From Wigner-Dyson to Pearcey: Universality of the Local Eigenvalue Statistics of Random Matrices at the Cusp

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What can be said about the statistical properties of the eigenvalues of a large random matrix?

Do some universal patterns emerge?

Eugene Wigner (1954)

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix} \quad \Longrightarrow \quad (\lambda_1, \lambda_2, \dots, \lambda_N) \quad \text{eigenvalues?}$$

N = size of the matrix, will go to infinity.

Analogy: Central limit theorem: $\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$

$H = (h_{jk})_{1 \le j,k \le N}$ complex hermitian or real symmetric $N \times N$ matrix

$h_{jk} = \bar{h}_{kj}$ (for $j < k$) are indep. random variables with normalization

$$\mathbf{E}\, h_{jk} = 0, \qquad \mathbf{E}\, |h_{jk}|^2 = \frac{1}{N}.$$

The eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N$ are of order one: (on average)

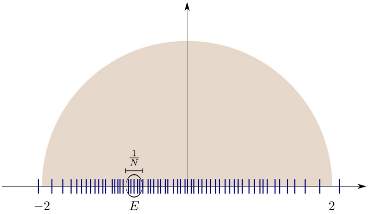
$$\mathbf{E}\, \frac{1}{N} \sum_i \lambda_i^2 = \mathbf{E}\, \frac{1}{N} \operatorname{Tr} H^2 = \frac{1}{N} \sum_{ij} \mathbf{E}\, |h_{ij}|^2 = 1$$

If h_{ij} is Gaussian, then GUE, GOE – (hard) explicit calculations are possible. No exact formula beyond Gaussian.

Goal: Statistics of eigenvalues

Size of the spectrum O(1). Typical ev. spacing (gap) $\approx \frac{1}{N}$ (bulk)

- Macro scale: Semicircle Law
- Meso scale: LLN holds in the entire mesoscopic regime ($\Longrightarrow$ local semicircle law)
- Micro scale: How do individual eigenvalues behave on the scale of spacing? (Wigner-Dyson universality)



Very different behavior than for independent (Poisson) points. Indicates that eigenvalues form a strongly correlated point process.

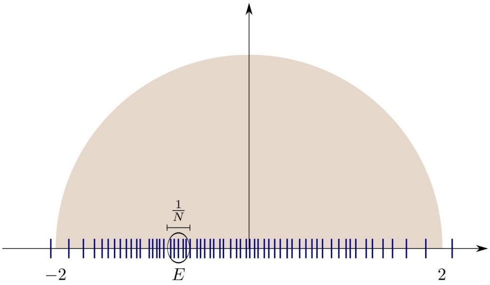
Let H be a Wigner matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$.

Density of eigenvalues on the global (macroscopic) scale:

$$\frac{1}{N}\#\{\lambda_i \in [a,b]\} \to \int_a^b \varrho(x)\mathrm{d}x, \qquad \varrho(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$$

holds for any fixed $[a,b]$ interval.

What about shrinking intervals as $\frac{1}{N} \ll |b-a| \ll 1$? Local Law !



Def: Let μ be a probability measure on $\mathbb{R}$. Its Stieltjes transform at spectral parameter $z \in \mathbb{C}_+$ is given by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x - z}$$

Meaning: $m_\mu(E + i\eta)$ resolves the measure μ around E on scale η

$$\frac{1}{\pi} \, \mathrm{Im}\, m_\mu(z) := (\delta_\eta \star \mu)(E) = \int_{\mathbb{R}} \delta_\eta(x - E)\mathrm{d}\mu(x)$$

where δ_η is an approximate delta fn. on scale η

$$\delta_\eta(x) := \frac{1}{\pi} \, \frac{\eta}{x^2 + \eta^2}, \qquad \int \delta_\eta(x)\mathrm{d}x = 1$$

Inversion formula

$$\lim_{\eta \to 0+} \frac{1}{\pi} \, \mathrm{Im}\, m_\mu(E + i\eta) = \mu(E)\mathrm{d}E \qquad \text{(weakly)}$$

Obvious: The trace of the resolvent G of a hermitian matrix H is the Stieltjes transform of its empirical spectral density:

$$\varrho_N(E) := \frac{1}{N}\sum_{\alpha=1}^{N}\delta(\lambda_\alpha - E), \qquad \frac{1}{N}\operatorname{Tr} G(z) = \frac{1}{N}\sum_\alpha \frac{1}{\lambda_\alpha - z} = m_{\varrho_N}(z)$$

Recall: $\eta = \operatorname{Im} z$ is the resolution scale.

Fact: $\frac{1}{N}\operatorname{Tr} G(z)$ becomes deterministic as long as $\eta \gg$ (ev spacing).
In fact, it holds for $G(z)$ itself [Mesoscopic LLN or local law]

This limit is the Stieltjes transform of the density of states (DOS)

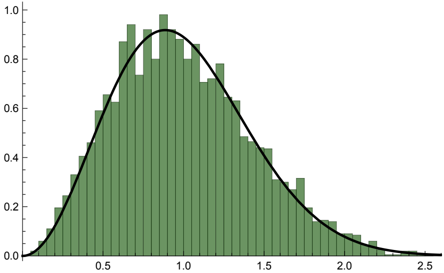
Local law tells us that the eigenvalues are uniformly distributed down to the smallest possible scales η above the local ev spacing:

$$\eta \gg N^{-1} \quad \text{[bulk]}, \qquad \eta \gg N^{-2/3} \quad \text{[edge]}$$

$$\mathbb{P}_{GOE}\Big(N\varrho(\lambda_i)(\lambda_{i+1}-\lambda_i) = s + \mathrm{d}s\Big) \simeq \frac{\pi}{2}\, s\, \mathrm{e}^{-\pi s^2/4}\,\mathrm{d}s$$

$$\mathbb{P}_{GUE}\Big(N\varrho(\lambda_i)(\lambda_{i+1}-\lambda_i) = s + \mathrm{d}s\Big) \simeq \frac{32}{\pi^2}\, s^2\, \mathrm{e}^{-4s^2/\pi}\,\mathrm{d}s\,,$$

for λ_i in the bulk (repulsive correlation!)



Histogram of the rescaled gaps for GUE matrix with $N = 3000$

Wigner-Dyson universality

Wigner's revolutionary observation: The global density may be model dependent, but the gap statistics (i.e. micro-scale fluctuation) is very robust, it depends only on the symmetry class (complex hermitian or real symmetric) and not on other details of the RM ensemble.

In particular, it can be determined from the Gaussian case (GUE/GOE).

Physically the Wigner matrix is a special toy model, but it is the basic prototype of a disordered quantum system in the mean field regime

The universality of microscopic ev. gap fluctuation is expected to hold for models very far from mean field, most prominent example is the Anderson model in the conducting regime:

$$H = \Delta_x + \sum_{i\in\Lambda} v_i \delta(x-i) \qquad \text{on } \ell^2(\Lambda), \text{ with } \Lambda = [-L, L]^d \cap \mathbb{Z}^d$$

where $\{v_i \,:\, i \in \Lambda\}$ is i.i.d., $L \to \infty$. Big open question!

