5 Cluster Expansion

5.1 Introduction

The cluster expansion is a powerful tool in the rigorous study of statistical mechanics. It was introduced by Mayer during the early stages of the study of the phenomenon of condensation and remains widely used nowadays. In particular, it remains at the core of the implementation of many renormalization arguments in mathematical physics, yielding rigorous results that no other methods have yet been able to provide.

Simply stated, the cluster expansion provides a method for studying the logarithm of a partition function. We will use it in various situations, for instance to obtain new analyticity results for the Ising model in the thermodynamic limit.

In a first application, we will obtain new results on the pressure $h \mapsto \psi_\beta(h)$, completing those of Chapter 3. There, we saw that the pressure is analytic in the half space $\{\mathfrak{Re}\, h > 0\}$, but the techniques we used did not provide further quantitative information. Here we will use the cluster expansion to compute the coefficients of the expansion of $\psi_\beta(h) - h$, in terms of the variable $z = e^{-2h}$:

$$\psi_\beta(h) - h = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots, \qquad (\mathfrak{Re}\, h \text{ large}).$$

In our second application, we will fix $h = 0$, and study the analyticity of $\beta \mapsto \psi_\beta(0)$. We will obtain, when β is sufficiently *small*, an expansion in terms of the variable $z = \tanh(\beta)$,

$$\psi_\beta(0) - d\log(\cosh\beta) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots, \qquad (\beta \text{ small}).$$

One might hope that this series converges for all $\beta < \beta_c(d)$. Unfortunately, the method developed in this chapter will guarantee analyticity only when $\beta < \beta_0$, where $\beta_0 = \beta_0(d)$ is some number, strictly smaller than the critical value $\beta_c(d)$. We will call a regime such as $\beta < \beta_0$ a regime of *very high temperature* to distinguish it from the *high temperature* regime $\beta < \beta_c$ used in earlier chapters.

Similarly, we will also obtain, when β is sufficiently *large*, an expansion in terms of the variable $z = e^{-2\beta}$,

$$\psi_\beta(0) - \beta d = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \qquad (\beta \text{ large}).$$

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Once again, this series will be guaranteed to converge at very low temperature, that is for all $\beta > \beta_0'$, where $\beta_0' > \beta_c(d)$.

Of course, the cluster expansion is not limited to the study of the pressure in these different regimes and we will show how it can be used to extract additional information on other quantities of interest. In particular, at very low temperatures, we will derive a series expansion for the spontaneous magnetization and prove exponential decay of the truncated 2-point function.

We hope that this sample of applications will convince the reader that the cluster expansion is a versatile tool that, even though applicable only in restricted regions of the space of parameters, provides there precious information, which is often unavailable when using other techniques.

Remark 5.1. The cluster expansion will also be used in other parts of this book. We will use it in Chapter 6 to derive uniqueness of the infinite-volume Gibbs state at sufficiently high temperatures for a rather large class of models and it will play a central role in the Pirogov–Sinai theory exposed in Chapter 7. ⋄

5.2 Polymer models

The cluster expansion applies when the model under consideration has a partition function that can be written in a particular form, already encountered earlier in the book. For instance, remember from Section 3.7.2 that the configurations of the Ising model at low temperature were conveniently described using extended geometric objects, the *contours*, rather than the individual spins; namely (see (3.32)):

1. each configuration was set in one-to-one correspondence with a family of *pairwise disjoint* contours;

2. once expressed in terms of contours, the Boltzmann weight split into a *product of weights* associated to the contours.

Relying on this geometric representation, the Peierls argument allowed us to prove positivity of the spontaneous magnetization at sufficiently low temperature.

Later, when studying the Ising model at high temperature, a different representation of the partition function was used. Although the objects involved were of a different nature (especially in higher dimensions, see (3.45)), they also satisfied some geometric compatibility condition, namely that of being pairwise disjoint. Moreover, the Boltzmann weight again factorized as a product of the weights associated to these objects.

The description of a system in terms of geometrical objects (rather than the original microscopic components, such as Ising spins) turns out to be common in equilibrium statistical mechanics; the resulting class of models, usually called *polymer models*, is precisely the one for which the cluster expansion will be developed. The corresponding partition functions often have a common structure that can be exploited to provide, under suitable hypotheses, detailed information on their logarithm.

Consider a finite set Γ, the elements of which are called **polymers** and usually denoted by $\gamma \in \Gamma$. In specific situations, polymers can be complicated objects, but in this abstract setting we only need two main ingredients:

1. To each polymer $\gamma \in \Gamma$ is associated a **weight** (or **activity**) $\mathrm{w}(\gamma)$, which can be a real or complex number.

2. The interaction between polymers is **pairwise** and is encoded in a function $\delta : \Gamma \times \Gamma \to \mathbb{R}$, which is assumed to be symmetric (that is, $\delta(\gamma,\gamma') = \delta(\gamma',\gamma)$) and to satisfy the following two conditions:

$$\delta(\gamma,\gamma) = 0\,, \qquad \forall \gamma \in \Gamma\,, \tag{5.1}$$

$$|\delta(\gamma,\gamma')| \le 1\,, \qquad \forall \gamma,\gamma' \in \Gamma\,. \tag{5.2}$$

Definition 5.2. *The* ***(polymer) partition function*** *is defined by*

$$\Xi \stackrel{\text{def}}{=} \sum_{\Gamma' \subset \Gamma} \Big\{ \prod_{\gamma \in \Gamma'} \mathrm{w}(\gamma) \Big\} \Big\{ \prod_{\{\gamma,\gamma'\} \subset \Gamma'} \delta(\gamma,\gamma') \Big\}\,, \tag{5.3}$$

where the sum is over all finite subsets of Γ.

Of course, each pair $\{\gamma,\gamma'\}$ appears only once in the product. We allow $\Gamma' = \varnothing$, in which case the products are, as usual, defined to be 1.

The polymers will always be geometric objects of finite size living on $\mathbb{Z}^d$ (or, possibly, on the dual lattice) and their interaction will be related to pairwise geometric compatibility conditions between the polymers; these conditions will usually be *local*, that is, the compatibility of two polymers can be checked by inspecting their "neighborhood" on $\mathbb{Z}^d$.

5.3 The formal expansion

The cluster expansion provides an explicit expansion for $\log \Xi$, in the form of a series. To obtain the coefficients of this expansion, we will perform a sequence of operations on Ξ, leading to an expression of the form

$$\Xi = \exp(\cdots)\,.$$

As a first step, the sum over $\Gamma' \subset \Gamma$ can be decomposed according to the number $|\Gamma'|$ of polymers contained in Γ':

$$\sum_{\Gamma' \subset \Gamma} \big(\cdots \big) = 1 + \sum_{n \ge 1} \sum_{\substack{\Gamma' \subset \Gamma: \\ |\Gamma'| = n}} \big(\cdots \big)\,.$$

For convenience, we now transform the second sum over $\Gamma' \subset \Gamma$ into a sum over *ordered* n-tuples. So let $G_n = (V_n, E_n)$ be the complete graph on $V_n = \{1,2,\dots,n\}$. That is, G_n is the simple undirected graph in which there is precisely one edge $\{i,j\} \in E_n$ for each pair of distinct vertices $i,j \in V_n$ (see Figure 2.1). We can then write

$$\Xi = 1 + \sum_{n \ge 1} \frac{1}{n!} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \Big\{ \prod_{i \in V_n} \mathrm{w}(\gamma_i) \Big\} \Big\{ \prod_{\{i,j\} \in E_n} \delta(\gamma_i,\gamma_j) \Big\}\,. \tag{5.4}$$

Notice that, the sum being now over all ordered n-tuples $(\gamma_1,\dots,\gamma_n) \in \Gamma^{V_n}$, we had to introduce a factor $\frac{1}{n!}$ to avoid overcounting. Observe that only collections in which all polymers $\gamma_1,\dots,\gamma_n$ are distinct contribute to the sum, since $\delta(\gamma_i,\gamma_j) = 0$

whenever $\gamma_i = \gamma_j$. This means that only a finite number of terms, in this sum over $n \geq 1$, are non-zero.

The next step is the following: rather than working with a sum over the n-tuples of Γ^{V_n}, we will work with a sum over suitable subgraphs of G_n. We write $G \subset G_n$ to indicate that G is a subgraph of G_n with the same set of vertices V_n and with a set of edges which is a subset of E_n. Given a graph $G = (V, E)$, we will often write $i \in G$, respectively $e \in G$, instead of $i \in V$, respectively $e \in E$.

Subgraphs of G_n can be introduced if one uses the "$+1-1$" trick to expand the product containing the interactions between the polymers (see Exercise 3.22). Letting

$$\zeta(\gamma,\gamma') \stackrel{\text{def}}{=} \delta(\gamma,\gamma') - 1\,,$$

we get

$$\prod_{\{i,j\}\in E_n} \delta(\gamma_i,\gamma_j) = \prod_{\{i,j\}\in E_n} (1+\zeta(\gamma_i,\gamma_j)) = \sum_{E\subset E_n} \prod_{\{i,j\}\in E} \zeta(\gamma_i,\gamma_j)\,.$$

Since a set $E \subset E_n$ can be put in one-to-one correspondence with the subgraph $G \subset G_n$ defined by $G \stackrel{\text{def}}{=} (V_n, E)$, we can interpret the sum over $E \subset E_n$ as a sum over $G \subset G_n$. We thus obtain

$$\begin{aligned}\Xi &= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{G\subset G_n} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \Big\{ \prod_{i\in V_n} \mathrm{w}(\gamma_i) \Big\} \Big\{ \prod_{\{i,j\}\in E} \zeta(\gamma_i,\gamma_j) \Big\} \\ &= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{G\subset G_n} Q[G]\,, \end{aligned} \tag{5.5}$$

where we have introduced, for a graph $G = (V, E)$,

$$Q[G] \stackrel{\text{def}}{=} \sum_{\gamma_1} \cdots \sum_{\gamma_{|V|}} \Big\{ \prod_{i\in V} \mathrm{w}(\gamma_i) \Big\} \Big\{ \prod_{\{i,j\}\in E} \zeta(\gamma_i,\gamma_j) \Big\}\,.$$

Let us now define the **Ursell functions** φ on ordered families $(\gamma_1,\dots,\gamma_m)$ by $\varphi(\gamma_1) \stackrel{\text{def}}{=} 1$, when $m = 1$, and

$$\varphi(\gamma_1,\dots,\gamma_m) \stackrel{\text{def}}{=} \frac{1}{m!} \sum_{\substack{G\subset G_m \\ \text{connected}}} \prod_{\{i,j\}\in G} \zeta(\gamma_i,\gamma_j)\,,$$

when $m \geq 2$.

Proposition 5.3.

$$\Xi = \exp\Big(\sum_{m\geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_m} \varphi(\gamma_1,\dots,\gamma_m) \prod_{i\in V_m} \mathrm{w}(\gamma_i) \Big)\,. \tag{5.6}$$

Observe that, even if Ξ is a finite sum, the resulting series in (5.6) is infinite, since a given polymer can appear several times in the same collection, without the Ursell function necessarily vanishing. In the next section, we will state conditions that ensure that the series is actually absolutely convergent, which will justify the rearrangements done in the proof below. For the time being, however, we are only interested in the structure of its coefficients, so the series in (5.6) should (temporarily) only be considered as formal.

