Corrections

We list here typos and errors that have been found in the published version of the book. We welcome any additional corrections you may find! We indicate page and line numbers for the published version of the book; negative line numbers are counted from the bottom of the page. Most typos have already been corrected in the version available on our web page (the address is given in the footer). When this has not been done, the relevant pages are indicated in brackets.

- **p. 6, l. 5:** $\big|_{\widetilde{U}=\overline{U}_1}$ should be $\big|_{\widetilde{U}_1=\overline{U}_1}$.
- **p. 10, l. −9:** $-\frac{R}{v}\mathrm{d}v$ should be $+\frac{R}{v}\mathrm{d}v$.
- **p. 10, l. −7:** $s(u,v)$ should be $s(e,v)$ and $-R\log(v/v_0)$ should be $+R\log(v/v_0)$.
- **p. 20, l. −1:** litterature should be literature.
- **p. 30, l. −1:** Ω_{Λ,N_k} should be Ω_{Λ_k,N_k}.
- **p. 31, l. 11:** see **the** exercise below.
- **p. 32, l. 11:** the equation should read

$$\lim \frac{1}{2V}\mathsf{S}_{\mathrm{Boltz}}(\Lambda;N) = \mathsf{s}^{\mathrm{hard}}_{\mathrm{Boltz}}(\overline{\rho}) \stackrel{\mathrm{def}}{=} -\overline{\rho}\log\overline{\rho} - (1-\overline{\rho})\log(1-\overline{\rho})\,.$$

- **p. 65, l. 4, 8 and 13:** min should be inf.
- **p. 66, (2.11):** The factor $N^{1/2}$ can be removed from the upper bound. The same is true when applying this bound just after Exercise 2.1.
- **p. 66, l. −2:** min should be inf.

We thank all readers who have sent us their corrections, among which (our sincere apologies to those we forgot): Costanza Benassi, Quentin Berthet, Adolfo Camacho Melo, Tecla Cardilli, Loren Coquille, Yair Daon, Margherita Disertori, Hugo Duminil-Copin, Arved Niklas Fanta, Maëllie Godard, Christian Hansch, Qidong He, Roman Kotecký, Benjamin Lees, Antoine Maier, Mirmukhsin Makhmudov, Mauro Mariani, Philippe Moreillon, Sébastien Ott, Ron Peled, Elias Reutelsterz, Costanza Rojas-Molina, Sylvie Roelly, Vedran Sohinger, Daniel Ueltschi, Zichun Ye.

- **p. 70, Fig. 2.4:** $-h/\beta$ should be $-h/(2d\beta)$.
- **p. 71, l. −3:** ψ_β should be ψ_β^{CW}.
- **p. 72, l. 5:** min should be inf.
- **p. 75, l. −8:** $2/N$ can be replaced by $1/N$. In the following line, $4d\beta/N$ can be replaced by $2d\beta/N$.
- **p. 83, l. −2:** "such that $\sum_{r=1}^d |(i_r - j_r) \mod n|$" should be "such that $i-j$ has only one nonzero component and the latter is equal to ± 1 modulo n".
- **p. 90, l. 3:** k_0 is actually independent of h.
- **p. 91, l. 8:** the notation $\mathscr{H}_\Lambda^\eta$ has never been introduced (although its meaning should be pretty clear from the context).
- **p. 92, l. 1:** $\Lambda \subset \mathbb{Z}^d$ should be $\Lambda \Subset \mathbb{Z}^d$.
- **p. 93, l. −10:** magnetization should be magnetization density.
- **p. 103, l. −1:** “prove that $\langle \sigma_A \rangle^+_{\Lambda;\mathbf{J},\mathbf{h}}$ is increasing” should be “prove that $\langle \sigma_A \rangle^+_{\Lambda;\mathbf{J},\mathbf{h}}$ and $\langle \sigma_A \rangle^\varnothing_{\Lambda;\mathbf{J},\mathbf{h}}$ are nondecreasing”.
- **p. 103, l. −1:** increasing should be nondecreasing.
- **p. 108, l. 10:** $\Delta_{2k-1} \subset \{\Lambda_n^1 : n \ge 1\}$, $\Delta_{2k} \subset \{\Lambda_n^2 : n \ge 1\}$ should be $\Delta_{2k-1} \in \{\Lambda_n^1 : n \ge 1\}$, $\Delta_{2k} \in \{\Lambda_n^2 : n \ge 1\}$.
- **p. 112, l. −5:** $|\langle \sigma_0 \rangle^+_{\Lambda_n;\beta,h}| \le 1$ should be $|\langle \sigma_i \rangle^+_{\Lambda_n;\beta,h}| \le 1$.
- **p. 116:** The code for Figure 3.10 has some bug that make it looks rather poor.
- **p. 121, l. 1:** “There exists” should be “There exist”.
- **p. 122, l. −6:** it should read “$\log\mu \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log C_n$”.
- **p. 129, l. −3:** “with the free boundary condition” should be “with free boundary condition”.
- **p. 130, l. 8:** “Observing that” should be “Since”.
- **p. 136, l. −7:** “For any $A, B \subset \Lambda$” should be “For any $A \subset \Lambda$”. Moreover, the claim in this exercise (Exercise 3.31) can be strengthened, see the additional exercises for Chapter 3 below.
- **p. 144, l. −9:** This should be replaced by

