Universality for the dimer model

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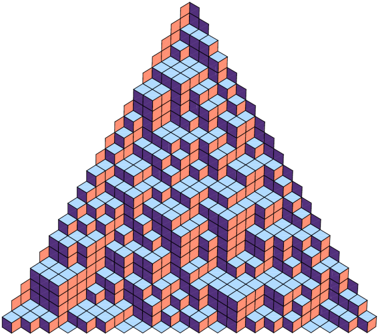
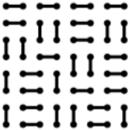
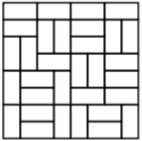


Les Diablerets, February 2017

The dimer model

Definition

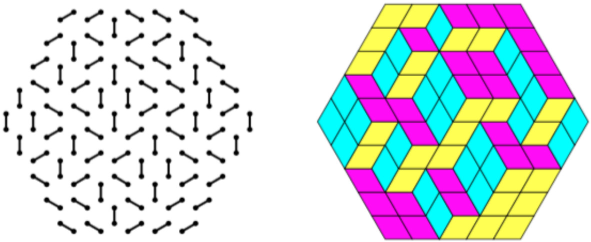
G = bipartite finite graph, planar
Dimer configuration = perfect matching on G:
each vertex incident to one edge
Dimer model: uniformly chosen configuration

On square lattice, equivalent to domino tiling.

Dimer model as a random surface

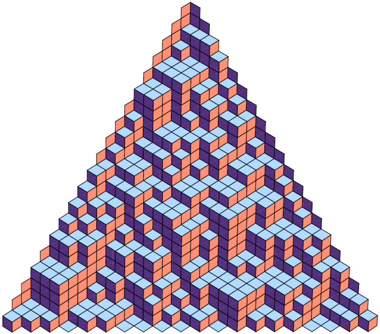
Can describe the dimer model through a height function.
Hence view as random surface.



Example: honeycomb lattice

Dimer = lozenge tiling
Equivalently: stack of 3d cubes.

Large scale behaviour?



Main Question:

What is large scale behaviour of height function?

Background

Classical model of statistical mechanics:

Kasteleyn, Temperley–Fisher 1960s
Kenyon, Propp, Okounkov, Sheffield, Dubédat,... 1990s+

"Exactly Solvable": determinantal structure

$$\text{e.g.,}\quad Z_{m,n} = \prod_{j=1}^{m}\prod_{k=1}^{n} \left|2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right|^{1/2}$$

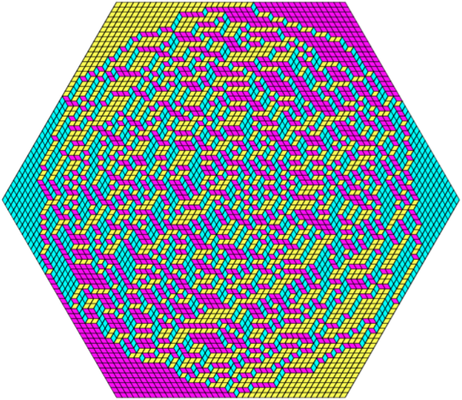
Analysis via: discrete complex analysis, Schur polynomials, Young tableaux, algebraic geometry... + Connection to SLE

Mapping to other models:

Tilings, 6-vertex, XOR Ising, **Uniform Spanning Trees** (UST)

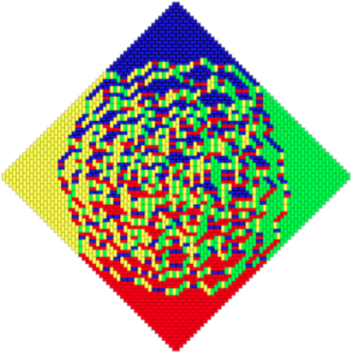
Arctic circle phenomenon

Some regions can be frozen, other liquid (temperate)
Depends on boundary conditions in sensitive way
Interface between frozen / liquid = arctic circle