Proof of Proposition 5.3: Notice that, if $G'_1,\dots,G'_k$ are the (maximal) connected components of G, then

$$Q[G]=\prod_{r=1}^{k} Q[G'_r]\,.$$

Now, observe that $Q[G]=Q[G']$ whenever G and G' are isomorphic [1]. One can thus replace the vertex set V'_i of G'_i by $\{1,\dots,m_i\}$, where $m_i=|V'_i|$. Therefore,

$$\begin{aligned}\sum_{G\subset G_n} Q[G] &= \sum_{k=1}^{n} \sum_{\substack{G\subset G_n\\ G=(G'_1,\dots,G'_k)}} \prod_{r=1}^{k} Q[G'_r]\\ &= \sum_{k=1}^{n}\frac{1}{k!}\sum_{\substack{m_1,\dots,m_k\\ m_1+\dots+m_k=n}} \frac{n!}{m_1!\cdots m_k!}\sum_{\substack{G'_1\subset G_{m_1}\\ \text{connected}}}\cdots\sum_{\substack{G'_k\subset G_{m_k}\\ \text{connected}}}\prod_{r=1}^{k} Q[G'_r]\,, \end{aligned}\tag{5.7}$$

where, in the second identity, the coefficient $n!/(m_1!\cdots m_k!)$ takes into account the number of ways of partitioning V_n into k disjoint subsets of respective cardinalities $m_1,\dots,m_k\ge 1$. Observe that, at least formally,

$$\sum_{n\ge1}\sum_{k=1}^{n}\sum_{\substack{m_1,\dots,m_k\\ m_1+\dots+m_k=n}}(\cdots)=\sum_{k\ge1}\sum_{n\ge k}\sum_{\substack{m_1,\dots,m_k\\ m_1+\dots+m_k=n}}(\cdots)=\sum_{k\ge1}\sum_{m_1,\dots,m_k}(\cdots)\,,\tag{5.8}$$

which leads to

$$\begin{aligned}\Xi &= 1+\sum_{k\ge1}\frac{1}{k!}\sum_{m_1,\dots,m_k}\prod_{r=1}^{k}\Big\{\frac{1}{m_r!}\sum_{\substack{G'_r\subset G_{m_r}\\ \text{connected}}} Q[G'_r]\Big\}\\ &= 1+\sum_{k\ge1}\frac{1}{k!}\Big(\sum_{m\ge1}\sum_{\gamma_1}\cdots\sum_{\gamma_m}\varphi(\gamma_1,\dots,\gamma_m)\prod_{j=1}^{m}\mathsf{w}(\gamma_j)\Big)^k\,,\end{aligned}$$

which is (5.6). □

Let us emphasize the delicate point ignored in the above computation. In a first step, in (5.5)*, Ξ was written with the help of a sum $\sum_{n\ge1} a_n$, where n indexes the size of the complete graph G_n, and where the a_n are all equal to zero when n is sufficiently large. In a second step (see* (5.7)*), each a_n was decomposed as $a_n=\sum_{k=1}^{n} b_{k,n}$, where the index k denotes the number of connected components of the subgraph $G\subset G_n$. Since a_n vanishes for large n, this means that important cancellations occur among the $b_{k,n}$ (when summed over k). The main formal computation that requires justification was done in* (5.8)*, when we interchanged the summations over n and k:*

$$\sum_{n\ge1} a_n=\sum_{n\ge1}\sum_{k=1}^{n} b_{k,n}=\sum_{k\ge1}\sum_{n\ge k} b_{k,n}\,.$$

Namely, the interchange is allowed only if each of the series $\sum_{n\ge k} b_{k,n}$ is known to converge, and this is not guaranteed in general. ◇

[1] Two graphs $G=(V,E)$ and $G=(V',E')$ are **isomorphic** if there exists a bijection $f:V\to V'$ such that an edge $e=\{x,y\}$ belongs to E if and only if $e'=\{f(x),f(y)\}$ belongs to E'.

Proving that Ξ has a well-defined logarithm implies in particular that $\Xi \neq 0$. As we saw when studying uniqueness in the Ising model, the absence of zeros of the partition function on a complex domain (for each Λ along a sequence $\Lambda \Uparrow \mathbb{Z}^d$) entails in fact uniqueness of the infinite-volume Gibbs measure of this model. This indicates that guaranteeing the absolute convergence of the series for $\log \Xi$ is non-trivial in general, and the latter will usually hold only for some restricted range of values of the parameters of the underlying model.

5.4 A condition ensuring convergence

We now impose conditions on the weights that ensure that the series in (5.6) converges absolutely:

$$\sum_{k\geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_k} |\varphi(\gamma_1,\dots,\gamma_k)| \prod_{i=1}^{k} |\mathrm{w}(\gamma_i)| < \infty. \tag{5.9}$$

The main ingredient is the following:

Theorem 5.4. *Assume that* (5.2) *holds and that there exists* $a : \Gamma \to \mathbb{R}_{>0}$ *such that, for each* $\gamma_* \in \Gamma$,

$$\sum_{\gamma} |\mathrm{w}(\gamma)| e^{a(\gamma)} |\zeta(\gamma,\gamma_*)| \leq a(\gamma_*). \tag{5.10}$$

Then, for all $\gamma_1 \in \Gamma$,

$$1 + \sum_{k\geq 2} k \sum_{\gamma_2} \cdots \sum_{\gamma_k} |\varphi(\gamma_1,\gamma_2,\dots,\gamma_k)| \prod_{j=2}^{k} |\mathrm{w}(\gamma_j)| \leq e^{a(\gamma_1)}. \tag{5.11}$$

In particular, (5.9) *holds.*

Remark 5.5. In this chapter, we always assume that $|\Gamma| < \infty$. Nevertheless, this restriction is not necessary. When it is not imposed, in addition to (5.10), one has to require that

$$\sum_{\gamma} |\mathrm{w}(\gamma)| e^{a(\gamma)} < \infty.$$

◇

The series in (5.10) *should remind the reader of those considered when implementing Peierls' argument, such as* (3.37). *Actually, verifying that these conditions hold in a specific situation usually amounts to a similar energy-entropy argument.*

◇

Exercise 5.1. *Verify that* (5.11) *implies* (5.9).

Proof of Theorem 5.4: We fix $\gamma_1 \in \Gamma$ and show that, for all $N \geq 2$,

$$1 + \sum_{k=2}^{N} k \sum_{\gamma_2} \cdots \sum_{\gamma_k} |\varphi(\gamma_1,\gamma_2,\dots,\gamma_k)| \prod_{j=2}^{k} |\mathrm{w}(\gamma_j)| \leq e^{a(\gamma_1)}. \tag{5.12}$$

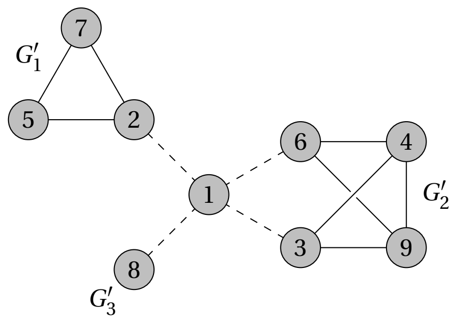
Clearly, letting $N \to \infty$ in (5.12) yields (5.11). The proof of (5.12) is done by induction over N.

For $N = 2$, the only connected graph $G \subset G_2$ is the one with one edge connecting 1 and 2, and so $\varphi(\gamma_1, \gamma_2) = \frac{1}{2!}\zeta(\gamma_1, \gamma_2)$. Therefore, the left-hand side of (5.12) is

$$1 + 2\sum_{\gamma_2} |\varphi(\gamma_1, \gamma_2)||\mathrm{w}(\gamma_2)| = 1 + \sum_{\gamma_2} |\zeta(\gamma_1, \gamma_2)||\mathrm{w}(\gamma_2)| \le e^{a(\gamma_1)},$$

where we used $1 \le e^{a(\gamma_2)}$, (5.10) and $1 + x \le e^x$. This proves (5.12) for $N = 2$. We now show that if (5.12) holds for N, then it also holds for $N + 1$.

To do that, consider the left-hand side of (5.12) with $N + 1$ in place of N, take some $k \le N + 1$, and consider any connected graph $G \subset G_k$ appearing in the sum defining $\varphi(\gamma_1, \gamma_2, \dots, \gamma_k)$. Let E' denote the non-empty set of edges of G with an endpoint at 1. The graph G', obtained from G by removing 1 together with each edge of E', splits into a set of connected components $G'_1, \dots, G'_l$.



We can thus see G as obtained by (i) partitioning the set $\{2, 3, \dots, k\}$ into subsets $V'_1, \dots, V'_l$, $l \le k - 1$, (ii) associating to each V'_i a connected graph G'_i, and (iii) connecting 1 in all possible ways to at least one point in each connected component V'_i. Accordingly,

$$\varphi(\gamma_1, \gamma_2, \dots, \gamma_k) = \tag{5.13}$$
$$\frac{1}{k!}\sum_{l=1}^{k-1}\frac{1}{l!}\sum_{V'_1,\dots,V'_l}\prod_{i=1}^{l}\Big\{\sum_{\substack{G'_i : V(G'_i) = V'_i\\ \text{connected}}}\ \prod_{\{i',j'\}\in G'_i}\zeta(\gamma_{i'}, \gamma_{j'})\Big\}\Big\{\sum_{\substack{K_i \subset V'_i\\ K_i \ne \varnothing}}\ \prod_{j' \in K_i}\zeta(\gamma_1, \gamma_{j'})\Big\}.$$

The next step is to specify the number of points in each V'_i. If $|V'_i| = m_i$,

$$\sum_{\substack{G'_i : V(G'_i) = V'_i\\ \text{connected}}}\ \prod_{\{i',j'\}\in G'_i}\zeta(\gamma_{i'}, \gamma_{j'}) = m_i!\,\varphi\big((\gamma_{j'})_{j' \in V'_i}\big).$$

Moreover,

$$\sum_{\substack{K_i \subset V'_i\\ K_i \ne \varnothing}}\ \prod_{j' \in K_i}\zeta(\gamma_1, \gamma_{j'}) = \Big\{\prod_{j' \in V'_i}\big(1 + \zeta(\gamma_1, \gamma_{j'})\big)\Big\} - 1. \tag{5.14}$$

Exercise 5.2. *Assuming $|1 + \alpha_k| \le 1$ for all $k \ge 1$, show that*

$$\Big|\prod_{k=1}^{n}(1 + \alpha_k) - 1\Big| \le \sum_{k=1}^{n}|\alpha_k|. \tag{5.15}$$

Since (5.2) guarantees that $|1+\zeta| \le 1$, (5.14) and (5.15) yield

$$\Bigl|\sum_{\substack{K_i \subset V'_i\\ K_i \neq \varnothing}} \prod_{j' \in K_i} \zeta(\gamma_1, \gamma_{j'})\Bigr| \le \sum_{j' \in V'_i} |\zeta(\gamma_1, \gamma_{j'})|.$$

