Asymptotics of even-even correlations in the Ising model

(joint work with Sébastien Ott)

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Introduction and results

Ising model on $\mathbb{Z}^d$

Formal Hamiltonian: $$\mathcal{H}_\Lambda = -\sum_{\{i,j\}\subset\mathbb{Z}^d} J_{j-i}\,\sigma_i\sigma_j$$

- *ferromagnetism:* $J_x \geq 0$ for all $x \in \mathbb{Z}^d$;
- *symmetry:* $J_x = J_{-x}$ for all $x \in \mathbb{Z}^d$;
- *finite-range:* $\exists R < \infty$ such that $J_x = 0$ whenever $\|x\|_2 \geq R$;
- *irreducibility:* $J_x > 0$ for all $x \in \mathbb{Z}^d$ with $\|x\| = 1$.

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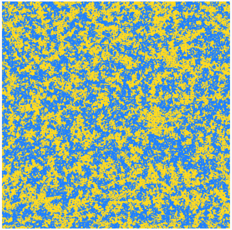
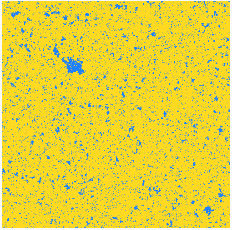
We assume that $d \geq 2$ and $\beta < \beta_c(d)$, and let μ_β be the unique Gibbs measure.



$\beta < \beta_c(2)$



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Decay of correlations

Let $Cov_\beta(f,g) = \mu_\beta(fg) - \mu_\beta(f)\mu_\beta(g)$.
Let $[x] \in \mathbb{Z}^d$ be the coordinate-wise integer part of $x \in \mathbb{Z}^d$.

Theorem [Aizenman, Barsky, Fernández 1987]

For all $\beta < \beta_c(d)$ and any unit-vector $\mathbf{u}$ in $\mathbb{R}^d$, the **inverse correlation length**

$$\xi_\beta(\mathbf{u}) = -\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Cov}_\beta(\sigma_0, \sigma_{[n\mathbf{u}]})$$

exists and is positive.

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What about covariances of more general functions?

Asymptotics of correlations

Let f, g be two *local* functions and denote by θ_x the translation by $x \in \mathbb{Z}^d$.

What is the asymptotic behavior of

$$\mathrm{Cov}_\beta(f, \theta_{[n\mathbf{u}]} g)$$

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Let $\sigma_A = \prod_{i \in A} \sigma_i$. Writing

$$f = \sum_{A \subset \mathrm{supp}(f)} \hat{f}_A \sigma_A, \qquad g = \sum_{B \subset \mathrm{supp}(g)} \hat{g}_B \sigma_B,$$

yields

$$\mathrm{Cov}_\beta(f, \theta_{[n\mathbf{u}]} g) = \sum_{\substack{A \subset \mathrm{supp}(f) \\ B \subset \mathrm{supp}(g)}} \hat{f}_A \hat{g}_B \, \mathrm{Cov}_\beta(\sigma_A, \sigma_{B+n\mathbf{u}}).$$

This motivates the following

Main question

What is the asymptotic behavior of

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Of course, by symmetry, $\mu_\beta(\sigma_C) = 0$ whenever $|C|$ is odd.
In particular, if $|A| + |B|$ is odd, then $\mathrm{Cov}_\beta(\sigma_A, \sigma_B) = 0$.

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There are thus **two cases to consider:**

Odd-odd correlations	Even-even correlations
$\lvert A\rvert, \lvert B\rvert$ both odd	$\lvert A\rvert, \lvert B\rvert$ both even

Odd-odd correlations

Theorem [Campanino, Ioffe, V. 2004]

Let $d \geq 2$ and $\beta < \beta_c(d)$. Let $A, B \Subset \mathbb{Z}^d$ with $|A|$ and $|B|$ odd.
For any unit-vector $\mathbf{u}$, there exists a constant $0 < C < \infty$ (depending on $A, B, \mathbf{u}, \beta$) such that

$$\mathrm{Cov}_\beta(\sigma_A, \sigma_{B+[n\mathbf{u}]}) = \frac{C}{n^{(d-1)/2}} e^{-\xi_\beta(\mathbf{u})n} (1 + o(1)),$$

as $n \to \infty$.