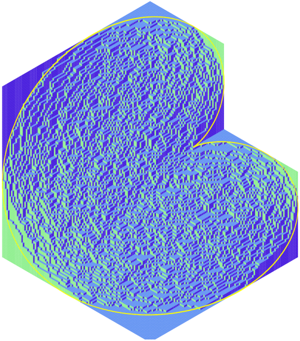
Algebraic curves

Aztec diamond:



Jockusch, Propp and Shor 1996

Cardioid:

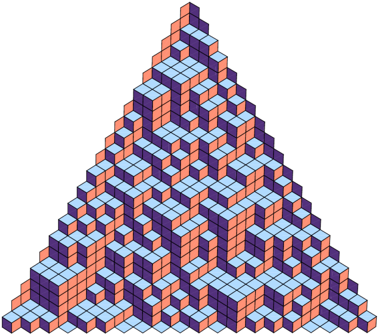


Kenyon–Okounkov–Sheffield 2006

Main questions (bis)

Fluctuations

Is there universality? (in the temperate region)
Is there conformally invariance?



Main theorem

Let $h^{\#\delta}$ = height function on hexagonal lattice, mesh-size = δ.

Theorem (B.–Laslier–Ray 2016)

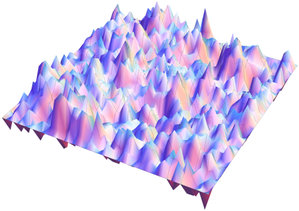
Assume D is Jordan domain and boundary conditions of height lie in plane $P \subset \mathbb{R}^3$.

$$\frac{h^{\#\delta} - \mathbb{E}(h^{\#\delta})}{\delta} \circ \ell \xrightarrow[\delta \to 0]{} \frac{1}{\chi} h_{\mathsf{GFF}},$$

where ℓ = linear map
h_{GFF} = Gaussian free field with Dirichlet boundary conditions.
$\chi = 1/\sqrt{2}$.
(Convergence in distribution in $H^{-1-\varepsilon}$.)

What is the Gaussian free field?

Informally, $\mathbb{P}(f) = \frac{1}{Z} \exp\left(-\frac{1}{2}\int_D |\nabla f|^2\right) df$



GFF = canonical random function on D.
But too rough to be a function
Rigorously: in Sobolev space H^{-s}, $\forall s > 0$

$$(h_{\mathrm{GFF}}, f) \sim \mathcal{N}\left(0, \iint_D G_D(x,y) f(x) f(y) dx dy\right)$$

where $G_D(\cdot,\cdot) = -\Delta^{-1}$ Green's function in D.

Novelty of approach

Universality of fluctuations

Insight as to why GFF universal?
Needed: SRW $\to$ BM on certain graph.
Does not fundamentally rely on exact solvability
Instead: imaginary geometry and SLE

Robustness

Recover Kenyon 2000 (flat case with smooth D)
Extends to Dimer Model on isoradial graphs (extends Li 2014)
Dimer model in random environment
Work in progress: compact Riemann surfaces with no boundary
etc.

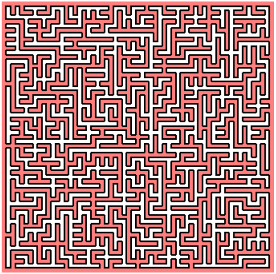
Temperley's bijection

Benoit explained: dimer configurations with given slope $\iff$ UST in associated T-graph.

Dimer configurations $\leftrightarrow$ UST on T-graph
Height function $\leftrightarrow$ Winding of branches in tree

New goal:

Study winding of branches
in Uniform Spanning Trees.



Question

How much do you wind around in a random maze?

Winding in UST

Question

How much do you wind around in a random maze?
Answer: the GFF !

Let $h^{\#\delta}$ = winding of branches in UST.

