

# A Notes

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## Chapter 1

[1] (p. 3) The property described in (1.1) is usually referred to as *additivity* rather than extensivity. Extensivity of the energy is usually valid and equivalent to additivity in the thermodynamic limit, at least for systems with finite-range interactions, as usually considered in this book. For systems with long-range interactions, extensivity does not always hold.

[2] (p. 19) This terminology was introduced by Gibbs [137], but the statistical ensembles were first introduced by Boltzmann under a different name (*ergode* for the microcanonical ensemble and *holode* for the canonical).

[3] (p. 19) We adopt here the following point of view explained by Jaynes in [181]:

This problem of specification of probabilities in cases where little or no information is available, is as old as the theory of probability. Laplace's "Principle of Insufficient Reason" was an attempt to supply a criterion of choice, in which one said that two events are to be assigned equal probabilities if there is no reason to think otherwise.

Of course, some readers might not consider such a point of view to be fully satisfactory. In particular, one might dislike the interpretation of a probability distribution as a description of a state of knowledge, rather than as a quantity intrinsic to the system. After all, *there is* a more fundamental theory and it would be satisfactory to *derive* this probability distribution from the latter. Many attempts have been done, but no fully satisfactory derivation has been obtained. We will not discuss such issues further here, but refer the interested reader to the extensive literature on this topic; see for example [130].

[4] (p. 21) Historically, the entropy of a probability density had already been introduced by Gibbs in [137].

[5] (p. 35) In our brief description of a ferromagnet and its basic properties, we are neglecting many physically important aspects of the corresponding phenomena. Our goal is not to provide a faithful account, but rather to provide the uninitiated reader with an idea of what ferromagnetic and paramagnetic behaviors correspond to. We refer readers who would prefer a more thorough description to any of the many books on condensed matter physics, such as [356, 14, 1, 65].

[6] (p. 37). Let us briefly recall a famous anecdote originally reported by Uhlenbeck (see [260]). In November 1937, during the Van der Waals Centenary Conference, a morning-long debate took place about the following question: does the partition function contain the information necessary to describe a sharp phase transition? As the debate turned out to be inconclusive, Kramers, who was the chairman, put the question to a vote, the result of which was nearly a tie (the “yes” winning by a small margin).

[7] (p. 37) The importance of Peierls’ contribution was not immediately recognized. Rather, it was the groundbreaking mathematical analysis by Lars Onsager in 1944 that convinced the physics community. In particular, Onsager’s formula for the pressure of the two-dimensional Ising model in the thermodynamic limit showed explicitly the existence of a singularity of this function. Moreover, and perhaps even more importantly, it showed that the behavior at the transition was completely different from what all of the former approximation schemes were predicting. The ensuing necessity of developing more refined approximation methods triggered the development of the modern theory of critical phenomena, in which the Ising model played a central role.

[8] (p. 37). In this book, we only provide brief and very qualitative physical motivations and background information for the Ising model. Much more can be found in many statistical physics textbooks aimed at physicists, such as [298, 264, 165, 299, 331]; see also the (old) review by Fisher [105]. An interesting and detailed description of the major role played by this model in the development of statistical mechanics in the 20th century is given in the series of papers [255, 256, 257], while a shorter one can be found in [55].

[9] (p. 45). The determination of the explicit expression for the spontaneous magnetization of the two-dimensional Ising model given in (1.51) is due to Onsager and Kaufman and was announced by Onsager in 1949. However, they did not publish their result since they still had to work out “how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that” [159]. The first published proof appeared in 1952 and is due to Yang [352]. See [15, 16] for more information.

[10] (p. 48) As an example, let us cite this passage from Peierls’ famous paper [266]:

In the meantime it was shown by Heisenberg that the forces leading to ferromagnetism are due to electron exchange. Therefore the energy function is of a more complicated nature than was assumed by Ising; it depends not only on the arrangement of the elementary magnets, but also on the speed with which they exchange their places. *The Ising model is therefore now only of mathematical interest* [emphasis added].

[11] (p. 48) In the words of Fisher [105]:

[I]t is appropriate to ask what the main aim of theory should be. This is sometimes held (implicitly or explicitly) to be the calculation of the observable properties of a system from first principles using the full microscopic quantum-mechanical description of the constituent electrons, protons and neutrons. Such a calculation, however, even if feasible for a many-particle system which undergoes a phase transition need not and, in all probability, would not increase one’s understanding of the observed behaviour of the system. Rather, the aim of the

theory of a complex phenomenon should be to elucidate which general features of the Hamiltonian of the system lead to the most characteristic and typical observed properties. Initially one should aim at a broad qualitative understanding, successively refining one's quantitative grasp of the problem when it becomes clear that the main features have been found.

