 0)\,,$$

so that $\lim_{n\to\infty} \mathcal{E}(\boldsymbol{\Psi}) = 0$ if and only if X is recurrent.

Mermin-Wagner theorem: extension

Even though smoothness of W (and thus V) played a crucial role in the above proof, and in all alternative proofs, it turns out that it can be substantially weakened:

Theorem (Ioffe, Shlosman, YV, CMP, 2001)

The previous theorem remains true when W is only piecewise continuous.

Decay of correlations

▶ A weakness of the above approach is that it does not provide very good estimates on decay of correlations in its regime of validity. The best that can be extracted from it is: for all limiting measures μ,

$$\mu(\vec{S}_i \cdot \vec{S}_j) \leq c\,(\log \|j - i\|)^{-1/2}\,,$$

uniformly in $i \neq j$ in $\mathbb{Z}^2$.

▶ In the **finite-range case** ($\exists R < \infty$ s.t. $p(i,j) = 0$ when $\|j - i\| > R$), Dobrushin and Shlosman proved that

$$\mu(\vec{S}_i \cdot \vec{S}_j) \leq c_1\,\|j - i\|^{-c_2}\,,$$

for W continuously differentiable, and this was extended to piecewise continuous W in [Ioffe, Shlosman, YV, CMP, 2001].

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▶ When W is **analytic**, an alternative approach, due to McBryan and Spencer and extended by Messager, Miracle-Sole and Ruiz, allows one to prove that this is still true as long as

$$p(i,j) \leq c\,\|j - i\|^{-\alpha} \quad \text{for some } \alpha > 4.$$

(Note: correlations do not always decay when $\beta \gg 1$ and $\alpha < 4$.)

Decay of correlations

▶ The above is optimal in general, in the sense that Fröhlich and Spencer proved a lower bound with the same behavior for the $2d$ XY model at large enough β.

▶ It is however expected to be quite poor for the O(N) models with $N \geq 3$, since in that case it is conjectured that

$$\mu(\vec{S}_i \cdot \vec{S}_j) \leq c_1 \, e^{-c_2 \|j-i\|}$$

at all temperatures.

Decay of correlations: Proof for the n.n. XY model

Let us prove the upper bound in the simplest case of the $2d$ XY model. We follow the approach of McBryan and Spencer. Details can again be found in Chapter 8 of our book with S. Friedli.

Again, it is convenient to work in terms of the angles $\boldsymbol{\theta}$. We fix an arbitrary b.c. $\boldsymbol{\theta}^*$ outside the box Λ_n.

To prove: $\lim_{n\to\infty} \mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\big(\cos(\theta_k - \theta_0)\big) \leq c_1 \, \|k\|^{-c_2}$ for all $k \neq 0$.

$$\begin{aligned}\left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\big(\cos(\theta_k - \theta_0)\big)\right| &\leq \left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\big(e^{\mathrm{i}(\theta_k - \theta_0)}\big)\right| \\ &= \frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}} \int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n} \exp\Big\{\mathrm{i}(\theta_k - \theta_0) + \beta \sum_{\substack{\{i,j\}\cap\Lambda_n \neq \emptyset \\ i\sim j}} \cos(\theta_i - \theta_j)\Big\}\,.\end{aligned}$$

Decay of correlations: Proof for the n.n. XY model

$$\left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\left(\cos(\theta_k-\theta_0)\right)\right| = \frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}} \int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n} \exp\Big\{\mathrm{i}(\theta_k-\theta_0)+\beta \sum_{\substack{\{i,j\}\cap\Lambda_n\neq\emptyset \\ i\sim j}} \cos(\theta_i-\theta_j)\Big\}\,.$$

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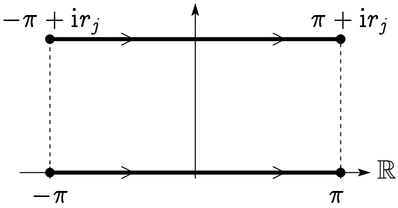
Decay of correlations: Proof for the n.n. XY model

$$\left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\left(\cos(\theta_k-\theta_0)\right)\right| = \frac{1}{\mathbf{Z}_{\Lambda_n}^{\boldsymbol{\theta}^*}} \int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n} \exp\Big\{\mathrm{i}(\theta_k-\theta_0)+\beta \sum_{\substack{\{i,j\}\cap\Lambda_n\neq\emptyset \\ i\sim j}} \cos(\theta_i-\theta_j)\Big\}\,.$$

The integrand being analytic, Cauchy theorem allows one, for each integration variable θ_j, to shift its path of integration from $(-\pi,\pi]$ to $\mathrm{i}\,r_j+(-\pi,\pi]$, where $\mathbf{r}=(r_j)_{j\in\mathbb{Z}^d}$ will be chosen later and will satisfy $r_j=0$ for all $j\notin\Lambda_n$.



