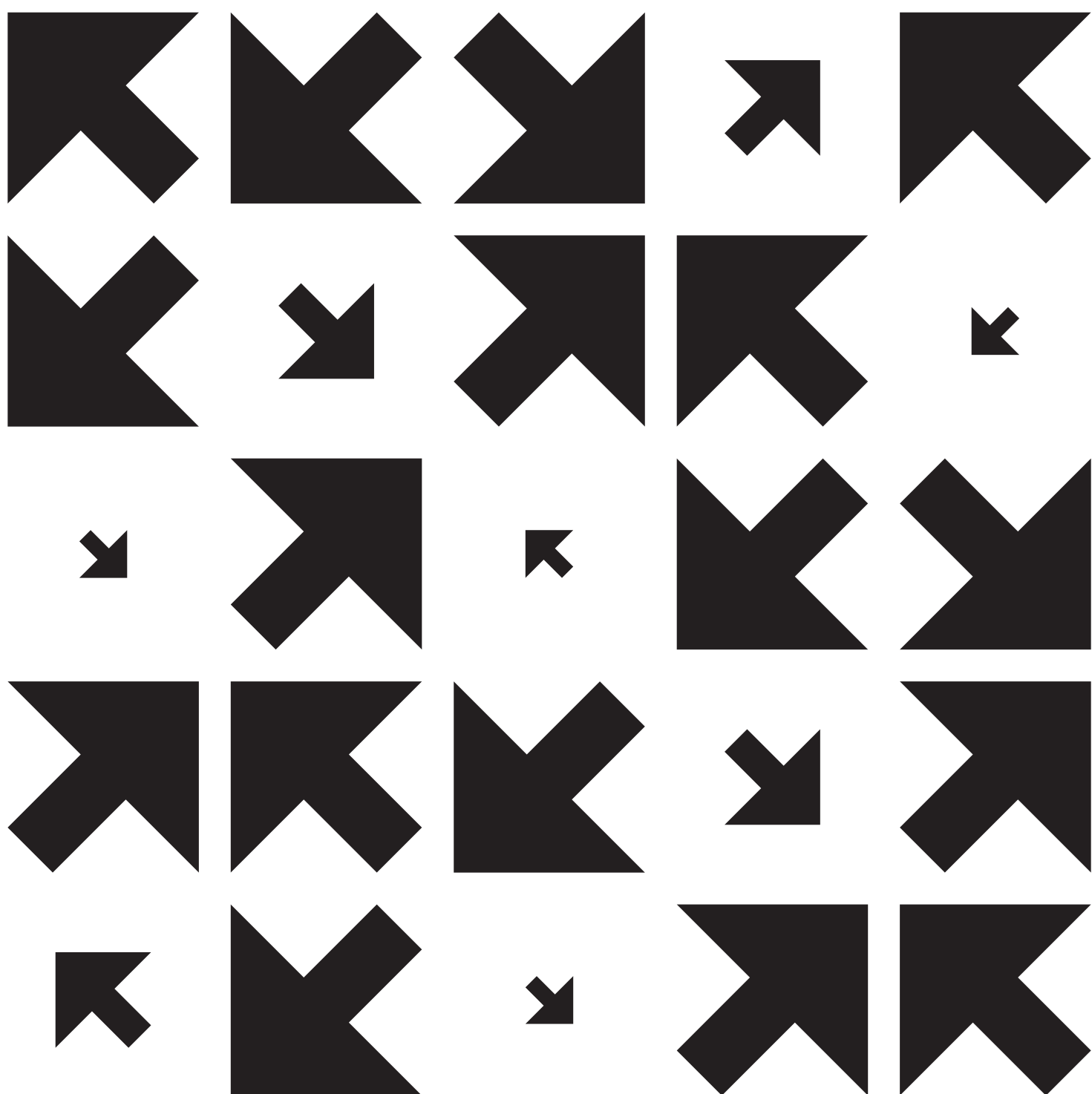


# Statistical Mechanics of Lattice Systems

S. Friedli and Y. Velenik

## A Concrete Mathematical Introduction





*For the sunshine and smiles: Janet, Jean-Pierre, Mimi and Kathryn.*

*Aos amigos e colegas do Departamento de Matemática.*

*À Agnese, Laure et Alexandre, ainsi qu'à mes parents.*



# Preface

---

Equilibrium statistical mechanics is a field that has existed for more than a century. Its origins lie in the search for a microscopic justification of equilibrium thermodynamics, and it developed into a well-established branch of mathematics in the second half of the twentieth century. The ideas and methods that it introduced to treat systems with many components have now permeated many areas of science and engineering, and have had an important impact on several branches of mathematics.

There exist many good introductions to this theory designed for physics undergraduates. It might however come as a surprise that textbooks addressing it from a *mathematically rigorous* standpoint have remained rather scarce. A reader looking for an introduction to its more advanced mathematical aspects must often either consult highly specialized monographs or search through numerous research articles available in peer-reviewed journals. It might even appear as if the mastery of certain techniques has survived from one generation of researchers to the next only by means of oral communication, through the use of chalk and blackboard...

It seems a general opinion that pedagogical introductory mathematically rigorous textbooks simply do not exist. This book aims at starting to bridge this gap. Both authors graduated in physics before turning to mathematical physics. As such, we have witnessed this lack from the student's point of view, before experiencing it, a few years later, from the teacher's point of view. Above all, this text aims to provide the material we would have liked to have at our disposal when entering this field.

Although our hope is that it will also be of interest to students in theoretical physics, this is in fact a book on *mathematical physics*. There is no general consensus on what the latter term actually refers to. In rough terms, what it means for us is: the analysis of problems originating in physics, at the level of rigor associated to mathematics. This includes the introduction of concepts and the development of tools enabling such an analysis. It is unfortunate that mathematical physics is often held in rather low esteem by physicists, many of whom see it as useless nitpicking and as dealing mainly with problems that they consider to be already fully understood. There are however very good reasons for these investigations. First, such an approach allows a very clear separation between the assumptions (the basic principles of the underlying theory, as well as the particulars of the model analyzed)