- Wigner matrix: i.i.d. entries, $s_{ij} := \mathbf{E}\,|h_{ij}|^2$ are constant $(= \frac{1}{N})$
  (Density = semicircle; $G \approx$ diagonal, $G_{xx} \approx G_{yy}$)
  [E-Schlein-Yau-Yin, 2009–2011], [Tao-Vu, 2009]
  (Same if $\sum_j s_{ij} = 1$, for all i – [E-Yau-Yin, '11] [E-Knowles-Yau-Yin, '12])

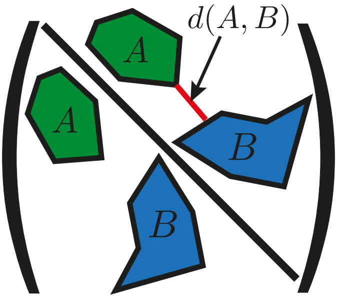
- Wigner type matrix: indep. entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \approx$ diagonal, $G_{xx} \not\approx G_{yy}$)
  [Ajanki-E-Krüger, 2015]

- Correlated Wigner matrix: correlated entries, s_{ij} arbitrary
  (Density $\neq$ semicircle; $G \not\approx$ diagonal)
  [Ajanki-E-Krüger '15-'16] [Che '16], [E-Krüger-Schröder '17], [Alt-E-Krüger-Schröder '18]

Typical correlation structure: like 2d random field

$$\mathrm{Cov}\big(\phi(W_A), \psi(W_B)\big) \leqslant \frac{C_K \|\nabla\phi\|_\infty \|\nabla\psi\|_\infty}{[1+\mathrm{dist}(A,B)]^{12}}$$

for any $A, B \subset S \times S$, assuming the usual metric on the set $S = \{1, 2, \dots, N\}$ of indices. Here $W_A = \{W_{ij} \;:\; (i,j) \in A\}$.

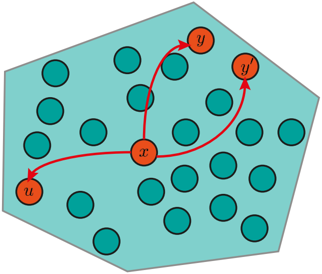


+ Matching bounds for higher cumulants (smallest spanning tree)

H is viewed as a $\Sigma \times \Sigma$ matrix (operator) acting on $\ell^2(\Sigma)$.

Equip the configuration space Σ with a metric to have "nearby" states.

It is reasonable to assume that h_{xy} and $h_{xy'}$ are correlated if y and y' are close with a decaying correlation as $\mathrm{dist}(y, y')$ increases.



Non-trivial spatial structure changes the density of states but not the micro statistics!

Other extensions of the original Wigner model

- Invariant ensembles: $P(H) \sim \exp\big[-\beta N \,\mathrm{Tr}\, V(H)\big]$

  Deift et. al., Valko-Virag, Pastur-Shcherbina, Bourgade-E-Yau, Bekerman-Guionnet-Figalli

- Low moment assumptions, heavy tails

  Johansson, Guionnet-Bordenave, Götze-Naumov-Tikhomirov, Benaych-Peche, Aggarwal

- Deformed models, general expectation

  O'Rourke-Vu, Lee-Schnelli-Stetler-Yau, He-Knowles-Rosenthal

- Sparse matrices, Erdős-Rényi and d-regular graphs

  E-Knowles-Yau-Yin, Huang-Landon-Yau, Bauerschmidt-Huang-Knowles-Yau, etc.

- Nonhermitian matrices

  Girko, Bai, Tao-Vu-Krishnapur, Bordenave-Chafai, Fyodorov, Bourgade-Yau-Yin, Alt-E-Kruger, E-Kruger-Renfrew, Bourgade-Dubach

- Band matrices

  Fyodorov-Mirlin, Disertori-Pinson-Spencer, Schenker, Sodin, E-Knowles-Yau, T. Shcherbina, Bourgade-E-Yau-Yin, E-Bao, Bourgade-Yau-Yin etc.

Many other directions and references are left out, apologies...

$G(z) = (H - z)^{-1}$, where $H = H^*$ has a correlation structure given

$$\mathcal{S}[R] := \mathbf{E}(H - A)R(H - A), \qquad A := \mathbf{E}\, H$$

Theorem [AEK, EKS] In the bulk spectrum, $\varrho(\Re z) \geq c$, we have

$$|G_{xy}(z) - M_{xy}(z)| \lesssim \frac{1}{\sqrt{N\ \operatorname{Im} z}}, \quad \big|\langle G(z) - M(z)\rangle\big| \lesssim \frac{1}{N\ \operatorname{Im} z}$$

with very high probability, where $\langle A\rangle := \frac{1}{N}\operatorname{Tr} A$.
M is given by the solution of the Matrix Dyson Equation (MDE)

$$-\frac{1}{M} = z - A + \mathcal{S}[M], \qquad \operatorname{Im} M := \frac{M - M^*}{2i} \geq 0, \quad \operatorname{Im} z > 0$$

Self-consistent DOS

$$\rho(E) := \pi^{-1}\langle \operatorname{Im} M(E + i0)\rangle$$

Depends only on the first two moments of H.

Local law on optimal scale implies rigidity of the eigenvalues on the optimal scale, i.e. that eigenvalues are close to the corresponding quantiles of the density.

Given a density ρ and $x \in \mathbb{R}$, define the local spacing $\eta_f(x)$ as

$$\int_{x-\eta_f}^{x+\eta_f} \rho(y)\mathrm{d}y = \frac{2}{N}$$

Let $\gamma_i = \gamma_i^{(N)}$ be the i-th N-quantile of ρ:

$$\int_{-\infty}^{\gamma_i} \rho(y)\mathrm{d}y = \frac{i}{N}$$

Lemma: Optimal local law (in averaged form) implies

$$|\lambda_i - \gamma_i| \leq N^{\varepsilon} \eta_f(\gamma_i)$$

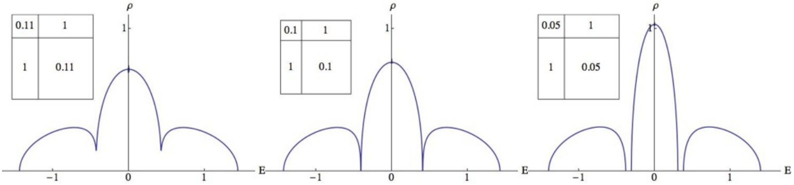
with very high probability.

Theorem [Ajanki-E-Krüger, Alt-E-Krüger] Let $\mathcal{S}$ be flat ($\sim$ mean field):

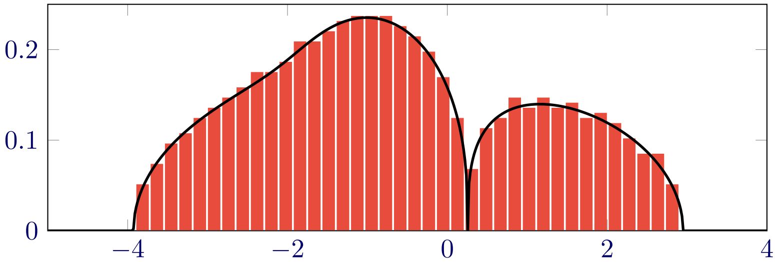
$$c\langle R\rangle \le \mathcal{S}[R] \le C\langle R\rangle, \quad \forall R \ge 0, \qquad \text{and} \quad \|A\| \le C$$

Then the DOS is compactly supported, real analytic, $1/3$-Hölder continuous. Only two types of singularities occur: square root edge or cubic root cusp. The density profile near the cusp is universal.