$$\frac{\sum_{\omega\in\Omega_\Lambda^+} \omega_0 \exp\bigl\{h\sum_{j\in\Lambda}\sigma_j + \beta\sum_{\{j,k\}\in\mathscr{E}_\Lambda^{\mathrm{b}}\setminus\{0,i\}} \omega_j\omega_k\bigr\}\bigl(1+\omega_0\omega_i\tanh\beta\bigr)}{\sum_{\omega\in\Omega_\Lambda^+} \exp\bigl\{h\sum_{j\in\Lambda}\sigma_j + \beta\sum_{\{j,k\}\in\mathscr{E}_\Lambda^{\mathrm{b}}\setminus\{0,i\}} \omega_j\omega_k\bigr\}\bigl(1+\omega_0\omega_i\tanh\beta\bigr)}.$$

- **p. 147, l. −5:** $\tilde\omega_j(n)$ should be $\tilde\omega_j$.
- **p. 148, l. 4:** $\tilde\omega_j(n)$ should be $\tilde\omega_j$ and $\omega_j(n)$ should be ω_j.
- **p. 151, l. 18:** $2^{N_\Lambda^{\mathrm{w}}(E)}$ should be $2^{N_\Lambda^{\mathrm{w}}(E)-1}$.

- **p. 155:** In the equation following (3.82), $\{i,j\}$ should be $\{i\}$ (three times) and $i \xleftrightarrow{\mathbf{m}} j$ should be $i \xleftrightarrow{\mathbf{m}} \partial^{\mathrm{ex}}\Lambda$.
- **p. 156, l. 6:** $\mathbb{P}_{\Lambda;\beta}$ should be $\mathbb{P}^{+}_{\Lambda;\beta}$ (6 times).
- **p. 161, l. 11:** Theorem 3.59 should be Theorem 3.58. A couple of lines below, $[\beta_{\mathrm{c}}(d-1),\beta_{\mathrm{c}}(d))$ should be $(\beta_{\mathrm{c}}(d),\beta_{\mathrm{c}}(d-1)]$ and $[\beta_{\mathrm{c}}(2),\beta_{\mathrm{c}}(3))$ should be $(\beta_{\mathrm{c}}(3),\beta_{\mathrm{c}}(2)]$.
- **p. 164, l. 4:** $e^{-c_1 n}$ should be $e^{-c_1 R}$.
- **p. 181, l. 9:** In (4.5), $f_{\Lambda_n;\beta}(\rho)$ should be $f_{\Lambda_n;\beta}(N_n/|\Lambda_n|)$. 3 lines below, one might specify explicitly that $\rho \mapsto f_\beta(\rho)$ is continuous and convex on $[0,1]$.
- **p. 183, l. 5:** $\sqrt{2\pi\frac{N}{|\Lambda|}(1-\frac{N}{|\Lambda|})}$ should be $\sqrt{2\pi N(1-\frac{N}{|\Lambda|})}$.
- **p. 183, l. −3:** $\log(\frac{1-\rho}{\rho})$ should be $|\log(\frac{1-\rho}{\rho})|$.
- **p. 184, l. 15:** Proposition B.9 should be Lemma B.11.
- **p. 184, l. −4:** In (4.23), $\beta N_\Lambda(\eta)$ should be $\mu N_\Lambda(\eta)$.
- **p. 185, l. 3:** "β" should be "$\beta \geq 0$".
- **p. 185, l. −8** $\frac{\partial p_\beta}{\partial\mu}$ should be $\frac{\partial p_\beta}{\partial\mu}(\mu)$.
- **p. 186, l. 12, 14, 15:** $c(1-c)$ should be c^2.
- **p. 189, l. 5:** "interval" should be "closed interval".
- **p. 189, l. 13:** Theorem 4.5 should be Theorem 4.13.
- **p. 191, l. 13:** The sets in $\mathscr{D}_\alpha(\Lambda)$ should be translates of each other.
- **p. 192, l. 9:** $N'_{\max}$ should be $N''_{\max}$.
- **p. 196, l. −7:** (4.1) should be "the exponent in (4.22)".
- **p. 198, l. 12:** $\frac{1}{4}\beta_{\mathrm{c}}(d)$ should be $4\beta_{\mathrm{c}}(d)$.
- **p. 207:** The paragraph following (4.62) should read as follows:

  Once more, for fixed β, once v is taken large enough, $\widehat{p}^{\mathrm{vW}}_\beta(v)$ is well approximated by the solution to

  $$\Big(p + \big(\tfrac{1}{2} - \tfrac{1}{2\beta}\big)\frac{1}{v^2}\Big)(v-1) = \beta^{-1},$$

  which is essentially van der Waals' expression (1.23), with $a = \frac{1}{2} - \frac{1}{2\beta}$ and $b = 1$.

- **p. 207, l. −8:** s_β should be $s^{\mathrm{l.g.}}$.
- **p. 208, Fig. 4.16:** The maximizers should be expressed in terms of the variable x, not the density ρ (so, ρ_g should be x_g, etc.).

- **p. 208, l. 3:** f_β should be f_β^{vW} (twice).
- **p. 208, l. −4:** $\rho_\beta(\mu)$ should be $\rho_\beta^{\mathrm{vW}}(\mu)$.
- **p. 210, l. 4:** p_β should be p_β^{vW}. After (4.66), s should be $s^{\mathrm{l.g.}}$ (9 times!).
- **p. 214, l. 13**: "a cube of sidelength $\ell 2^n$, given by" should be "a cube given by".
- **p. 221, l. 15:** $e^{-\mathscr{H}_{\Lambda;\mu_*}(\eta)}$ should be $e^{-\beta\mathscr{H}_{\Lambda;\mu_*}(\eta)}$ (twice).
- **p. 260, l. 9 and 11:** $X \ni v$ should be $\overline{X} \ni v$ (twice).
- **p. 262, l. 5:** the semicolon should be a comma.
- **p. 264, l. 15:** decribe should be describe.
- **p. 265:** In the second part of Notation 6.2, Λ is a subset of S.
- **p. 266:** In Exercise 6.2, countably should be at most countably.
- **p. 267:** In Lemma 6.3, one should add that φ is measurable.
- **p. 276, l. 15:** $A \in \mathscr{F}_\Lambda$ should be $A \in \mathscr{F}$.
- **p. 277, l. −11:** the comma should be a colon.
- **p. 279, l. 14 and 16:** the sign before the term $\mathscr{H}_\Delta(\eta_\Delta \tau''_{\Lambda\setminus\Delta}\omega_{\Lambda^c})$ is incorrect (twice).
- **p. 280:** The two lines before Definition 6.18 should belong to a new paragraph.
- **p. 284 [p. 265]:** In the caption of Fig. 6.1, one can assume, for simplicity, that A has its support in Λ.
- **p. 287, l. −1:** it would be better to use η_i rather than ω_i in the right-hand side.
- **p. 290, l. −2:** $k \geq i$ should be $k > i$.
- **p. 291, l. 2:** the last identity should be an inequality (since we are taking the supremum over i.)
- **p. 291, l. 4:** the factor a_k should not be present.
- **p. 291, l. 13:** Theorem 6.35 should be Theorem 6.31.
- **p. 293, l. 6:** (6.46) should be (6.37).
- **p. 301, l. 8:** One should replace the assumption that $\mathscr{G}(\pi) \neq \varnothing$ by the assumption that π is quasilocal. This guarantees that $\mathscr{G}(\pi)$ is a closed subset of $\mathscr{M}_1(\Omega)$ (Lemma 6.27). (The condition $\mathscr{G}(\pi) \neq \varnothing$ is not needed anymore by Theorem 6.26.)
- **p. 308, l. 12:** Corollary 3.60 should be Theorem 3.60.
- **p. 389, l. 6:** $\mathbf{Z}^\#(\Lambda) = e^{-\mathrm{e}^\#|\Lambda|}\Xi^\#(\Lambda)$ should be $\mathbf{Z}^\#(\Lambda) = e^{-\beta\mathrm{e}^\#|\Lambda|}\Xi^\#(\Lambda)$. The same is true 9 lines below.