and the actual derivation: once the proper framework is set, the entire analysis is done without further assumptions or approximations. This is essential in order to ensure that the phenomenon that has been derived is indeed a consequence of the starting hypotheses and not an artifact of the approximations made along the way. Second, to provide a complete mathematical analysis requires us to understand the phenomenon of interest in a much deeper and detailed way. In particular, it forces one to provide precise definitions and statements. This is highly useful in clarifying issues that are sometimes puzzling for students and, occasionally, researchers.

Let us emphasize two central features of this work.

- The first has to do with content. Equilibrium statistical mechanics has become such a rich and diverse subject that it is impossible to cover more than a fraction of it in a single book. Since our driving motivation is to provide an easily accessible introduction in a form suitable for self-study, our first decision was to focus on some of the most important and relevant examples rather than to present the theory from a broad point of view. We hope that this will help the reader build the necessary intuition, in concrete situations, as well as provide background and motivation for the general theory. We also refrained from introducing abstractions for their own sake and have done our best to keep the technical level as low as possible.
- The second central feature of this book is related to our belief that the main value of the proof of a theorem is measured by the extent to which it enhances understanding of the phenomena under consideration. As a matter of fact, the concepts and methods introduced in the course of a proof are often at least as important as the claim of the theorem itself. The most useful proof, for a beginner, is thus not necessarily the shortest or the most elegant one. For these reasons, we have strived to provide, throughout the book, the arguments we personally consider the most enlightening in the most simple manner possible.

These two features have shaped the book from its very first versions. (They have also contributed, admittedly, to the lengthiness of some chapters.) Together with the numerous illustrations and exercises, we hope that they will help the beginner to become familiarized with some of the central concepts and methods that lie at the core of statistical mechanics.

