The Fibonacci Hamiltonian

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Spectra and L^2-Invariants

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The Fibonacci Hamiltonian

The Fibonacci Hamiltonian is the bounded self-adjoint operator

$$[H_{\lambda,\omega}^{(\mathrm{Fib})}\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha+\omega \mod 1)\psi(n)$$

in $\ell^2(\mathbb{Z})$, with the coupling constant $\lambda > 0$ and the phase $\omega \in \mathbb{T}$.
The frequency is given by $\alpha = \frac{\sqrt{5}-1}{2}$. This operator has been studied in a large number of papers since the early 1980's.
It is of interest for a variety of reasons:

- It is the central model in the study of one-dimensional quasicrystals.
- It is one of the few explicit Schrödinger operators with non-trivial spectral analysis that can be carried out more or less completely.
- It builds a bridge between spectral theory and dynamical systems.
- The orthogonal polynomials display unusual behavior.

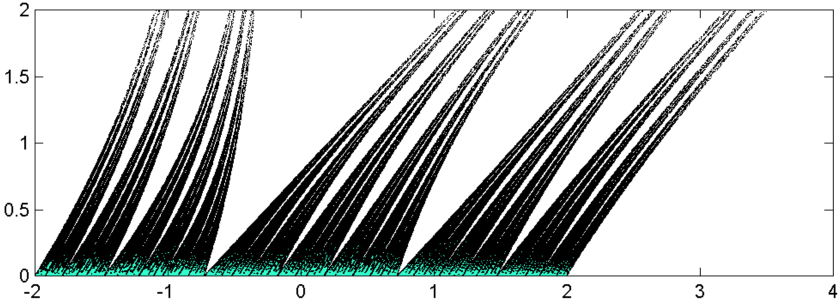
The Spectrum

The spectrum $\sigma(H_{\lambda,\omega}^{(\mathrm{Fib})}) \subset \mathbb{R}$ is easily seen to be independent of ω and may therefore be denoted by Σ_λ.

Theorem (Sütő 1989)

For every $\lambda > 0$, Σ_λ is a Cantor set of zero Lebesgue measure.

Here is a plot of (a numerical approximation of) Σ_λ for $0 \leq \lambda \leq 2$:



The Trace Map Formalism

Recall that the potential takes the form

$$V_{\lambda,\omega}(n) = \lambda \chi_{[1-\alpha,1)}(n\alpha + \omega \mod 1)$$

Let us consider the case $\omega = 0$ and form the standard transfer matrices

$$M(n, E) = \begin{pmatrix} E - V_{\lambda,0}(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_{\lambda,0}(1) & -1 \\ 1 & 0 \end{pmatrix}$$

Considering these matrices at Fibonacci sites,

$$M_k = M_k(E) = M(F_k, E)$$

the following remarkable identity holds:

$$M_{k+1} = M_{k-1} M_k$$

The Trace Map Formalism

Passing to half-traces, $x_k = \frac{1}{2}\mathrm{tr} M_k$, it then follows that

$$x_{k+1} = 2x_k x_{k-1} - x_{k-2}$$

We are naturally led to the definition of the Fibonacci *trace map*, which is given by

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ T(x,y,z) = (2xy - z, x, y)$$