We now use (5.13) to bound the sum on the left-hand side of (5.12) (with $N+1$ in place of N). The sum over the sets V'_i will be made as in the proof of Proposition 5.3: the number of partitions of $\{2,3,\dots,k\}$ into $(V'_1,\dots,V'_l)$, with $|V'_i| = m_i$, $m_1+\cdots+m_l = k-1$, is equal to $\frac{(k-1)!}{m_1!\cdots m_l!}$. But, since the summands are nonnegative, we can bound

$$\begin{aligned}\sum_{k=2}^{N+1}\sum_{l=1}^{k-1}\sum_{\substack{m_1,\dots,m_l:\\ m_1+\cdots+m_l=k-1}}(\cdots) &= \sum_{l=1}^{N}\sum_{k=l+1}^{N+1}\sum_{\substack{m_1,\dots,m_l:\\ m_1+\cdots+m_l=k-1}}(\cdots)\\ &\le \sum_{l=1}^{N}\sum_{m_1=1}^{N}\cdots\sum_{m_l=1}^{N}(\cdots),\end{aligned}$$

which leaves us with

$$\begin{aligned}&\sum_{k=2}^{N+1} k \sum_{\gamma_2}\cdots\sum_{\gamma_k} |\varphi(\gamma_1,\gamma_2,\dots,\gamma_k)| \prod_{j=2}^{k} |\mathrm{w}(\gamma_j)|\\ &\le \sum_{l\ge 1}\frac{1}{l!}\prod_{i=1}^{l}\Bigl\{\sum_{m_i=1}^{N}\sum_{\gamma'_1}\cdots\sum_{\gamma'_{m_i}} |\varphi(\gamma'_1,\dots,\gamma'_{m_i})| \prod_{j'=1}^{m_i} |\mathrm{w}(\gamma'_{j'})| \sum_{j'=1}^{m_i} |\zeta(\gamma_1,\gamma'_{j'})|\Bigr\}. \qquad (5.16)\end{aligned}$$

Lemma 5.6. *If* (5.12) *holds, then, for all* $\gamma_* \in \Gamma$,

$$\sum_{k=1}^{N}\sum_{\gamma_1}\cdots\sum_{\gamma_k}\Bigl\{\sum_{i=1}^{k}|\zeta(\gamma_*,\gamma_i)|\Bigr\}|\varphi(\gamma_1,\dots,\gamma_k)|\prod_{j=1}^{k}|\mathrm{w}(\gamma_j)| \le a(\gamma_*). \qquad (5.17)$$

Proof. We fix $\gamma_* \in \Gamma$, and multiply both sides of (5.12) by $|\zeta(\gamma_*,\gamma_1)| \cdot |\mathrm{w}(\gamma_1)|$, and sum over γ_1. Using (5.10), the right-hand side of the expression obtained can be bounded by $a(\gamma_*)$, whereas the left-hand side becomes

$$\sum_{k=1}^{N} k \sum_{\gamma_1}\cdots\sum_{\gamma_k} |\zeta(\gamma_*,\gamma_1)||\varphi(\gamma_1,\dots,\gamma_k)| \prod_{j=1}^{k} |\mathrm{w}(\gamma_j)|.$$

But clearly, for all $i \in \{2,\dots,k\}$,

$$\begin{aligned}&\sum_{\gamma_1}\cdots\sum_{\gamma_k} |\zeta(\gamma_*,\gamma_1)||\varphi(\gamma_1,\dots,\gamma_k)| \prod_{j=1}^{k} |\mathrm{w}(\gamma_j)|\\ &\qquad = \sum_{\gamma_1}\cdots\sum_{\gamma_k} |\zeta(\gamma_*,\gamma_i)||\varphi(\gamma_1,\dots,\gamma_k)| \prod_{j=1}^{k} |\mathrm{w}(\gamma_j)|,\end{aligned}$$

which proves the claim. □

Using (5.17), we can bound (5.16) by $\sum_{l\ge 1}\frac{1}{l!}a(\gamma_1)^l = e^{a(\gamma_1)} - 1$. This concludes the proof of Theorem 5.4. □

The determination of a suitable function $a(\gamma)$ for (5.10) will depend on the problem considered. As we will see in the applications below, $a(\gamma)$ will usually be naturally related to some measure of the size of γ.

Example 5.7. As the most elementary application of the previous lemma, let us consider the expansion of $\log(1+z)$ for small $|z|$.

The function $1+z$ can be seen as a particularly simple example of polymer partition function: one with a single polymer, $\Gamma = \{\gamma\}$, and a weight $\mathrm{w}(\gamma) = z$. Indeed, in this case, there are only two terms in the right-hand side of (5.3) ($\Gamma' = \varnothing$ and $\Gamma' = \{\gamma\}$) and the partition function reduces to

$$\Xi = 1 + z\,.$$

Condition (5.10) then becomes

$$|z|e^{a} \le a\,,$$

where $a > 0$ is a constant we can choose. Since $a \mapsto ae^{-a}$ is maximal when $a = 1$, the best possible choice for a is $a = 1$ and the condition for convergence becomes

$$|z| \le e^{-1}\,.$$

Theorem 5.4 then guarantees convergence of the cluster expansion for $\log\Xi$ for all such values of z: by (5.6),

$$\log(1+z) = \log\Xi = \sum_{m\ge 1} \varphi_m z^m\,,$$

where we have introduced

$$\varphi_m \overset{\text{def}}{=} \varphi\underbrace{(\gamma,\dots,\gamma)}_{m \text{ copies}} = \frac{1}{m!}\sum_{k=0}^{\binom{m}{2}} (-1)^k\,|\mathscr{G}_{m,k}|\,,$$

and $\mathscr{G}_{m,k}$ is the set of all connected subgraphs of the complete graph G_m with m vertices and k edges.

It is instructive to compare the above result with the classical Taylor expansion

$$\log(1+z) = \sum_{m\ge 1} \frac{(-1)^{m-1}}{m}\, z^m\,.$$

First, we see that the condition in Theorem 5.4 is not optimal, since the latter series actually converges whenever $|z| < 1$. Moreover, identifying the coefficients of z^n in both expansions, we obtain the following nontrivial combinatorial identity:

$$\sum_{k=0}^{\binom{m}{2}} (-1)^k|\mathscr{G}_{m,k}| = (-1)^{m-1}(m-1)!\,. \qquad \diamond$$

5.5 When the weights depend on a parameter

The convergence of the cluster expansion is very often used to prove analyticity of the pressure in the thermodynamic limit. So let us assume that the weights of the polymers depend on some complex parameter:

$$z \mapsto \mathrm{w}_z(\gamma)\,, \qquad z \in D\,,$$

where D is a domain of $\mathbb{C}$. When each weight depends smoothly (for example, analytically) on z, it can be useful to determine whether this smoothness extends to $\log\Xi$.

Theorem 5.8. *Assume that $z \mapsto \mathsf{w}_z(\gamma)$ is analytic on D, for each $\gamma \in \Gamma$, and that there exists a real weight $\overline{\mathsf{w}}(\gamma) \geq 0$ such that*

$$\sup_{z \in D} |\mathsf{w}_z(\gamma)| \leq \overline{\mathsf{w}}(\gamma), \quad \forall \gamma \in \Gamma, \tag{5.18}$$

and such that (5.10) *holds with* $\overline{\mathsf{w}}(\gamma)$ *in place of* $\mathsf{w}(\gamma)$. *Then,* (5.6) *and* (5.9) *hold with* $\mathsf{w}_z(\gamma)$ *in place of* $\mathsf{w}(\gamma)$*, and $z \mapsto \log \Xi$ is analytic on D.*

Proof. Let us write the expansion as $\log \Xi = \sum_{n \geq 1} f_n(z)$, where

$$f_n(z) \stackrel{\text{def}}{=} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \varphi(\gamma_1, \dots, \gamma_n) \prod_{i=1}^{n} \mathsf{w}_z(\gamma_i).$$

Since $|\Gamma| < \infty$, f_n is a sum containing only a finite number of terms; it is therefore analytic in D. If we can verify that the series $\sum_n f_n$ is uniformly convergent on compact sets $K \subset D$, Theorem B.27 will imply that it represents an analytic function on D. We therefore compute

$$\begin{aligned} \sup_{z \in K} \Big| \sum_{n \geq 1} f_n(z) - \sum_{n=1}^{N} f_n(z) \Big| &\leq \sup_{z \in K} \sum_{n > N} |f_n(z)| \\ &\leq \sum_{n > N} \sup_{z \in K} |f_n(z)| \\ &\leq \sum_{n > N} \sum_{\gamma_1} \cdots \sum_{\gamma_n} |\varphi(\gamma_1, \dots, \gamma_n)| \prod_{i=1}^{n} \overline{\mathsf{w}}(\gamma_i). \end{aligned} \tag{5.19}$$

By our assumptions, Theorem 5.4 implies that (5.9) holds, with $\overline{\mathsf{w}}(\cdot)$ in place of $|\mathsf{w}(\cdot)|$. This implies that (5.19) goes to zero when $N \to \infty$. The fact that (5.11) holds is immediate. □

5.6 The case of hard-core interactions

Up to now, we have considered fairly general interactions. But often in practice, and in all cases treated in this book, δ takes the particularly simple form of a **hard-core** interaction, that is,

$$\delta(\gamma, \gamma') \in \{0, 1\} \qquad \text{for all } \gamma, \gamma' \in \Gamma.$$

In such a case, two polymers γ and γ' will be said to be **compatible** if $\delta(\gamma, \gamma') = 1$ and **incompatible** if $\delta(\gamma, \gamma') = 0$. Obviously, only collections of pairwise compatible polymers yield a non-zero contribution to the partition function Ξ in (5.4).

Let us now turn to the series (5.6) for $\log \Xi$. We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ is **decomposable** if it is possible to express it as a disjoint union of two non-empty sets, in such a way that each γ_i in the first set is compatible with each γ_j in the second. It follows immediately from the definition of the Ursell functions that

$$\varphi(\gamma_1, \dots, \gamma_n) = 0 \quad \text{if } \{\gamma_1, \dots, \gamma_n\} \text{ is decomposable.}$$

In particular, the non-zero contributions to $\log \Xi$ in (5.6) therefore come from the *non-decomposable* collections. An unordered, non-decomposable collection $X = \{\gamma_1, \dots, \gamma_n\}$ is called a **cluster**. Note that X is actually a *multiset*, that is, the same

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polymer can appear multiple times. We denote by $n_X(\gamma)$ the number of times the polymer $\gamma \in \Gamma$ appears in X. We can write

$$\log \Xi = \sum_{n\geq 1} \sum_{\gamma_1} \cdots \sum_{\gamma_n} \varphi(\gamma_1,\dots,\gamma_n) \prod_{i=1}^{n} \mathrm{w}(\gamma_i) = \sum_X \Psi(X),$$

where the sum is over all clusters of polymers in Γ and, for a cluster $X = \{\tilde{\gamma}_1,\dots,\tilde{\gamma}_n\}$,

$$\Psi(X) \stackrel{\text{def}}{=} \Big\{\prod_{\gamma\in\Gamma} \frac{1}{n_X(\gamma)!}\Big\} \Big\{ \sum_{\substack{G\subset G_n \\ \text{connected}}} \prod_{\{i,j\}\in G} \zeta(\tilde{\gamma}_i,\tilde{\gamma}_j)\Big\} \prod_{i=1}^{n} \mathrm{w}(\tilde{\gamma}_i). \tag{5.20}$$

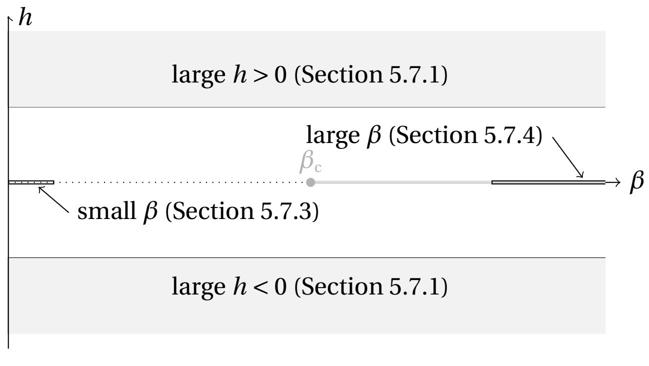
Indeed, given a cluster $X = \{\tilde{\gamma}_1,\dots,\tilde{\gamma}_n\}$, there are $\frac{n!}{\prod_{\gamma\in\Gamma} n_X(\gamma)!}$ distinct ways of assigning the polymers $\tilde{\gamma}_1,\dots,\tilde{\gamma}_n$ to the summation variables $\gamma_1,\dots,\gamma_n$ above.

