Monomer-dimer model and Neumann GFF

Nathanaël Berestycki*
Universität Wien
with Marcin Lis (Vienna) and Wei Qian (Cambridge)

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∗ on leave from Cambridge

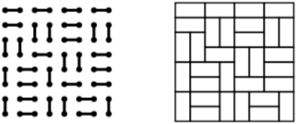
The dimer model

Definition

G = bipartite finite graph, planar
Dimer configuration = perfect matching on G:
each vertex incident to one edge
Dimer model: uniformly chosen configuration

More generally, weight w_e on each edge,

$$\mathbb{P}(\mathbf{m}) \propto \prod_{e \in \mathbf{m}} w_e.$$



On square lattice, equivalent to domino tiling.

Monomer-dimer model

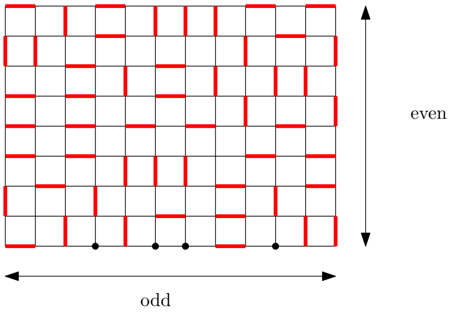
Now allow monomers on a part of the boundary, call it ∂_m.
Let $z > 0$ and define

$$\mathbb{P}(\mathbf{m}) \propto z^{\#\text{monomers}}.$$

Assumptions

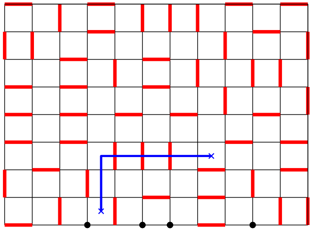
(1) G is dimerable (so partition function is > 0).
(2) $|\partial_m|$ is *odd* (a technical assumption).

Example:



Height function

Paths between faces avoid monomers:



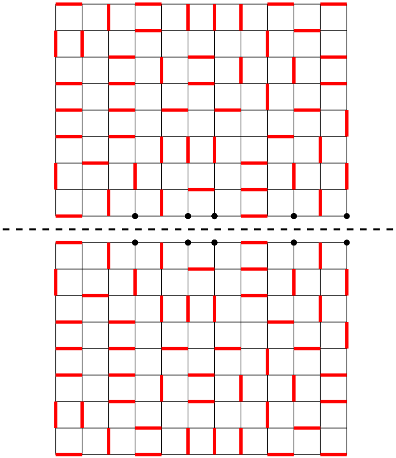
so height function still defined.

Main question

What is scaling limit of centered height function?
Conformal invariance?

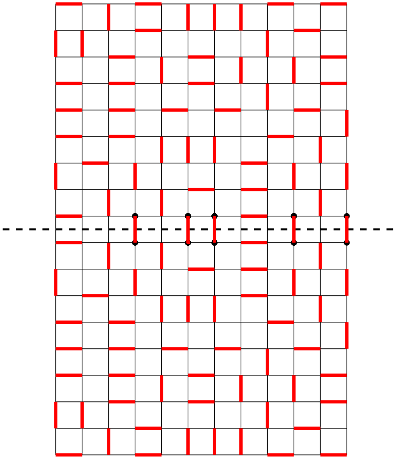
Reflection symmetry

Suppose $G \subset \mathbb{H}$ and $\partial_m \subset \mathbb{R}$. Then apply reflection:



Get a dimer configuration on G_{double}.

Height function



MD-height function is restriction of dimer height function to $\mathbb{H}$.
Note that height function is then *even*: $h(z) = h(\bar{z})$.