and in particular satisfies

$$T(x_k, x_{k-1}, x_{k-2}) = (x_{k+1}, x_k, x_{k-1})$$

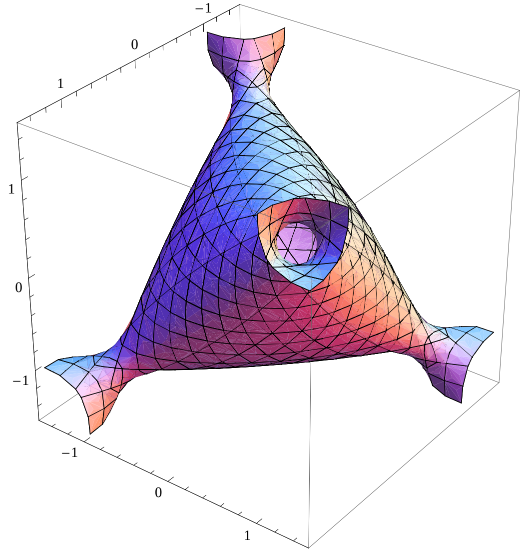
The function

$$I(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1$$

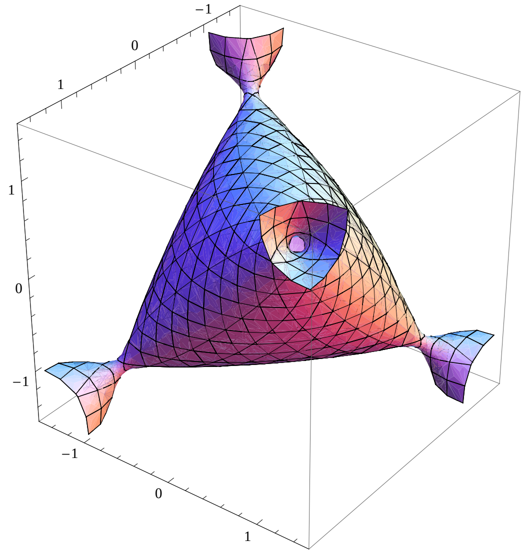
is invariant under the action of T and hence T preserves the surfaces

$$S_I = \left\{(x,y,z) \in \mathbb{R}^3 : I(x,y,z) = I\right\}$$

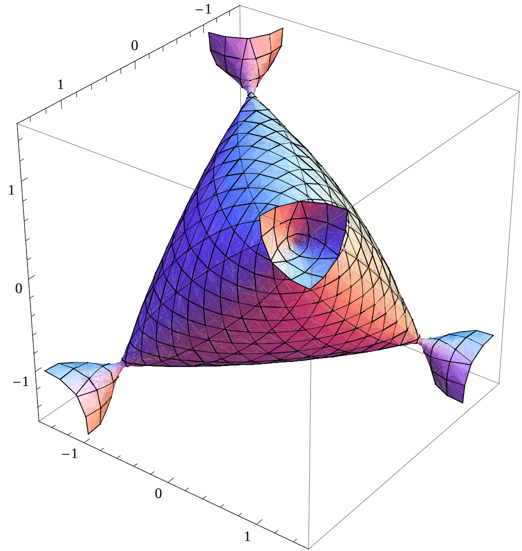
The Surface $S_{0.5}$



The Surface $S_{0.2}$



The Surface $S_{0.1}$



The Trace Map as a Surface Diffeomorphism

It is therefore natural to consider the restriction T_I of the trace map T to the invariant surface S_I. That is, $T_I : S_I \to S_I$, $T_I = T|_{S_I}$.

We denote by Λ_I the set of points in S_I whose full orbits under T_I are bounded.

Denote by ℓ_λ the line

$$\ell_\lambda = \left\{ \left(\frac{E-\lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}$$

It is easy to check that $\ell_\lambda \subset S_{\frac{\lambda^2}{4}}$.

Λ_I is a Locally Maximal Hyperbolic Set

Let us recall that an invariant closed set Λ of a diffeomorphism $f : M \to M$ is *hyperbolic* if there exists a splitting of the tangent space $T_x M = E^s_x \oplus E^u_x$ at every point $x \in \Lambda$ such that this splitting is invariant under Df, the differential Df exponentially contracts vectors from the stable subspaces $\{E^s_x\}$, and the differential of the inverse, Df^{-1}, exponentially contracts vectors from the unstable subspaces $\{E^u_x\}$.

A hyperbolic set Λ of a diffeomorphism $f : M \to M$ is *locally maximal* if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$$

It is known (Casdagli 1986, Damanik-Gorodetski 2009, Cantat 2009) that for $I > 0$, the set Λ_I is a locally maximal hyperbolic set of $T_I : S_I \to S_I$.

Spectrum and Bounded Trace Map Orbits

The key to the fundamental connection between the spectral properties of the Fibonacci Hamiltonian and the dynamics of the trace map is the following result:

Proposition (Sütő 1987)

An energy $E \in \mathbb{R}$ belongs to the spectrum of the discrete Fibonacci Hamiltonian $H_{\lambda,\omega}^{(\mathrm{Fib})}$ if and only if the positive semiorbit of the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.

In terms of the structure coming from the hyperbolicity of the set Λ_I, this can be rephrased as follows:

An energy $E \in \mathbb{R}$ belongs to the spectrum of the discrete Fibonacci Hamiltonian $H_{\lambda,\omega}^{(\mathrm{Fib})}$ if and only if the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ belongs to the stable manifold of a point in $\Lambda_{\frac{\lambda^2}{4}}$.