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as $n \to \infty$.

This result has a **long history**. Some milestones:

- *Ornstein–Zernike 1914, Zernike 1916:* $|A| = |B| = 1$ non-rigorous
- *Abraham–Kunz 1977, Paes-Leme 1978:* $|A| = |B| = 1$ $\beta \ll 1$
- *Bricmont–Fröhlich 1985, Minlos-Zhizhina 1988, 1996:* $|A|, |B|$ odd $\beta \ll 1$
- *Campanino–Ioffe–V. 2003:* $|A| = |B| = 1$ $\beta < \beta_c(d)$

Even-even correlations

Substantially more delicate!

The analysis started with the case $|A| = |B| = 2$. Physicists quickly understood that

$$\mathrm{Cov}_\beta(\sigma_A, \sigma_{B+[n\mathbf{u}]}) = e^{-2\xi_\beta(\mathbf{u})n\,(1+o(1))}.$$

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However, concerning the prefactor, two conflicting predictions were put forward:

Polyakov 1969		Camp–Fisher 1971	
n^{-2}	$d = 2$		
$(n \log n)^{-2}$	$d = 3$	n^{-d}	for all $d \geq 2$
$n^{-(d-1)}$	$d \geq 4$		

(Note that both coincide with the exact computation when $d = 2$.)

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It turns out that Polyakov was right. This was first shown in

- *Bricmont–Fröhlich 1985:* $|A| = |B| = 2$ $\beta \ll 1$ $d \geq 4$
- *Minlos-Zhizhina 1988, 1996:* $|A|, |B|$ even $\beta \ll 1$ $d \geq 2$

Even-even correlations

Let $\quad \Xi(n) = \begin{cases} n^2 & \text{when } d = 2, \\ (n \log n)^2 & \text{when } d = 3, \\ n^{d-1} & \text{when } d \geq 4. \end{cases}$

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Our main result is

Theorem [Ott, V. 2018]

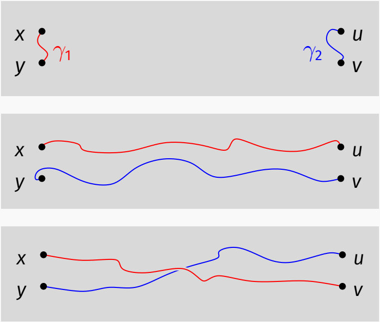
Let $d \geq 2$ and $\beta < \beta_c(d)$. Let $A, B \Subset \mathbb{Z}^d$ with $|A|$ and $|B|$ even.
For any unit vector $\mathbf{u}$ in $\mathbb{R}^d$, there exist constants $0 < C_- \leq C_+ < \infty$ (depending on $A, B, \mathbf{u}, \beta$) such that, for all n large enough,

$$\frac{C_-}{\Xi(n)} e^{-2\xi_\beta(\mathbf{u})n} \leq \mathrm{Cov}_\beta(\sigma_A, \sigma_{B+[n\mathbf{u}]}) \leq \frac{C_+}{\Xi(n)} e^{-2\xi_\beta(\mathbf{u})n}.$$

Heuristics

Heuristics

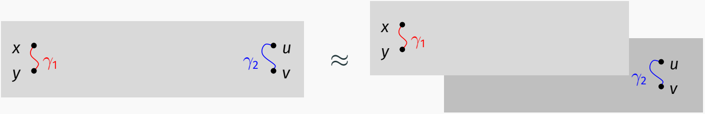
Let $A = \{x, y\}$ and $B + [n\mathbf{u}] = \{u, v\}$. **High-temperature expansion** of $\mu_\beta(\sigma_x\sigma_y\sigma_u\sigma_v)$ yields 3 types of configurations:



High-temperature expansion of $\mu_\beta(\sigma_x\sigma_y)\mu_\beta(\sigma_u\sigma_v)$ yields

Heuristics (and difficulties)