Even height functions

Conversely

Take dimer model on G_{double} with weight $z > 0$ for $\mathbb{R}$-edges
Condition to be symmetric (or height to be *even*)

Get monomer-dimer model by restricting to $\mathbb{H}$

Guessing the scaling limit

In $\mathbb{C}$, dimer height function $\to$ full plane GFF (de Tiliere 2005):

Full plane GFF

Consider $\tilde{\mathcal{D}}$ = smooth test functions with compact support and

$$\int_{\mathbb{C}} \rho(z)dz = 0.$$

Scalar product

$$(\rho_1, \rho_2)_\nabla = \frac{1}{2\pi} \int_{\mathbb{C}} \nabla \rho_1 \cdot \nabla \rho_2,$$

$\tilde{H}$ = completion of $\tilde{\mathcal{D}}$ under $(\cdot, \cdot)_\nabla$. $\tilde{f}_n$ = orthonormal basis.

$$h_{\mathbb{C}} = \sum_n X_n \tilde{f}_n.$$

Can integrate $\tilde{h}$ against fixed $\rho \in \bar{\mathcal{D}}$: defined up to constant.

Even/odd decomposition

Question

What is a (full plane) GFF conditioned to be even?

Any $\rho \in \tilde{\mathcal{D}}$ can be written uniquely as

$$\rho = \rho_{\text{odd}} + \rho_{\text{even}}$$

where $\rho_{\text{odd}}/\rho_{\text{even}}$ Dirichlet/Neumann boundary conditions. Moreover

$$(\rho_{\text{odd}}, \rho_{\text{even}})_{\nabla} = 0$$

so

$$H = H_{\text{odd}} \oplus H_{\text{even}}.$$

and

$$h = h_{\text{odd}} + h_{\text{even}}$$

where $h_{\text{odd}}, h_{\text{even}}$ are independent Dirichlet/Neumann GFF.

Conjecture and main result

Hence "$h_{\mathbb{C}}$ conditioned to be even": simply a Neumann GFF.

Conjecture:

The centered monomer-dimer height function on D converges to a GFF with Neumann boundary conditions on ∂_m and Dirichlet boundary conditions on $\partial D \setminus \partial_m$.

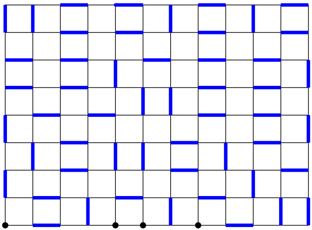
Theorem (B.-Lis-Qian, 2019+)

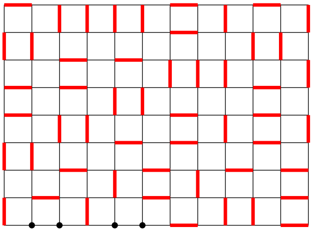
When $D_n \uparrow \mathbb{H}$ there is a local (inf. volume) limit. Furthermore, in the scaling limit, the height function converges to Neumann GFF.

Remark: also true on infinite strips.
Note: first time the limit doesn't have Dirichlet b.c.

Double monomer-dimer model

Superposition of two independent realisations of monomer-dimer model:





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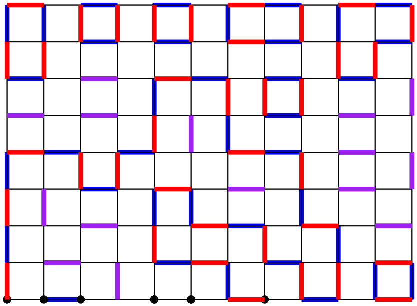

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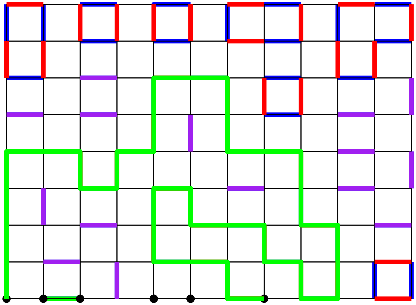


Get a collection of Green arcs connecting ∂_m to ∂_m.

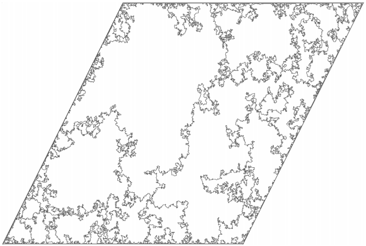
Double monomer-dimer model

Question:

In the scaling limit, what is the law of these arcs?