Real main theorem

Assume $(\star)$.

$$h^{\#\delta} - \mathbb{E}(h^{\#\delta}) \xrightarrow[\delta \to 0]{} \frac{1}{\chi} h_{\mathsf{GFF}},$$

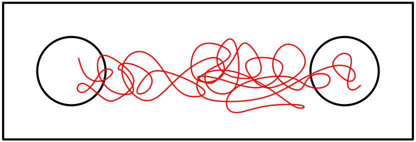
h_{GFF} = Gaussian free field (Dirichlet boundary conditions).
$\chi = 1/\sqrt{2}$.

Note: $\mathbb{E}(h^{\#\delta})$ itself is **not** universal, only fluctuations!

Assumptions for the theorem

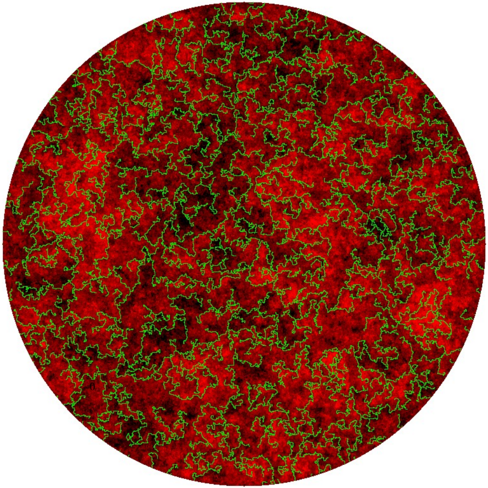
Holds under very general assumptions:

(1) Simple Random Walk on $G^{\#\delta}$ converges to Brownian motion
(2) Uniform crossing condition:



("Russo–Seymour–Welsh" estimate)
(3) Bounded density of vertices; edges have bounded winding

Ideas for the proof: working in the continuum



Scaling limit of Uniform Spanning Tree

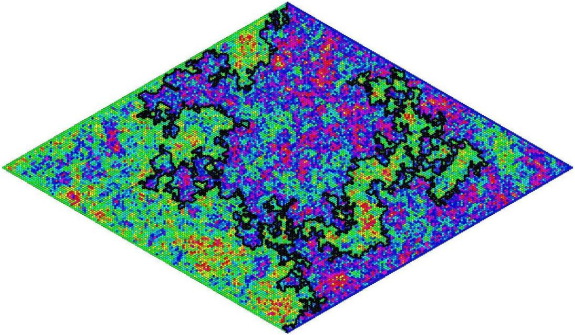
Theorem (Lawler, Schramm, Werner '03, Schramm '00)

$D \subset \mathbb{C}$

- *Uniform spanning tree on $D \cap \delta\mathbb{Z}^2 \to$ "A continuum tree" (continuum uniform spanning tree).*
- *Branches of the continuum tree are SLE_2 curves.*

Yadin–Yehudayoff 2010: universality (assuming convergence of SRW to BM).

Relations between SLE and GFF



Theorem (Schramm–Sheffield)

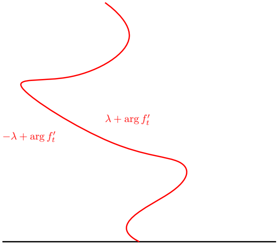
"Level lines" of the GFF are given by SLE_4 curves.

Imaginary Geometry

Miller–Sheffield: "flow lines of GFF/χ are SLE_κ curves", provided:

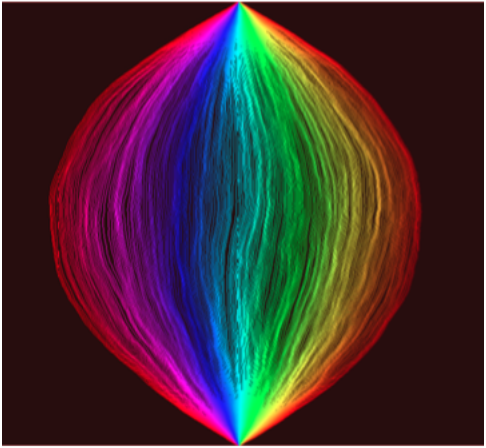
$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

