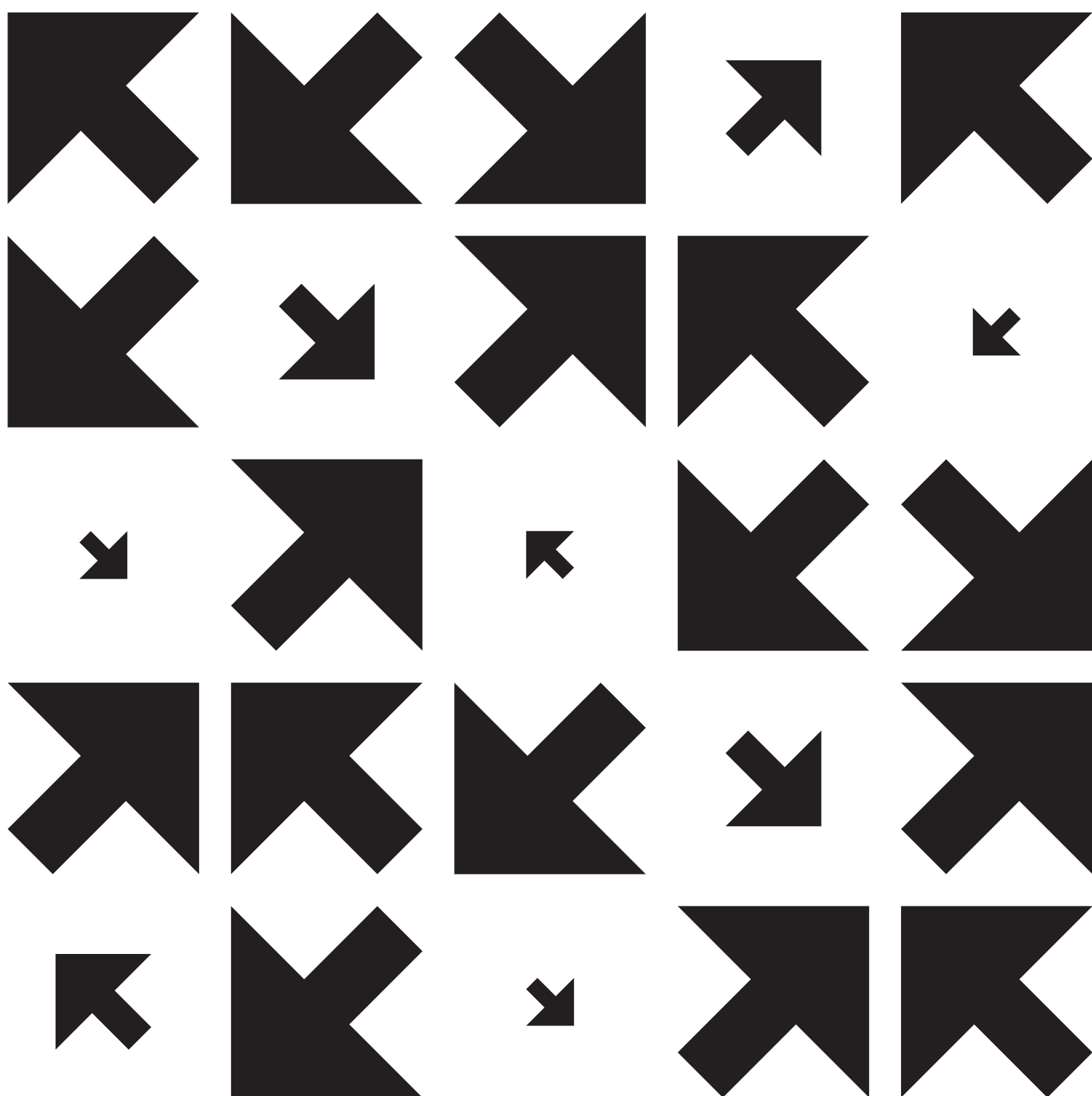


Statistical Mechanics of Lattice Systems

S. Friedli and Y. Velenik

A Concrete Mathematical Introduction



For the sunshine and smiles: Janet, Jean-Pierre, Mimi and Kathryn.

Aos amigos e colegas do Departamento de Matemática.

À Agnese, Laure et Alexandre, ainsi qu'à mes parents.

Preface

Equilibrium statistical mechanics is a field that has existed for more than a century. Its origins lie in the search for a microscopic justification of equilibrium thermodynamics, and it developed into a well-established branch of mathematics in the second half of the twentieth century. The ideas and methods that it introduced to treat systems with many components have now permeated many areas of science and engineering, and have had an important impact on several branches of mathematics.