Example (Wigner type): Support splits via cusps:



(Matrices in the pictures represent the variance matrix)



Histogram of eigenvalues of a Wigner type matrix with nontrivial expectation in the diagonal.

Solid line: (self-consistent) density of states computed from Dyson equation.

$p_N(x_1, x_2, \dots, x_N)$ is the (symmetrized) joint prob. density of the ev's

Local statistics is expressed via the k-point correlation functions

$$p_N^{(k)}(x_1, \dots x_k) = \int p_N(x_1, \dots x_k, x_{k+1}, \dots, x_N) \mathrm{d}x_{k+1} \dots \mathrm{d}x_N$$

Gap etc. distribution follows from this information.

In most Gaussian cases, the correlation functions are determinantal , i.e.

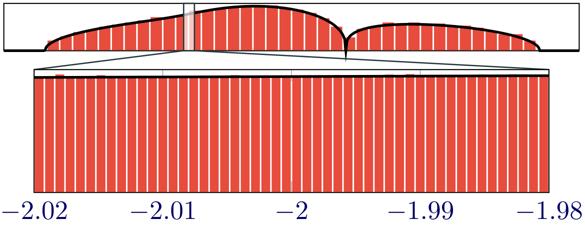
$$p_N^{(k)}(x_1, \dots x_k) = \det \big[K_N(x_i, x_j) \big]_{i,j=1}^{k}$$

with a kernel K_N. After appropriate rescaling (micro scale around a fixed energy E with density $\rho := \rho(E)$), it has a limit, e.g. in the GUE bulk

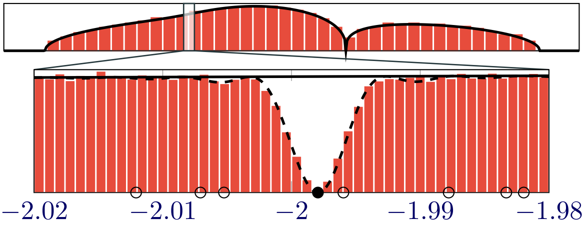
$$\frac{1}{\rho^2} K_N\Big(E + \frac{x}{\rho N}, E + \frac{y}{\rho N} \Big) \ \to \ K_{\text{sine}}(x, y) := \frac{\sin \pi (x-y)}{\pi (x-y)}$$

$$p_N^{(2)}\Big(E + \frac{x}{\rho N}, E + \frac{y}{\rho N} \Big) \to 1 - \Big(\frac{\sin \pi (x-y)}{\pi (x-y)} \Big)^2$$

Interpreted as conditional prob. to find an ev at x assuming there is at y.

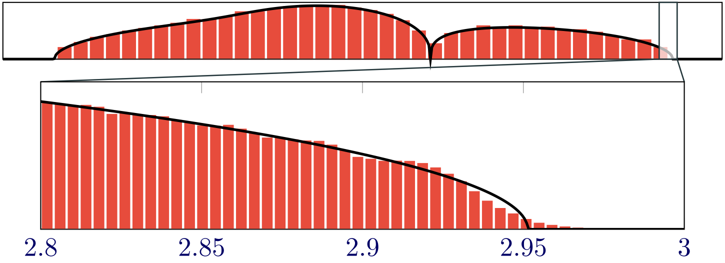


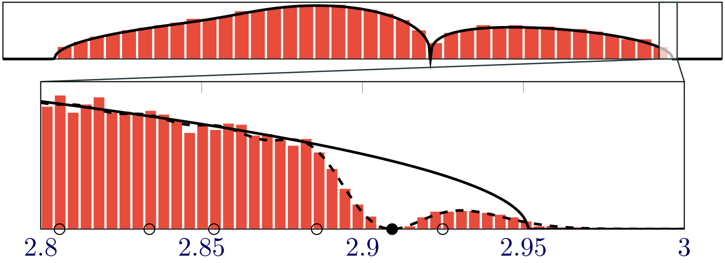
Local density (1 point correlation function)

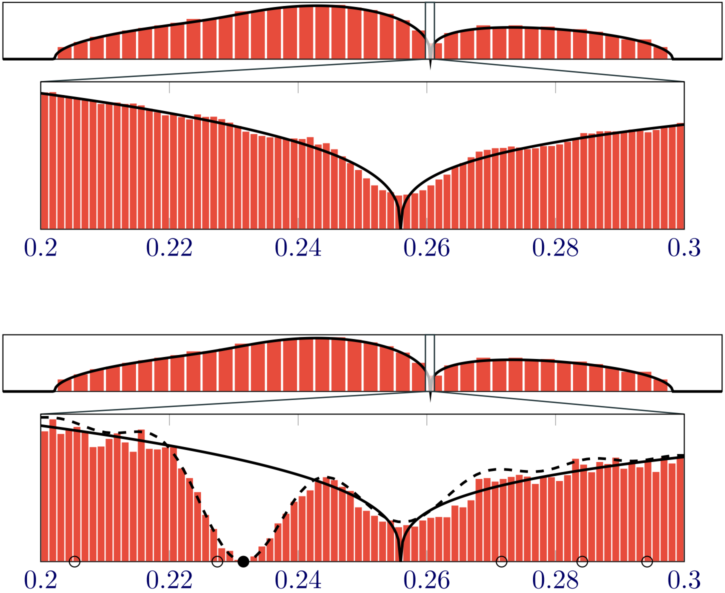


Conditional density of all other evalues, if there is an e.v. at the black dot.

Local 1 and 2 point correlation functions at the edge






0.2
0.22
0.24
0.26
0.28
0.3
0.2
0.22
0.24
0.26
0.28
0.3

Resolution of the Wigner-Dyson-Mehta universality conjecture

Theorem. The gap distribution of a Wigner matrix in the bulk spectrum is universal, it depends only on the symmetry type and is independent of the distribution of the matrix elements. In particular, it coincides with that of the computable Gaussian case.

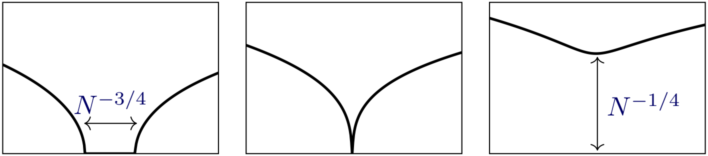
[Johansson, 2000] Hermitian case with large Gaussian component
[E-Peche-Ramirez-Schlein-Yau, 2009] Hermitian case with smoothness
[Tao-Vu, 2009] Hermitian case via moment matching
[E-Schlein-Yau, 2009] Symmetric and hermitian cases via Dyson Brownian motion
[E-Schlein-Yin-Yau, 2010] Generalized Wigner matrices
[E-Knowles-Yin-Yau, 2012] Sparse matrices, Erdős-Rényi graphs.
[E-Yau, 2013] Single gap universality
[Bourgade-E-Yin-Yau, 2014] Fixed energy universality
[Ajanki-E-Krüger, 2015] General variance profile – beyond semicircle
[Ajanki-E-Krüger, 2016] Beyond independence: general correlations included.
[E-Krüger-Schröder, 2017] Long range correlation

Similar development at the edge and very recently at the cusp:

[E-Krüger-Schröder, 2018] Pearcey (cusp) universality for complex hermitian
[Cipolloni-E-Krüger-Schröder, 2018] Cusp universality for real symm.