Spectrum and Bounded Trace Map Orbits

This suggests that a dynamical-spectral connection may be established via the stable manifolds. That is, the structure of the Cantor set $\Lambda_{\frac{\lambda^2}{4}} \subset S_{\frac{\lambda^2}{4}}$, including local and global fractal dimensions, should be connected to the structure of the Cantor set $\Sigma_\lambda \subset \mathbb{R}$. Similarly, there should be a connection between dynamically defined measures on these sets.

In order to actually implement this connection, the following result is crucial:

Theorem (D.-Gorodetski-Yessen 2016)

The stable manifold of a point in $\Lambda_{\frac{\lambda^2}{4}}$ intersects the line

$$\ell_\lambda = \left\{ \left(\frac{E-\lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}$$

transversally.

The Dynamical-Spectral Dictionary and Applications

With this transversality result in hand, DGY were able to establish several results about the Fibonacci Hamiltonian. Let us mention a few:

Theorem (D.-Gorodetski-Yessen 2016)

For every $\lambda > 0$, Σ_λ is a dynamically defined Cantor set. In particular, for every $E \in \Sigma_\lambda$ and every $\varepsilon > 0$, we have

$$\begin{aligned}\dim_H \left((E - \varepsilon, E + \varepsilon) \cap \Sigma_\lambda \right) &= \dim_B \left((E - \varepsilon, E + \varepsilon) \cap \Sigma_\lambda \right) \\ &= \dim_H \Sigma_\lambda \\ &= \dim_B \Sigma_\lambda .\end{aligned}$$

The Dynamical-Spectral Dictionary and Applications

Before we state further results, let us recall the following definitions. By the spectral theorem, there are Borel probability measures $\mu_{\lambda,\omega}$ on $\mathbb{R}$ such that

$$\langle \delta_0, g(H_{\lambda,\omega})\delta_0 \rangle = \int g(E)\, d\mu_{\lambda,\omega}(E)$$

for all bounded measurable functions g. The *density of states measure* ν_λ is given by the ω-average of these measures with respect to Lebesgue measure, that is,

$$\int_{\mathbb{T}} \langle \delta_0, g(H_{\lambda,\omega})\delta_0 \rangle \, d\omega = \int g(E)\, d\nu_\lambda(E)$$

for all bounded measurable functions g.

The distribution function of the density of states measure ν_λ is called the integrated density of states and denoted by N_λ.

The Dynamical-Spectral Dictionary and Applications

By general principles, the density of states measure is non-atomic and its topological support is Σ_λ. The fact that Σ_λ has zero Lebesgue measure therefore implies that ν_λ is singular continuous for every $\lambda > 0$.

The density of states measure can also be obtained by counting the number of eigenvalues per unit volume, in a given energy region, of restrictions of the operator to finite intervals (which explains the terminology). Indeed, for any real $a < b$,

$$\nu_\lambda(a, b) = \lim_{L \to \infty} \frac{1}{L} \#\{\text{eigenvalues of } H_{\lambda,\omega}|_{[1,L]} \text{ that lie in } (a, b)\},$$

uniformly in ω. Here, for definiteness, $H_{\lambda,\omega}|_{[1,L]}$ is defined with Dirichlet boundary conditions.

The Dynamical-Spectral Dictionary and Applications

Finally, in this case it is also true that the density of states measure ν_λ is the equilibrium measure associated with the spectrum Σ_λ in the sense of logarithmic potential theory.

We will also be interested in the optimal Hölder exponent γ_λ of ν_λ. That is, γ_λ is the unique number in $[0, 1]$ such that the following two properties hold.

1. For any $\gamma < \gamma_\lambda$ and any sufficiently small interval $I \subset \mathbb{R}$, we have $\nu_\lambda(I) < |I|^\gamma$;
2. For any $\tilde{\gamma} > \gamma_\lambda$ and any $\varepsilon > 0$, there exists an interval $I \subset \mathbb{R}$ such that $|I| < \varepsilon$ and $\nu_\lambda(I) > |I|^{\tilde{\gamma}}$.

