Nevanlinna Domains with Large Boundaries

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Nevanlinna Domains

Definition

A bounded simply connected domain $G \subset \mathbb{C}$ is said to be a Nevanlinna domain, if there exist functions $u, v \in H^\infty(G)$ such that the equality

$$\overline{z} = \frac{u(z)}{v(z)},$$

holds on ∂G a. e. in sense of conformal mappings.

That means that

$$\overline{\varphi(\zeta)} = \frac{u(\varphi(\zeta))}{v(\varphi(\zeta))}, \text{ a.e. on } \mathbb{T}$$

for some(any) conformal mapping $\varphi : \mathbb{D} \mapsto G$.

Question

How large can be (accessible) boundary of Nevanlinna domain?

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Examples

Nevanlinna domains: unit disk $\mathbb{D}$, Neumann's oval (image of ellipse with center at origin under $z \mapsto \frac{1}{z}$).

Non- Nevanlinna domains: ellipse, polygon.

$$G_{a,b} = \left\{ z = x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \qquad b < a.$$

Boundary of $G_{a,b}$

$$\bar{z} = S_{a,b}(z), \qquad S_{a,b}(z) = \frac{(a^2 + b^2)z - 2ab\sqrt{z^2 - c^2}}{c^2},$$

$$c = \sqrt{a^2 - b^2}.$$

Motivations

- Polyanalytic approximation
- Quadratutre domains. Schwarz functions;
- Univalent functions in model subspaces of Hardy space

Polyanalytic approximation

Definition

We will say that function f is n-analytic if

$$f(z) = \overline{z}^{n-1} f_{n-1}(z) + \dots + \overline{z} f_1(z) + f_0(z),$$

where f_k are holomorphic functions.

X - compact subset of $\mathbb{C}$

$$\mathcal{A}_n(X) := \{f \in C(X) : f - \text{n-analytic in } \operatorname{Int} X\},$$

$$\mathcal{P}_n(X) = \mathrm{Clos}_{C(X)}\{P : P - \text{n-analytic polynomial}\}.$$

Question

For which X

$$\mathcal{P}_n(X) = \mathcal{A}_n(X)?$$

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Theorem (Mergelyan 1952)

$\mathcal{P}_1(X) = \mathcal{A}_1(X)$ if and only if the set $\mathbb{C} \setminus X$ is connected.

Theorem (Carmona 1985)

If $\mathbb{C} \setminus X$ is connected, then $\mathcal{P}_m(X) = \mathcal{A}_m(X)$ for any $m \geq 2$.

Theorem (Carmona, Paramonov, Fedorovskiy 2002)

Let X be Caratheodory compact set, $m \geq 2$. We have $\mathcal{P}_m(X) = \mathcal{A}_m(X)$ if and only if every bounded component of $\mathbb{C} \setminus X$ is not a Nevanlinna domain.

For arbitrary compact X answer may depend on m.

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Quadrature domains

Definition

A bounded domain Ω is a classical quadrature domain if there exists a finite set of points $\{z_k\} \subset \Omega$ such that

$$\int_\Omega f(z)dxdy = \sum_{j=1}^{n} \sum_{s=0}^{n_j-1} a_{js} f^{(s)}(z_j)$$

for every summable analytic function f.

Every quadrature domain is a Nevanlinna domain. Moreover nodes correspond to the poles of function u/v

Any boundary of quadrature domain (even in wide sense) Ω admits a one-sided Schwarz function

$$\bar{z} = S(z) \text{ on } \partial\Omega, S \in C(\overline{\Omega}), S - \text{ analytic in } \Omega \setminus K.$$

Theorem (Sakai 1991)

If Ω admits a one-sided Schwarz function, then $\partial\Omega$ consists of finitely many analytic curves.

The function u/v seems to be a rather weak generalization of the concept of a one-sided Schwarz function, since we are dealing with the equality of angular boundary values only for almost all points on $\mathbb{T}$.

Model subspaces of Hardy space

Pseudocontinuation

A domain G is a Nevanlinna domain if and only if a conformal mapping f of the unit disc $\mathbb{D}$ onto G admits a Nevanlinna-type pseudocontinuation, so that there exist two functions $f_1, f_2 \in H^\infty(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ such that $f(\zeta) = f_1(\zeta)/f_2(\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

Model subspaces of Hardy space, Θ-inner functions in $\mathbb{D}$,

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

Parametrization

Let G be a bounded simply connected domain and let f be some conformal mapping from $\mathbb{D}$ onto G. If G is a Nevanlinna domain, then there exists an inner function Θ such that $f \in K_\Theta$. Reciprocally, if Θ is an inner function, then any bounded univalent function from the space K_Θ maps $\mathbb{D}$ conformally onto some Nevanlinna domain.

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Factorization of Θ.

$$\Theta(z) = \alpha B(z) S(z), \quad B(z) := \prod_{n=1}^{\infty} \frac{\overline{a_n}}{a_n} \cdot \frac{z - a_n}{\overline{a_n} z - 1},$$

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_S(\zeta) \right).$$

Fedorovkiy, B.