Conjecture

Converges to ALE_4 aka $A_{-\lambda,\lambda}$ (cf. Aru–Lupu–Sepulveda)



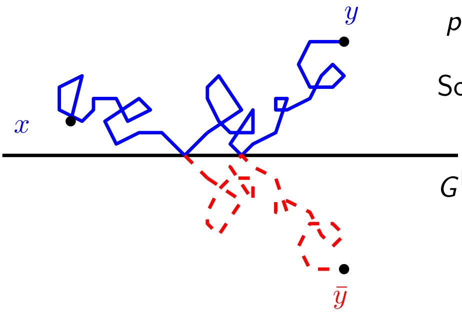
Indeed, ALE = boundary touching level lines of Neumann GFF (Qian–Werner, CMP 2018).

More about Neumann GFF

Dirichlet GFF: "pointwise correlation" $=$ Dirichlet Green function

$$\mathbb{E}[h^{\mathrm{Dir}}(x)h^{\mathrm{Dir}}(y)] = G^{\mathrm{Dir}}(x,y) = \pi \int_0^\infty p_t^{\mathrm{Dir}}(x,y)dt$$

In $\mathbb{H}$ (more integrable):



$$p_t^{\mathrm{Dir}}(x,y) = p_t^{\mathbb{C}}(x,y) - p_t^{\mathbb{C}}(x,\bar{y})$$

So

$$G^{\mathrm{Dir}}(x,y) = -\log|x-y| + \log|x-\bar{y}|$$
$$= \log\left|\frac{x-\bar{y}}{x-y}\right|.$$

Correlation of Neumann GFF in $\mathbb{H}$

Neumann GFF == free boundary conditions:
e.g. scaling limit of DGFF with free b.c.

$$\mathbb{E}[h^{\text{Neu}}(x)h^{\text{Neu}}(y)] = G^{\text{Neu}}(x,y) = \int_0^\infty p_t^{\text{Neu}}(x,y)dt$$

In $\mathbb{H}$:

$$p_t^{\text{Neu}}(x,y) = p_t^{\mathbb{C}}(x,y) + p_t^{\mathbb{C}}(x,\bar{y})$$

So

$$\begin{aligned} G^{\text{Neu}}(x,y) &= -\log|x-y| - \log|x-\bar{y}| \\ &= -\log|(x-\bar{y})(x-y)| . \end{aligned}$$

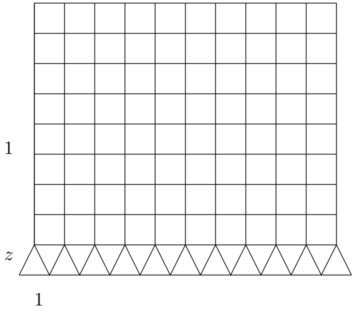
Only defined up to constant so G^{Neu} not unique. Instead:

$$\mathbb{E}[(h(a_i)-h(b_i))(h(a_j)-h(b_j))] = \log\left|\frac{(a_i-a_j)(b_i-b_j)(\bar{a}_i-a_j)(\bar{b}_i-b_j)}{(a_i-b_j)(b_i-a_j)(\bar{a}_i-b_j)(\bar{b}_i-a_j)}\right|$$

Sketch of proof of main result

Bijection to non-bipartite dimer

Giuliani, Jauslin, Lieb: Pfaffian formula for correlations.
In fact, bijection dimer model



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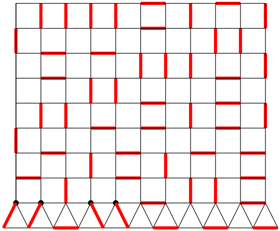


Lemma

If $|\partial_m|$ odd, and # monomers even, then unique way to associate dimer configuration on augmented graph.

Kasteleyn theory

Problem:

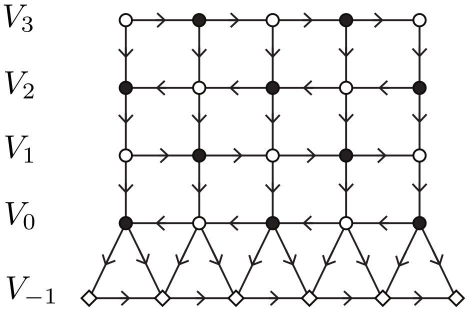
Graph becomes non-bipratite.