5.7 Applications

The cluster expansion can be applied in many situations. Our main systematic use of it will be in Chapter 7, when developing the Pirogov–Sinai theory. We will also use it to obtain a uniqueness criterion for infinite-volume Gibbs measures, in Section 6.5.4.

Before that, we apply it in various ways to the Ising model (and to the corresponding nearest-neighbor lattice gas). We will see that to different regions of the phase diagram correspond different well-suited polymer models. The cluster expansion can then be used to extract useful information on the model for parameters in these regions.

When checking Condition (5.10), we will see that the regions in which the cluster expansion converges for those polymer models are all far from the point $(\beta, h) = (\beta_c, 0)$:



5.7.1 The Ising model in a strong magnetic field

Consider the Ising model with a complex magnetic field $h \in \mathbb{C}$, at an arbitrary inverse temperature $\beta \geq 0$. The Lee–Yang Circle theorem proved in Chapter 3 yields existence and analyticity of the pressure in the half planes $H^+ \stackrel{\text{def}}{=} \{h \in \mathbb{C} : \mathfrak{Re}\, h > 0\}$ and $H^- \stackrel{\text{def}}{=} \{h \in \mathbb{C} : \mathfrak{Re}\, h < 0\}$. Here, we will use the cluster expansion to obtain a weaker result, namely that analyticity holds in the regions $\{h \in \mathbb{C} : \mathfrak{Re}\, h > x_0 > 0\}$

and $\{h \in \mathbb{C} : \mathfrak{Re}\, h < -x_0 < 0\}$ (see below for the value of x_0). Although theses regions are proper subsets of the half-planes H^+ and H^-, the convergent expansion provides a wealth of additional information on the pressure in these regions, not provided by the Lee–Yang approach.

We will consider the case $\mathfrak{Re}\, h > 0$. As seen in Chapter 3, when $h \in \mathbb{R}$, the pressure does not depend on the boundary condition used in the thermodynamic limit, and we can thus choose the most convenient one. In this section, this turns out to be the + boundary condition. The first step is to define a polymer model that is well suited for the analysis of the Ising model with a large magnetic field.

When the magnetic field $h > 0$ is large, there is a very strong incentive for spins to take the value $+1$. It is therefore natural to describe configurations by only keeping track of the negative spins. ◇

We emphasize the role of the negative spins by writing the Hamiltonian as follows:

$$\begin{aligned}\mathscr{H}_{\Lambda;\beta,h} &= -\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} \sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \\ &= -\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| - h|\Lambda| - \beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} (\sigma_i\sigma_j - 1) - h\sum_{i\in\Lambda}(\sigma_i - 1)\,. \end{aligned} \tag{5.21}$$

Let $\omega \in \Omega^+_\Lambda$. Introducing the set

$$\Lambda^-(\omega) \stackrel{\text{def}}{=} \{i \in \Lambda : \omega_i = -1\}\,, \tag{5.22}$$

we can write

$$\mathscr{H}_{\Lambda;\beta,h}(\omega) = -\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| - h|\Lambda| + 2\beta|\partial_e\Lambda^-(\omega)| + 2h|\Lambda^-(\omega)|\,,$$

where we remind the reader that $\partial_e A \stackrel{\text{def}}{=} \big\{\{i,j\} : i \sim j,\, i \in A,\, j \not\in A\big\}$. Notice that $\mathscr{H}_{\Lambda;\beta,h}$ has a unique ground state, namely the constant configuration η^+ (in which all spins equal $+1$), for which $\Lambda^-(\eta^+) = \varnothing$ and

$$\mathscr{H}_{\Lambda;\beta,h}(\eta^+) = -\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| - h|\Lambda|\,.$$

We can then write the partition function by emphasizing that configurations ω with $\Lambda^-(\omega) \neq \varnothing$ represent deviations from the ground state:

$$\begin{aligned}\mathbf{Z}^+_{\Lambda;\beta,h} &= e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| + h|\Lambda|} \sum_{\Lambda^-\subset\Lambda} e^{-2\beta|\partial_e\Lambda^-| - 2h|\Lambda^-|} \\ &= e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| + h|\Lambda|}\Big\{1 + \sum_{\substack{\Lambda^-\subset\Lambda:\\ \Lambda^-\neq\varnothing}} e^{-2\beta|\partial_e\Lambda^-| - 2h|\Lambda^-|}\Big\}.\end{aligned}$$

Let us declare two vertices $i, j \in \Lambda^-$ to be connected if $d_1(i,j) \stackrel{\text{def}}{=} \|j - i\|_1 = 1$. We can then decompose Λ^- into maximal connected components (see Figure 5.1):

$$\Lambda^- = S_1 \cup \cdots \cup S_n\,.$$

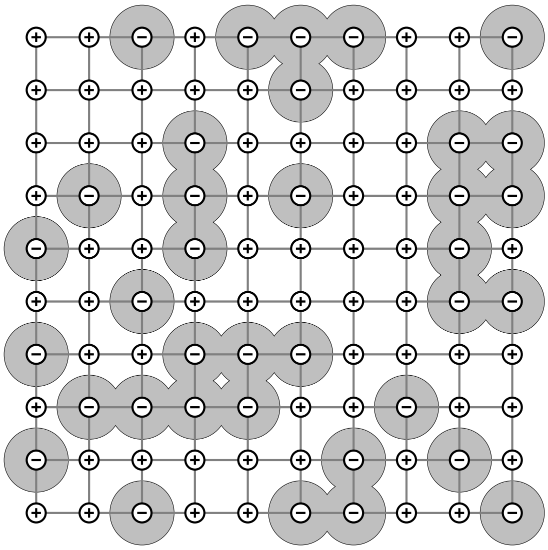


Figure 5.1: A configuration of the Ising model. Each connected component of the shaded area delimits one of the polymers $S_1, \dots, S_{17}$.

By definition, $d_1(S_i, S_j) \stackrel{\text{def}}{=} \inf\{d_1(k,l) : k \in S_i, l \in S_j\} > 1$ if $i \neq j$. The components S_i play the role of the polymers in the present application. Since $|\partial_e \Lambda^-| = \sum_{i=1}^n |\partial_e S_i|$ and $|\Lambda^-| = \sum_{i=1}^n |S_i|$, we can write

$$\mathbf{Z}^+_{\Lambda;\beta,h} = e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| + h|\Lambda|}\, \Xi^{\mathrm{LF}}_{\Lambda;\beta,h}\,, \tag{5.23}$$

where the **large-field polymer partition function** is

$$\Xi^{\mathrm{LF}}_{\Lambda;\beta,h} \stackrel{\text{def}}{=} 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{S_1 \subset \Lambda} \cdots \sum_{S_n \subset \Lambda} \Big\{ \prod_{i=1}^n \mathsf{w}_h(S_i) \Big\} \Big\{ \prod_{1\leq i<j\leq n} \delta(S_i, S_j) \Big\}. \tag{5.24}$$

Each sum $\sum_{S_i \subset \Lambda}$ is over non-empty connected subsets of Λ (from now on, all sets denoted by the letter S, with or without a subscript, will be considered as non-empty and connected), the weights are

$$\mathsf{w}_h(S_i) \stackrel{\text{def}}{=} e^{-2\beta|\partial_e S_i| - 2h|S_i|}\,,$$

and the interactions are of hard-core type:

$$\delta(S_i, S_j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } d_1(S_i, S_j) > 1, \\ 0 & \text{otherwise.} \end{cases}$$

We will now show that there exists a function $a(S) \geq 0$ such that (5.10) holds when $\mathfrak{Re}\, h$ is taken sufficiently large. In the present context, this condition becomes

$$\forall S_* \subset \Lambda, \quad \sum_{S \subset \Lambda} |\mathsf{w}_h(S)| e^{a(S)} |\zeta(S, S_*)| \leq a(S_*)\,, \tag{5.25}$$

where we remind the reader that $\zeta(S, S_*) \stackrel{\text{def}}{=} \delta(S, S_*) - 1$. Observe that $\zeta(S, S_*) \neq 0$ if and only if $S \cap [S_*]_1 \neq \varnothing$, where

$$[S_*]_1 \stackrel{\text{def}}{=} \Big\{ j \in \mathbb{Z}^d : d_1(j, S_*) \leq 1 \Big\}\,.$$

Therefore, the sum in (5.25) can be bounded by

$$\sum_{S\subset\Lambda}|\mathrm{w}_h(S)|e^{a(S)}|\zeta(S,S_*)|\le|[S_*]_1|\max_{j\in[S_*]_1}\sum_{S\ni j}|\mathrm{w}_h(S)|e^{a(S)},$$

where now the sum over $S\ni j$ is an infinite sum over all finite connected subsets of $\mathbb{Z}^d$ that contain the point j. Let us define, for all S,

$$a(S)\stackrel{\text{def}}{=}|[S]_1|.$$

Since both the weights and $a(\cdot)$ are invariant under translations,

$$\max_{j\in[S_*]_1}\sum_{S\ni j}|\mathrm{w}_h(S)|e^{|[S]_1|}=\sum_{S\ni 0}|\mathrm{w}_h(S)|e^{|[S]_1|}.$$

Therefore, guaranteeing that

$$\sum_{S\ni 0}|\mathrm{w}_h(S)|e^{|[S]_1|}\le 1 \tag{5.26}$$

ensures that (5.25) is satisfied. The weight $\mathrm{w}_h(S)$ contains two terms: a **surface term** $e^{-2\beta|\partial_e S|}$, and a **volume term** $e^{-2h|S|}$. Observe that $e^{|[S]_1|}$ is also a volume term, since

$$|S|\le|[S]_1|\le(2d+1)|S|.$$

We therefore see that, in order for the series in (5.26) to converge and be smaller or equal to 1, the real part of the magnetic field will need to be taken sufficiently large for $e^{-2\mathfrak{Re}\,h|S|}$ to compensate $e^{(2d+1)|S|}$. It will also be necessary to compensate for the number of sets $S\ni 0$ as a function of their size, since the latter also grows exponentially fast with $|S|$. The surface term, on the other hand, will be of no help and will be simply bounded by 1. So, grouping the sets $S\ni 0$ by size,

$$\begin{aligned}\sum_{S\ni 0}|\mathrm{w}_h(S)|e^{|[S]_1|}&=\sum_{k\ge1}e^{-2k\mathfrak{Re}\,h}\sum_{\substack{S\ni 0\\|S|=k}}e^{-2\beta|\partial_e S|}e^{|[S]_1|}\\&\le\sum_{k\ge1}e^{-(2\mathfrak{Re}\,h-2d-1)k}\#\{S\ni 0:|S|=k\}.\end{aligned}$$

Exercise 5.3. *Using Lemma 3.38, show that*

$$\#\{S\ni 0:|S|=k\}\le(2d)^{2k}. \tag{5.27}$$

Using (5.27), we get

$$\sum_{S\ni 0}|\mathrm{w}_h(S)|e^{|[S]_1|}\le\eta(\mathfrak{Re}\,h,d), \tag{5.28}$$

where $\eta(x,d)\stackrel{\text{def}}{=}\sum_{k\ge1}e^{-(2x-2d-1-2\log(2d))k}$. If we define

$$x_0=x_0(d)\stackrel{\text{def}}{=}\inf\{x>0:\eta(x,d)\le1\}$$

and let

$$H^+_{x_0}\stackrel{\text{def}}{=}\{h\in\mathbb{C}:\mathfrak{Re}\,h>x_0\},$$

then, for all $h\in H^+_{x_0}$, the cluster expansion

$$\log\Xi^{\mathrm{LF}}_{\Lambda;\beta,h}=\sum_{n\ge1}\sum_{S_1\subset\Lambda}\cdots\sum_{S_n\subset\Lambda}\varphi(S_1,\dots,S_n)\prod_{i=1}^n\mathrm{w}_h(S_i)$$

converges absolutely. As seen in Section 5.6, the contributions to the expansion come from the *clusters* $X = \{S_1, \dots, S_n\}$. We define the **support** of $X = \{S_1, \dots, S_n\}$ by $\overline{X} \stackrel{\text{def}}{=} S_1 \cup \dots \cup S_n$. With these notations,

$$\sum_{n \geq 1} \sum_{S_1 \subset \Lambda} \cdots \sum_{S_n \subset \Lambda} \varphi(S_1, \dots, S_n) \prod_{i=1}^{n} \mathrm{w}_h(S_i) = \sum_{X : \overline{X} \subset \Lambda} \Psi(X),$$

where $\Psi(\cdot)$ was defined in (5.20).