As underlined by many authors, one of the main purposes of writing a book should be one's own pleasure. Indeed, leading this project to its conclusion was by and large a very enjoyable albeit long journey! But, beyond that, the positive feedback we have already received from students and from colleagues who have used early drafts in their lectures, indicates that it may yet reach its goal, which is to help beginners enter this beautiful field...

**Acknowledgements.** This book benefited both directly and indirectly from the help and support of many colleagues. First and foremost, we would like to thank Charles Pfister who, as a PhD advisor, introduced both authors to this field of research many years ago. We have also learned much of what we know from our various co-authors during the last two decades. In particular, YV would like to express his thanks to Dima Ioffe, for a long, fruitful and very enjoyable ongoing collaboration.

Our warmest thanks also go to Aernout van Enter who has been a constant source of support and feedback and whose enthusiasm for this project has always been highly appreciated!

We are very grateful to all the people who called to our attention various errors they found in preliminary versions of the book — in particular, Costanza Benassi, Quentin Berthet, Tecla Cardilli, Loren Coquille, Margherita Disertori, Hugo Duminil-Copin, Mauro Mariani, Philippe Moreillon, Sébastien Ott, Ron Peled, Sylvie Roelly, Constanza Rojas-Molina and Daniel Ueltschi.

We also thank Claudio Landim, Vidas Sidoravicius and Augusto Teixeira for their support and comments. Our warm thanks to Maria Eulalia Vares, for her constant encouragement since the earliest drafts of this work.

SF thanks the Departamento de Matemática of the Federal University of Minas Gerais, for its long term support, Prof. Hans-Jörg Ruppen (CMS, EPFL), as well as the Section de Mathématiques of the University of Geneva for hospitality and financial support during countless visits during which large parts of this work were written. Both authors are also grateful to the Swiss National Science Foundation for its support, in particular through the NCCR SwissMAP.

Finally, writing this book would have been considerably less enjoyable without the following fantastic pieces of open source software: bash, GNU/Linux (openSUSE and Ubuntu flavors), GCC, GIMP, git, GNOME, gnuplot, Inkscape, KDE, Kile,  $\LaTeX 2_{\epsilon}$ , PGF/Tikz, POV-Ray, Processing, Python, Sketch (for  $\LaTeX$ ), TeXstudio, Vim and Xfig.

Geneva, February 2017

*Sacha Friedli*  
*Yvan Velenik*





<b>Preface</b>	<b>iii</b>
<b>Conventions</b>	<b>xv</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Equilibrium Thermodynamics . . . . .	2
1.1.1 On the description of macroscopic systems . . . . .	2
1.1.2 The thermodynamic entropy . . . . .	4
1.1.3 Conjugate intensive quantities and equations of state . . . . .	8
1.1.4 Densities . . . . .	8
1.1.5 Alternative representations; thermodynamic potentials . . . . .	10
1.1.6 Condensation and the Van der Waals–Maxwell Theory . . . . .	13
1.2 From Micro to Macro: Statistical Mechanics . . . . .	17
1.2.1 The microcanonical ensemble . . . . .	19
1.2.2 The canonical ensemble . . . . .	20
1.2.3 The grand canonical ensemble . . . . .	23
1.2.4 Examples: Two models of a gas. . . . .	24
1.3 Linking Statistical Mechanics and Thermodynamics . . . . .	26
1.3.1 Boltzmann’s Principle and the thermodynamic limit . . . . .	27
1.3.2 Deriving the equation of state of the ideal gas . . . . .	34
1.3.3 The basic structure . . . . .	35
1.4 Magnetic systems . . . . .	35
1.4.1 Phenomenology: Paramagnets vs. Ferromagnets . . . . .	35
1.4.2 A simple model for a magnet: the Ising model . . . . .	37
1.4.3 Thermodynamic behavior . . . . .	39
1.5 Some general remarks . . . . .	46
1.5.1 The role of the thermodynamic limit . . . . .	46
1.5.2 On the role of simple models . . . . .	48
1.6 About this book . . . . .	49
1.6.1 Contents, chapter by chapter . . . . .	49
1.6.2 The existing literature . . . . .	53
<b>2 The Curie–Weiss Model</b>	<b>57</b>
2.1 The mean-field approximation . . . . .	57
2.2 The behavior for large $N$ when $h = 0$ . . . . .	59
2.3 The behavior for large $N$ when $h \neq 0$ . . . . .	65
2.4 Bibliographical references . . . . .	69
2.5 Complements and further reading . . . . .	69
2.5.1 The “naive” mean-field approximation. . . . .	69
2.5.2 Alternative approaches to analyze the Curie–Weiss model. . . . .	70
2.5.3 Critical exponents . . . . .	72
2.5.4 Links with other models on $\mathbb{Z}^d$ . . . . .	76
<b>3 The Ising Model</b>	<b>79</b>
3.1 Finite-volume Gibbs distributions . . . . .	80
3.2 Thermodynamic limit, pressure and magnetization . . . . .	83
3.2.1 Convergence of subsets . . . . .	83
3.2.2 Pressure . . . . .	83
3.2.3 Magnetization . . . . .	87
3.2.4 A first definition of phase transition . . . . .	89