Let Θ be an inner function in $\mathbb{D}$. The space K_Θ contains a bounded univalent functions if and only if one of the following two conditions satisfied:
i) Θ has zero in $\mathbb{D}$;
ii) $\Theta = S$ is a singular inner function and measure μ_S is such that $\mu_S(E) > 0$ for some Carleson set $E \subset \mathbb{T}$, which means that $\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E) d\zeta > -\infty$.

Let Θ be a Blashke product. If $f \in K_\Theta$, then

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{1 - \overline{a_n} z}. \quad (1)$$

Almost unrectifiable boundary.

Theorem (Fedorovskiy 2006)

For any $\alpha \in (0, 1)$ there exists a Nevanlinna domain with boundary in the class C^1 but not in the class $C^{1,\alpha}$.

Theorem (Baranov, Fedorovskiy 2011)

There exits an univalent (in $\mathbb{D}$) function f of the form (1) such that $f' \notin H^p$ for every $p > 1$.

Boundaries of Nevanlinna domains

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Hedgehog domains.

Theorem (Mazalov 2015)

There exists a Nevanlinna domain with unrectifiable boundary.

Theorem (Mazalov 2017)

There exists a Nevanlinna domain G such that

$$\dim_H(\partial G) = \log_2 3.$$

For a given bounded simply connected domain G let us define the set $\partial_a G \subset \partial G$, which consists of all points of ∂G being accessible from G by some curve.

Question

How large can be accessible boundary of Nevanlinna domain?

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Main results

Hedgehog with needles on needles.

Theorem (Borichev, Fedorovskiy, B. 2018)

For each $\beta \in [1,2]$ there exists a Nevanlinna domain G such that $\dim_H(\partial_a G) = \beta$. This domain has the form $G = f(\mathbb{D})$, where f is some function of the form (1) univalent in $\mathbb{D}$.

Univalent functions from Bernstein class (correponds to the case when $\mu_S = \delta_1$).

Theorem (Borichev, Fedorovskiy, B. 2018)

For each $\beta \in [1,2]$ there exists a Nevanlinna domain G such that $\dim_H(\partial G) = \beta$ and $G = f(\mathbb{C}^+)$, where f is some univalent function from Bernstein class $\mathcal{B}_{[0,1]}$.

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Nevanlinna domains of finite order

Let $\mathcal{RU}_n$ be a set of all rational functions of degree n which is univalent in $\mathbb{D}$. We know that $R(\mathbb{D})$ is a Nevanlinna domain for $R \in \mathcal{RU}_n$.

Let

$$\gamma_0 = \limsup_{n\to\infty} \sup_{R\in\mathcal{RU}_n, \|R\|_\infty \le 1} \frac{\log \ell(R)}{\log n}, \qquad \ell(R) := \frac{1}{2\pi}\int_{\mathbb{T}} |R'(\zeta)||d\zeta|.$$

Theorem (Baranov, Fedorovskiy 2013)

$$B_b(1) < \gamma_0 \le 1/2.$$

$B_b(1)$ is the integral means spectrum for bounded univalent functions. It is known that (Smirnov, Belyaev, Shimorin, Hedenmalm)

$$0,23 < B_b(1) \le 0,46.$$

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Snake domain

Theorem (Borichev, Fedorovskiy, B. 2018)

For every $R \in \mathcal{RU}_n$, $\|R\|_\infty \le 1$ we have

$$\frac{\sqrt{n}}{6\pi} \le \ell(R) \le 6\pi\sqrt{n}.$$

So, $\gamma_0 = 1/2$.

Theorem (Dolzhenko 1978, Spijker 1991 ($E = \mathbb{T}$))

Let R be a rational function of degree n with poles outside $\overline{\mathbb{D}}$. For any measurable set $E \subset \mathbb{T}$ of positive mesure the estimate

$$\int_{\mathbb{T}} |R'(\zeta)||d\zeta| \le n\|R\|_{\infty,E}$$

holds and is sharp.

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Idea of the proof of some results

Put $\varepsilon = 10^{-9}$. Let $\{w_n\}_{n=0}^{\infty}$ be a bounded sequence. Put

$$a_n = w_{n+1} - w_n, \qquad Q_n^{\pm} = conv\{w_n, w_{n+1}, w_n \pm 2ia_n, w_{n+1} \pm 2ia_n\},$$

$$T_n^{\pm} = conv\{w_{n+1}, w_{n+1} \pm 2ia_{n+1}, w_{n+1} \pm 2ia_{n+1}\}.$$

We will assume that $|w_n| < 1$, $w_0 = 0$,

$$1 - \varepsilon < |a_{n+1}|/|a_n| < 1 + \varepsilon, \qquad |\arg a_{n+1}\overline{a_n}| \le \varepsilon,$$

$Q_n^{\pm} \cap Q_m^{\pm} = \emptyset, Q_n^{\pm} \cap T_m^{\pm} = \emptyset$ for $|n - m| > 1$.
Put

$$L = \cup_n [w_n, w_{n+1}], \qquad \Omega_L = \cup_n Q_n^{\pm} \cup T_n^{\pm}.$$

There exists a meromorphic function f which is univalent in $\mathbb{C}^+$ and $L \subset f(\mathbb{C}^+) \subset \Omega_L$.