Nevanlinna Domains with Large Boundaries

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Nevanlinna Domains

Definition
A bounded simply connected domain $G \subset \mathbb{C}$ is said to be a Nevanlinna domain, if there exist functions $u, v \in H^\infty(G)$ such that the equality

$$\overline{z} = \frac{u(z)}{v(z)},$$

holds on $\partial G$ a. e. in sense of conformal mappings.

That means that

$$\phi(\zeta) = \frac{u(\phi(\zeta))}{v(\phi(\zeta))}, \text{ a.e. on } \mathbb{T}$$

for some(any) conformal mapping $\phi : \mathbb{D} \mapsto G$.

Question
How large can be (accessible) boundary of Nevanlinna domain?
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Nevanlinna domains: unit disk $\mathbb{D}$, Neumann’s oval (image of ellipse with center at origin under $z \mapsto \frac{1}{z}$).

Non-Nevanlinna domains: ellipse, polygon.

$$G_{a,b} = \left\{ z = x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \quad b < a.$$  

Boundary of $G_{a,b}$

$$\bar{z} = S_{a,b}(z), \quad S_{a,b}(z) = \frac{(a^2 + b^2)z - 2ab\sqrt{z^2 - c^2}}{c^2},$$

$$c = \sqrt{a^2 - b^2}.$$
Motivations

- Polyanalytic approximation
- Quadrature domains. Schwarz functions;
- Univalent functions in model subspaces of Hardy space
Polyanalytic approximation

**Definition**

We will say that function $f$ is $n$-analytic if

$$f(z) = \overline{z}^{n-1}f_{n-1}(z) + \ldots + \overline{z}f_1(z) + f_0(z),$$

where $f_k$ are holomorphic functions.

$X$ - compact subset of $\mathbb{C}$

$$\mathcal{A}_n(X) := \{ f \in C(X) : f \text{ is } n\text{-analytic in } \text{Int } X \},$$

$$\mathcal{P}_n(X) = \text{Clos}_{C(X)}\{ P : P \text{ is } n\text{-analytic polynomial} \}.$$ 

**Question**

For which $X$

$$\mathcal{P}_n(X) = \mathcal{A}_n(X)?$$
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Polyanalytic approximation

**Theorem (Mergelyan 1952)**

\[ P_1(X) = \mathcal{A}_1(X) \] if and only if the set \( \mathbb{C} \setminus X \) is connected.

**Theorem (Carmona 1985)**

If \( \mathbb{C} \setminus X \) is connected, then \( P_m(X) = \mathcal{A}_m(X) \) for any \( m \geq 2 \).

**Theorem (Carmona, Paramonov, Fedorovskiy 2002)**

Let \( X \) be Caratheodory compact set, \( m \geq 2 \). We have \( P_m(X) = \mathcal{A}_m(X) \) if and only if every bounded component of \( \mathbb{C} \setminus X \) is not a Nevanlinna domain.

For arbitrary compact \( X \) answer may depend on \( m \).
Polyanalytic approximation

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For arbitrary compact \( X \) answer may depend on \( m \).
A bounded domain $\Omega$ is a classical quadrature domain if there exists a finite set of points $\{z_k\} \subset \Omega$ such that

$$\int_{\Omega} f(z) \, dx \, dy = \sum_{j=1}^{n} \sum_{s=0}^{n_j-1} a_{js} f^{(s)}(z_j)$$

for every summable analytic function $f$.

Every quadrature domain is a Nevanlinna domain. Moreover, nodes correspond to the poles of function $u/v$. 
Any boundary of quadrature domain (even in wide sense) \( \Omega \) admits a one-sided Schwarz function

\[
\overline{z} = S(z) \text{ on } \partial \Omega, \ S \in C(\overline{\Omega}), \ S - \text{ analytic in } \Omega \setminus K.
\]

**Theorem (Sakai 1991)**

If \( \Omega \) admits a one-sided Schwarz function, then \( \partial \Omega \) consists of finitely many analytic curves.

The function \( u/v \) seems to be a rather weak generalization of the concept of a one-sided Schwarz function, since we are dealing with the equality of angular boundary values only for almost all points on \( \mathbb{T} \).
Model subspaces of Hardy space

Pseudocontinuation
A domain $G$ is a Nevanlinna domain if and only if a conformal mapping $f$ of the unit disc $\mathbb{D}$ onto $G$ admits a Nevanlinna-type pseudocontinuation, so that there exist two functions $f_1, f_2 \in H^\infty(\overline{\mathbb{C}} \setminus \mathbb{D})$ such that $f(\zeta) = f_1(\zeta)/f_2(\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

Model subspaces of Hardy space, $\Theta$-inner functions in $\mathbb{D}$,

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$  

Parametrization
Let $G$ be a bounded simply connected domain and let $f$ be some conformal mapping from $\mathbb{D}$ onto $G$. If $G$ is a Nevanlinna domain, then there exists an inner function $\Theta$ such that $f \in K_\Theta$. Reciprocally, if $\Theta$ is an inner function, then any bounded univalent function from the space $K_\Theta$ maps $\mathbb{D}$ conformally onto some Nevanlinna domain.
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Factorization of $\Theta$.

$$\Theta(z) = \alpha B(z) S(z), \quad B(z) := \prod_{n=1}^{\infty} \frac{\overline{a_n}}{a_n} \cdot \frac{z - a_n}{\overline{a_n} z - 1},$$

$$S(z) = \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_S(\zeta) \right).$$

Fedorovkiy, B.