Decay of correlations: Proof for the n.n. XY model

Since

$$\left|e^{\mathrm{i}(\theta_k+\mathrm{i}r_k-\theta_0-\mathrm{i}r_0)}\right| = e^{-(r_k-r_0)},$$

$$\left|e^{\cos(\theta_i+\mathrm{i}r_i-\theta_j-\mathrm{i}r_j)}\right| = e^{\cosh(r_i-r_j)\cos(\theta_i-\theta_j)},$$

we obtain

$$\begin{aligned}
&\left|\mu^{\boldsymbol{\theta}^*}_{\Lambda_n}\big(\cos(\theta_k-\theta_0)\big)\right| \\
&\leq \frac{e^{-(r_k-r_0)}}{\mathbf{Z}^{\boldsymbol{\theta}^*}_{\Lambda_n}} \int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n} \exp\Big\{\beta\sum_{i\sim j}\cosh(r_i-r_j)\cos(\theta_i-\theta_j)\Big\} \\
&= e^{-(r_k-r_0)} \int \mathrm{d}\boldsymbol{\theta}_{\Lambda_n} \exp\Big\{\beta\sum_{i\sim j}\big(\cosh(r_i-r_j)-1\big)\cos(\theta_i-\theta_j)\Big\} \frac{e^{-\mathcal{H}_{\Lambda_n}(\boldsymbol{\theta})}}{\mathbf{Z}^{\boldsymbol{\theta}^*}_{\Lambda_n}} \\
&= e^{-(r_k-r_0)}\, \mu^{\boldsymbol{\theta}^*}_{\Lambda_n}\Big(\exp\Big\{\beta\sum_{i\sim j}\big(\cosh(r_i-r_j)-1\big)\cos(\vartheta_i-\vartheta_j)\Big\}\Big) \\
&\leq e^{-(r_k-r_0)} \exp\Big\{\beta\sum_{i\sim j}\big(\cosh(r_i-r_j)-1\big)\Big\}.
\end{aligned}$$

Decay of correlations: Proof for the n.n. XY model

We now need to find a suitable candidate for $\mathbf{r}$.
Assume that r can be chosen in such a way that, for some C,

$$|r_i - r_j| \leq C/\beta\,, \quad \forall i \sim j\,.$$

Then, for any fixed $\epsilon > 0$,

$$\cosh(r_i - r_j) - 1 \leq \tfrac{1}{2}(1+\epsilon)(r_i - r_j)^2, \qquad \forall i \sim j\,,$$

provided that β is large enough. Therefore

$$\sum_{i\sim j}\bigl(\cosh(r_i - r_j) - 1\bigr) \leq \tfrac{1}{2}(1+\epsilon)\sum_{i\sim j}(r_i - r_j)^2 \equiv (1+\epsilon)\mathcal{E}(\mathbf{r})\,.$$

We thus have

$$\left|\mu_{\Lambda_n}^{\boldsymbol{\theta}^*}\bigl(\cos(\theta_k - \theta_0)\bigr)\right| \leq \exp\{-\mathcal{D}(\mathbf{r})\}\,,$$

where, setting $\beta' = (1+\epsilon)\beta$,

$$\mathcal{D}(\mathbf{r}) = r_k - r_0 - \beta'\mathcal{E}(r)\,.$$

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The choice of $\mathbf{r}$ minimizing $\mathcal{D}$ is the unique solution to

$$(\Delta \mathbf{r})_i = (1_{\{i=0\}} - 1_{\{i=k\}})/\beta', \quad i \in \Lambda_n ,$$

and can be expressed explicitly as

$$r_i = \big(G_{\Lambda_n}(k,i) - G_{\Lambda_n}(0,i)\big)/(4\beta'), \quad i \in \Lambda_n ,$$

where $G_{\Lambda_n}(\cdot,\cdot)$ is the Green function of the simple random walk in Λ_n. Since, for this choice of $\mathbf{r}$,

$$\mathcal{D}(\mathbf{r}) = \tfrac{1}{2}(r_k - r_0)\,,$$

the conclusion follows from the well-known asymptotics of the Green function. □

Decay of correlations: Main result

Our main result is the following

Theorem (Gagnebin, YV, CMP, 2014)

Assume that $d = 2$, V *is continuous and* $p(i,j) \leq c\,\|j-i\|^{-\alpha}$ *for some* $\alpha > 4$. *Then,*

$$\mu(\vec{S}_i \cdot \vec{S}_j) \leq c_1\,\|j-i\|^{-c_2}\,,$$

uniformly in $i \neq j$.

Thanks
for your attention!