In contrast to bulk and edge, the "cusp" covers an entire regime characterized by eigenvalue spacing $\sim N^{-3/4}$:

Three cases of physical cusp: small gap, exact cusp, small minimum



Three parameters: location, "slope", deviation from exact cusp
After shifting/scaling, the universality class is described by a one parameter family.

Theorem [E-Krüger-Schröder 2018] Let $H = A + W$ be complex Hermitian Wigner type matrix with variance profile $s_{ij} = \mathbf{E}\,|w_{ij}|^2$ and diagonal expectation $A = \mathbf{E}\,H = \mathrm{diag}(\mathbf{a})$ s.t.

- S is flat, i.e. $\frac{c}{N} \le s_{ij} \le \frac{C}{N}$, $|a_{ii}| \le C$
- W is genuinely complex: $|\,\mathbf{E}\,\mathrm{Re}\, w\, \mathrm{Im}\, w|^2 \le (1-\varepsilon)\,\mathbf{E}\,\mathrm{Re}^2\, w\,\mathbf{E}\,\mathrm{Im}^2\, w^2$
- DOS ρ has a physical cusp: local min of size $\rho(\mathfrak{m}) \lesssim N^{-1/4}$ or a small gap $[\mathfrak{e}^-, \mathfrak{e}^+]$ of size $\Delta := \mathfrak{e}^+ - \mathfrak{e}^- \lesssim N^{-3/4}$
- $m(z)$ is bounded in the vicinity of the physical cusp.

Then the k point correlation function $p_N^{(k)}$ satisfies

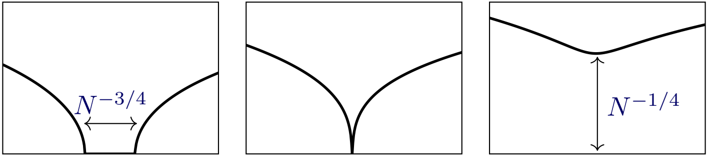
$$\int_{\mathbb{R}^k} F(x)\Big[\frac{N^{k/4}}{\gamma^k} p_N^{(k)}\Big(\mathfrak{b} + \frac{x}{\gamma N^{3/4}}\Big) - \det\big(K_\alpha(x_i, x_j)\big)_{i,j=1}^k\Big]\mathrm{d}x = O(N^{-c(k)})$$

where

$$\mathfrak{b} := \begin{cases} \mathfrak{m} \\ \frac{1}{2}(\mathfrak{e}^+ + \mathfrak{e}^-) \end{cases}, \qquad \alpha := \begin{cases} -(\pi\rho(\mathfrak{m})/\gamma)^2 N^{1/2} \\ 3(\gamma\Delta/4)^{2/3} N^{1/2} \end{cases}$$

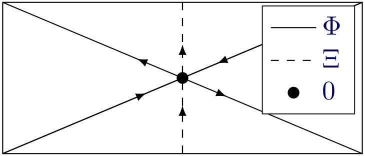
where γ is the slope parameter and K_α is the 1-parameter family of Pearcey kernels.

Three cases of physical cusp. Note: $N^{-3/4}$ is the eigenvalue spacing



Pearcey kernel ($\alpha = 0$ is the exact cusp)

$$K_\alpha(x,y) = \frac{1}{(2\pi i)^2}\int_\Xi\int_\Phi \frac{\exp(-\frac{1}{4}w^4 + \frac{\alpha}{2}w^2 - yw + \frac{1}{4}z^4 - \frac{\alpha}{2}z^2 + xz)}{w-z}\mathrm{d}w\,\mathrm{d}z$$



[Pearcey 1940', Brézin-Hikami 1998: GUE $+\mathrm{diag}(1,\ldots,1,-1,\ldots-1)$]

Theorem [Cipolloni-E-Krüger-Schröder 2018] Let $H = A + W$ be real symmetric Wigner type matrix with flat variance profile $s_{ij} = \mathbf{E}\,|w_{ij}|^2$ and diagonal expectation as before. Then the local correlation functions at the physical cusp are universal in a sense that they coincide with those of a Gaussian reference model

$$GOE + \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots -1)$$

Key points

- Optimal local law in both symmetry classes
- In complex Hermitian case we use Brézin-Hikami formula + contour integration
- In real symmetric case no Brézin-Hikami. In fact, no explicit "Pearcey" formula is known.
- We use Dyson Brownian motion adapting the edge analysis of [Landon-Yau, 2017] to the cusp (Second lecture)

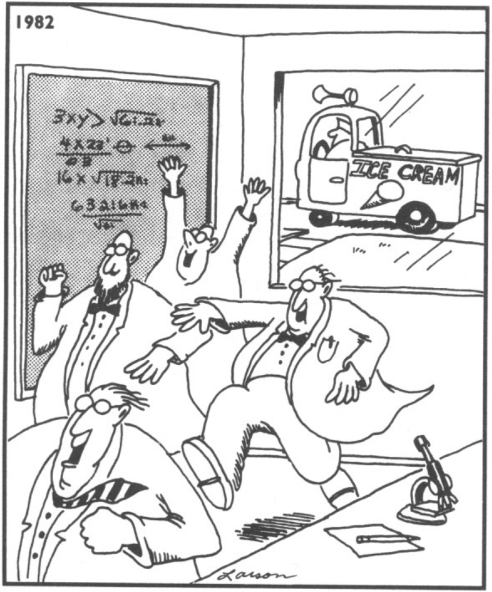
1. Prove optimal local law just above the optimal scale.
2. Use the fast local equilibration property of the Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component
3. Use perturbation theory to remove the Gaussian component

Step 2 and Step 3 need Step 1 as an apriori bound for input.

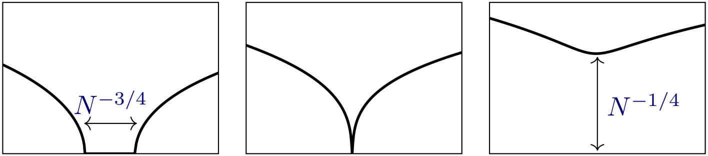
Step 2 needs a lower bound on the size of the Gaussian component, while Step 3 needs an upper bound. Surprisingly, there is quite a room to match.

Reason: DBM is very efficient on small scales.

Thanks for the attention!



$H = \mathrm{diag}(a) + W$ has indep. elements with $s_{ij} = \mathbf{E}\,|w_{ij}|^2 \sim \frac{1}{N}$ with a self-consistent DOS with a physical cusp (ev. spacing $\sim N^{-3/4}$) with Pearcey parameter α.



GOAL: Local statistics of ev's $\lambda = \mathrm{Spec}(H)$ coincide with those of μ, ev's of the canonical reference ensemble

$$\mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) + (1 - \alpha N^{-1/2})^{1/2} GOE.$$

Real symmetric class is much harder (no contour integral) but also more conceptual.

Recapitulaton: Three step strategy

1. Prove optimal local law just above the optimal scale.
2. Use the fast local equilibration property of the Dyson Brownian motion to prove universality for matrices with a tiny Gaussian component
3. Use perturbation theory to remove the Gaussian component

Originally developed for the bulk [E-Schlein-Yau-Yin, 2009], then extended to the edge [Bourgade-E-Yau, 2013]. Methods have changed over the years.