Let $\Theta$ be an inner function in $\mathbb{D}$. The space $K_{\Theta}$ contains a bounded univalent functions if and only if one of the following two conditions satisfied:

i) $\Theta$ has zero in $\mathbb{D}$;

ii) $\Theta = S$ is a singular inner function and measure $\mu_S$ is such that $\mu_S(E) > 0$ for some Carleson set $E \subset \mathbb{T}$, which means that

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, E) d\zeta > -\infty.$$
Let $\Theta$ be a Blashke product. If $f \in K_{\Theta}$, then

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{1 - a_n z}.$$ (1)

Almost unrectifiable boundary.

**Theorem (Fedorovskiy 2006)**

For any $\alpha \in (0, 1)$ there exists a Nevanlinna domain with boundary in the class $C^1$ but not in the class $C^{1,\alpha}$.

**Theorem (Baranov, Fedorovskiy 2011)**

There exists an univalent (in $D$) function $f$ of the form (1) such that $f' \not\in H^p$ for every $p > 1$. 
Let $\Theta$ be a Blaschke product. If $f \in K_\Theta$, then

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Boundaries of Nevanlinna domains

Hedgehog domains.

**Theorem (Mazalov 2015)**
There exists a Nevanlinna domain with unrectifiable boundary.

**Theorem (Mazalov 2017)**
There exists a Nevanlinna domain $G$ such that

$$\dim_H(\partial G) = \log_2 3.$$  

For a given bounded simply connected domain $G$ let us define the set $\partial_a G \subset \partial G$, which consists of all points of $\partial G$ being accessible from $G$ by some curve.

**Question**
How large can be accessible boundary of Nevanlinna domain?
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Main results

Hedgehog with needles on needles.

**Theorem (Borichev, Fedorovskiy, B. 2018)**

For each $\beta \in [1, 2]$ there exists a Nevanlinna domain $G$ such that $\dim_H(\partial a G) = \beta$. This domain has the form $G = f(\mathbb{D})$, where $f$ is some function of the form (1) univalent in $\mathbb{D}$.

Univalent functions from Bernstein class (corresponds to the case when $\mu_S = \delta_1$).

**Theorem (Borichev, Fedorovskiy, B. 2018)**

For each $\beta \in [1, 2]$ there exists a Nevanlinna domain $G$ such that $\dim_H(\partial G) = \beta$ and $G = f(\mathbb{C}^+)$, where $f$ is some univalent function from Bernstein class $B_{[0,1]}$. 
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Let $\mathcal{RU}_n$ be a set of all rational functions of degree $n$ which is univalent in $\mathbb{D}$. We know that $R(\mathbb{D})$ is a Nevanlinna domain for $R \in \mathcal{RU}_n$.

Let

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\gamma_0 = \limsup_{n \to \infty} \sup_{R \in \mathcal{RU}_n, \|R\|_\infty \leq 1} \frac{\log \ell(R)}{\log n}, \quad \ell(R) := \frac{1}{2\pi} \int_{\mathbb{T}} |R'(\zeta)||d\zeta|.
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Theorem (Baranov, Fedorovskiy 2013)

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B_b(1) < \gamma_0 \leq 1/2.
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$B_b(1)$ is the integral means spectrum for bounded univalent functions. It is known that (Smirnov, Belyaev, Shimorin, Hedenmalm)

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Nevanlinna domains of finite order

Snake domain

**Theorem (Borichev, Fedorovskiy, B. 2018)**

For every \( R \in \mathcal{R}\mathcal{U}_n \), \( \|R\|_\infty \leq 1 \) we have

\[
\frac{\sqrt{n}}{6\pi} \leq \ell(R) \leq 6\pi \sqrt{n}.
\]

So, \( \gamma_0 = 1/2 \).

**Theorem (Dolzhenko 1978, Spijker 1991 (\( E = \mathbb{T} \)))**

Let \( R \) be a rational function of degree \( n \) with poles outside \( \overline{\mathbb{D}} \). For any measurable set \( E \subset \mathbb{T} \) of positive measure the estimate

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\int_{\mathbb{T}} |R'(\zeta)| d\zeta \leq n \|R\|_\infty, E
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holds and is sharp.
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holds and is sharp.
Put $\varepsilon = 10^{-9}$. Let $\{w_n\}_{n=0}^{\infty}$ be a bounded sequence. Put

$$a_n = w_{n+1} - w_n, \quad Q_n^\pm = \text{conv}\{w_n, w_{n+1}, w_n \pm 2ia_n, w_{n+1} \pm 2ia_n\},$$

$$T_n^\pm = \text{conv}\{w_{n+1}, w_{n+1} \pm 2ia_{n+1}, w_{n+1} \pm 2ia_{n+1}\}.$$  

We will assume that $|w_n| < 1$, $w_0 = 0$,

$$1 - \varepsilon < |a_{n+1}|/|a_n| < 1 + \varepsilon, \quad |\arg a_{n+1}\overline{a_n}| \leq \varepsilon,$$

$$Q_n^\pm \cap Q_m^\pm = \emptyset, \quad Q_n^\pm \cap T_m^\pm = \emptyset \text{ for } |n - m| > 1.$$

Put

$$L = \bigcup_n [w_n, w_{n+1}], \quad \Omega_L = \bigcup_n Q_n^\pm \cup T_n^\pm.$$  

There exists a meromorphic function $f$ which is univalent in $\mathbb{C}^+$ and $L \subset f(\mathbb{C}^+) \subset \Omega_L$. 