We will now see how to use this to extract the volume and surface contributions to the pressure in Λ. First, notice that, when defining x_0 above, we have actually guaranteed that the sum in (5.28) converges even if the sum is over *all* connected subsets $S \ni 0$ (not only over those contained in Λ). This allows us to bound series containing clusters of all sizes whose support includes a given vertex: using Theorem 5.4 for the terms $n \geq 2$,

$$\begin{aligned} \sum_{X : \overline{X} \ni i} |\Psi(X)| &\leq \sum_{n \geq 1} n \sum_{S_1 \ni i} \sum_{S_2} \cdots \sum_{S_n} |\varphi(S_1, \dots, S_n)| \prod_{k=1}^{n} |\mathrm{w}_h(S_k)| \\ &\leq \sum_{S_1 \ni i} |\mathrm{w}_h(S_1)| e^{|[S_1]_1|} \leq \eta(\mathfrak{Re}\, h, d) \leq 1 . \end{aligned} \tag{5.29}$$

We can then rearrange the terms of the cluster expansion in Λ as follows. Since $\frac{1}{|\overline{X}|} \sum_{i \in \Lambda} \mathbf{1}_{\{\overline{X} \ni i\}} = 1$ for any $\overline{X} \subset \Lambda$,

$$\begin{aligned} \sum_{X : \overline{X} \subset \Lambda} \Psi(X) &= \sum_{i \in \Lambda} \sum_{\substack{X: \\ i \in \overline{X} \subset \Lambda}} \frac{1}{|\overline{X}|} \Psi(X) \\ &= \sum_{i \in \Lambda} \Big\{ \sum_{\substack{X: \\ i \in \overline{X}}} \frac{1}{|\overline{X}|} \Psi(X) - \sum_{\substack{X: \\ i \in \overline{X} \not\subset \Lambda}} \frac{1}{|\overline{X}|} \Psi(X) \Big\} . \end{aligned} \tag{5.30}$$

The difference between the two series is well defined, since both are absolutely convergent. Notice that both of them contain clusters of unbounded sizes. By translation invariance, the first sum over X in the right-hand side of (5.30) does not depend on i, and thus yields a constant contribution. The second sum is a boundary term. Indeed, whenever $i \in \overline{X} \not\subset \Lambda$, there must exist at least one component $S_k \in X$ which intersects the boundary of Λ: $\overline{X} \cap \partial^{\text{ex}} \Lambda \neq \varnothing$. Therefore, using (5.29) for the second inequality,

$$\Big| \sum_{i \in \Lambda} \sum_{\substack{X: \\ i \in \overline{X} \not\subset \Lambda}} \frac{1}{|\overline{X}|} \Psi(X) \Big| \leq |\partial^{\text{ex}} \Lambda| \max_{j \in \partial^{\text{ex}} \Lambda} \sum_{X : \overline{X} \ni j} |\Psi(X)| \leq |\partial^{\text{ex}} \Lambda| .$$

We thus obtain

$$\frac{1}{|\Lambda|} \log \mathbf{Z}^{+}_{\Lambda; \beta, h} = \beta \frac{|\mathscr{E}^{\text{b}}_{\Lambda}|}{|\Lambda|} + h + \sum_{X : \overline{X} \ni 0} \frac{1}{|\overline{X}|} \Psi(X) + \frac{O(|\partial^{\text{ex}} \Lambda|)}{|\Lambda|} . \tag{5.31}$$

We now fix $h \in H^{+}_{x_0}$ and take the thermodynamic limit in (5.31) along the sequence of boxes $\mathsf{B}(n)$. In this limit, the boundary term vanishes and $|\mathscr{E}^{\text{b}}_{\mathsf{B}(n)}| / |\mathsf{B}(n)| \to d$, yielding

$$\psi_\beta(h) = \beta d + h + \sum_{X : \overline{X} \ni 0} \frac{1}{|\overline{X}|} \Psi(X), \qquad \mathfrak{Re}\, h > x_0 . \tag{5.32}$$

Remark 5.9. Inserting (5.32) into (5.31), we can write, for any fixed region Λ,

$$\mathbf{Z}^{+}_{\Lambda;\beta,h} = e^{\psi_\beta(h)|\Lambda| + O(|\partial^{\mathrm{ex}}\Lambda|)}, \tag{5.33}$$

which provides a direct access to the finite-volume corrections to the pressure (the boundary term can of course be written down explicitly, as was done above). Thus, the cluster expansion provides a tool to study systematically *finite-size effects*, at least in perturbative regimes. This plays a particularly important role when extracting information about thermodynamic behavior from (finite-volume) numerical simulations. Such a decomposition will also be used repeatedly in Chapter 7. ⋄

The cluster expansion of the pressure, in (5.32), describes the contributions to the pressure when $\mathfrak{Re}\, h$ is large. Namely, the term $\beta d + h$ corresponds to the **energy density** of the ground state η^+:

$$\lim_{n\to\infty} \frac{-\mathscr{H}_{\mathsf{B}(n);\beta,h}(\eta^+)}{|\mathsf{B}(n)|} = \beta d + h.$$

The contributions due to the excitations away from η^+ are added successively by considering terms of the series associated to larger and larger clusters. The contribution of a cluster $X = \{S_1, \dots, S_n\}$ is of order $e^{-2h(|S_1|+\cdots+|S_n|)}$. Thanks to the absolute summability of the series (5.32), we can regroup all terms coming from clusters contributing to the same order e^{-2nh}, $n \geq 1$. In this way, we obtain an absolutely convergent series for $\psi_\beta(h) - \beta d - h$ in the variable e^{-2h}.

Lemma 5.10. *When $h \in H^+_{x_0}$, the pressure of the Ising model on $\mathbb{Z}^d$ satisfies, with $z = e^{-2h}$,*

$$\psi_\beta(h) - \beta d - h = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \tag{5.34}$$

where

$$a_1 = e^{-4d\beta},$$
$$a_2 = d e^{-(8d-4)\beta} - (\tfrac{1}{2} + d) e^{-8d\beta}.$$

Proof. As pointed out above, the contribution of a cluster $X = \{S_1, \dots, S_n\}$ is of order $e^{-2h(|S_1|+\cdots+|S_n|)}$. The associated combinatorial factor can be read from (5.20) and (5.32), namely

$$\underbrace{\frac{1}{|\overline{X}|}}_{A} \underbrace{\prod_{S\in\Gamma} \frac{1}{n_X(S)!}}_{B} \underbrace{\Big\{ \sum_{\substack{G\subset G_n \\ \text{connected}}} \prod_{\{i,j\}\in G} \zeta(S_i,S_j) \Big\}}_{C} \prod_{i=1}^{n} \mathrm{w}_h(S_i),$$

where Γ is here the set of all connected components in $\mathbb{Z}^d$. The only cluster contributing to a_1 is the cluster composed of the single polymer $\{0\}$. In this case, $A = 1$, $B = 1$ and $C = 1$; this yields the first coefficient since $\partial_e\{0\} = 2d$.

There are two types of clusters contributing to a_2: the clusters composed of a single polymer of size 2 containing 0, and the clusters made of two polymers of size 1, at least one of which is $\{0\}$.

Let us first consider the former: there are exactly $2d$ polymers of size 2 containing the origin and, for each such polymer S, $\partial_e S = 2(2d-1)$, $A = \frac{1}{2}$, $B = 1$ and $C = 1$; this yields a contribution

$$2d \cdot \frac{1}{2} \cdot e^{-2(2d-1)2\beta} = d e^{-(8d-4)\beta}.$$

Let us now turn to the clusters made up of two polymers of size 1, at least one of which is $\{0\}$. The first possibility is that both polymers are $\{0\}$, and therefore $A = 1$, $B = \frac{1}{2}$ and $C = -1$; this yields the contribution

$$-\tfrac{1}{2}\, e^{-2\cdot 4d\beta} = -\tfrac{1}{2}\, e^{-8d\beta}.$$

The second possibility is that $X = \{S_1, S_2\} = \{\{0\},\{i\}\}$, with $i \sim 0$. There are $2d$ ways of choosing i, and for each of those, $A = \frac{1}{2}$, $B = 1$ and $C = -1$; we thus obtain a contribution of

$$2d \cdot (-\tfrac{1}{2}) \cdot e^{-2\cdot 4d\beta} = -d e^{-8d\beta}. \qquad \square$$

It is of course possible to compute the coefficients to arbitrary order, but the computations become tricky when the order gets large.

Exercise 5.4. *Show that*

$$a_3 = ((2d+1)d + \tfrac{1}{3})e^{-12d\beta} - 4d^2 e^{-(12d-4)\beta} + d(2d-1)e^{-(12d-8)\beta}.$$

Remark 5.11. The expansion obtained in (5.34) converges when $|z| < e^{-2x_0}$. Remember that the Lee–Yang theorem (Theorem 3.43) implies analyticity of the pressure (as a function of z) in the whole open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. By the uniqueness of the Taylor coefficients, this means that the series in (5.34) converges not only for $\mathfrak{Re}\, h > x_0$, but for all $\mathfrak{Re}\, h > 0$. ⋄

Using the + boundary condition was quite convenient, but the same analysis could have been done with any other boundary condition, with slight changes, and would have led to the same expansion (5.32), only the boundary term in (5.30) being affected by the choice of boundary condition.

Exercise 5.5. *Prove that last statement. What changes must be made if one uses non-constant boundary conditions? Conclude that, when $|\mathfrak{Re}\, h|$ is large, the thermodynamic limit for the pressure exists for arbitrary boundary conditions.*

To summarize, we have seen that considering the Ising model with $\mathfrak{Re}\, h > 0$ large allows one to see the regions of $-$ spins as perturbations of the ground-state η^+. These perturbations are under control whenever the cluster expansion converges. This led us to a series expansion for the pressure of the model in the variable e^{-2h}.

5.7.2 The virial expansion for the lattice gas

(In order for him to get motivation and notation for the material in this section, we strongly recommend the reader to have a look at Chapter 4.)