Step 2 used to rely on stoch. particle system ideas (entropy, Dirichlet form), more recently it became PDE flavored. The core is a 1+1 dim parabolic equation with time dependent, singular and long range coeff's [E-Yau, Bourgade-Yau, E-Schnelli, Landon-Yau, Landon-Huang].

For a while I will explain bulk/edge/cusp together, then specialize to cusp.

Especially important predecessor is [Landon-Yau, 2017] at the edge.

Cusp is much more involved than edge. We heavily rely on the fine shape analysis of the MDE [Alt-E-Krüger, 18] + Rigidity [E-Krüger-Schröder]

Let H be a correlated matrix with first two moments given by

$$A = \mathbf{E}\, H, \qquad \mathcal{S}[R] = \mathbf{E}(H-A)R(H-A)$$

Consider the following matrix SDE (OU-flow)

$$\mathrm{d}H_s = -\frac{1}{2}(H_s - A)\mathrm{d}s + \Sigma^{1/2}[\mathrm{d}B_s], \qquad \Sigma[R] := \mathbf{E}(H-A)\,\mathrm{Tr}\left[(H-A)R\right]$$

where B_s is a matrix of standard BM's of the same symmetry class as H.

Fact: The flow preserves the expectation A and variance $\mathcal{S}$, hence it preserves the solution M to the MDE and the self consistent DOS ρ.

Theorem of Step 3: The local ev. stat of H and H_T coincide if

$$T \ll N^{-1/2} \quad \text{[bulk]}, \qquad T \ll N^{-1/4} \quad \text{[cusp]}, \qquad T \ll N^{-1/6} \quad \text{[edge]}$$

Proof: Perturbation theory: higher order correlations of H and H_T differ, but only by an order $O(T)$.

Step 3: Perturbation theory of OU flow

Correlation functions are expressible in terms of expectations of products of

$$X(E) := \eta \,\mathrm{Im}\,\mathrm{Tr}\, G(E+i\eta) = \sum_\alpha \frac{\eta^2}{(\lambda_\alpha - E)^2 + \eta^2}$$

if η is below the scale of the local eigenvalue spacing η_f:

$$\eta_f = N^{-1} \quad \text{[bulk]}, \qquad \eta_f = N^{-3/4} \quad \text{[cusp]}, \qquad \eta_f = N^{-2/3} \quad \text{[edge]}$$

Study

$$\mathbf{E}\prod_j X_T(E_j) - \mathbf{E}\prod_j X_0(E_j) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\prod_j X_s(E_j)\mathrm{d}s$$

via Ito calc. using that $G_s(z) = (H_s - z)^{-1}$ is a (quite singular) fn. of H_s.

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\, f(H) = \mathbf{E}\Big[-\frac{1}{2}\sum_\alpha h_\alpha \partial_\alpha f + \frac{1}{2}\sum_{\alpha,\beta}\kappa(\alpha,\beta)\partial_\alpha\partial_\beta f\Big], \qquad \alpha = (x,y)$$

and use cumulant expansion. After cancellations, only third and higher order cumulants/derivatives matter.

$$\mathbf{E}\prod_j X_T(E_j) - \mathbf{E}\prod_j X_0(E_j) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}\prod_j X_s(E_j)\mathrm{d}s \qquad (*)$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}\,X_s \sim \eta\,\mathrm{Im}\,\mathbf{E}\,\mathrm{Tr}\,GGGG + \dots$$

Estimate each factor G at scale $\eta \ll \eta_f$ by G at scale $\eta' = N^\varepsilon \eta_f$, where local law is available. This (morally) guarantees the bound

$$G_{xy}(E+i\eta) \lesssim \frac{\eta'}{\eta} G_{xy}(E+i\eta') \lesssim N^\varepsilon \frac{1}{\sqrt{N\eta'}} \qquad \text{[bulk]}$$

The rest is a careful power counting and if T is not too big then (*) is affordably small. □

Step 2: Universality with small Gaussian component

$$\mathrm{d}H_s = -\frac{1}{2}(H_s - A)\mathrm{d}s + \Sigma^{1/2}[\mathrm{d}B_s], \qquad \Sigma[R] := \mathbf{E}(H-A)\,\mathrm{Tr}\left[(H-A)R\right]$$

Since $\mathcal{S}$ is flat, $\mathcal{S}[R] \geq c\langle R\rangle$, OU flow adds a Gauss component of size T:

$$H_T \stackrel{d}{=} \widetilde{H}_T + \sqrt{T}U, \qquad U \sim \text{GUE/GOE indep of } \widetilde{H}_T$$

where $\widetilde{H}_T$ is a correlated matrix with $\widetilde{\mathcal{S}}_T = \mathcal{S} - T\langle\cdot\rangle$, still flat, hence optimal local law and rigidity holds for $\widetilde{H}_T$.

Thm of Step 2: Suppose the evalues of some $H^\#$ are optimally rigid and

$$T \gg N^{-1} \quad \text{[bulk]}, \qquad T \gg N^{-1/2} \quad \text{[cusp]}, \qquad T \gg N^{-1/3} \quad \text{[edge]}$$

then the eigenvalue statistics of $H^\# + \sqrt{T}U$ are universal.

$H^\#$ can even be deterministic and one may assume it is diagonal. The entire stochastic effect comes from the GOE/GUE component U.

Step 2 & Step 3 $\Longrightarrow$ univ. of the original H (there is room to choose T)

Consider the DBM matrix flow (not the previous OU flow!)

$$\mathrm{d}H_s = \mathrm{d}B_s, \qquad H_0 := H^{\#} \quad \Longrightarrow \quad H_s \stackrel{d}{=} H^{\#} + \sqrt{s}U$$

Fact [Dyson]: The eigenvalues $\lambda_i = \lambda_i(s)$ satisfy the Dyson Brownian Motion (DBM) with initial condition $\lambda_i(0) = \lambda_i(H^{\#})$:

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j}\mathrm{d}t, \qquad i = 1, 2, \dots, N$$

where $(b_1, b_2, \dots, b_N)$ are indep. standard Brownian motions ($\beta = 1, 2$).

Key property: Local equilibration ($\Longrightarrow$ universality) happens when

$$\Big|\int_0^T \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i\Big| \sim \sqrt{\frac{T}{N}} \quad \text{is much bigger than} \quad \eta_f \quad \text{[local spacing]}$$

E.g. at the cusp $\sqrt{T/N} \gg N^{-3/4}$ gives the choice $T = N^{-1/2+\varepsilon}$.

Coupling of two DBM

We consider two DBM, $\lambda(s)$ starting from $\lambda(H^{\#})$ and $\mu(s)$ starting from a reference Gaussian ensemble, coupled via the same Brownian motions!
[Bourgade-E-Yau-Yin, 2014]

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\mathrm{d}t,$$

$$\mathrm{d}\mu_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{\mu_i-\mu_j}\mathrm{d}t,$$

The difference $\nu_i := \lambda_i - \mu_i$ satisfies a deterministic parabolic equation

$$\frac{\mathrm{d}\nu_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j\neq i}\frac{\nu_j-\nu_i}{(\lambda_i-\lambda_j)(\mu_i-\mu_j)} =: \sum_j \mathcal{B}_{ij}(\nu_j-\nu_i) := (\mathcal{B}\nu)_i$$

with time dependent coefficients and with very good contraction properties.
"Long time" heat kernel bound forces λ's to trail μ's so that their local statistics asymptotically coincide.
To control this, we describe the evolution of the density near the cusp.

