Sarnak-Xue and Applications

Amitay Kamber

joint work with Konstantin Golubev

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Introduction

We will describe some generalizations and applications of the work of Sarnak and Xue on limit multiplicities.
The idea is that their limit on multiplicity can be used as a replacement to the Ramanujan property to prove "optimal" results.

Limit Multiplicities

- G a real or p-adic s.s. algebraic group, $\Gamma_1 \subset G$ cocompact lattice, $\Gamma_N \subset \Gamma_1$ a sequence of f.i. subgroups.
- $V_N = Vol\left(\Gamma_N \backslash G\right) \asymp [\Gamma_1 : \Gamma_N] \to \infty$.

$$L^2\left(\Gamma_N \backslash G\right) \cong \oplus_{\pi \in \hat{G}} m\left(\pi, \Gamma_N\right)$$

Various results concerning the limit of $m\left(\pi, \Gamma_N\right)$ as $N \to \infty$, following Degeorge-Wallach(1978).
Sauvageot (1997)- assuming Benjamini-Schramm convergence (e.g. increasing injective radius)

$$\mu_N = \frac{1}{V_N} \sum_{\pi \in \hat{G}} m\left(\pi, \Gamma_N\right) \delta_\pi \to \mu_{\text{pl}}$$

Simple result:

$$m(\pi, \Gamma_N) \ll V_N$$

Sarnak-Xue Hypothesis

Let $p(\pi) = \inf \{p : \text{K -finite matrix coeff. are in } L^p(G)\}$.
If $p(\pi) > 2$ π is called **non-tempered**. Then it is not in the support of the Plancherel measure.
Benjamini-Schramm implies that for $p(\pi) > 2$, π non-trivial,

$$m(\pi, \Gamma) \to 0.$$

Naive Ramanujan Hypothesis says that for $p(\pi) > 2$, π non-trivial,

$$m(\pi, \Gamma) = 0.$$

Definition (Sarnak-Xue 1991)

$\{\Gamma_N\}$ satisfies **Sarnak-Xue (pointwise) hypothesis** if

$$m(\pi, \Gamma_N) \ll_{\pi,\epsilon} V_N^{\frac{2}{p(\pi)}+\epsilon}.$$

Sarnak-Xue Hypothesis

Definition (Sarnak-Xue 1991)

$\{\Gamma_N\}$ satisfies **Sarnak-Xue (pointwise) hypothesis** if for every $\pi \in \hat{G}$, $\epsilon > 0$,

$$m(\pi, \Gamma_N) \ll_{\pi,\epsilon} V_N^{\frac{2}{p(\pi)}+\epsilon}.$$

Theorem (Sarnak-Xue)

The pointwise hypothesis holds for (cocompact) principal congruence subgroups of arithmetic subgroups of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

Conjecture (Sarnak-Xue)

The pointwise hypothesis holds for cocompact principal congruence subgroups of general arithmetic lattices of Lie groups.

Sarnak's Density Hypothesis

For $A \subset \hat{G}$ compact let

$$m(A, \Gamma_N, p) = \sum_{\pi \in A, p(\pi) \geq p} m(\pi, \Gamma_N)$$

Definition (Sarnak 2018)

$\{\Gamma_N\}$ satisfies **Sarnak's density hypothesis** if for $A \subset \hat{G}$ compact,

$$m(A, \Gamma_N, p) \ll_{A,\epsilon} V_N^{\frac{2}{p}+\epsilon}.$$

Conjecture (Sarnak 2018)

The density hypothesis holds for cocompact congruence subgroups of arithmetic lattices of Lie groups.

Spherical Density

A **spherical representation** of G is a representation having a non-trivial K-fixed vectors. Its K-fixed vectors appear in the spectral decomposition of $L^2(\Gamma_N \backslash G/K)$.
For G of rank 1 or p-adic, the non-tempered spherical representations are easily described and are pre-compact in the unitary dual $\hat{G}$.

Definition

(G p-adic or rank 1)- $\{\Gamma_N\}$ satisfies **spherical density** if

$$m\left(\hat{G}_{\mathrm{sph}}, \Gamma_N, p\right) \ll_\epsilon V_N^{\frac{2}{p}+\epsilon}.$$

Sarnak and Xue actually proved spherical density for principal congruence subgroups of arithmetic subgroups of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

Theorem (Golubev-K,2018)

If G is p-adic then $\{\Gamma_N\}$ satisfies Sarnak's density hypothesis if and only if it satisfies spherical density.

Density for Graphs and Hyperbolic Surfaces

For a family of $q+1$ regular graphs $\{X_N\}$, A_N the adjacency matrix:

- X_N is **Ramanujan** if

$$\mathrm{spec} A_N \subset [-2\sqrt{q}, 2\sqrt{q}] \cup \{\pm (q+1)\}$$

- $\{X_N\}$ satisfies **Sarnak-Xue density** if

$$\# \left\{ \lambda \in \mathrm{spec} A_N : |\lambda| \geq q^{\frac{1}{p}} + q^{1-\frac{1}{p}} \right\} \ll_\epsilon |X_N|^{\frac{2}{p}+\epsilon}.$$

  The spherical density for **LPS graphs** was proved (implicitly) using elementary methods by Davidoff-Sarnak-Vallete.

For a family of hyperbolic surfaces X_N, Δ_N the Laplacian:

- X_N is **Ramanujan** (or satisfies Selberg's conjecture) if

$$\mathrm{spec}\Delta_N \subset \{0\} \cup \left[\frac{1}{4}, \infty\right).$$

- $\{X_n\}$ satisfies **Sarnak-Xue density** if

$$\# \left\{ \lambda \in \mathrm{spec}\Delta_N : |\lambda| \geq \frac{1}{4} - \left(\frac{1}{2} - p^{-1}\right)^2 \right\} \ll_\epsilon V_N^{\frac{2}{p}+\epsilon}.$$

Diameter of Graphs

In recent years there have been a number of results about optimal behavior of Ramanujan graphs. Let us recall some classical results.