The Dynamical-Spectral Dictionary and Applications

Theorem (D.-Gorodetski-Yessen 2016)

For every $\lambda > 0$, all gaps allowed by the gap labeling theorem are open. That is,

$$\{N_\lambda(E) : E \in \mathbb{R} \setminus \Sigma_\lambda\} = \{\{m\alpha\} : m \in \mathbb{Z}\} \cup \{1\}.$$

There are many similarities between the Fibonacci Hamiltonian and the critical almost Mathieu operator, which has the potential $V(n) = 2\cos(2\pi(n\alpha + \omega))$. For the latter, a result like the theorem above is conjectured, but still not known.

Avila-Bochi-D. showed in 2012 that all gaps allowed by the gap labeling theorem are generically open in the C^0 topology.

The Dynamical-Spectral Dictionary and Applications

Theorem (D.-Gorodetski-Yessen 2016)

For every $\lambda > 0$, the density of states measure ν_λ is exact-dimensional. Namely, for every $\lambda > 0$, the limit (called the scaling exponent of ν_λ at E)

$$\lim_{\varepsilon \downarrow 0} \frac{\log \nu_\lambda(E - \varepsilon, E + \varepsilon)}{\log \varepsilon}$$

ν_λ-almost everywhere exists and is constant.

The Dynamical-Spectral Dictionary and Applications

Theorem (D.-Gorodetski-Yessen 2016)

For every $\lambda > 0$, we have

$$\dim_H \Sigma_\lambda = \frac{h_{\mu_\lambda}}{\mathrm{Lyap}^u \mu_\lambda}$$

$$\dim_H \nu_\lambda = \dim_H \mu_{\lambda,\max} = \frac{h_{\mathrm{top}}(T_\lambda)}{\mathrm{Lyap}^u \mu_{\lambda,\max}} = \frac{\log(1+\alpha)}{\mathrm{Lyap}^u \mu_{\lambda,\max}}$$

$$\gamma_\lambda = \frac{\log(1+\alpha)}{\sup_{p \in Per(T_\lambda)} \mathrm{Lyap}^u(p)}$$

In this theorem, $\mu_{\lambda,\max}$ denotes the measure of maximal entropy of $T_\lambda|_{\Lambda_\lambda}$ and μ_λ denotes the equilibrium measure of $T_\lambda|_{\Lambda_\lambda}$ that corresponds to the potential $-\dim_H \Sigma_\lambda \cdot \log \| DT_\lambda|_{E^u} \|$.

The Dynamical-Spectral Dictionary and Applications

The identities from the previous theorem enable one to determine the asymptotics of the various quantities as $\lambda \to \infty$ or 0.

Theorem

We have

$$\lim_{\lambda\to\infty} \dim_H \Sigma_\lambda \cdot \log \lambda = \log(1+\sqrt{2}) \approx 1.83156 \log(1+\alpha)$$

$$\lim_{\lambda\to\infty} \dim_H \nu_\lambda \cdot \log \lambda = \frac{5+\sqrt{5}}{4} \log(1+\alpha) \approx 1.80902 \log(1+\alpha)$$

$$\lim_{\lambda\to\infty} \gamma_\lambda \cdot \log \lambda = 1.5 \log(1+\alpha)$$

$$\lim_{\lambda\to 0} \dim_H \Sigma_\lambda = 1$$

$$\lim_{\lambda\to 0} \dim_H \nu_\lambda = 1$$

$$\lim_{\lambda\to 0} \gamma_\lambda = \frac{1}{2}$$

The Dynamical-Spectral Dictionary and Applications

The asymptotics suggest strict inequalities, which can actually be proved, again via the dynamical-spectral connection, for every value of the coupling constant:

Theorem (D.-Gorodetski-Yessen 2016)

For every $\lambda > 0$, we have

$$\gamma_\lambda < \dim_H \nu_\lambda < \dim_H \Sigma_\lambda$$

The second inequality was conjectured by Simon.

Further Related Developments

- One can consider multi-dimensional Schrödinger operators with separable potentials built from the Fibonacci potential. This leads to the consideration of sums of $1d$ spectra and convolutions of $1d$ measures. Studying such objects is highly non-trivial, but leads to very interesting results.
- One can consider other types of operators such as Jacobi matrices, CMV matrices, and continuum Schrödinger operators. There are new phenomena since in these cases, the curve of initial conditions is no longer a line and, more importantly, no longer lies in a single invariant surface. This necessitates the use of partially hyperbolic dynamics and one loses certain properties. For example, the spectrum is no longer a dynamically defined Cantor set, and local dimensions are no longer constant throughout the spectrum.