The (self-consistent) DOS ρ_s of the DBM matrix flow

$$\mathrm{d}H_s = \mathrm{d}B_s, \qquad H_0 := H^{\#} \quad \Longrightarrow \quad H_s \stackrel{d}{=} H^{\#} + \sqrt{s}U$$

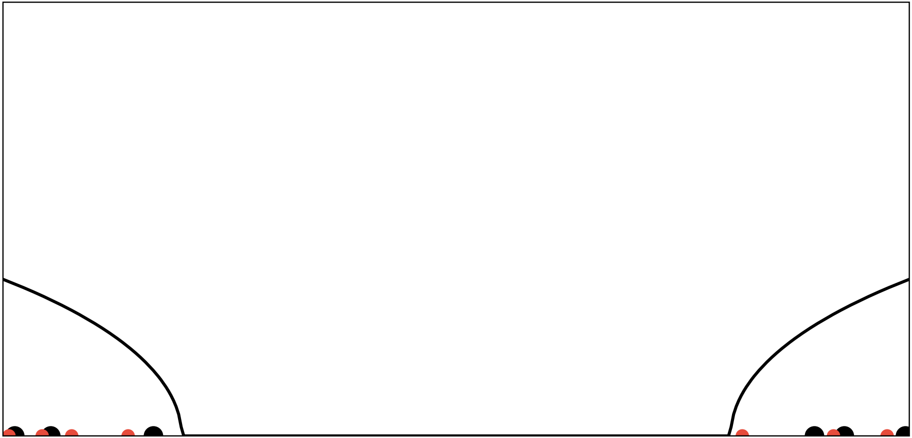
is the free convolution (fc) of the initial density $\rho_0 = \rho^{\#}$ with a semicircle law of variance s:

$$\rho_s = \rho \boxplus \sqrt{t}\rho_{\rm sc}, \qquad m_s^{\rm fc}(\zeta) = m(\zeta + s m_s^{\rm fc}(\zeta)), \qquad \zeta \in \mathbb{C}_+, \quad s \geq 0.$$

This flow is analyzed in great details at the cusp formation. Starting with a small gap $[E_0^-, E_0^+]$ with length $\Delta_0 = E_0^+ - E_0^-$ at time 0 and developing a cusp at time T, ρ_s evolves on three scales.

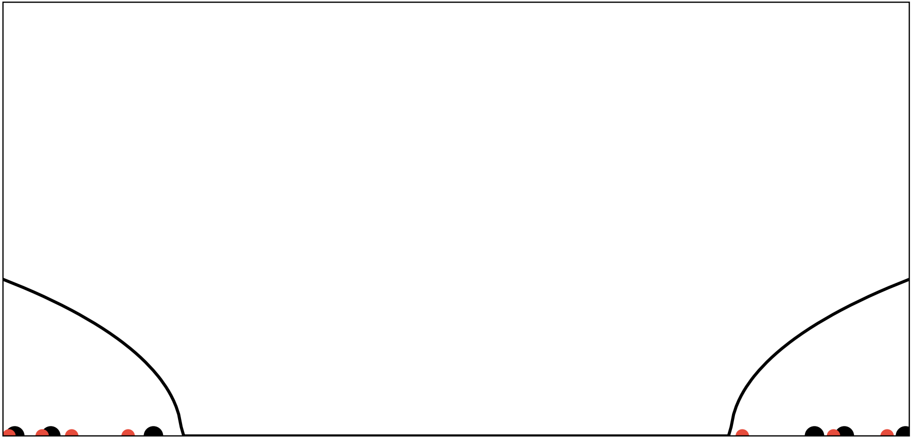
Similarly, we have precise description of the quantiles (cusp local law!)

Shift/scale $\Longrightarrow$ both ensembles develop cusp at T ($\sim N^{-1/2+\varepsilon}$).
For presentation simplicity, we assume that locally the two SCflows are very close (needs nontriv adjustment). See movie!



[Special thank to Dominik Schröder for the movie]

Evolution of two coupled DBM near the cusp



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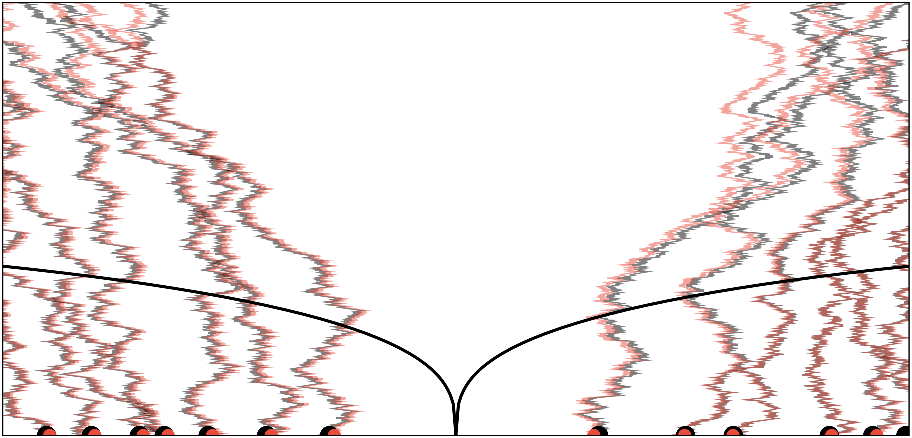
Evolution of two coupled DBM near the cusp



[Special thank to Dominik Schröder for the movie]

$$\frac{d\nu_i}{dt} = \frac{1}{N}\sum_{j\neq i}\frac{\nu_j-\nu_i}{(\lambda_i-\lambda_j)(\mu_i-\mu_j)} =: \sum_j \mathcal{B}_{ij}(\nu_j-\nu_i) := (\mathcal{B}\nu)_i$$

Clearly $\mathcal{B}$ is a symmetric negative operator with quadratic form

$$\langle \nu, \mathcal{B}\nu\rangle = -\frac{1}{2}\sum_{ij}\mathcal{B}_{ij}(\nu_i-\nu_j)^2$$

Assuming rigidity for both λ, μ we have

$$\mathcal{B}_{ij} \gtrsim \frac{N}{|i-j|^2} \text{ [bulk]}, \quad \mathcal{B}_{ij} \gtrsim \frac{N^{1/2}}{|i^{\frac{3}{4}}-j^{\frac{3}{4}}|^2} \text{ [cusp]}, \quad \mathcal{B}_{ij} \gtrsim \frac{N^{1/3}}{|i^{\frac{2}{3}}-j^{\frac{2}{3}}|^2} \text{ [edge]}$$

indicating the [very naive] sizes, N, $N^{1/2}$ and $N^{1/3}$, of these operators.

The equilibration times are exactly the inverses: $T \gg N^{-1}, N^{-1/2}, N^{-1/3}$.