Theorem

Let X be a $(q+1)$-regular graph.
If X is a (λ_0-)expander then

$$diam(X) \leq C_{\lambda_0} \log_q(|X|)$$

(LPS) If X is Ramanujan then

$$diam(X) \leq (2 + o(1)) \log_q(|X|)$$

This is the best known bound for the diameter of LPS graphs - twice the optimal value $\log_q(|X|)$.
(Sardari, 2015)- The diameter for LPS graphs is at least $(\frac{4}{3} + o(1)) \log_q(|X|)$, and is therefore not optimal.

Almost-Diameter of Graphs

Optimal Diameter and Almost-Diameter of a Family

A family $\{X_N\}$ of graphs has:
Optimal Diameter if:

$$\forall x, y \in X_N \ d(x,y) \le (1+o(1)) \log_q (|X_N|)$$

Optimal Almost-Diameter if:

$$\forall x \in X_N \ \# \left\{ y \in X : d(x,y) > (1+o(1)) \log_q (|X_N|) \right\} < o(|X_N|)$$

Optimal Average-Distance if:

$$\# \left\{ x, y \in X : d(x,y) > (1+o(1)) \log_q (|X_N|) \right\} < o\left(|X_N|^2\right)$$

For Cayley graphs, Optimal Average-Distance and Optimal Almost-Diameter are the same.

Almost-Diameter of Ramanujan graphs

Theorem (Lubetzky-Peres 2015, Sardari 2015)

If $\{X_N\}$ is a family of Ramanujan graphs, then it has optimal almost-diameter.

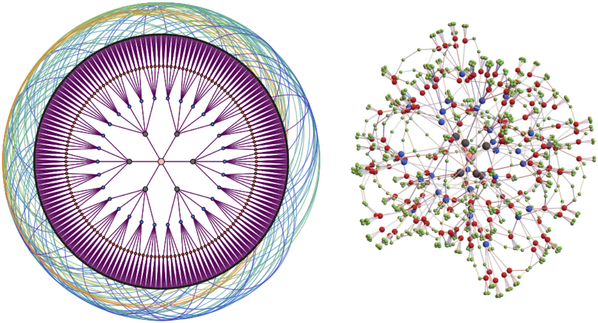

FIGURE 1. A ball of radius 4 in the Lubotzky–Phillips–Sarnak 6-regular Ramanujan graph on $n = 12180$ vertices via $\mathrm{PSL}(2, \mathbb{F}_{29})$.

Similar results, in a slightly different context, came up in the work of Parzanchevski-Sarnak on Golden-Gates.

Almost-Diameter and Density

Theorem (Golubev-K (2018)

If $\{X_N\}$ is an expander family of graphs satisfying Sarnak-Xue density, then it has optimal average-distance.
In particular, if the graphs are Cayley, the family has optimal almost-diameter.

As a matter of fact, optimal diameter, almost-diameter and average distance can be defined for a general family of quotients $\Gamma_N \backslash G/K/$, once a metric is chosen.

Theorem (Golubev-K (2018)

Assume that G is p-adic or rank 1. If $\{\Gamma_N\}$ is a family with a spectral gap which satisfies Sarnak-Xue spherical density, then the quotients $\Gamma_N \backslash G/K$ have optimal average-distance.
In particular, if $\Gamma_N \subset \Gamma_1$ is normal, then the quotients have optimal almost-diameter.

Almost-Diameter and Density

Applying the last theorem to principal congruence subgroup of $SL_2(\mathbb{Z})$ (and playing a little with the definitions), we get the following theorem. Sarnak called it **optimal strong approximation**.
The spherical density for this case is a result of Huxley from 1984. Note that the lattice $SL_2(\mathbb{Z})$ is not cocompact, but in $SL_2(\mathbb{R})$ it is not a problem.

Theorem (Sarnak 2015)

For all but $o\left(SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)\right)$ of $g \in SL_2\left(\mathbb{Z}/N\mathbb{Z}\right)$ there exists a lift $\tilde{g} \in SL_2\left(\mathbb{Z}\right)$ with $\|\tilde{g}\| \ll_\epsilon N^{\frac{3}{2}+\epsilon}$.

The exponent $3/2$ is optimal, as otherwise there will not be enough elements of $\mathrm{SL}_2\left(\mathbb{Z}\right)$.

Cutoff (Lubetzky-Peres)

Let X be a $(q+1)$-regular graph and $x_0 \in X$. Consider the distribution $A^m \delta_{x_0}$ of the random walk starting at x_0.
Notice that $\frac{q-1}{q+1}$ is the rate of divergence on the tree, so one needs $\frac{q+1}{q-1} \log_q(|X|)$ steps to reach almost all of X.

Theorem - Cutoff (Lubetzky-Peres)

Assume that X is a Ramanujan graph.
For $m < (1-\epsilon)\frac{q+1}{q-1}\log_q(|X|)$ we have

$$\|A^m \delta_{x_0} - \pi\|_1 = 2 - o(1).$$

For $m > (1+\epsilon)\frac{q+1}{q-1}\log_q(|X|)$ we have

$$\|A^m \delta_{x_0} - \pi\|_1 = o(1).$$

This behavior of the random walk is called Cutoff.
For Cayley graphs, one may replace here the Ramanujan assumption by Sarnak-Xue density.

Thank You!