Meaning: there is a coupling (h, η) such that $h = GFF$, $\eta = \mathrm{SLE}_\kappa$, such that



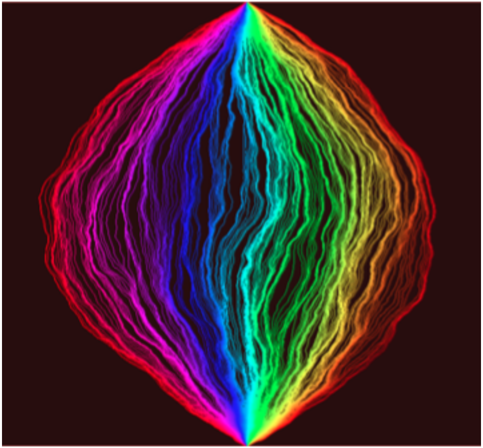
In other words, the values of the GFF along the curve records "winding" of the SLE.

Flow lines of GFF: $e^{ih/\chi}$.



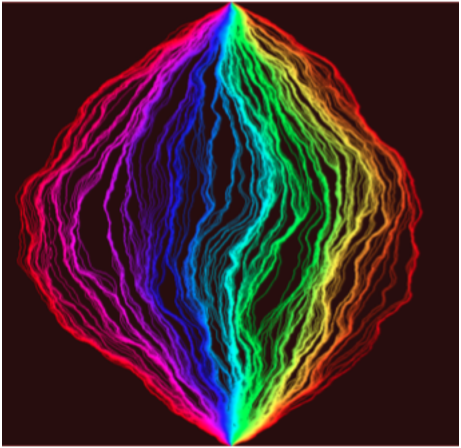
$\chi = 31.97\ldots$, flow lines = $\mathrm{SLE}_{1/256}$ (Miller–Sheffield).

Flow lines of GFF: $e^{ih/\chi}$.



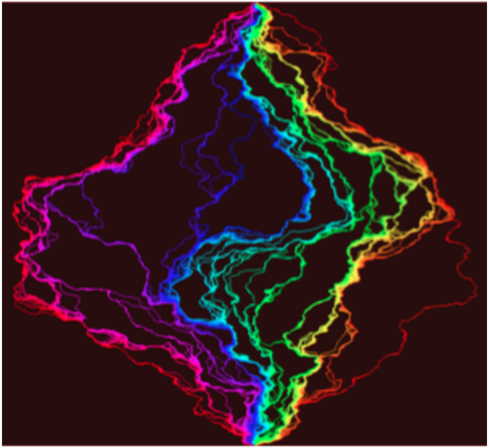
$\chi = 11.23...$, flow lines $=$ $\mathrm{SLE}_{1/32}$ (Miller–Sheffield).

Flow lines of GFF: $e^{ih/\chi}$.



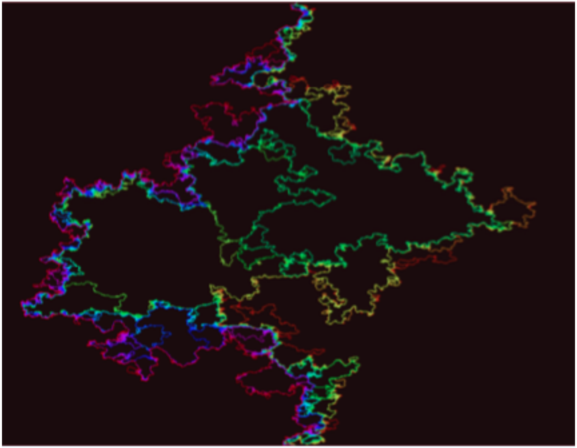
$\chi = 7.88...$, flow lines $=$ $SLE_{1/16}$ (Miller–Sheffield).

Flow lines of GFF: $e^{ih/\chi}$.



$\chi = 2.47...$, flow lines $=$ $\mathrm{SLE}_{1/2}$ (Miller–Sheffield).