### Chapter 3

[1] (p. 87). It is known that subadditivity is not sufficient to prove convergence along arbitrary sequences  $\Lambda_n \uparrow \mathbb{Z}^d$  and that it has to be replaced by *strong subadditivity*, see [148]. Subadditivity is however sufficient to prove convergence *in the sense of Fisher*, that is, for sequences  $\Lambda_n \uparrow \mathbb{Z}^d$  such that, for all  $n \geq 1$ , there exists a cube  $K_n$  such that  $\Lambda_n \subset K_n$  and  $\sup_n |K_n|/|\Lambda_n| < \infty$ .

[2] (p. 104). The statement of Theorem 3.25 does not indicate what happens at the critical point  $(\beta, h) = (\beta_c(d), 0)$ . In that case, one can prove that, in all dimensions  $d \geq 2$ , uniqueness holds. In dimension 2, this can be proved in many ways; see, for example, [350]. In dimension  $d \geq 4$ , the proof is due to Aizenman and Fernández [7]. The case of dimension 3 was treated recently by Aizenman, Duminil-Copin and Sidoravicius [8]. Both are based on the random-current representation, a geometric representation of the Ising model which we briefly present in Section 3.10.6.

[3] (p. 108). Much is known about the decay of correlations in the Ising model. In two dimensions, explicit computations show that  $\langle \sigma_0 \sigma_x \rangle_{\beta,0}$  decays exponentially in  $\|x\|_2$  for all  $\beta \neq \beta_c(2)$  (with a rate that can be determined) and that  $\langle \sigma_0 \sigma_x \rangle_{\beta_c(2),0} \approx \|x\|_2^{-1/4}$ ; see, for instance, [239, 261].

In any dimension, Aizenman, Barsky and Fernández have proved that there is exponential decay of the 2-point function  $\langle \sigma_0 \sigma_x \rangle_{\beta,0}$  for all  $\beta < \beta_c(d)$  [5]. In the same regime, it is actually possible to prove [60] that the 2-point function has *Ornstein–Zernike* behavior:  $\langle \sigma_0 \sigma_x \rangle_{\beta,0} \simeq \Psi_\beta(x/\|x\|_2) \|x\|_2^{-(d-1)/2} e^{-\xi_\beta(x/\|x\|_2)\|x\|_2}$ , as  $\|x\|_2 \rightarrow \infty$ , where  $\Psi_\beta$  and  $\xi_\beta$  are positive, analytic functions.

Sakai proved [292] that  $\langle \sigma_0 \sigma_x \rangle_{\beta_c(d),0} \sim c_d \|x\|_2^{2-d}$ , for some constant  $c_d$ , in large enough dimensions  $d$ .

In the remaining cases, the 2-point function remains uniformly bounded away from 0 (by the FKG inequality), and the relevant quantity is the truncated 2-point function  $\langle \sigma_i; \sigma_j \rangle_{\beta,h}^+ \stackrel{\text{def}}{=} \langle \sigma_i \sigma_j \rangle_{\beta,h}^+ - \langle \sigma_i \rangle_{\beta,h}^+ \langle \sigma_j \rangle_{\beta,h}^+$ .

It is known that  $\langle \sigma_i; \sigma_j \rangle_{\beta,0}^+$  decays exponentially for all  $\beta > \beta_c(d)$  in dimension 2 [67]. A proof for sufficiently low temperatures is given in Section 5.7.4.

Finally,  $\langle \sigma_i; \sigma_j \rangle_{\beta,h}^+$  decays exponentially for all  $h \neq 0$  [95]; a simple geometric proof relying on the random-current representation can be found in [172].

[4] (p. 121). Even this statement should be qualified, since procedures allowing an experimental observation of the effect of complex values of physical parameters have recently been proposed and implemented. We refer the interested readers to [268] for more information.

[5] (p. 151). These two conjectures are supported by proofs of a similar behavior for a simplified model of the interface, known as the SOS (or Solid-On-Solid) model. For the latter, the analogue of the first claim above is already highly nontrivial, and was proved in [122], while the second claim was proved in [52].

[6] (p. 151). In sufficiently large dimensions (conjecturally: for all dimensions  $d \geq 3$ ), there exists  $\beta_p(d) > \beta_c(d)$  such that the  $-$  spins percolate under  $\mu_{\beta,0}^+$  for all

$\beta \in (\beta_c(d), \beta_p(d))$  [6]. As a consequence, Peierls contours, and in particular the interface as defined in Section 3.10.7, are not very relevant anymore. One should then consider analogous objects defined on a coarser scale. For example, one might partition  $\mathbb{Z}^d$  into blocks of  $R \times R$  spins, with  $R \uparrow \infty$  as  $\beta \downarrow \beta_c(d)$ . A block would then be said to be of type + if the corresponding portion of configuration is “typical of the + phase”, of type – if it is “typical of the – phase”, and of type 0 otherwise. Provided one defines these notions in a suitable way, then + and – blocks are necessarily separated by 0 blocks, and one can define contours as connected components of 0 blocks. If  $R$  diverges fast enough as  $\beta \downarrow \beta_c(d)$ , then this notion of contours makes sense for all  $\beta > \beta_c(d)$ . We refer to [276] for an explicit example of such a construction.