To do it properly, we use Nash method for $\ell^1 \to \ell^\infty$ heat kernel bounds.

$$\partial_t \nu = \mathcal{B}\nu \qquad \Longrightarrow \qquad \partial_t \|\nu\|_2^2 = \langle \nu, \mathcal{B}\nu \rangle \le 0$$

Sobolev (Gagliardo-Nirenberg) inequality [bulk]

$$-\langle \nu, \mathcal{B}\nu \rangle \gtrsim N \|\nu\|_4^4 \|\nu\|_2^{-2}$$

Thus we have

$$\partial_t \|\nu\|_2^2 \lesssim -N \|\nu\|_4^4 \|\nu\|_2^{-2} \le -N \|\nu\|_2^4 \|\nu\|_1^{-2} \qquad \text{[Hölder: } \|\nu\|_2 \le \|\nu\|_1^{1/3} \|\nu\|_4^{2/3}]$$

Integrating from 0 to t and use the ℓ^1-contraction, $\|\nu(t)\|_1 \le \|\nu(0)\|_1$, get

$$\|\nu(t)\|_2 \lesssim (tN)^{-1/2} \|\nu(0)\|_1$$

By duality we get $\ell^2 \to \ell^\infty$ bound and combining them gives

$$\|\nu(t)\|_\infty \lesssim (Nt)^{-1} \|\nu(0)\|_1$$

Gives contraction of $\max_i |\lambda_i - \mu_i|$ after time $t \gg N^{-1}$. [E-Yau, 2013]
Global ℓ^1 norm is too big ($\sim O(1)$), need to localize by finite speed of propagation. Also: μ can be a good model of λ only locally.

Let $z_i(t) = \lambda_i(t)$ or $\mu_i(t)$ DBM processes of the two sets of ev's.

$$\mathrm{d}z_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{z_i - z_j}\mathrm{d}t,$$

The long range interaction is replaced by its mean field value, ie. define

$$\mathrm{d}\widehat{z}_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \Big[\frac{1}{N}\sum_{|i-j|\leq L}\frac{1}{\widehat{z}_i - \widehat{z}_j} + \int_{\mathcal{I}_i^z(t)^c}\frac{\rho_t^z(x)\mathrm{d}x}{\widehat{z}_i - x}\Big]\mathrm{d}t, \quad \widehat{z}(0) = z(0)$$

where $\mathcal{I}_i^z = [\gamma_{i-L}^z, \gamma_{i+L}^z]$, recalling that γ_i^z are the quantiles of ρ^z.
[Landon-Yau, E-Schnelli 2015]

Lemma [Caricature]: Assuming some rigidity along the flow,

$$\max_i |z_i(t) - \gamma_i^z(t)| \leq N^{-\frac{3}{4}+a}, \qquad t \leq T = N^{-1/2+\varepsilon}$$

then the short range approx $\widehat{z}$ is good if $L \geq N^{\varepsilon'}$ is sufficiently large:

$$|z_i(t) - \widehat{z}_i(t)| \ll N^{-3/4}, \qquad t \leq T$$

Proof. Subtracting the two equations, we get for $w_i = z_i - \widehat{z}_i$

$$\partial_t w = \mathcal{L}w + \zeta := \widetilde{\mathcal{B}}w + \mathcal{V}w + \zeta \qquad (1)$$

$$(\widetilde{\mathcal{B}}w)_i := \sum_j \widetilde{\mathcal{B}}_{ij}(w_j - w_i), \qquad (\mathcal{V}w)_i := -\int_{\mathcal{I}_i^z} \frac{\rho_t^z(x)\mathrm{d}x}{(\widehat{z}_i - x)(z_i - x)}$$

again with a time dependent positive kernel

$$\widetilde{\mathcal{B}}_{ij} := \frac{\mathbf{1}(|i-j| \le L)}{N(z_i - z_j)(\widehat{z}_i - \widehat{z}_j)} \ge 0$$

and

$$\zeta_i := \frac{1}{N} \sum_{|j-i| \ge L} \frac{1}{z_i - z_j} - \int_{\mathcal{I}_i^z} \frac{\rho_t^z(x)\mathrm{d}x}{z_i - x} \ll N^{-1/4}$$

by shape analysis of ρ_t^z and rigidity since $|\mathcal{I}_i^z| \gg N^{-3/4}$ [rigidity scale].

$$\|w(t)\|_\infty = \Big| \int_0^t \mathcal{U}^{\mathcal{L}}(s,t)\zeta(s)\mathrm{d}s \Big|_\infty \le t \cdot \sup_s \|\zeta(s)\|_\infty$$

by Duhamel ($\mathcal{U}^{\mathcal{L}}$ is the propagator of (1)). Many fine details left out. □

We replace z $(=\lambda,\mu)$ with $\widehat{z}$ and repeat the comparison for $\widehat{\nu} := \widehat{\lambda} - \widehat{\mu}$.

$$\frac{\mathrm{d}\widehat{\nu}_i}{\mathrm{d}t} = \frac{1}{N} \sum_{|i-j|\le L} \frac{\widehat{\nu}_j - \widehat{\nu}_i}{(\widehat{\lambda}_i - \widehat{\lambda}_j)(\widehat{\mu}_i - \widehat{\mu}_j)} + \widehat{\mathcal{V}}_i \widehat{\nu}_i =: (\widehat{\mathcal{B}}\widehat{\nu})_i + \widehat{\mathcal{V}}_i \widehat{\nu}_i =: (\widehat{\mathcal{L}}\widehat{\nu})_i$$

The kernel $\widehat{\mathcal{B}}$ has a short range of size $L \sim N^{\varepsilon}$ so for short time the offdiag elements of the propagator are very small. [E-Yau 2013]

Lemma [Caricature]: Assuming some rigidity along the flow,

$$\max_i |\widehat{z}_i(t) - \gamma_i^z(t)| \le N^{-\frac{3}{4}+a}, \qquad t \le T = N^{-1/2+\varepsilon}$$

and assume $L \gg N^{4\varepsilon}$, then for $|a| \ge 2L$ and $|b| \le L$ we have

$$\sup_{0\le s\le t\le T} \mathcal{U}_{ab}^{\widehat{\mathcal{L}}}(s,t) \le N^{-100}$$

Finite speed of propagation. Cont'd

Proof. Let f solve

$$\partial_t f = \widehat{\mathcal{L}} f, \qquad f(0) = \delta_a \qquad \Longrightarrow \quad \mathcal{U}^{\widehat{\mathcal{L}}}_{ab}(0,t) = f_b(t),$$

and set

$$F(t) := \sum_b f_b(t)^2 e^{c|\widehat{z}_b - \gamma_a|}$$

Compute dF [long] and find that $\sup_{t\le T} F(t) \le C$, giving exponential decay for f_b away from a. □

Almost correct....

Recall

$$\widehat{\mathcal{B}}_{ij} := \frac{\mathbf{1}(|i-j| \le L)}{N(\widehat{\lambda}_i - \widehat{\lambda}_j)(\widehat{\nu}_i - \widehat{\nu}_j)} \tag{2}$$

By rigidity we have a lower bound on $\widehat{\mathcal{B}}_{ij}$ with high prob. This is sufficient for Sobolev, Nash, heat kernel decay etc.

However, for finite speed of prop., one also needs some upper bound on $\widehat{\mathcal{B}}_{ij}$. We have level repulsion, but not in high probability – very cumbersome. If instead of (2) we had

$$\frac{\mathbf{1}(|i-j| \le L)}{N(\widehat{\lambda}_i - \widehat{\lambda}_j)^2}$$

then large $\widetilde{\mathcal{B}}_{ij}$ would not be a problem; there is a nice cancellation in $\mathrm{d}F$ [Bourgade-Yau 2014]

Repeat everything for a continuous interpolation between λ and ν.