3.3	The one-dimensional Ising model . . . . .	90
3.4	Infinite-volume Gibbs states . . . . .	93
3.5	Two families of local functions. . . . .	96
3.6	Correlation inequalities . . . . .	97
3.6.1	The GKS inequalities. . . . .	97
3.6.2	The FKG inequality. . . . .	98
3.6.3	Consequences . . . . .	99
3.7	Phase Diagram . . . . .	103
3.7.1	Two criteria for (non)-uniqueness . . . . .	104
3.7.2	Spontaneous symmetry breaking at low temperatures . . . . .	109
3.7.3	Uniqueness at high temperature . . . . .	116
3.7.4	Uniqueness in nonzero magnetic field . . . . .	119
3.7.5	Summary of what has been proved . . . . .	126
3.8	Proof of the Correlation Inequalities . . . . .	127
3.8.1	Proof of the GKS inequalities . . . . .	127
3.8.2	Proof of the FKG inequality . . . . .	128
3.9	Bibliographical references . . . . .	130
3.10	Complements and further reading . . . . .	132
3.10.1	Kramers–Wannier duality . . . . .	132
3.10.2	Mean-field bounds . . . . .	134
3.10.3	An alternative proof of the FKG inequality . . . . .	136
3.10.4	Transfer matrix and Markov chains . . . . .	139
3.10.5	The Ising antiferromagnet . . . . .	140
3.10.6	Random-cluster and random-current representations. . . . .	141
3.10.7	Non-translation-invariant Gibbs states and interfaces. . . . .	146
3.10.8	Gibbs states and local behavior in large finite systems . . . . .	152
3.10.9	Absence of analytic continuation of the pressure. . . . .	156
3.10.10	Metastable behavior in finite systems. . . . .	159
3.10.11	Critical phenomena. . . . .	160
3.10.12	Exact solution . . . . .	164
3.10.13	Stochastic dynamics. . . . .	165
<b>4</b>	<b>Liquid-Vapor Equilibrium</b> . . . . .	<b>167</b>
4.1	The lattice gas approximation . . . . .	168
4.2	Canonical ensemble and free energy . . . . .	170
4.3	Grand canonical ensemble and pressure . . . . .	173
4.4	Equivalence of ensembles . . . . .	176
4.5	An overview of the rest of the chapter . . . . .	177
4.6	Concentration and typical configurations . . . . .	177
4.6.1	Typical densities . . . . .	177
4.6.2	Strict convexity and spatial homogeneity . . . . .	179
4.7	The hard-core lattice gas . . . . .	181
4.7.1	Parenthesis: equivalence of ensembles at the level of measures	183
4.8	The nearest-neighbor lattice gas . . . . .	184
4.8.1	The pressure . . . . .	185
4.8.2	The free energy . . . . .	187
4.8.3	Typical densities . . . . .	188
4.8.4	The pressure as a function of $\rho$ and $\nu$ . . . . .	188
4.9	The van der Waals lattice gas . . . . .	190
4.9.1	(Non-)convexity of the free energy. . . . .	192