Flow lines of GFF: $e^{ih/\chi}$.



$\chi = \sqrt{2}$, flow lines = SLE_2 (Miller–Sheffield).
Suggests winding of continuum UST is $(1/\chi)$ GFF.

Proof of convergence, 1/4

$$\begin{array}{ccc} \text{UST} & \xrightarrow{\text{winding}} & h^{\#\delta} \\ \downarrow & & \downarrow ? \\ \text{Continuum UST} & \xrightarrow{\text{winding}} & h_{\text{GFF}} \end{array}$$

Step 1: Making sense of intrinsic winding of rough curves.

Let $\gamma : [0,1] \to \mathbb{C}$ smooth, *simple* curve. Let

$$W(\gamma, z) = \text{topological winding around } z$$

and let

$$W_{\text{int}}(\gamma) = \text{ intrinsic winding of } \gamma \ = \int_0^1 \arg \gamma'(s) ds$$
$$= \frac{\pi}{2}(\ \# \text{ left turns - } \# \text{ right turns in discrete}).$$

Proof of convergence, 1/4.

Lemma

$W_{int}(\gamma) = W(\gamma, \gamma(0)) + W(\gamma, \gamma(1))$.

Let $h_t(z)$ = intrinsic winding of branch to z, truncated at capacity t, followed by segment connecting to z.

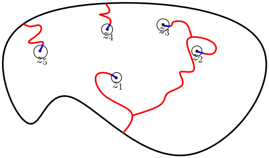
Theorem (B.–Laslier–Ray)

$$h_t - \mathbb{E}(h_t) \to \frac{1}{\chi} h_{\mathsf{GFF}}$$

(almost surely in $H^{-1-\varepsilon}$).

Relies on Miller–Sheffield,
+ deformation of intrinsic winding under conformal maps.

Proof of convergence, 2/4



Step 2: The blue parts are roughly independent.

Multiscale coupling, based on Schramm's finiteness theorem:
Fix $k \geq 1$. We show UST in small neighbourhoods of $z_1, \ldots, z_k$ can be coupled to independent full-planes UST (with good probability).

Proof of convergence, 3/4

Step 3: method of moments

Overall, if coupling successful:

$$h^{\#\delta} \approx h_t + e_t \approx h_{\mathrm{GFF}} + e_t$$

$e_t - \mathbb{E}(e_t) \approx$ independent from point to point with mean zero.

Fix test function f,

$$(h^{\#\delta}, f)^k = \int \dots \int h^{\#\delta}(z_1) f(z_1) \dots h^{\#\delta}(z_k) f(z_k) dz_1 \dots dz_k$$

Convergence of $\mathbb{E}$ ok if coupling successful.

Proof of convergence, 4/4

Step 4: a priori winding estimates

When coupling fails, need a priori bounds on winding, eg:

Lemma (Stretched exponential tails for winding of LERW)

Fix $t > -10\log(|v - \partial D|)$. Then

$$\mathbb{P}\Big(\sup_{t \le t_1, t_2 \le t+1} |h^{\#\delta}_{t_1}(v) - h^{\#\delta}_{t_2}(v)| > n \Big) < Ce^{-cn^{\alpha}}.$$

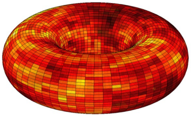
Uses only uniform crossing assumption (RSW).

Then separate argument for tightness in $H^{-1-\varepsilon}$. □

Robustness

Work in progress

Compact Riemann surfaces with no boundary, eg torus.



To infinity and beyond...

On torus, height function $\to$ compactified GFF.
This answers question by Dubédat–Gheissari
Universal limit for Cycle-Rooted Spanning Forest (extends Kassel–Kenyon)
A theory of Imaginary Geometry on Riemann surfaces

Robustness

Future work:

General boundary conditions; multiply connected case etc.
Interacting dimers and space-filling $SLE_{\kappa'}$.

Pictures acknowledgements: Kenyon, Miller, Sheffield, Lee, Ray ...