We have seen, in Section 4.8, that the pressure $p_\beta(\mu)$ of the nearest-neighbor lattice gas is analytic everywhere except at μ_*, where it has a discontinuous derivative if the temperature is sufficiently low. To express the pressure as a function of the particle density $\rho \in [0,1]$, we inverted the relation $\rho = \frac{\partial p_\beta}{\partial \mu}$, to obtain $\mu_\beta = \mu_\beta(\rho)$, and defined $\tilde{p}_\beta(\rho) \stackrel{\text{def}}{=} p_\beta(\mu_\beta(\rho))$. The latter function was shown to be analytic on the gas branch $(0, \rho_g)$, constant on the coexistence plateau $[\rho_g, \rho_l]$, and again analytic on the liquid branch $(\rho_l, 1)$ (see Figure 4.11 and Exercise 4.10).

In this section, we go one step further. We will consider the behavior of the model on the gas branch, for small values of the density, and obtain a representation of $\widetilde{p}_\beta$ as a convergent series, called the **virial expansion**:

$$\beta\widetilde{p}_\beta(\rho) = b_1\rho + b_2\rho^2 + b_3\rho^3 + \cdots \qquad (\rho \text{ small}),$$

with (in principle) explicit expressions for the **virial coefficients** b_k, $k \geq 1$.

The canonical lattice gas at low density corresponds, in the grand canonical ensemble, to large negative values of the chemical potential μ (remember Exercise 4.29). We have also seen in Section 4.8 that the nearest-neighbor lattice gas can be mapped, via $\eta_i \mapsto 2\eta_i - 1$, to the Ising model with an inverse temperature $\beta' = \frac{1}{4}\beta$ and magnetic field $h' = \frac{\beta}{2}(2d+\mu)$; in particular, their pressures are related by

$$\beta p_\beta(\mu) = \psi_{\beta'}(h') + \frac{\beta\mu}{2} + \frac{\beta\kappa}{8}. \tag{5.35}$$

Since a large negative chemical potential corresponds to a large negative magnetic field, we can derive the virial expansion from the results for the Ising model at large values of $\mathfrak{Re}\, h$ which were obtained in the previous section. Namely, using the symmetry $\psi_\beta(-h) = \psi_\beta(h)$ and using the expansion (5.34), in terms of the variable $z' = e^{2h'}$, with $\mathfrak{Re}\, h' < -x_0$:

$$\psi_{\beta'}(h') = \beta' d - h' + a_1 z' + a_2 z'^2 + a_3 z'^3 + \ldots \tag{5.36}$$

(Remember that each a_n should be used with β' instead of β.) This gives

$$\beta p_\beta(\mu) = \sum_{n\geq 1} a_n z'^n, \tag{5.37}$$

which is called the **Mayer expansion**. The Mayer series is absolutely convergent and can therefore be differentiated term by term with respect to μ. Since $\frac{\partial z'^n}{\partial\mu} = n\beta z'^n$, this yields

$$\rho = \frac{\partial p_\beta}{\partial\mu} = \sum_{n\geq 1} n a_n z'^n \stackrel{\text{def}}{=} \sum_{n\geq 1} \tilde{a}_n z'^n \stackrel{\text{def}}{=} \phi(z').$$

We will obtain the virial expansion by inverting this last expression, obtaining $z' = \phi^{-1}(\rho)$, and injecting the result into (5.37). Since $\frac{\mathrm{d}\phi}{\mathrm{d}z}(0) = a_1 = e^{-4d\beta} > 0$, the analytic Implicit Function Theorem (Theorem B.28) implies that ϕ can indeed be inverted on a small disk $D \subset \mathbb{C}$ centered at the origin, and the inverse is analytic on that disk. We write the Taylor expansion of the inverse by $\phi^{-1}(\rho) = \sum_k c_k\rho^k$. Assuming that the coefficients c_k are known (they will be computed below), we can write down the virial expansion. Namely,

$$\begin{aligned}
\beta\widetilde{p}_\beta(\rho) &= \sum_{n\geq 1} a_n\{\phi^{-1}(\rho)\}^n \\
&= \sum_{n\geq 1} a_n \sum_{k_1\geq 1}\cdots\sum_{k_n\geq 1}\prod_{i=1}^n c_{k_i}\rho^{k_i} \\
&= \sum_{n\geq 1} a_n \sum_{m\geq n}\sum_{\substack{k_1,\ldots,k_n\geq 1\\ k_1+\cdots+k_n=m}}\prod_{i=1}^n c_{k_i}\rho^{k_i} = \sum_{m\geq 1}\tilde{b}_m\rho^m,
\end{aligned}$$

where

$$\tilde{b}_m \stackrel{\text{def}}{=} \sum_{n=1}^{m} a_n \sum_{\substack{k_1,\dots,k_n \geq 1 \\ k_1+\cdots+k_n=m}} \prod_{i=1}^{n} c_{k_i}\,.$$

A similar computation can be used in the following exercise.

Exercise 5.6 (Computing the Taylor coefficients of an inverse function)**.** *Let $\phi(z) = \sum_{k\geq 1} \tilde{a}_k z^k$ be convergent in a neighborhood of $z = 0$, with $\tilde{a}_1 \neq 0$ (in particular, ϕ is invertible in a neighborhood of $z = 0$). Write its compositional inverse ϕ^{-1} as $\phi^{-1}(z) = \sum_{k\geq 1} c_k z^k$. Show that $c_1 = \tilde{a}^{-1}$ and that, for all $m \geq 2$,*

$$\sum_{n=1}^{m} \tilde{a}_n \sum_{\substack{k_1,\dots,k_n \geq 1 \\ k_1+\cdots+k_n=m}} \prod_{i=1}^{n} c_{k_i} = 0\,.$$

Using this, compute the first few coefficients of ϕ^{-1}:

$$c_2 = -\frac{\tilde{a}_2}{\tilde{a}_1^3}\,, \quad c_3 = 2\frac{\tilde{a}_2^2}{\tilde{a}_1^5} - \frac{\tilde{a}_3}{\tilde{a}_1^4}\,, \quad \textit{etc.}$$

As can be verified, using the coefficients a_k computed in Lemma 5.10,

$$\tilde{b}_1 = 1\,, \quad \tilde{b}_2 = -\frac{a_2}{a_1^2} = \tfrac{1}{2} + d - de^{\beta}\,, \quad \text{etc.}$$

We have thus shown

Theorem 5.12. *At low densities, the pressure of the nearest-neighbor lattice gas satisfies*

$$\beta\tilde{p}_\beta(\rho) = \rho + (\tfrac{1}{2} + d - de^{\beta})\rho^2 + O(\rho^3)\,.$$

5.7.3 The Ising model at high temperature ($h = 0$)

In this section, we consider again the pressure of the Ising model but in another regime: $h = 0$ and $\beta \ll 1$. In the latter, thermal fluctuations are so strong that the spins behave nearly independently from each other.

We choose the free boundary condition, as it is the most convenient one in the high-temperature regime. Proceeding as in Section 3.7.3, we express the partition function as in Exercise 3.23:

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathscr{E}_\Lambda|} \sum_{E\in\mathfrak{E}^{\text{even}}_\Lambda} (\tanh\beta)^{|E|}\,, \tag{5.38}$$

where the sum is over all subsets of edges $E \subset \mathscr{E}_\Lambda$ such that the number of edges of E incident to each vertex $i \in \Lambda$ is even.

Each set $E \in \mathfrak{E}^{\text{even}}_\Lambda$ can be identified with a graph, by simply considering it together with the endpoints of each of its edges. This graph can be decomposed into (maximal) connected components, which play the role of polymers. In terms of edges, this decomposition can be written $E = E_1 \cup \cdots \cup E_n$, so that we obtain

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,0} = 2^{|\Lambda|}(\cosh\beta)^{|\mathscr{E}_\Lambda|}\,\Xi^{\text{HT}}_{\Lambda;\beta,0}\,,$$

with

$$\Xi^{\mathrm{HT}}_{\Lambda;\beta,0} \stackrel{\mathrm{def}}{=} 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{E_1 \subset \mathscr{E}_\Lambda} \cdots \sum_{E_n \subset \mathscr{E}_\Lambda} \Big\{ \prod_{i=1}^{n} (\tanh\beta)^{|E_i|} \Big\} \prod_{1\leq i<j\leq n} \delta(E_i, E_j),$$

where $E_i \in \mathfrak{E}^{\mathrm{even}}_\Lambda$, for all i, and

$$\delta(E_i, E_j) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & \text{if } E_i \text{ and } E_j \text{ have no vertex in common,} \\ 0 & \text{otherwise,} \end{cases}$$

is again of hard-core type.

The above representation is well suited to the high-temperature regime, since the weight $(\tanh\beta)^{|E_i|}$*, associated to a polymer* E_i*, decays fast when* β *is small.* ⋄

Proceeding as in Section 5.7.1, we can show that the conditions for the convergence of the cluster expansion are satisfied when β is sufficiently small, thus proving that the pressure behaves analytically at high temperature:

Theorem 5.13. *There exists* $r_0 > 0$ *such that* $\beta \mapsto \psi_\beta(0)$ *is analytic in the disk* $\{\beta \in \mathbb{C} : |\beta| < r_0\}$.

Exercise 5.7. *Prove Theorem 5.13, and compute the first few terms of the expansion of* $\psi_\beta(0) - d\log(\cosh\beta) - \log 2$ *as a power series in the variable* $z = \tanh\beta$.

Remark 5.14. Even though we have only considered analyticity of the pressure as a function of β here, it is possible to extract a lot of additional information on the model in this regime. In Section 6.5.4, we will use a variant of the above approach to prove uniqueness of the infinite volume Gibbs measure at all sufficiently high temperatures, for a large class of models. ⋄

It follows from the results of Chapter 3 that $h \mapsto \psi_\beta(h)$ is continuously differentiable at $h = 0$ when $\beta < \beta_c(d)$. In the next exercise, the reader is asked to adapt the high-temperature representation to show that it is in fact analytic in a neighborhood of $h = 0$, at least when β is sufficiently small.

Exercise 5.8. *Show that there exists* $\beta_0 = \beta_0(d) > 0$ *such that, for all* $0 \leq \beta \leq \beta_0$, $h \mapsto \psi_\beta(h)$ *is analytic at* $h = 0$.

5.7.4 The Ising model at low temperature ($h = 0$)

We now consider the Ising model on $\mathbb{Z}^d$, $d \geq 2$, at very low temperature and in the absence of a magnetic field. Our goals, in this regime, are (i) to establish analyticity of the pressure $\beta \mapsto \psi_\beta(0)$, (ii) to derive an explicit series expansion for the magnetization and (iii) to prove exponential decay of the truncated 2-point correlation function $\langle \sigma_i ; \sigma_j \rangle^+_{\beta,0}$ as $\|i - j\|_2 \to \infty$.

We know from Section 3.7.2 that, when $h = 0$ and β is large, the relevant objects for the description of configurations are the *contours* separating the regions of + and − spins. In dimension 2, we used the deformation rule of Figure 3.11. Since that deformation was specific to $d = 2$, we will here define contours in a slightly different manner. The description will be used in any dimension $d \geq 2$, in a large box Λ, with either + or − boundary condition.