4.9.2	An expression for the pressure; Maxwell's construction . . . . .	193
4.10	Kač interactions and the van der Waals limit . . . . .	198
4.10.1	van der Waals limit of the thermodynamic potentials . . . . .	200
4.11	Bibliographical references . . . . .	205
4.12	Complements and further reading . . . . .	206
4.12.1	The phase separation phenomenon . . . . .	206
4.12.2	Kač interactions when $\gamma$ is small but fixed . . . . .	212
4.12.3	Condensation, metastability and the analytic structure of the isotherms . . . . .	213
<b>5</b>	<b>Cluster Expansion</b> . . . . .	<b>219</b>
5.1	Introduction . . . . .	219
5.2	Polymer models . . . . .	220
5.3	The formal expansion . . . . .	221
5.4	A condition ensuring convergence . . . . .	224
5.5	When the weights depend on a parameter . . . . .	227
5.6	The case of hard-core interactions . . . . .	228
5.7	Applications . . . . .	229
5.7.1	The Ising model in a strong magnetic field . . . . .	229
5.7.2	The virial expansion for the lattice gas . . . . .	235
5.7.3	The Ising model at high temperature ( $h = 0$ ) . . . . .	237
5.7.4	The Ising model at low temperature ( $h = 0$ ) . . . . .	238
5.8	Bibliographical references . . . . .	244
<b>6</b>	<b>Infinite-Volume Gibbs Measures</b> . . . . .	<b>245</b>
6.1	The problem with infinite systems . . . . .	247
6.2	Events and probability measures on $\Omega$ . . . . .	248
6.2.1	The DLR approach . . . . .	252
6.3	Specifications and measures . . . . .	254
6.3.1	Kernels vs. conditional probabilities . . . . .	257
6.3.2	Gibbsian specifications . . . . .	258
6.4	Existence . . . . .	261
6.4.1	Convergence on $\Omega$ . . . . .	262
6.4.2	Convergence on $\mathcal{M}_1(\Omega)$ . . . . .	264
6.4.3	Existence and quasilocality . . . . .	264
6.5	Uniqueness . . . . .	267
6.5.1	Uniqueness vs. sensitivity to boundary conditions . . . . .	267
6.5.2	Dobrushin's Uniqueness Theorem . . . . .	268
6.5.3	Application to Gibbsian specifications . . . . .	271
6.5.4	Uniqueness at high temperature via cluster expansion . . . . .	274
6.5.5	Uniqueness in one dimension . . . . .	274
6.6	Symmetries . . . . .	276
6.6.1	Measures compatible with a G-invariant specification . . . . .	277
6.7	Translation invariant Gibbs measures . . . . .	278
6.7.1	Translation invariant specifications . . . . .	280
6.8	Convexity and Extremal Gibbs measures . . . . .	280
6.8.1	Properties of extremal Gibbs measures . . . . .	281
6.8.2	Extremal Gibbs measures and the thermodynamic limit . . . . .	286
6.8.3	More on $\mu_{\beta,h}^+$ , $\mu_{\beta,h}^-$ and $\mathcal{G}(\beta, h)$ . . . . .	287
6.8.4	Extremal decomposition . . . . .	291