Kasteleyn theory

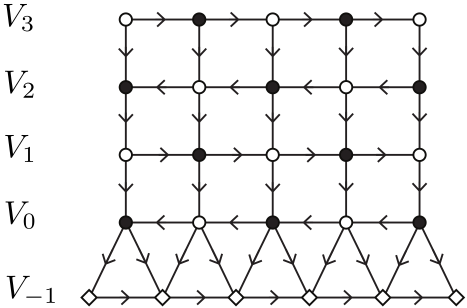
Kasteleyn orientation: going cclw on every face, odd number of clw arrows
Gauge transform: weight of every edge coming of a vertex v $\to \times\lambda_v$, with $\lambda_v \in \mathbb{C}, |\lambda_v| = 1$.
Kasteleyn matrix: $K(u, v) =$ signed weight of edge (u, v) (so K antisymmetric).

Then correlations are given by $Pf(K^{-1})$.

Kasteleyn orientation

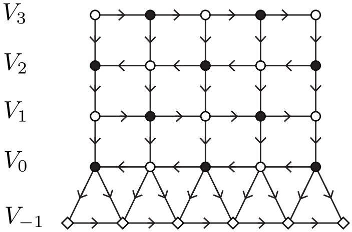


Gauge transform



Even rows multiplied by i:
$\implies$ each edge $e \in E_{\text{even}}$ multiplied by -1;
and each vertical edge has weight i.

Kasteleyn matrix in bulk



Kenyon: consider $D = K^* K$.
K = n.n. so D nonzero only from $W \to W$, $B \to B$.
Diagonal contributions vanish
So really $W_0 \to W_0, \ldots B_1 \to B_1$.
Then D = discrete Laplacian on each four sublattices.
Temperleyan b.c.: $\implies$ D has Dirichlet b.c. on B_0.

Scaling limit in bipartite setup

From the relation $D = K^*K$ we get

$$D^{-1} = K^{-1}(K^*)^{-1}$$

and so

$$K^{-1} = D^{-1}K^*.$$

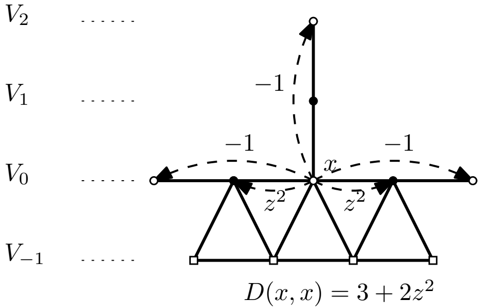
Moreover D^{-1} = Green function and K^* = discrete derivative.
By Kasteleyn's theorem and since graph is bipartite,

$$\mathbb{P}(e_1, \ldots, e_n \in \mathbf{m}) = \det(K^{-1}(e_i, e_j)_{1 \le i,j \le n})$$

so leads to scaling limit for n-point correlation function.

Kasteleyn matrix near monomers

At rows 1, 0, -1 the above analysis breaks down:



but $-\sum_{y \sim x} D(x,y) = 3 - 2z^2$.
Diagonal terms still vanish.

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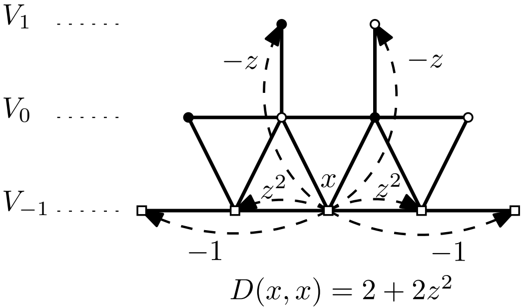

but $-\sum_{y \sim x} D(x,y) = 2 - 2z^2 + 2z$.
Diagonal terms still vanish.

Dealing with negative rates

Let $P(x,y) = -D(x,y)/D(x,x)$: can be signed, don't sum to 1...

Question

Can we still make sense of Green function?