- **p. 390, l. 6:** $e^{-\beta\|\gamma\|}$ should be $e^{-\beta\|\gamma'\|}$.
- **p. 391, l. 1:** $\widehat{\psi}^{\#}_{\text{stable}}$ should be $\widehat{\psi}^{\#}_{n,\text{stable}}$.
- **p. 434, l. −4:** Replace "Let $\epsilon > 0$ be such that $e^{\epsilon}/(1+m^2) < 1$ and n be large enough to ensure that $|\eta_i| \le e^{\epsilon n}$ for all $i \in \partial^{\text{ext}}\mathsf{B}(n)$. Then," by "Let $\epsilon > 0$ be such that $e^{\epsilon}/(1+m^2) < 1$ and, for all n large enough, $|\eta_i| \le e^{\epsilon n}$ for all $i \in \partial^{\text{ext}}\mathsf{B}(n)$. Then, for all such n,".
- **p. 462, l. −15:** the exterior-most expectation is missing the proper indices $\langle\cdot\rangle^{\eta}_{\mathsf{B}(N)}$ (twice).
- **p. 462, l. −7:** $\tilde{J}_{oj}, J_{oj}$ should be $\tilde{J}_{0j}, J_{0j}$.
- **p. 470, l. 17:** the comma before Namely should be a colon.
- **p. 480, l. 2:** $J^m(i,\Theta(i))$ should be $J^m_{i,\Theta(i)}$.
- **p. 499-500:** In (10.41) and in the statement of Theorem 10.25, $\dfrac{\nu}{4d}$ should be $\dfrac{\nu}{4d(2\pi)^d}$ (three occurrences). Moreover, in (10.41), $\dfrac{1}{|\mathbb{T}_L|}$ should be $\dfrac{(2\pi)^d}{|\mathbb{T}_L|}$.
- **p.521, l. −2 [p. 484, l. −3]:** The functions $(f_n)_{n\ge1}$ should be uniformly strongly convex, in the sense that $\inf_{n\ge1}\inf_{x\in\mathbb{R}} f_n''(x) > 0$.
- **p. 528, l. −3:** One should add that φ satisfies $\varphi(\omega_0) = z_0$.
- **p. 574, l. −6:** $c(1-c)$ should be c^2.
- **p. 579, l. 8:** $2d\sum_{k\ge r} k^{d-1}2^{-k}$ should be $2d\sum_{k>r}(2k+1)^{d-1}2^{-k}$.
- **p. 579, l. −17:** remove "a sequence $(\delta_n)_{n\ge1}$ decreasing to 0".
- **p. 580, l. −4:** Exercise 6.65 should be Exercise 6.23.
- **p. 582, l. 6:** Theorem 6.27 should be Lemma 6.27.
- **General:** We incorrectly (for reasons we don't remember) write Marc Kac's name as Kać in the book.

Additional exercises

Here, we suggest a variety of additional relevant exercises. To facilitate using the latter for teaching, we do not plan on providing solutions.

Chapter 3

▶ The following two exercises are complements to Section 3.3.

Exercise. Consider the one-dimensional Ising model on the torus $\mathbb{T}_n$. Show that

$$\langle\sigma_0\rangle^{\mathrm{per}}_{V_n;\beta,h} = \frac{\partial}{\partial h}\psi^{\mathrm{per}}_{V_n}(\beta,h)$$

and deduce an expression for $\langle\sigma_0\rangle_{\beta,h} = \lim_{n\to\infty}\langle\sigma_0\rangle^{\mathrm{per}}_{V_n;\beta,h}$.

Exercise. Let $\Lambda_n = \{-n,\dots,n\}$. Using the transfer matrix, compute $\psi^{+}_{\Lambda_n}(\beta,h)$, $\langle\sigma_0\rangle^{+}_{\Lambda_n;\beta,h}$ and $\langle\sigma_0\sigma_i\rangle^{+}_{\Lambda_n;\beta,h}$, as well as the corresponding limits as $n\to\infty$.

▶ The following exercises and remark are complements to Lemma 3.31:

Exercise. Prove that, in any dimension d, the function $h\mapsto\langle\sigma_0\rangle^{+}_{\beta,h}$ is strictly increasing on $\mathbb{R}$ and that, for all $\Lambda\subset\mathbb{Z}^d$ containing 0, all $\beta\geq 0$ and all $h\in\mathbb{R}$,

$$\tanh(h)\leq\langle\sigma_0\rangle^{+}_{\Lambda;\beta,h}\leq\tanh(h+4d\beta).$$

Hint: Use FKG for the upper bound. Then, show that $\frac{\partial}{\partial h}\langle\sigma_0\rangle^{+}_{\Lambda;\beta,h}\geq 1-(\langle\sigma_0\rangle^{+}_{\Lambda;\beta,h})^2$. Deduce from that both the lower bound and the strict monotonicity.