6.9	The variational principle . . . . .	297
6.9.1	Formulation in the finite case . . . . .	297
6.9.2	Specific entropy and energy density . . . . .	299
6.9.3	Variational principle for Gibbs measures . . . . .	302
6.10	Continuous spins . . . . .	305
6.10.1	General definitions . . . . .	305
6.10.2	DLR formalism for compact spin space . . . . .	307
6.10.3	Symmetries . . . . .	308
6.11	A criterion for non-uniqueness . . . . .	308
6.12	Some proofs . . . . .	311
6.12.1	Proofs related to the construction of probability measures . . . . .	311
6.12.2	Proof of Theorem 6.5 . . . . .	312
6.12.3	Proof of Theorem 6.6 . . . . .	312
6.12.4	Proof of Theorem 6.24 . . . . .	313
6.12.5	Proof of Proposition 6.39 . . . . .	313
6.13	Bibliographical references . . . . .	317
6.14	Complements and further reading . . . . .	318
6.14.1	The equivalence of ensembles . . . . .	318
6.14.2	Pathologies of transformations and weaker notions of Gibbsianness. . . . .	319
6.14.3	Gibbs measures and the thermodynamic formalism. . . . .	320
<b>7</b>	<b>Pirogov–Sinai Theory</b> . . . . .	<b>321</b>
7.1	Introduction . . . . .	322
7.1.1	A modified Ising model . . . . .	323
7.1.2	Models with three or more phases . . . . .	323
7.1.3	Overview of the chapter . . . . .	325
7.1.4	Models with finite-range translation invariant interactions . . . . .	325
7.2	Ground states and Peierls' Condition . . . . .	327
7.2.1	Boundaries of a configuration . . . . .	329
7.2.2	m-potentials . . . . .	333
7.2.3	Lifting the degeneracy . . . . .	335
7.2.4	A glimpse of the rest of this chapter . . . . .	338
7.2.5	From finite-range interactions to interactions of range one . . . . .	338
7.2.6	Contours and their labels . . . . .	339
7.3	Boundary conditions and contour models . . . . .	341
7.3.1	Extracting the contribution from the ground state . . . . .	342
7.3.2	Representing probabilities involving external contours . . . . .	346
7.4	Phase diagram of the Blume–Capel model . . . . .	347
7.4.1	Heuristics . . . . .	347
7.4.2	Polymer models with $\tau$ -stable weights . . . . .	351
7.4.3	Truncated weights and pressures, upper bounds on partition functions . . . . .	355
7.4.4	Construction of the phase diagram . . . . .	363
7.4.5	Results for the pressure . . . . .	366
7.4.6	The Gibbs measures at low temperature . . . . .	367
7.5	Bibliographical references . . . . .	373
7.6	Complements and further reading . . . . .	374
7.6.1	Completeness of the phase diagram . . . . .	374
7.6.2	Generalizations . . . . .	375