There exist many good introductions to this theory designed for physics undergraduates. It might however come as a surprise that textbooks addressing it from a *mathematically rigorous* standpoint have remained rather scarce. A reader looking for an introduction to its more advanced mathematical aspects must often either consult highly specialized monographs or search through numerous research articles available in peer-reviewed journals. It might even appear as if the mastery of certain techniques has survived from one generation of researchers to the next only by means of oral communication, through the use of chalk and blackboard...

It seems a general opinion that pedagogical introductory mathematically rigorous textbooks simply do not exist. This book aims at starting to bridge this gap. Both authors graduated in physics before turning to mathematical physics. As such, we have witnessed this lack from the student's point of view, before experiencing it, a few years later, from the teacher's point of view. Above all, this text aims to provide the material we would have liked to have at our disposal when entering this field.

Although our hope is that it will also be of interest to students in theoretical physics, this is in fact a book on *mathematical physics*. There is no general consensus on what the latter term actually refers to. In rough terms, what it means for us is: the analysis of problems originating in physics, at the level of rigor associated to mathematics. This includes the introduction of concepts and the development of tools enabling such an analysis. It is unfortunate that mathematical physics is often held in rather low esteem by physicists, many of whom see it as useless nitpicking and as dealing mainly with problems that they consider to be already fully understood. There are however very good reasons for these investigations. First, such an approach allows a very clear separation between the assumptions (the basic principles of the underlying theory, as well as the particulars of the model analyzed)

and the actual derivation: once the proper framework is set, the entire analysis is done without further assumptions or approximations. This is essential in order to ensure that the phenomenon that has been derived is indeed a consequence of the starting hypotheses and not an artifact of the approximations made along the way. Second, to provide a complete mathematical analysis requires us to understand the phenomenon of interest in a much deeper and detailed way. In particular, it forces one to provide precise definitions and statements. This is highly useful in clarifying issues that are sometimes puzzling for students and, occasionally, researchers.

Let us emphasize two central features of this work.

- The first has to do with content. Equilibrium statistical mechanics has become such a rich and diverse subject that it is impossible to cover more than a fraction of it in a single book. Since our driving motivation is to provide an easily accessible introduction in a form suitable for self-study, our first decision was to focus on some of the most important and relevant examples rather than to present the theory from a broad point of view. We hope that this will help the reader build the necessary intuition, in concrete situations, as well as provide background and motivation for the general theory. We also refrained from introducing abstractions for their own sake and have done our best to keep the technical level as low as possible.
- The second central feature of this book is related to our belief that the main value of the proof of a theorem is measured by the extent to which it enhances understanding of the phenomena under consideration. As a matter of fact, the concepts and methods introduced in the course of a proof are often at least as important as the claim of the theorem itself. The most useful proof, for a beginner, is thus not necessarily the shortest or the most elegant one. For these reasons, we have strived to provide, throughout the book, the arguments we personally consider the most enlightening in the most simple manner possible.

These two features have shaped the book from its very first versions. (They have also contributed, admittedly, to the lengthiness of some chapters.) Together with the numerous illustrations and exercises, we hope that they will help the beginner to become familiarized with some of the central concepts and methods that lie at the core of statistical mechanics.

As underlined by many authors, one of the main purposes of writing a book should be one's own pleasure. Indeed, leading this project to its conclusion was by and large a very enjoyable albeit long journey! But, beyond that, the positive feedback we have already received from students and from colleagues who have used early drafts in their lectures, indicates that it may yet reach its goal, which is to help beginners enter this beautiful field...

Acknowledgements. This book benefited both directly and indirectly from the help and support of many colleagues. First and foremost, we would like to thank Charles Pfister who, as a PhD advisor, introduced both authors to this field of research many years ago. We have also learned much of what we know from our various co-authors during the last two decades. In particular, YV would like to express his thanks to Dima Ioffe, for a long, fruitful and very enjoyable ongoing collaboration.

Our warmest thanks also go to Aernout van Enter who has been a constant source of support and feedback and whose enthusiasm for this project has always been highly appreciated!

We are very grateful to all the people who called to our attention various errors they found in preliminary versions of the book — in particular, Costanza Benassi, Quentin Berthet, Tecla Cardilli, Loren Coquille, Margherita Disertori, Hugo Duminil-Copin, Mauro Mariani, Philippe Moreillon, Sébastien Ott, Ron Peled, Sylvie Roelly, Constanza Rojas-Molina and Daniel Ueltschi.