Now, since $\beta < \beta_c(d)$, one may expect the paths γ_1 and γ_2 to stay far from each other, so that the expectation factorizes and



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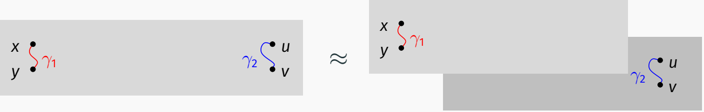
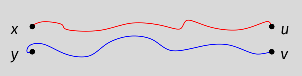


Then, $\mathrm{Cov}_\beta(\sigma_A, \sigma_{B+[n\mathbf{u}]})$ would be dominated by the contributions of



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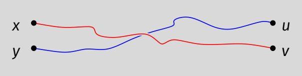

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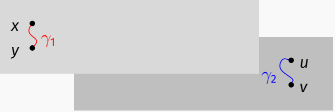

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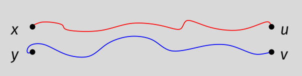

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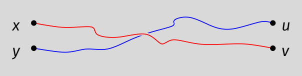

Then, neglecting the interactions between γ_1, γ_2 would yield

$$\mathrm{Cov}_\beta(\sigma_A, \sigma_{B+[n\mathbf{u}]}) \approx \mu_\beta(\sigma_x\sigma_u)\mu_\beta(\sigma_y\sigma_v) + \mu_\beta(\sigma_x\sigma_v)\mu_\beta(\sigma_y\sigma_u) \approx n^{-(d-1)}e^{-2\xi_\beta(\mathbf{u})n},$$

which is what we want when $d \geq 4$. Assuming that the paths behave as random walk bridges and taking into account the non-intersection constraint would then yield the correct behavior also when $d = 2$ or 3...

1. In fact,

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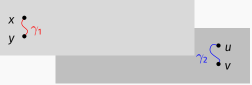

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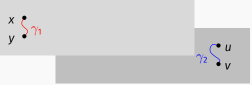

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2. The paths γ_1, γ_2 have long-distance (self)interactions. In particular, $\mathrm{w}(\gamma_1, \gamma_2) \neq \mathrm{w}(\gamma_1)\mathrm{w}(\gamma_2)$.

Problems with this argument

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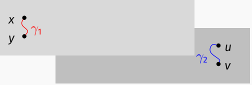
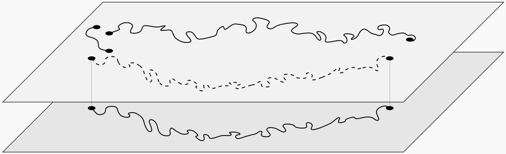


$$= e^{-2\xi_\beta(\mathbf{u})n\,(1+o(1))}$$

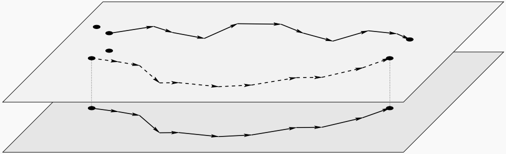
   and thus cannot be neglected!
2. The paths γ_1, γ_2 have long-distance (self)interactions. In particular, $\mathrm{w}(\gamma_1, \gamma_2) \neq \mathrm{w}(\gamma_1)\mathrm{w}(\gamma_2)$.
3. It is not at all obvious why the non-intersection constraint should yield the same behavior as if γ_1 and γ_2 were random walk bridges.

To solve these problems, we use

- The random-current & high-temperature, or the FK representations, in order to reduce to two **independent** objects (HT paths or FK clusters), conditioned on not intersecting:



- The Ornstein-Zernike theory (Campanino–Ioffe–V. 2003, 2008 and Ott–V. 2017), in order to approximate these objects using directed random walks on $\mathbb{Z}^d$:



Sketch of the lower bound

Sketch of the lower bound: Step 1

The first step is to prove that, for any $x, y \in A$ and $u, v \in B$ with $x \neq y$ and $u \neq v$, the following bound holds:

$$\frac{\mathrm{Cov}_\beta(\sigma_A, \sigma_B)}{\mu_\beta(\sigma_x \sigma_u)\, \mu_\beta(\sigma_y \sigma_v)} \geq \sum_{\substack{C_1 \ni x,u \\ C_2 \ni y,v}} \mathbb{1}_{\{C_1 \cap C_2 = \varnothing\}} \mathbb{P}^{\mathrm{FK}}(C_{x,u} = C_1 \,|\, x \leftrightarrow u)\, \mathbb{P}^{\mathrm{FK}}(C_{y,v} = C_2 \,|\, y \leftrightarrow v),$$

where the sum is over pairs of disjoint FK-clusters containing, respectively x, u and y, z, and $C_{x,u}$ denotes the common cluster of x and u, and similarly for $C_{y,v}$.

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where the sum is over pairs of disjoint FK-clusters containing, respectively x, u and y, z, and $C_{x,u}$ denotes the common cluster of x and u, and similarly for $C_{y,v}$.

Note that the RHS is precisely the probability that the two clusters $C_{x,u}$ and $C_{y,v}$, sampled **independently** from $\mathbb{P}^{\mathrm{FK}}(\cdot \,|\, x \leftrightarrow u)$ and $\mathbb{P}^{\mathrm{FK}}(\cdot \,|\, y \leftrightarrow v)$, are disjoint.

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Note also that the denominator in the LHS provides the main “squared OZ” behavior. The RHS can then be used to find the corrections due to the non-intersection constraint.

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- It is well know that Ising correlation functions can be expressed in FK terms as

$$\mu_\beta(\sigma_C) = \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_C),$$

where $\mathfrak{E}_C$ is the event that all FK-clusters contain an even number of vertices of C.

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$$\mathrm{Cov}_\beta(\sigma_A, \sigma_B) \quad = \quad \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A \cup B}) - \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_A)\, \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_B)$$

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&\geq \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B} \cap \mathfrak{E}_A^{\mathrm{c}}) \qquad \text{(by FKG)}
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- Therefore,

$$\begin{aligned} \mathrm{Cov}_\beta(\sigma_A, \sigma_B) &= \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B}) - \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_A)\,\mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_B) \\ &= \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B} \cap \mathfrak{E}_A^{\mathrm{c}}) + \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B} \cap \mathfrak{E}_A) - \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_A)\,\mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_B) \\ &= \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B} \cap \mathfrak{E}_A^{\mathrm{c}}) + \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_A \cap \mathfrak{E}_B) - \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_A)\,\mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_B) \\ &\geq \mathbb{P}^{\mathrm{FK}}(\mathfrak{E}_{A\cup B} \cap \mathfrak{E}_A^{\mathrm{c}}) \qquad \text{(by FKG)} \\ &\geq \mathbb{P}^{\mathrm{FK}}(x \leftrightarrow u, y \leftrightarrow v, x \not\leftrightarrow y). \end{aligned}$$

Sketch of the lower bound: Step 1

Next, we partition the event in the last expression according to the realizations of clusters C_1, C_2 such that $x, u \in C_1$, $y, v \in C_2$ and $C_1 \cap C_2 = \varnothing$.

$$\mathbb{P}^{\mathrm{FK}}(x \leftrightarrow u, y \leftrightarrow v, x \not\leftrightarrow y) = \sum_{C_1, C_2} \mathbb{P}^{\mathrm{FK}}(C_{x,u} = C_1, C_{y,v} = C_2)$$

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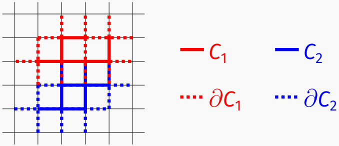

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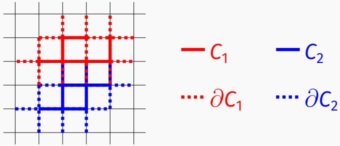

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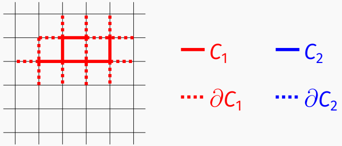

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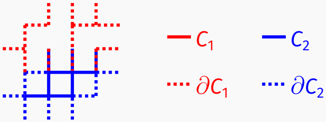