7.6.3	Large- $\beta$ asymptotics of the phase diagram . . . . .	375
7.6.4	Other regimes. . . . .	376
7.6.5	Finite-size scaling. . . . .	377
7.6.6	Complex parameters, Lee–Yang zeroes and singularities. . . . .	377
<b>8</b>	<b>The Gaussian Free Field on <math>\mathbb{Z}^d</math></b>	<b>379</b>
8.1	Definition of the model . . . . .	380
8.1.1	Overview . . . . .	382
8.2	Parenthesis: Gaussian vectors and fields . . . . .	383
8.2.1	Gaussian vectors . . . . .	384
8.2.2	Gaussian fields and the thermodynamic limit. . . . .	385
8.3	Harmonic functions and Green Identities . . . . .	387
8.4	The massless case . . . . .	389
8.4.1	The random walk representation . . . . .	390
8.4.2	The thermodynamic limit . . . . .	393
8.5	The massive case . . . . .	397
8.5.1	Random walk representation . . . . .	398
8.5.2	The thermodynamic limit . . . . .	400
8.5.3	The limit $m \downarrow 0$ . . . . .	402
8.6	Bibliographical references . . . . .	406
8.7	Complements and further reading . . . . .	406
8.7.1	Random walk representations . . . . .	406
8.7.2	Gradient Gibbs states . . . . .	407
8.7.3	Effective interface models . . . . .	407
8.7.4	Continuum GFF . . . . .	407
8.7.5	A link to discrete spin systems . . . . .	407
<b>9</b>	<b>Models with Continuous Symmetry</b>	<b>411</b>
9.1	$O(N)$ -symmetric models . . . . .	411
9.1.1	Overview . . . . .	412
9.2	Absence of continuous symmetry breaking . . . . .	414
9.2.1	Heuristic argument . . . . .	416
9.2.2	Proof of the Mermin–Wagner Theorem for $N = 2$ . . . . .	419
9.2.3	Proof of the Mermin–Wagner theorem for $N \geq 3$ . . . . .	424
9.3	Digression on gradient models . . . . .	424
9.4	Decay of correlations . . . . .	426
9.4.1	One-dimensional models . . . . .	427
9.4.2	Two-dimensional models . . . . .	428
9.5	Bibliographical references . . . . .	433
9.6	Complements and further reading . . . . .	433
9.6.1	The Berezinskii–Kosterlitz–Thouless phase transition . . . . .	433
9.6.2	Generalizations. . . . .	435
<b>10</b>	<b>Reflection Positivity</b>	<b>437</b>
10.1	Motivation: some new results for $O(N)$ -type models . . . . .	437
10.2	Models defined on the torus. . . . .	438
10.3	Reflections . . . . .	439
10.3.1	Reflection positive measures . . . . .	440
10.3.2	Examples of reflection positive measures . . . . .	442
10.4	The chessboard estimate . . . . .	444

10.4.1	Proof of the estimate . . . . .	444
10.4.2	Application: the Ising model in a large magnetic field . . . . .	450
10.4.3	Application: the two-dimensional anisotropic $XY$ model . . . . .	451
10.5	The infrared bound . . . . .	459
10.5.1	Models to be considered . . . . .	459
10.5.2	Application: long-range order in the $O(N)$ model, $d \geq 3$ . . . . .	459
10.5.3	Gaussian domination and the infrared bound . . . . .	463
10.6	Bibliographical remarks . . . . .	466
<b>A</b>	<b>Notes</b>	<b>469</b>
<b>B</b>	<b>Mathematical Appendices</b>	<b>477</b>
B.1	Real analysis . . . . .	478
B.1.1	Elementary Inequalities . . . . .	478
B.1.2	Double sequences . . . . .	478
B.1.3	Subadditive sequences . . . . .	479
B.1.4	Functions defined by series . . . . .	479
B.2	Convex functions . . . . .	480
B.2.1	Convexity vs. continuity . . . . .	481
B.2.2	Convexity vs. differentiability . . . . .	482
B.2.3	The Legendre transform . . . . .	485
B.2.4	Legendre transform of non-differentiable functions . . . . .	487
B.3	Complex analysis . . . . .	488
B.4	Metric spaces . . . . .	491
B.5	Measure Theory . . . . .	491
B.5.1	Measures and probability measures . . . . .	491
B.5.2	Measurable functions . . . . .	493
B.6	Integration . . . . .	494
B.6.1	Product spaces . . . . .	496
B.7	Lebesgue measure . . . . .	496
B.8	Probability . . . . .	497
B.8.1	Random variables and vectors . . . . .	497
B.8.2	Independence . . . . .	498
B.8.3	Moments and cumulants of random variables . . . . .	499
B.8.4	Characteristic function . . . . .	500
B.8.5	Conditional Expectation . . . . .	500
B.8.6	Conditional probability . . . . .	503
B.8.7	Random vectors . . . . .	503
B.9	Gaussian vectors and fields . . . . .	504
B.9.1	Basic definitions and properties . . . . .	504
B.9.2	Convergence of Gaussian vectors . . . . .	505
B.9.3	Gaussian fields and independence . . . . .	506
B.10	The total variation distance . . . . .	506
B.11	Shannon's Entropy . . . . .	507
B.12	Relative entropy . . . . .	510
B.12.1	Definition, basic properties . . . . .	510
B.12.2	Two useful inequalities . . . . .	512
B.13	The symmetric simple random walk on $\mathbb{Z}^d$ . . . . .	513
B.13.1	Stopping times and the strong Markov property . . . . .	513
B.13.2	Local Limit Theorem . . . . .	513