We also thank Claudio Landim, Vidas Sidoravicius and Augusto Teixeira for their support and comments. Our warm thanks to Maria Eulalia Vares, for her constant encouragement since the earliest drafts of this work.

SF thanks the Departamento de Matemática of the Federal University of Minas Gerais, for its long term support, Prof. Hans-Jörg Ruppen (CMS, EPFL), as well as the Section de Mathématiques of the University of Geneva for hospitality and financial support during countless visits during which large parts of this work were written. Both authors are also grateful to the Swiss National Science Foundation for its support, in particular through the NCCR SwissMAP.

Finally, writing this book would have been considerably less enjoyable without the following fantastic pieces of open source software: bash, GNU/Linux (openSUSE and Ubuntu flavors), GCC, GIMP, git, GNOME, gnuplot, Inkscape, KDE, Kile, $\LaTeX 2_{\epsilon}$, PGF/Tikz, POV-Ray, Processing, Python, Sketch (for \LaTeX), TeXstudio, Vim and Xfig.

Geneva, February 2017

Sacha Friedli
Yvan Velenik

Preface	iii
Conventions	xv
1 Introduction	1
1.1 Equilibrium Thermodynamics	2
1.1.1 On the description of macroscopic systems	2
1.1.2 The thermodynamic entropy	4
1.1.3 Conjugate intensive quantities and equations of state	8
1.1.4 Densities	8
1.1.5 Alternative representations; thermodynamic potentials	10
1.1.6 Condensation and the Van der Waals–Maxwell Theory	13
1.2 From Micro to Macro: Statistical Mechanics	17
1.2.1 The microcanonical ensemble	19
1.2.2 The canonical ensemble	20
1.2.3 The grand canonical ensemble	23
1.2.4 Examples: Two models of a gas.	24
1.3 Linking Statistical Mechanics and Thermodynamics	26
1.3.1 Boltzmann’s Principle and the thermodynamic limit	27
1.3.2 Deriving the equation of state of the ideal gas	34
1.3.3 The basic structure	35
1.4 Magnetic systems	35
1.4.1 Phenomenology: Paramagnets vs. Ferromagnets	35
1.4.2 A simple model for a magnet: the Ising model	37
1.4.3 Thermodynamic behavior	39
1.5 Some general remarks	46
1.5.1 The role of the thermodynamic limit	46
1.5.2 On the role of simple models	48
1.6 About this book	49
1.6.1 Contents, chapter by chapter	49
1.6.2 The existing literature	53
2 The Curie–Weiss Model	57
2.1 The mean-field approximation	57
2.2 The behavior for large N when $h = 0$	59
2.3 The behavior for large N when $h \neq 0$	65
2.4 Bibliographical references	69
2.5 Complements and further reading	69
2.5.1 The “naive” mean-field approximation.	69
2.5.2 Alternative approaches to analyze the Curie–Weiss model.	70
2.5.3 Critical exponents	72
2.5.4 Links with other models on \mathbb{Z}^d	76
3 The Ising Model	79
3.1 Finite-volume Gibbs distributions	80
3.2 Thermodynamic limit, pressure and magnetization	83
3.2.1 Convergence of subsets	83
3.2.2 Pressure	83
3.2.3 Magnetization	87
3.2.4 A first definition of phase transition	89