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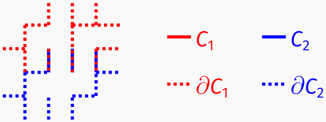

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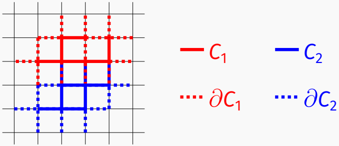
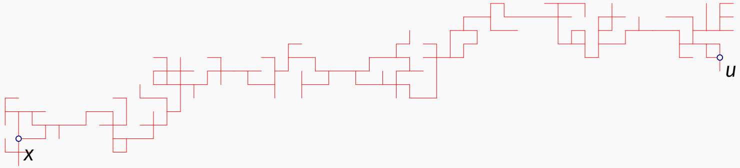


Sketch of the lower bound: Step 2

- We now only need to understand the asymptotic behavior of the probability that the two clusters do not intersect.
- Under both measures $\mathbb{P}^{\mathrm{FK}}(\cdot \mid x \leftrightarrow u)$ and $\mathbb{P}^{\mathrm{FK}}(\cdot \mid y \leftrightarrow v)$, the OZ theory can be applied to approximate the cluster by a suitable directed random walk.



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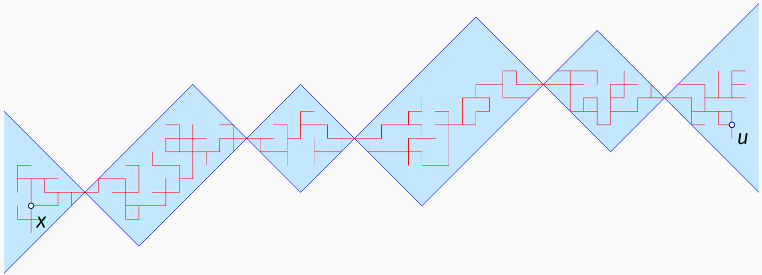

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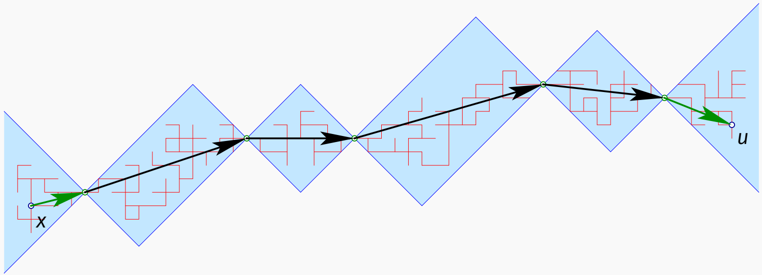
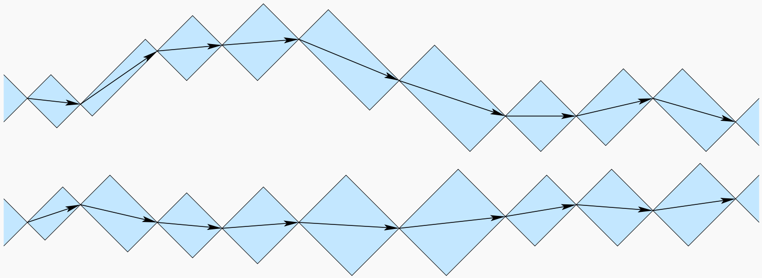


The resulting directed random walk has increments with exponential tails and thus approximates well the original cluster.

Sketch of the lower bound: Step 2

▶ Whenever the two “necklaces” are disjoint, the corresponding clusters are also necessarily disjoint.
⇝ Lower bound in terms of a random walk event, with the same asymptotic behavior as non-crossing constraint.



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Conclusion

- Another nice example of the power and versatility of the OZ theory!
- It would be nice to obtain *sharp* asymptotics, but this seems difficult. Maybe by developing a version of OZ applicable directly in the (double)random-current representation...
- Extension to models with richer symmetry group seems interesting (even just a classification of possible behaviors). We could not find literature on the subject.

Thank you for your attention!