We again write the Hamiltonian in a way that emphasizes the role played by pairs of neighboring spins with opposite signs:

$$\mathscr{H}_{\Lambda;\beta,0} = -\beta|\mathscr{E}_\Lambda^{\mathrm{b}}| - \beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda^{\mathrm{b}}} (\sigma_i\sigma_j - 1). \tag{5.39}$$

We consider the + boundary condition in a region $\Lambda \Subset \mathbb{Z}^d$. Given $\omega \in \Omega_\Lambda^+$, we use again $\Lambda^-(\omega)$ to denote the set of vertices i at which $\omega_i = -1$. Rather than $\Lambda^-(\omega)$ itself (which was relevant when considering a large magnetic field), we will be interested only in its boundary.

We associate to each $i \in \mathbb{Z}^d$ the closed unit cube of $\mathbb{R}^d$ centered at i:

$$\mathscr{S}_i \stackrel{\text{def}}{=} i + [-\tfrac{1}{2}, \tfrac{1}{2}]^d,$$

and let

$$\mathscr{M}(\omega) \stackrel{\text{def}}{=} \bigcup_{i\in\Lambda^-(\omega)} \mathscr{S}_i. \tag{5.40}$$

We can then consider the (maximal) connected components of $\partial\mathscr{M}(\omega)$ (here, the boundary ∂ is in the sense of the Euclidean topology of $\mathbb{R}^d$):

$$\Gamma'(\omega) \stackrel{\text{def}}{=} \{\gamma_1, \dots, \gamma_n\}.$$

Each γ_i is called a **contour** of ω. (Note that when $d = 2$, this notion slightly differs from the one used in Chapter 3.) In $d = 2$ (see Figure 3.10), contours can be identified with connected sets of dual edges. In higher dimensions, contours are connected sets of *plaquettes*, which are the $(d-1)$-dimensional faces of the d-dimensional hypercubes $\mathscr{S}_i$, $i \in \mathbb{Z}^d$. The number of plaquettes contained in γ_i will be denoted $|\gamma_i|$. Observe that there is a one-to-one mapping between the plaquettes of $\partial\mathscr{M}(\omega)$ and the edges of $\partial_e\Lambda^-(\omega)$ (associating to a plaquette the unique edge crossing it), and so $|\partial_e\Lambda^-(\omega)| = \sum_{i=1}^n |\gamma_i|$. We can thus write

$$\mathbf{Z}_{\Lambda;\beta,0}^+ = e^{\beta|\mathscr{E}_\Lambda^{\mathrm{b}}|} \sum_{\omega\in\Omega_\Lambda^+} \prod_{\gamma\in\Gamma'(\omega)} \mathrm{w}_\beta(\gamma),$$

where

$$\mathrm{w}_\beta(\gamma) \stackrel{\text{def}}{=} e^{-2\beta|\gamma|}. \tag{5.41}$$

The final step is to transform the summation over ω into a summation over families of contours. To this end, we introduce a few notions. Let $\Gamma_\Lambda \stackrel{\text{def}}{=} \{\gamma \in \Gamma'(\omega) : \omega \in \Omega_\Lambda^+\}$ denote the set of all possible contours in Λ.

A collection of contours $\Gamma' \subset \Gamma_\Lambda$ is **admissible** if there exists a configuration $\omega \in \Omega_\Lambda^+$ such that $\Gamma'(\omega) = \Gamma'$. We say that $\Lambda \subset \mathbb{Z}^d$ is c-**connected** if $\mathbb{R}^d \setminus \bigcup_{i\in\Lambda} \mathscr{S}_i$ is a connected subset of $\mathbb{R}^d$.

Exercise 5.9. *Assuming that Λ is* c*-connected, show that a collection $\Gamma' = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma_\Lambda$ is admissible if and only if its contours are pairwise disjoint: $\gamma_i \cap \gamma_j = \varnothing$ for all $i \neq j$. Why is this not necessarily true when Λ is not* c*-connected?*

Therefore, provided that Λ be c-connected,

$$\mathbf{Z}^{+}_{\Lambda;\beta,0} = e^{\beta|\mathscr{E}^{\mathrm{b}}_{\Lambda}|}\,\Xi^{\mathrm{LT}}_{\Lambda;\beta,0}\,, \tag{5.42}$$

where

$$\begin{aligned}\Xi^{\mathrm{LT}}_{\Lambda;\beta,0} &\stackrel{\mathrm{def}}{=} \sum_{\substack{\Gamma'\subset\Gamma_\Lambda\\ \text{admiss.}}} \prod_{\gamma\in\Gamma'} \mathrm{w}_\beta(\gamma)\\ &= 1 + \sum_{n\geq 1}\frac{1}{n!}\sum_{\gamma_1\in\Gamma_\Lambda}\cdots\sum_{\gamma_n\in\Gamma_\Lambda}\Big\{\prod_{i=1}^{n}\mathrm{w}_\beta(\gamma_i)\Big\}\prod_{1\leq i<j\leq n}\delta(\gamma_i,\gamma_j)\,,\end{aligned}$$

with interactions which are once again of hard-core type:

$$\delta(\gamma_i,\gamma_j) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & \text{if } \gamma_i\cap\gamma_j = \varnothing\,,\\ 0 & \text{otherwise.}\end{cases} \tag{5.43}$$

The above representation of the Ising model is adapted to the low temperature regime, since the weight $\mathrm{w}_\beta(\gamma_i) = e^{-2\beta|\gamma_i|}$ associated to γ_i, decays fast when β is large. This observation was, of course, at the core of Peierls' argument. ◇

For the rest of the section, we will allow β to take complex values. We first verify that (5.10) holds with $a(\gamma) \stackrel{\mathrm{def}}{=} |\gamma|$.

Exercise 5.10. *Prove that there exists $x_0 = x_0(d) > 0$ such that, for all β satisfying $\mathfrak{Re}\,\beta > x_0$ and for each $\gamma_* \in \Gamma_\Lambda$,*

$$\sum_{\gamma} |\mathrm{w}_\beta(\gamma)|\, e^{|\gamma|}\, |\zeta(\gamma,\gamma_*)| \leq |\gamma_*|\,. \tag{5.44}$$

Hint: *Use Lemma 3.38 to count the number of contours γ whose support contains a fixed point.*

Pressure

Observe that, when $d = 2$, the analyticity of $\beta \mapsto \psi_\beta(0)$ for large β can be deduced directly from the analyticity at small values of β (Theorem 5.13 above), using the duality transformation described in Section 3.10.1. However, there is no analogous transformation in $d \geq 3$.

We leave it as an exercise to provide the details of the proof of the following result:

Theorem 5.15. *($d \geq 2$). There exists $x_0 = x_0(d) > 0$ such that $\beta \mapsto \psi_\beta(0)$, is analytic on $\{\beta\in\mathbb{C} : \mathfrak{Re}\,\beta > x_0\}$. Moreover,*

$$\psi_\beta(0) = \beta d + e^{-4d\beta} + d e^{-4(2d-1)\beta} + O(e^{-8d\beta})\,.$$

Magnetization and decay of the truncated 2-point function

We now move on to the study of correlation functions at low temperature.

Let $\Lambda \Subset \mathbb{Z}^d$ be c-connected, and $A \subset \Lambda$. Remembering that $\sigma_A \stackrel{\text{def}}{=} \prod_{i\in A}\sigma_i$, we will express the correlation function

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \sum_{\omega\in\Omega^+_\Lambda}\sigma_A(\omega)\frac{e^{-\mathscr{H}_{\Lambda;\beta,0}(\omega)}}{\mathbf{Z}^+_{\Lambda;\beta,0}}$$

in a form suitable for an analysis based on the cluster expansion. The denominator, $\mathbf{Z}^+_{\Lambda;\beta,0}$, can be expressed using (5.42). To do the same for the numerator, we start with

$$\sum_{\omega\in\Omega^+_\Lambda}\sigma_A(\omega)e^{-\mathscr{H}_{\Lambda;\beta,0}(\omega)} = e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda|}\sum_{\omega\in\Omega^+_\Lambda}\sigma_A(\omega)\prod_{\gamma\in\Gamma'(\omega)}\mathrm{w}_\beta(\gamma)\,.$$

Let $\omega\in\Omega^+_\Lambda$ and let $\gamma\in\Gamma'(\omega)$ be one of its contours. Consider the configuration $\omega^\gamma\in\Omega^+_\Lambda$ which has γ as its unique contour: $\Gamma'(\omega^\gamma)=\{\gamma\}$. The **interior** of γ is defined by (see Figure 5.2)

$$\operatorname{Int}\gamma \stackrel{\text{def}}{=} \bigl\{i\in\Lambda : \omega^\gamma_i = -1\bigr\} = \Lambda^-(\omega^\gamma)\,.$$

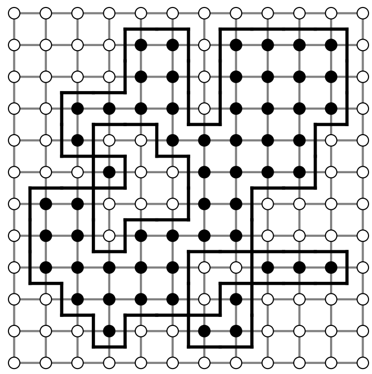


Figure 5.2: The interior of a (here two-dimensional) contour: the interior is the set of all black vertices.

The important observation is that, for any $\omega\in\Omega^+_\Lambda$,

$$\omega_i = (-1)^{\#\{\gamma\in\Gamma'(\omega):\, i\in\operatorname{Int}\gamma\}}\,,$$

that is, the sign of the spin at the vertex i is equal to $+1$ if and only if there is an even number of contours ***surrounding i*** (in the sense that i belongs to their interior). It follows from this observation that

$$\sigma_A(\omega) = (-1)^{\sum_{i\in A}\#\{\gamma\in\Gamma'(\omega):\, i\in\operatorname{Int}\gamma\}} = \prod_{\gamma\in\Gamma'(\omega)}(-1)^{\#\{i\in A:\, i\in\operatorname{Int}\gamma\}}\,.$$

We therefore get

$$\sigma_A(\omega)\prod_{\gamma\in\Gamma'(\omega)}\mathrm{w}_\beta(\gamma) = \prod_{\gamma\in\Gamma'(\omega)}\mathrm{w}^A_\beta(\gamma)\,,$$

where

$$\mathrm{w}^A_\beta(\gamma) \stackrel{\text{def}}{=} (-1)^{\#\{i\in A:\, i\in\operatorname{Int}\gamma\}}\mathrm{w}_\beta(\gamma)\,.$$

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We conclude that

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \frac{\sum_{\Gamma'\subset\Gamma_\Lambda,\text{admiss.}}\prod_{\gamma\in\Gamma'} \mathsf{w}^A_\beta(\gamma)}{\sum_{\Gamma'\subset\Gamma_\Lambda,\text{admiss.}}\prod_{\gamma\in\Gamma'} \mathsf{w}_\beta(\gamma)} \equiv \frac{\Xi^{\mathrm{LT},A}_{\Lambda;\beta,0}}{\Xi^{\mathrm{LT}}_{\Lambda;\beta,0}}. \tag{5.45}$$