3.3	The one-dimensional Ising model	90
3.4	Infinite-volume Gibbs states	93
3.5	Two families of local functions.	96
3.6	Correlation inequalities	97
3.6.1	The GKS inequalities.	97
3.6.2	The FKG inequality.	98
3.6.3	Consequences	99
3.7	Phase Diagram	103
3.7.1	Two criteria for (non)-uniqueness	104
3.7.2	Spontaneous symmetry breaking at low temperatures	109
3.7.3	Uniqueness at high temperature	116
3.7.4	Uniqueness in nonzero magnetic field	119
3.7.5	Summary of what has been proved	126
3.8	Proof of the Correlation Inequalities	127
3.8.1	Proof of the GKS inequalities	127
3.8.2	Proof of the FKG inequality	128
3.9	Bibliographical references	130
3.10	Complements and further reading	132
3.10.1	Kramers–Wannier duality	132
3.10.2	Mean-field bounds	134
3.10.3	An alternative proof of the FKG inequality	136
3.10.4	Transfer matrix and Markov chains	139
3.10.5	The Ising antiferromagnet	140
3.10.6	Random-cluster and random-current representations.	141
3.10.7	Non-translation-invariant Gibbs states and interfaces.	146
3.10.8	Gibbs states and local behavior in large finite systems	152
3.10.9	Absence of analytic continuation of the pressure.	156
3.10.10	Metastable behavior in finite systems.	159
3.10.11	Critical phenomena.	160
3.10.12	Exact solution	164
3.10.13	Stochastic dynamics.	165
4	Liquid-Vapor Equilibrium	167
4.1	The lattice gas approximation	168
4.2	Canonical ensemble and free energy	170
4.3	Grand canonical ensemble and pressure	173
4.4	Equivalence of ensembles	176
4.5	An overview of the rest of the chapter	177
4.6	Concentration and typical configurations	177
4.6.1	Typical densities	177
4.6.2	Strict convexity and spatial homogeneity	179
4.7	The hard-core lattice gas	181
4.7.1	Parenthesis: equivalence of ensembles at the level of measures	183
4.8	The nearest-neighbor lattice gas	184
4.8.1	The pressure	185
4.8.2	The free energy	187
4.8.3	Typical densities	188
4.8.4	The pressure as a function of ρ and ν	188
4.9	The van der Waals lattice gas	190
4.9.1	(Non-)convexity of the free energy.	192

4.9.2	An expression for the pressure; Maxwell's construction . . .	193
4.10	Kač interactions and the van der Waals limit	198
4.10.1	van der Waals limit of the thermodynamic potentials	200
4.11	Bibliographical references	205
4.12	Complements and further reading	206
4.12.1	The phase separation phenomenon	206
4.12.2	Kač interactions when γ is small but fixed	212
4.12.3	Condensation, metastability and the analytic structure of the isotherms	213
5	Cluster Expansion	219
5.1	Introduction	219
5.2	Polymer models	220
5.3	The formal expansion	221
5.4	A condition ensuring convergence	224
5.5	When the weights depend on a parameter	227
5.6	The case of hard-core interactions	228
5.7	Applications	229
5.7.1	The Ising model in a strong magnetic field	229
5.7.2	The virial expansion for the lattice gas	235
5.7.3	The Ising model at high temperature ($h = 0$)	237
5.7.4	The Ising model at low temperature ($h = 0$)	238
5.8	Bibliographical references	244
6	Infinite-Volume Gibbs Measures	245
6.1	The problem with infinite systems	247
6.2	Events and probability measures on Ω	248
6.2.1	The DLR approach	252
6.3	Specifications and measures	254
6.3.1	Kernels vs. conditional probabilities	257
6.3.2	Gibbsian specifications	258
6.4	Existence	261
6.4.1	Convergence on Ω	262
6.4.2	Convergence on $\mathcal{M}_1(\Omega)$	264
6.4.3	Existence and quasilocality	264
6.5	Uniqueness	267
6.5.1	Uniqueness vs. sensitivity to boundary conditions	267
6.5.2	Dobrushin's Uniqueness Theorem	268
6.5.3	Application to Gibbsian specifications	271
6.5.4	Uniqueness at high temperature via cluster expansion	274
6.5.5	Uniqueness in one dimension	274
6.6	Symmetries	276
6.6.1	Measures compatible with a G-invariant specification	277
6.7	Translation invariant Gibbs measures	278
6.7.1	Translation invariant specifications	280
6.8	Convexity and Extremal Gibbs measures	280
6.8.1	Properties of extremal Gibbs measures	281
6.8.2	Extremal Gibbs measures and the thermodynamic limit	286
6.8.3	More on $\mu_{\beta,h}^+$, $\mu_{\beta,h}^-$ and $\mathcal{G}(\beta, h)$	287
6.8.4	Extremal decomposition	291