[7] (p. 159) It can be shown, nevertheless, that the series (3.98) provides an **asymptotic expansion** for  $\psi_\beta$  at  $h = 0$ :

$$|\psi_\beta(h) - \sum_{k=0}^n a_k h^k| = o(h^n), \quad \forall n \geq 1.$$

## Chapter 4

[1] (p. 168) These were made in Van der Waals’ thesis [339].

[2] (p. 184) In Section 6.14.1, we give a sketch of one way by which equivalence can be approached for systems with interactions. We refer to the papers of Lanford [205] and of Lewis, Pfister and Sullivan [222, 223] for a much more complete and general treatment.

## Chapter 6

[1] (p. 252). This statement should be qualified. Indeed, there are very specific cases in which such an approach allows one to construct infinite-volume Gibbs measures. The main example concerns models on trees (instead of lattices such as  $\mathbb{Z}^d$ ,  $d \geq 2$ ). In such a case, the absence of loops in the graph makes it possible to compute explicitly the marginal of the field in a finite subset. Roughly speaking, it yields explicit (finite) sets of equations, each of whose solutions correspond to one possible compatible family of marginals. In this way, it is possible to have multiple infinite-volume measures, even though one is still relying on Kolmogorov’s extension theorem. A general reference for Gibbs measures on trees is [288]; see also [134, Chapter 12].

[2] (p. 263). The equivalence in Lemma 6.21 does not always hold if the single-spin space is not finite. When working on more general spaces, it turns out that quasilocal, rather than continuous, functions are the natural objects to consider. Note that the equivalences stated in Exercise 6.12 also fail to hold in general.

[3] (p. 266). In fact, the class of specifications that can be constructed in this way is very general. A specification  $\pi$  is **non-null** if, for all  $\Lambda \Subset \mathbb{Z}^d$  and  $\omega \in \Omega$ ,  $\pi(\eta_\Lambda | \omega) > 0$  for all  $\eta_\Lambda \in \Omega_\Lambda$ . (An alternative terminology, often used in the context of percolation models, is that the specification  $\pi$  has **finite energy**.) It can then be shown [134, Section 2.3] that, if  $\pi$  is quasilocal and non-null, then there exists an absolutely summable potential  $\Phi$  such that  $\pi^\Phi = \pi$ . This result is known as the *Kozlov–Sullivan* theorem.

[4] (p. 266) This counter-example was taken from [134].

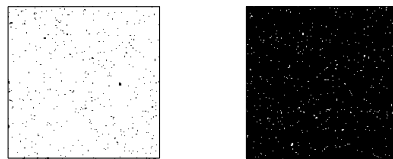
[5] (p. 275) The fact that a phase transition occurs in this model was proved by Dyson [99] when  $-1 < \epsilon < 0$  and by Fröhlich and Spencer [123] when  $\epsilon = 0$ .

[6] (p. 279). A proof can be found, for example, in [134, Section 14.A].

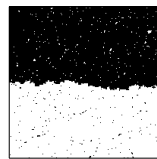
[7] (p. 283) The use of the operations  $r_\Lambda$  and  $t_\Lambda^\pi$  is taken from [278].

[8] (p. 286). The argument in Example 6.64 is due to Miyamoto and first appeared in his book [249] (in Japanese). The argument was rediscovered independently by Coquille [72].

[9] (p. 290). This statement should be slightly nuanced. For concreteness, let us consider the two-dimensional Ising model with  $\beta > \beta_c(2)$  and  $h = 0$ . On the one hand, when the free boundary condition is chosen, typical configurations show the box  $B(n)$  to be entirely filled with either the  $+$  phase or the  $-$  phase, both occurring with equal probability:



On the other hand, when Dobrushin boundary condition is applied (see the discussion in Section 3.10.7), typical configurations display coexistence of both  $+$  and  $-$  phases, separated by an interface:



Nevertheless, letting  $n \rightarrow \infty$ , both these sequences of finite-volume Gibbs distributions converge to the same Gibbs measure  $\frac{1}{2}\mu_{\beta,0}^+ + \frac{1}{2}\mu_{\beta,0}^-$ . So, even though all the physics in the latter measure is already present in the two extremal measures  $\mu_{\beta,0}^+$  and  $\mu_{\beta,0}^-$ , the physical mechanism leading to this particular non-extremal Gibbs state are very different. In this sense, there can be hidden physics behind the coefficients of the extremal decomposition.