Now, the polymer partition functions in the numerator and denominator in (5.45) differ only in the weight associated to contours that surround vertices in A. When $\mathfrak{Re}\,\beta > x_0$ (see Exercise 5.10), the cluster expansion for $\log\Xi^{\mathrm{LT}}_{\Lambda;\beta,0}$ converges and, since $|\mathsf{w}^A_\beta(\gamma)| = |\mathsf{w}_\beta(\gamma)|$ for all γ, the same holds for $\log\Xi^{\mathrm{LT},A}_{\Lambda;\beta,0}$. We thus obtain

$$\begin{aligned}\langle\sigma_A\rangle^+_{\Lambda;\beta,0} &= \exp\bigl\{\log\Xi^{\mathrm{LT},A}_{\Lambda;\beta,0} - \log\Xi^{\mathrm{LT}}_{\Lambda;\beta,0}\bigr\}\\ &= \exp\Bigl\{\sum_{X:\overline{X}\subset\Lambda}\Psi^A_\beta(X) - \sum_{X:\overline{X}\subset\Lambda}\Psi_\beta(X)\Bigr\},\end{aligned}$$

where the sums in the rightmost expression are over clusters of contours in Λ and $\Psi_\beta(X)$ and $\Psi^A_\beta(X)$ are defined as in (5.20) with weights w given by w_β and w^A_β respectively, and the support $\overline{X}$ of a cluster $X = \{\gamma_1,\dots,\gamma_n\}$ is defined as $\bigcup_{k=1}^n\gamma_k$ (of course, $\overline{X}\subset\Lambda$ means that, as subsets of $\mathbb{R}^d$, $\overline{X}\subset\bigcup_{i\in\Lambda}\mathscr{S}_i$). In particular, the contributions to both sums of all clusters containing no contour γ surrounding a vertex of A cancel each other, and we are left with

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \exp\Bigl\{\sum_{\substack{X\sim A:\\ \overline{X}\subset\Lambda}}(\Psi^A_\beta(X) - \Psi_\beta(X))\Bigr\},$$

where $X\sim A$ means that X contains at least one contour γ such that $A\cap\mathrm{Int}\,\gamma\neq\varnothing$. We leave it as an exercise to show that one can let $\Lambda\uparrow\mathbb{Z}^d$ in the above expression:

Exercise 5.11. *($d\geq 2$) Prove that*

$$\langle\sigma_A\rangle^+_{\beta,0} = \exp\Bigl\{\sum_{X\sim A}\bigl(\Psi^A_\beta(X) - \Psi_\beta(X)\bigr)\Bigr\}, \tag{5.46}$$

provided that $\mathfrak{Re}\,\beta$ is sufficiently large.

We now turn to two applications of this formula.

Magnetization at very low temperatures. In Section 3.7.2, we used Peierls' argument to obtain a lower bound on $\langle\sigma_0\rangle^+_{\beta,0}$ that tends to 1 as $\beta\to\infty$. We can use the cluster expansion to obtain an explicit expansion in $e^{-2\beta}$ for $\langle\sigma_0\rangle^+_{\beta,0}$, valid for large enough values of β. Namely, an application of (5.46) with $A=\{0\}$ yields

$$\langle\sigma_0\rangle^+_{\beta,0} = \exp\Bigl\{\sum_{X\sim\{0\}}\bigl(\Psi^{\{0\}}_\beta(X) - \Psi_\beta(X)\bigr)\Bigr\}, \tag{5.47}$$

where the condition $X\sim\{0\}$ now reduces to the requirement that at least one of the contours γ in X surrounds 0. It is then a simple exercise, proceeding as in the previous sections, to obtain the desired expansion.

Exercise 5.12. *($d\geq 2$) Prove that, for all sufficiently large values of β,*

$$\langle\sigma_0\rangle^+_{\beta,0} = 1 - 2e^{-4d\beta} - 4de^{-(8d-4)\beta} + O(e^{-8d\beta}).$$

Decay of the truncated 2-point function. As we saw in Exercises 3.23 and 3.24, the correlations of the Ising model decay exponentially fast at sufficiently high temperature (small β)

$$\langle\sigma_i\sigma_j\rangle_{\beta,0} \le e^{-c_{HT}(\beta)\|j-i\|_1}, \quad \forall i,j\in\mathbb{Z}^d.$$

In contrast, we know that at low temperature, $\beta>\beta_c$, the correlations do not decay anymore since, by the GKS inequalities, uniformly in i and j,

$$\langle\sigma_i\sigma_j\rangle^+_{\beta,0} \ge \langle\sigma_i\rangle^+_{\beta,0}\langle\sigma_j\rangle^+_{\beta,0} = (\langle\sigma_0\rangle^+_{\beta,0})^2 > 0. \tag{5.48}$$

Here, we will study the **truncated 2-point function**, which is the name usually given in physics to the covariance between the random variables σ_i and σ_j, in the Gibbs state $\langle\cdot\rangle^+_{\beta,0}$:

$$\langle\sigma_i;\sigma_j\rangle^+_{\beta,0} \stackrel{\text{def}}{=} \langle\sigma_i\sigma_j\rangle^+_{\beta,0} - \langle\sigma_i\rangle^+_{\beta,0}\langle\sigma_j\rangle^+_{\beta,0}.$$

Theorem 5.16. *($d\ge 2$) There exist $0<\beta_0<\infty$, $c>0$ and $C<\infty$ such that, for all $\beta\ge\beta_0$,*

$$0 \le \langle\sigma_i;\sigma_j\rangle^+_{\beta,0} \le C e^{-c\beta\|j-i\|_1}, \quad \forall i,j\in\mathbb{Z}^d. \tag{5.49}$$

This result shows that, at least at low enough temperatures, the correlation length of the Ising model on $\mathbb{Z}^d$, $d\ge 2$, is finite (and actually tends to 0 as $\beta\uparrow\infty$). In particular, the spins are only weakly correlated, even though there is long-range order.

Proof. The first inequality is just (5.48), so we only prove the second one. Let us write $\widetilde{\Psi}^A_\beta(X) \stackrel{\text{def}}{=} \Psi^A_\beta(X) - \Psi_\beta(X)$. On the one hand, by (5.47),

$$\langle\sigma_i\rangle^+_{\beta,0}\langle\sigma_j\rangle^+_{\beta,0} = \exp\Big\{\sum_{X\sim\{i\}}\widetilde{\Psi}^{\{i\}}_\beta(X) + \sum_{X\sim\{j\}}\widetilde{\Psi}^{\{j\}}_\beta(X)\Big\}.$$

On the other hand, by the general formula (5.46),

$$\langle\sigma_i\sigma_j\rangle^+_{\beta,0} = \exp\Big\{\sum_{X\sim\{i,j\}}\widetilde{\Psi}^{\{i,j\}}_\beta(X)\Big\}.$$

Clusters $X\sim\{i,j\}$ can be split into three disjoint classes:

$$\mathscr{C}_i \stackrel{\text{def}}{=} \big\{X : X\sim\{i\} \text{ but } X\not\sim\{j\}\big\}, \qquad \mathscr{C}_j \stackrel{\text{def}}{=} \big\{X : X\sim\{j\} \text{ but } X\not\sim\{i\}\big\},$$
$$\mathscr{C}_{i,j} \stackrel{\text{def}}{=} \big\{X : X\sim\{i\} \text{ and } X\sim\{j\}\big\}.$$

Observe now that $\Psi^{\{i,j\}}_\beta(X) = \Psi^{\{i\}}_\beta(X)$ for all $X\in\mathscr{C}_i$, and $\Psi^{\{i,j\}}_\beta(X) = \Psi^{\{j\}}_\beta(X)$ for all $X\in\mathscr{C}_j$. This implies that

$$\begin{aligned}\langle\sigma_i\sigma_j\rangle^+_{\beta,0} &= \exp\Big\{\sum_{X\sim\{i\}}\widetilde{\Psi}^{\{i\}}_\beta(X) + \sum_{X\sim\{j\}}\widetilde{\Psi}^{\{j\}}_\beta(X) \\ &\qquad\qquad + \sum_{X\in\mathscr{C}_{i,j}}\big(\widetilde{\Psi}^{\{i,j\}}_\beta(X) - \widetilde{\Psi}^{\{i\}}_\beta(X) - \widetilde{\Psi}^{\{j\}}_\beta(X)\big)\Big\} \\ &= \langle\sigma_i\rangle^+_{\beta,0}\langle\sigma_j\rangle^+_{\beta,0}\exp\Big\{\sum_{X\in\mathscr{C}_{i,j}}\big(\widetilde{\Psi}^{\{i,j\}}_\beta(X) - \widetilde{\Psi}^{\{i\}}_\beta(X) - \widetilde{\Psi}^{\{j\}}_\beta(X)\big)\Big\}.\end{aligned}$$

Now, for all A, $|\widetilde{\Psi}^A_\beta(X)| \le 2|\Psi_\beta(X)|$, and therefore

$$\langle\sigma_i\sigma_j\rangle^+_{\beta,0} \le \langle\sigma_i\rangle^+_{\beta,0}\langle\sigma_j\rangle^+_{\beta,0}\exp\Big\{6\sum_{X\in\mathscr{C}_{i,j}}|\Psi_\beta(X)|\Big\}.$$

The conclusion will thus follow once we prove that

$$\sum_{X\in\mathscr{C}_{i,j}}|\Psi_\beta(X)| \le C'e^{-c\beta\|j-i\|_1}, \qquad \forall i,j\in\mathbb{Z}^d,$$

for some constants $c>0$ and $C'<\infty$. To prove this claim, assume $\beta\ge 2x_0$. By (5.29), for any vertex $v\in\mathbb{R}^d$, since $|\overline{X}|\le\sum_{\gamma\in X}|\gamma|$,

$$\sum_{X:\overline{X}\ni v}|\Psi_\beta(X)|e^{\beta|\overline{X}|} \le \sum_{X:\overline{X}\ni v}|\Psi_{\beta/2}(X)| \le 1.$$

This implies that, for any $R>0$,

$$\sum_{\substack{X:\\ X\ni v,|\overline{X}|\ge R}}|\Psi_\beta(X)| \le e^{-\beta R}\sum_{X:\overline{X}\ni v}|\Psi_\beta(X)|e^{\beta|\overline{X}|} \le e^{-\beta R}.$$

Therefore, since each $X\in\mathscr{C}_{i,j}$ satisfies $|\overline{X}|\ge\|j-i\|_1$,

$$\begin{aligned}\sum_{X\in\mathscr{C}_{i,j}}|\Psi_\beta(X)| &\le \sum_{R\ge\|j-i\|_1}R^d\sum_{\substack{X:\\ X\ni v,|\overline{X}|=R}}|\Psi_\beta(X)|\\ &\le \sum_{R\ge\|j-i\|_1}R^d e^{-\beta R} \le C'e^{-c\beta\|j-i\|_1},\end{aligned}$$

uniformly in $i,j\in\mathbb{Z}^d$, for some $c=c(d)>0$ and all β large enough. □

5.8 Bibliographical references

The cluster expansion is one of the oldest tools of statistical mechanics. As already mentioned in Section 4.12.3, Mayer [236] started using it systematically in his analysis leading to the coefficients of the virial expansion of the pressure of a real gas. Groeneveld [153] was one of the first to provide a rigorous proof of its convergence.

Nowadays, there exist various approaches to the problem of convergence of the expansion, all leading more or less to the same conclusions. Adopting one is essentially a matter of personal taste. The proof of convergence we gave in Section 5.4 was taken from Ueltschi [335], since it is pretty straightforward and keeps the combinatorics elementary.

Some standard references on the subject include the following papers. Polymer models were introduced for the first time by Gruber and Kunz [155]. Kotecký and Preiss [196] gave the first inductive proof of the convergence of the cluster expansion, similar to the one used in Theorem 5.4. An interesting alternative approach, where the expansion is obtained as the result of a multi-variable Taylor expansion, was proposed by Dobrushin in [82]. A pedagagical description of the tree-graph approach that originated with the work of Penrose [269] can be found in Pfister [270]; see also the paper of Fernández and Procacci [103], where several of these methods are compared.