If $\|P\| < 1$ then

$$D^{-1}(x,y) = \frac{1}{D(y,y)} \sum_{\text{path } \pi : x \to y} w(\pi)$$

where

$$w(\pi) = \prod_{(u,v) \in \pi} P(u,v).$$

Monomer excursions

Paths still restricted even $\to$ even and odd $\to$ odd rows.
Decompose in excursions into V_{-1} or V_0.
Eg odd case (harder): let $u, v \in V_1$.

Associated vertices u_-, u_+ and v_-, v_+ in V_{-1}, two steps away.
Let $u_\bullet \in \{u_-, u_+\}$ and $v_\bullet \in \{v_-, v_+\}$.
Let $\pi : u_\bullet \to v_\bullet$. Parity fixed so

$$w(\pi) = (-1)^{v_\bullet - u_\bullet} \prod_{(x,y)\in\pi} p_{x,y}$$

where

$$p_{i,i\pm 1} = \frac{z^2}{2+2z^2} =: 1/2 - p, \qquad p_{i,i\pm 2} = \frac{1}{2+2z^2} =: p.$$

$\implies$ an honest RW on $V_{-1} \simeq \mathbb{Z}$ in limit!

Odd monomer excursions

So

$$\sum_{\pi : u_\bullet \to v_\bullet ; \pi \subset V_{-1}} w(\pi) = (-1)^{v_\bullet - u_\bullet} g_{u_\bullet, v_\bullet}$$

where $g_{x,y} = 1d$ Green function (with certain b.c.).

Sum over $u_\bullet \in \{u_-, u_+\}$, $v_\bullet \in \{v_-, v_+\}$, take local limit $D_n \uparrow \mathbb{H}$,

$$\sum_{\pi : u \to v ; \pi \subset V_{-1}} w(\pi) = C_z (-1)^k (2a_k - a_{k+1} - a_{k-1})$$

where a_k = Potential kernel of 1d walk; $k = \mathrm{Re}(v - u)$.

The miracle

Lemma

$(-1)^k \Delta a_k \geq 0$ for all $k \in \mathbb{Z}$
Moreover $\sum_{k\in\mathbb{Z}} C_z(-1)^k(2a_k - a_{k+1} - a_{k-1}) = 1$.

Gives an effective random walk on $V_1 \cup V_3 \cup \ldots \simeq \mathbb{H}$!

Lemma

$(-1)^k \Delta a_k$ decays exp. fast as $k \to \infty$.

So: reflection on boundary with jumps, but exponential tails!

Proof of oscillations

a_k solves a recurrence relation of order four.
Also by general theory [e.g. Lawler–Limic]:

$$a_x \sim \frac{|x|}{\sigma^2} \text{ as } |x| \to \infty$$

Hence

$$a_x = \frac{|x|}{\sigma^2} + A + B\gamma^{|x|}$$

where

$$1 = (1/2 - p)(\gamma + \gamma^{-1}) + p(\gamma^2 + \gamma^{-2}).$$

Hence let $s = \gamma + \gamma^{-1}$

$$1 = (1/2 - p)s + p(s^2 - 2).$$

Can solve s so $s = 2$ or $s = -1 - 1/(2p)$.
This implies $\gamma \in (-1, 0)$ so oscillations.

Towards scaling limit

Notice that $D^{-1}(u, v)$ not restricted to $B \to B, W \to W$:
However paths must go through boundary !
Eg: $e = (w, b); e' = (w', b')$

$$\mathbb{P}(e, e' \in \mathbf{m}) = \mathrm{Pf}\begin{pmatrix} 0 & K^{-1}(w,b) & K^{-1}(w,w') & K^{-1}(w,b') \\ & 0 & K^{-1}(b,w') & K^{-1}(b,b') \\ & & 0 & K^{-1}(w',b') \\ & & & 0 \end{pmatrix}$$

$$= \overbrace{K^{-1}(w,b)}^{\mathbb{P}(e \in \mathbf{m})}\overbrace{K^{-1}(w',b')}^{\mathbb{P}(e' \in \mathbf{m})} + K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

so

$$\mathrm{Cov}(1_{e \in \mathbf{m}}; 1_{e' \in \mathbf{m}}) = K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

Leads to scaling limit eventually...! □