For any $\alpha \in [0,1]$, set the initial condition

$$z_i(t=0,\alpha) := \alpha\lambda_i(0) + (1-\alpha)\nu_i(0)$$

for the DBM

$$\mathrm{d}z_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{1}{z_i - z_j}\mathrm{d}t, \qquad z_i = z_i(t,\alpha)$$

Clearly $z_i(t,\alpha=0) = \mu_i(t)$, $z_i(t,\alpha=1) = \lambda_i(t)$ and write

$$\lambda_i(t) - \mu_i(t) = \int_0^1 w_i(t,\alpha)\mathrm{d}\alpha, \qquad w := \frac{\mathrm{d}z}{\mathrm{d}\alpha}$$

w is a cont. interpolation of the difference process $\nu = \lambda - \mu$, satisfying

$$\partial_t w_i = \frac{1}{N}\sum_{j\neq i}\frac{w_j - w_i}{(z_i - z_j)^2} =: (\mathcal{L}w)_i \qquad \forall\alpha \in [0,1].$$

Let $\widehat{w}$ be its short range approximation dynamics with generator $\widehat{\mathcal{L}}$, which now really satisfies the finite speed of prop as well as the heat kernel contraction bounds.

Putting together

- $\|w(0)\|_\infty = \|\lambda - \mu\|_\infty \lesssim \eta_f := N^{-3/4}$ from rigidity for both λ and μ
- $\|w(t) - \widehat{w}(t)\|_\infty \ll \eta_f$ for $t \le T$, so enough to study $\widehat{w}$
- Finite speed of prop. for $\widehat{\mathcal{L}}$ and $t \le T$ gives

$$\widehat{w}_i(t) = \sum_j \mathcal{U}^{\widehat{\mathcal{L}}}_{ij}(t,0)\widehat{w}_j(0) = \sum_{|j-i|\le L} \mathcal{U}^{\widehat{\mathcal{L}}}_{ij}(t,0)\widehat{w}_j(0) + O(N^{-100})$$

- Heat kernel bound for $\widehat{\mathcal{L}}$ gives (for any fixed i)

$$|\widehat{w}_i(t)| \le \Big(\frac{1}{N^{1/2}t}\Big)^{2/p} \|w^\#(0)\|_p, \qquad w^\#_j := w_j \cdot \mathbf{1}(|i-j| \le L)$$

- $\|w^\#(0)\|_p \lesssim L^{1/p}\eta_f = N^{\varepsilon/p}\eta_f$ [caricature]
- Optimize p and ε for $T = N^{-1/2+\varepsilon}$ to get $|w_i(T)| \ll \eta_f$.

All epsilons are created equal, but not in this proof...

Fly in the ointment: missing rigidity for $z(t,\alpha)$

Optimal cusp rigidity was used in everywhere. It is proven [Dominik's talk] for any Wigner type matrix with flat variance profile, $\mathbf{E}\,|h_{ij}|^2 \sim 1/N$.

In particular it holds for $\lambda(t), \mu(t)$ and also for their interpolations.

But $z(t,\alpha)$ is the DBM-evolution of an interpolation $\neq$ interpolation of the DBM-evolutions of $\lambda(t), \mu(t)$

By the matrix interpretation of DBM, $z(t,\alpha)$ are the eigenvalues of

$$\mathrm{diag}[z(0,\alpha)] + \sqrt{t}U, \qquad U \sim \text{GOE (flat)}$$

with t tiny and $z(0,\alpha)$ are not known to be eigenvalues of a mean field RM.

Fix α, let $\overline{\gamma}(t) = \overline{\gamma}(t,\alpha)$ be the interpolation of the quantiles of ρ_t^λ, ρ_t^μ:

$$\overline{\gamma}_i(t) := \alpha\gamma_i^\lambda(t) + (1-\alpha)\gamma_i^\mu(t)$$

satisfying (recall m is the Stieltjes transform of the density)

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\gamma}_i(t) = -\Re\Big[\alpha m_t^\lambda(\gamma_i^\lambda(t)) + (1-\alpha)m_t^\mu(\gamma_i^\mu(t))\Big]$$

It is a badly singular ODE since m is only $\frac{1}{3}$-Hölder cont. around the cusp — it needs very precise analysis of the SCflow.

As usual, we have a parabolic equation for $u(t) := z(t,\alpha) - \overline{\gamma}(t)$

$$\mathrm{d}u_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{u_j - u_i}{(z_i - z_j)(\overline{\gamma}_j - \overline{\gamma}_i)}\mathrm{d}t + F_i(t)\mathrm{d}t$$

$$F_i := \frac{1}{N}\sum_{j\neq i}\frac{1}{\overline{\gamma}_i - \overline{\gamma}_j} + \Re\Big[\dots\Big] \quad \longrightarrow \text{Small after a lot of sweat}$$

The noncommutativity of the flow and interpolation is dealt with on the level of the quantiles!

$$\mathrm{d}u_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i + \frac{1}{N}\sum_{j\neq i}\frac{u_j - u_i}{(z_i - z_j)(\overline{\gamma}_j - \overline{\gamma}_i)}\mathrm{d}t + F_i(t)\mathrm{d}t, \qquad u = z - \overline{\gamma}$$

Duhamel formula, contraction etc. work here as well, but the stochastic term is $\sim \sqrt{T/N} = N^{-3/4+\varepsilon}$ too big for directly concluding rigidity.

But it is good enough to construct short range approximation, for which we have finite speed of propagation and effective heat kernel contraction.

However, the rigidity should get stronger away from the cusp, expect

$$|z_i - \overline{\gamma}_i| \lesssim \eta_f(\gamma_i) \sim \frac{1}{N^{3/4}|i|^{1/4}}$$

This improved factor is not accessible with $\ell^p \to \ell^\infty$ heat kernel estimates.

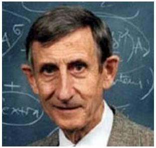
Couple z_i with y_{i-K}, $K = N^\varepsilon$, and use maximum principle for the deterministic dynamics for $z_i - y_{i-K}$ with $K = N^\varepsilon$. Since $y(t)$ is rigid and

$$y_{i-K}(t) \le z_i(t) \le y_{i+K}(t)$$

for all times, the process z_i, trailed by y remains rigid as well. □

- We proved cusp universality for Wigner type models in both symmetry classes: the last remaining universality in the Wigner-Dyson world.
- One parameter (α) family of Pearcey universalities.
- Analysis of the SCflow for long time through the cusp regime
- DBM theory developed at the cusp
- Rigidity via maximal principle

Eugene Wigner



Freeman Dyson

Craig Tracy

Harold Widom

Trevor Pearcey

The paper where the Pearcey kernel first appeared....

XXXI. *The Structure of an Electromagnetic Field in the Neighbourhood of a Cusp of a Caustic.*

By T. PEARCEY *.

[Received October 10, 1945.]

ABSTRACT.

It has been found possible to make a detailed mathematical and numerical study of the field structure at and near a line focus of a cylindrical electromagnetic wave train possessing any finite amount of cylindrical aberration of the first order. By a suitable choice of parameters all the possible wave-lengths and degrees of aberration can be expressed in terms of a single infinite integral of two independent parameters. Diagrams of the magnitude and phase of this integral are given, and only the final mathematical results are described.

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