Remark. An improved upper bound is provided in Theorem 3.53. When h is large, improved estimates follow from the expansion in Theorem 5.10. Moreover, related results are established in Theorem 4.12 and Proposition 10.16. ◇

Exercise. Prove that, in any dimension d, the function $\beta\mapsto\langle\sigma_0\rangle^{+}_{\beta,h}$ is strictly increasing when $h>0$ or when $h=0$ and $\beta>\beta_{\mathrm{c}}(d)$.

► The following exercise is a complement to Section 3.7.2. Remember that the q-state Potts model ($q \geq 2$) has spins taking value in the set $\{0, \dots, q-1\}$ and a Hamiltonian given by $-\beta \sum_{\{i,j\} \in \mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}} \delta_{\sigma_i \sigma_j}$. The corresponding Gibbs measure in $\mathsf{B}(n)$ with boundary condition identically equal to 0 is denoted by $\mu^{q-\mathrm{Potts},0}_{\mathsf{B}(n);\beta}$.

Exercise.

1. Check that the 2-state Potts model reduces to the Ising model at inverse temperature $\beta/2$.
2. Check that the Hamiltonian of the Potts model is invariant under permutation of the spin values.
3. Adapt Peierls' argument to the 2-dimensional Potts model, in order to show that, for all β large enough,
$$\liminf_{n\to\infty} \mu^{q-\mathrm{Potts},0}_{\mathsf{B}(n);\beta}(\sigma_0 = 0) > \frac{1}{q},$$
which shows that the symmetry under permutation is broken at low temperatures. Hint: One way to proceed is to use contours that keep some information on the values of the spins on each side.

► The following exercise is a complement to Exercise 3.31.

Exercise. Deduce from Exercise 3.31, under the same assumptions, that
$$|\langle \sigma_A \rangle_{\Lambda;\mathbf{K}}| \leq \langle \sigma_A \rangle_{\Lambda;\mathbf{K}'}.$$

Chapter 9

▶ The following exercise is a complement to Theorem 9.14.

Exercise. The goal of this exercise is to provide an alternative proof of exponential decay of the 2-point function of $\mathsf{O}(N)$ models at sufficiently high temperature. It relies on a suitable high-temperature expansion. Fix $0, k \in \mathbb{Z}^d$ and take n large enough to ensure that $0, k \in \mathsf{B}(n)$.

1. Writing $e^{\beta \mathbf{S}_i \cdot \mathbf{S}_j} = e^{-\beta}(e^{\beta(\mathbf{S}_i \cdot \mathbf{S}_j + 1)} - 1 + 1)$ and expanding the product over the edges of $\mathsf{B}(n)$, show that

$$|\langle \mathbf{S}_0 \cdot \mathbf{S}_k \rangle^{\varnothing}_{\mathsf{B}(n);\beta}| \leq \mathbb{Q}_{\mathsf{B}(n);\beta}(0 \longleftrightarrow k),$$

   where $\mathbb{Q}_{\mathsf{B}(n);\beta}$ is the probability measure on subsets of $\mathscr{E}_{\mathsf{B}(n)}$ defined by

$$\mathbb{Q}_{\mathsf{B}(n);\beta}(E) \propto \int_{\Omega_{\mathsf{B}(n)}} \prod_{\{u,v\} \in E} \bigl(e^{\beta(\mathbf{S}_u(\omega) \cdot \mathbf{S}_v(\omega) + 1)} - 1\bigr) \prod_{i \in \mathsf{B}(n)} \mathrm{d}\omega_i$$

   and $0 \longleftrightarrow k$ denotes the event that 0 and k are connected.

2. Let $\mathbf{E}$ be a random subset of $\mathscr{E}_{\mathsf{B}(n)}$ distributed according to $\mathbb{Q}_{\mathsf{B}(n);\beta}$. Show that, for any $e \in \mathscr{E}_{\mathsf{B}(n)}$ and $E \subset \mathscr{E}_{\mathsf{B}(n)} \setminus \{e\}$, $\mathbb{Q}_{\Lambda;\beta}\bigl(e \in \mathbf{E} \bigm| \mathbf{E} \setminus \{e\} = E\bigr) \leq e^{2\beta} - 1$.

3. Deduce from this that there exists $\beta_0 > 0$ such that, for any $\beta < \beta_0$,

$$\mathbb{Q}_{\mathsf{B}(n);\beta}(0 \longleftrightarrow k) \leq e^{-c\|k\|_2}$$

   for some $c = c(\beta) > 0$ uniformly in n.