6.9	The variational principle	297
6.9.1	Formulation in the finite case	297
6.9.2	Specific entropy and energy density	299
6.9.3	Variational principle for Gibbs measures	302
6.10	Continuous spins	305
6.10.1	General definitions	305
6.10.2	DLR formalism for compact spin space	307
6.10.3	Symmetries	308
6.11	A criterion for non-uniqueness	308
6.12	Some proofs	311
6.12.1	Proofs related to the construction of probability measures	311
6.12.2	Proof of Theorem 6.5	312
6.12.3	Proof of Theorem 6.6	312
6.12.4	Proof of Theorem 6.24	313
6.12.5	Proof of Proposition 6.39	313
6.13	Bibliographical references	317
6.14	Complements and further reading	318
6.14.1	The equivalence of ensembles	318
6.14.2	Pathologies of transformations and weaker notions of Gibbsianness.	319
6.14.3	Gibbs measures and the thermodynamic formalism.	320
7	Pirogov–Sinai Theory	321
7.1	Introduction	322
7.1.1	A modified Ising model	323
7.1.2	Models with three or more phases	323
7.1.3	Overview of the chapter	325
7.1.4	Models with finite-range translation invariant interactions	325
7.2	Ground states and Peierls' Condition	327
7.2.1	Boundaries of a configuration	329
7.2.2	m-potentials	333
7.2.3	Lifting the degeneracy	335
7.2.4	A glimpse of the rest of this chapter	338
7.2.5	From finite-range interactions to interactions of range one	338
7.2.6	Contours and their labels	339
7.3	Boundary conditions and contour models	341
7.3.1	Extracting the contribution from the ground state	342
7.3.2	Representing probabilities involving external contours	346
7.4	Phase diagram of the Blume–Capel model	347
7.4.1	Heuristics	347
7.4.2	Polymer models with τ -stable weights	351
7.4.3	Truncated weights and pressures, upper bounds on partition functions	355
7.4.4	Construction of the phase diagram	363
7.4.5	Results for the pressure	366
7.4.6	The Gibbs measures at low temperature	367
7.5	Bibliographical references	373
7.6	Complements and further reading	374
7.6.1	Completeness of the phase diagram	374
7.6.2	Generalizations	375

7.6.3	Large- β asymptotics of the phase diagram	375
7.6.4	Other regimes.	376
7.6.5	Finite-size scaling.	377
7.6.6	Complex parameters, Lee–Yang zeroes and singularities.	377
8	The Gaussian Free Field on \mathbb{Z}^d	379
8.1	Definition of the model	380
8.1.1	Overview	382
8.2	Parenthesis: Gaussian vectors and fields	383
8.2.1	Gaussian vectors	384
8.2.2	Gaussian fields and the thermodynamic limit.	385
8.3	Harmonic functions and Green Identities	387
8.4	The massless case	389
8.4.1	The random walk representation	390
8.4.2	The thermodynamic limit	393
8.5	The massive case	397
8.5.1	Random walk representation	398
8.5.2	The thermodynamic limit	400
8.5.3	The limit $m \downarrow 0$	402
8.6	Bibliographical references	406
8.7	Complements and further reading	406
8.7.1	Random walk representations	406
8.7.2	Gradient Gibbs states	407
8.7.3	Effective interface models	407
8.7.4	Continuum GFF	407
8.7.5	A link to discrete spin systems	407
9	Models with Continuous Symmetry	411
9.1	$O(N)$ -symmetric models	411
9.1.1	Overview	412
9.2	Absence of continuous symmetry breaking	414
9.2.1	Heuristic argument	416
9.2.2	Proof of the Mermin–Wagner Theorem for $N = 2$	419
9.2.3	Proof of the Mermin–Wagner theorem for $N \geq 3$	424
9.3	Digression on gradient models	424
9.4	Decay of correlations	426
9.4.1	One-dimensional models	427
9.4.2	Two-dimensional models	428
9.5	Bibliographical references	433
9.6	Complements and further reading	433
9.6.1	The Berezinskii–Kosterlitz–Thouless phase transition	433
9.6.2	Generalizations.	435
10	Reflection Positivity	437
10.1	Motivation: some new results for $O(N)$ -type models	437
10.2	Models defined on the torus.	438
10.3	Reflections	439
10.3.1	Reflection positive measures	440
10.3.2	Examples of reflection positive measures	442
10.4	The chessboard estimate	444