---

B.13.3	Recurrence and transience . . . . .	514
B.13.4	Discrete potential theory . . . . .	515
B.14	The isoperimetric inequality on $\mathbb{Z}^d$ . . . . .	516
B.15	A result on the boundary of subsets of $\mathbb{Z}^d$ . . . . .	518
<b>C</b>	<b>Solutions to Exercises</b>	<b>521</b>
	<b>References</b>	<b>543</b>
	<b>Index</b>	<b>563</b>





# Conventions

---

$a \stackrel{\text{def}}{=} b$	$a$ is defined as being $b$
$\mathbb{R}^d$	$d$ -dimensional Euclidean space
$\mathbb{Z}^d$	$d$ -dimensional cubic lattice
$\mathbb{R}_{\geq 0}$	nonnegative real numbers
$\mathbb{R}_{> 0}$	positive real numbers
$\mathbb{Z}_{\geq 0}$	nonnegative integers: $0, 1, 2, 3, \dots$
$\mathbb{N}, \mathbb{Z}_{> 0}$	positive integers: $1, 2, 3, \dots$
$i$	$\sqrt{-1}$
$\Re z, \Im z$	real and imaginary parts of $z \in \mathbb{C}$
$a \wedge b$	minimum of $a$ and $b$
$a \vee b$	maximum of $a$ and $b$
$\log$	natural logarithm, that is, in base $e = 2.718\dots$
$A \subset B$	$A$ is a (not necessarily proper) subset of $B$
$A \subsetneq B$	$A$ is a proper subset of $B$
$A \Delta B$	symmetric difference
$\#A,  A $	number of elements in the set $A$ (if $A$ is finite). At several places, also used to denote the Lebesgue measure.
$\delta_{m,n}$	Kronecker symbol: $\delta_{m,n} = 1$ if $m = n$ , $0$ otherwise
$\delta_x$	Dirac measure at $x$ : $\delta_x(A) = 1$ if $x \in A$ , $0$ otherwise
$\lfloor x \rfloor$	largest integer smaller or equal to $x$
$\lceil x \rceil$	smallest integer larger or equal to $x$

Asymptotic equivalence of functions will follow the standard conventions. For functions  $f, g$ , defined in the neighborhood of  $x_0$  (possibly  $x_0 = \infty$ ),

$f(x) \sim g(x)$	means $\lim_{x \rightarrow x_0} \frac{\log f(x)}{\log g(x)} = 1$ ,
$f(x) \approx g(x)$	means $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$ ,
$f(x) \approx g(x)$	means $0 < \liminf_{x \rightarrow x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow x_0} \frac{f(x)}{g(x)} < \infty$
$f(x) = O(g(x))$	means $\limsup_{x \rightarrow x_0} \left  \frac{f(x)}{g(x)} \right  < \infty$ ,
$f(x) = o(g(x))$	means $\lim_{x \rightarrow x_0} \left  \frac{f(x)}{g(x)} \right  = 0$ .

As usual,  $A^B$  is identified with the set of all maps  $f : B \rightarrow A$ . A sequence of elements  $a_n \in E$  will usually be denoted as  $(a_n)_{n \geq 1} \subset E$ . At several places, we will set  $0 \log 0 \stackrel{\text{def}}{=} 0$ . Sums or products over empty families are defined as follows:

$$\sum_{i \in \emptyset} a_i \stackrel{\text{def}}{=} 0 \quad \prod_{i \in \emptyset} a_i \stackrel{\text{def}}{=} 1.$$

Several important notations involving several geometrical notions on  $\mathbb{Z}^d$  will be defined at the end of the introduction and at the beginning of Chapter 3.