[10] (p. 308) The non-uniqueness criterion presented in Section 6.11 was inspired by [22].

[11] (p. 311) In dimension 2, this follows, for example, from the explicit expression (3.14) for the pressure at  $h = 0$ . For general dimensions, the claim follows from the fact that continuity of the magnetization implies differentiability of the pressure with respect to  $\beta$  [218] and the results on continuity of the magnetization [352, 7, 27, 8].

[12] (p. 311) This result can be found in [289, Theorem 5.6.2].

## Chapter 7

[1] (p. 323) A detailed analysis of this model can be found in [35].

[2] (p. 323) This model was first studied by Blume [25] and Capel [61].

[3] (p. 333) More specifically, the problem of determining the ground states of a lattice model can be shown to fall, in general, in the class of NP-hard problems. For a discussion of this notion in the context of statistical mechanics, we recommend the book [245].

[4] (p. 345) This trick is known as the “Minlos–Sinai trick”, and seems to have appeared first in [248].

## Chapter 9

[1] (p. 416). The first example of a two-dimensional  $O(N)$ -symmetric model with several infinite-volume Gibbs measures at low temperature was provided by Shlosman [304]. His model has  $N = 2$  and formal Hamiltonian

$$-\beta \sum_{\substack{i, j \in \mathbb{Z}^2 \\ \|j-i\|_2 = \sqrt{2}}} \cos(\vartheta_i - \vartheta_j) + \beta J \sum_{\substack{i, j \in \mathbb{Z}^2 \\ \|j-i\|_2 = 1}} \cos(2(\vartheta_i - \vartheta_j)),$$

where  $J$  is nonnegative, and we have written  $\mathbf{S}_i = (\cos \vartheta_i, \sin \vartheta_i)$  for the spin at  $i \in \mathbb{Z}^2$ . The crucial feature of this model is that, in addition to the  $SO(2)$ -invariance of the Hamiltonian, the latter is also preserved under the simultaneous transformation

$$\vartheta_i \mapsto \vartheta_i, \quad \vartheta_j \mapsto \vartheta_j + \pi,$$

for all  $i \in \mathbb{Z}_{\text{even}}^2 \stackrel{\text{def}}{=} \{i = (i_1, i_2) \in \mathbb{Z}^2 : i_1 + i_2 \text{ is even}\}$  and  $j \in \mathbb{Z}_{\text{odd}}^2 \stackrel{\text{def}}{=} \mathbb{Z}^2 \setminus \mathbb{Z}_{\text{even}}^2$ . It is this discrete symmetry that is spontaneously broken at low temperatures, yielding two Gibbs measures under which, in typical configurations, either most nearest-neighbor spins differ by approximately  $\pi/2$ , or most differ by approximately  $-\pi/2$ ; see Figure A.1.

[2] (p. 424), A proof can be found, for example, in [111].

## Chapter 10

[1] (p. 438) Extensions of Pirogov–Sinai theory covering some models with continuous spins can be found, for example, in [168], [87] and [355].

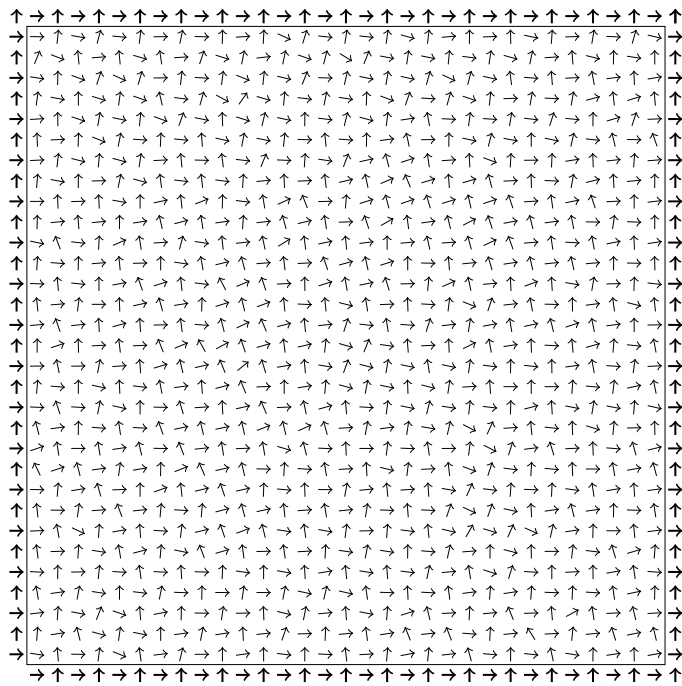


Figure A.1: A typical low-temperature configuration of the model in Note 1 of Chapter 9.