10.4.1	Proof of the estimate	444
10.4.2	Application: the Ising model in a large magnetic field	450
10.4.3	Application: the two-dimensional anisotropic XY model	451
10.5	The infrared bound	459
10.5.1	Models to be considered	459
10.5.2	Application: long-range order in the $O(N)$ model, $d \geq 3$	459
10.5.3	Gaussian domination and the infrared bound	463
10.6	Bibliographical remarks	466
A	Notes	469
B	Mathematical Appendices	477
B.1	Real analysis	478
B.1.1	Elementary Inequalities	478
B.1.2	Double sequences	478
B.1.3	Subadditive sequences	479
B.1.4	Functions defined by series	479
B.2	Convex functions	480
B.2.1	Convexity vs. continuity	481
B.2.2	Convexity vs. differentiability	482
B.2.3	The Legendre transform	485
B.2.4	Legendre transform of non-differentiable functions	487
B.3	Complex analysis	488
B.4	Metric spaces	491
B.5	Measure Theory	491
B.5.1	Measures and probability measures	491
B.5.2	Measurable functions	493
B.6	Integration	494
B.6.1	Product spaces	496
B.7	Lebesgue measure	496
B.8	Probability	497
B.8.1	Random variables and vectors	497
B.8.2	Independence	498
B.8.3	Moments and cumulants of random variables	499
B.8.4	Characteristic function	500
B.8.5	Conditional Expectation	500
B.8.6	Conditional probability	503
B.8.7	Random vectors	503
B.9	Gaussian vectors and fields	504
B.9.1	Basic definitions and properties	504
B.9.2	Convergence of Gaussian vectors	505
B.9.3	Gaussian fields and independence	506
B.10	The total variation distance	506
B.11	Shannon's Entropy	507
B.12	Relative entropy	510
B.12.1	Definition, basic properties	510
B.12.2	Two useful inequalities	512
B.13	The symmetric simple random walk on \mathbb{Z}^d	513
B.13.1	Stopping times and the strong Markov property	513
B.13.2	Local Limit Theorem	513

B.13.3	Recurrence and transience	514
B.13.4	Discrete potential theory	515
B.14	The isoperimetric inequality on \mathbb{Z}^d	516
B.15	A result on the boundary of subsets of \mathbb{Z}^d	518
C	Solutions to Exercises	521
	References	543

Conventions

$a \stackrel{\text{def}}{=} b$	a is defined as being b
\mathbb{R}^d	d -dimensional Euclidean space
\mathbb{Z}^d	d -dimensional cubic lattice
$\mathbb{R}_{\geq 0}$	nonnegative real numbers
$\mathbb{R}_{> 0}$	positive real numbers
$\mathbb{Z}_{\geq 0}$	nonnegative integers: $0, 1, 2, 3, \dots$
$\mathbb{N}, \mathbb{Z}_{> 0}$	positive integers: $1, 2, 3, \dots$
i	$\sqrt{-1}$
$\Re z, \Im z$	real and imaginary parts of $z \in \mathbb{C}$
$a \wedge b$	minimum of a and b
$a \vee b$	maximum of a and b
\log	natural logarithm, that is, in base $e = 2.718\dots$
$A \subset B$	A is a (not necessarily proper) subset of B
$A \subsetneq B$	A is a proper subset of B
$A \Delta B$	symmetric difference
$\#A, A $	number of elements in the set A (if A is finite). At several places, also used to denote the Lebesgue measure.
$\delta_{m,n}$	Kronecker symbol: $\delta_{m,n} = 1$ if $m = n$, 0 otherwise
δ_x	Dirac measure at x : $\delta_x(A) = 1$ if $x \in A$, 0 otherwise
$\lfloor x \rfloor$	largest integer smaller or equal to x
$\lceil x \rceil$	smallest integer larger or equal to x

Asymptotic equivalence of functions will follow the standard conventions. For functions f, g , defined in the neighborhood of x_0 (possibly $x_0 = \infty$),

$f(x) \sim g(x)$	means $\lim_{x \rightarrow x_0} \frac{\log f(x)}{\log g(x)} = 1$,
$f(x) \simeq g(x)$	means $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$,
$f(x) \approx g(x)$	means $0 < \liminf_{x \rightarrow x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow x_0} \frac{f(x)}{g(x)} < \infty$
$f(x) = O(g(x))$	means $\limsup_{x \rightarrow x_0} \left \frac{f(x)}{g(x)} \right < \infty$,
$f(x) = o(g(x))$	means $\lim_{x \rightarrow x_0} \left \frac{f(x)}{g(x)} \right = 0$.

As usual, A^B is identified with the set of all maps $f : B \rightarrow A$. A sequence of elements $a_n \in E$ will usually be denoted as $(a_n)_{n \geq 1} \subset E$. At several places, we will set $0 \log 0 \stackrel{\text{def}}{=} 0$. Sums or products over empty families are defined as follows:

$$\sum_{i \in \emptyset} a_i \stackrel{\text{def}}{=} 0 \quad \prod_{i \in \emptyset} a_i \stackrel{\text{def}}{=} 1.$$

Several important notations involving several geometrical notions on \mathbb{Z}^d will be defined at the end of the introduction and at the beginning of Chapter 3.