Scaling limits for planar aggregation with subcritical fluctuations

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Joint work with
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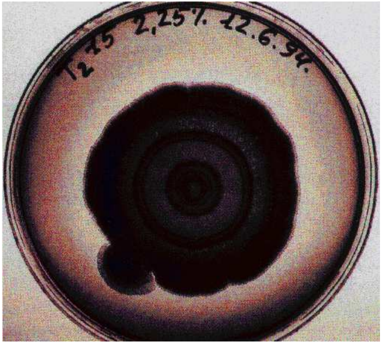
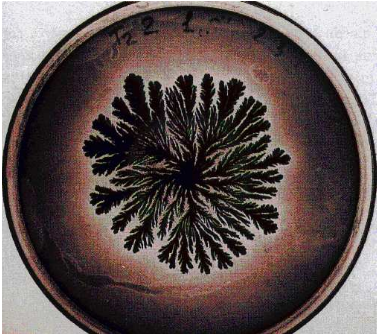
Bacterial growth in increasingly stressed conditions





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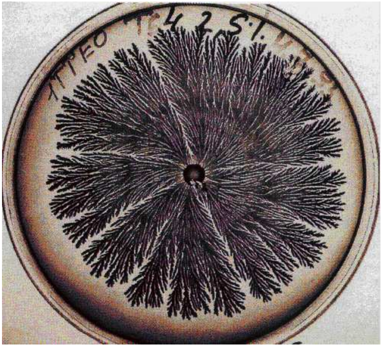

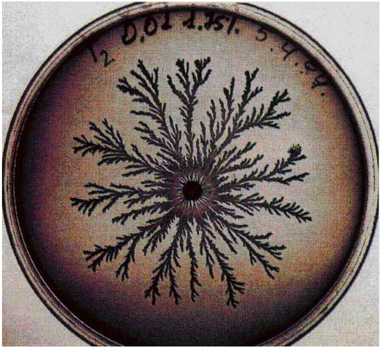

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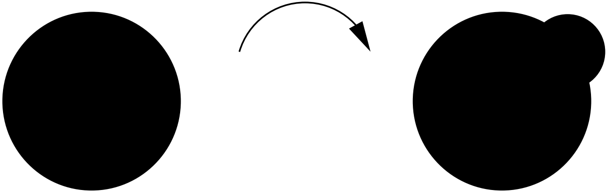
Conformal mapping representation of single particle

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$ and P denote a particle of logarithmic capacity c and attachment angle θ.

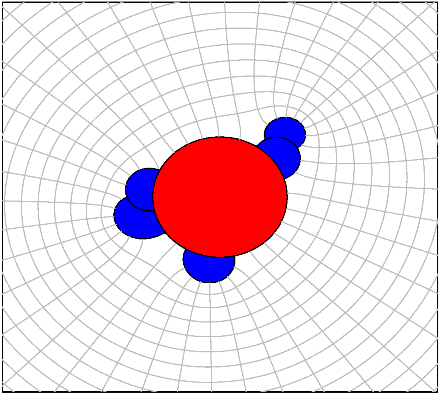
Use the unique conformal mapping $f_c^\theta : D_0 \to D_0 \setminus P$ that fixes ∞ as a mathematical description of the particle.



Conformal mapping representation of a cluster

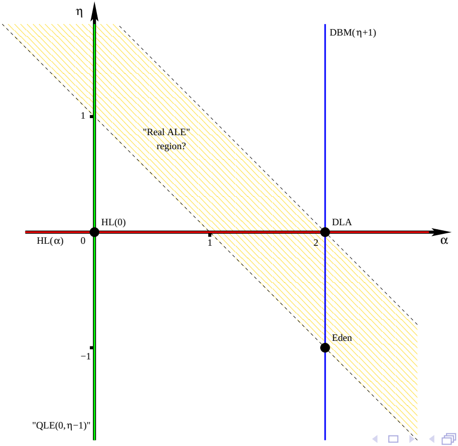
- Suppose $P_1, P_2, \ldots$ is a sequence of particles, where P_n has capacity c_n and attachment angle θ_n, $n = 1, 2, \ldots$.
  - Set $\Phi_0(z) = z$.
  - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ f_{c_n}^{\theta_n}(z)$, for $n = 1, 2, \ldots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.
- By varying the sequences $\{\theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.

Cluster formed by iteratively composing conformal mappings



Aggregate Loewner Evolution, ALE(α, η, σ)

- θ_n distributed $\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$; $\quad c_n = c|\Phi'_{n-1}(e^{\sigma+i\theta_n})|^{-\alpha}$.



Previous results

- Almost all previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.
    - Norris and T. (2012) showed scaling limit of HL(0) is a growing disk with a branching structure related to the Brownian web.
    - Silvestri (2017) showed fluctuations converge to a log-correlated Fractional Gaussian Field.
- Very few results for HL(α) with $\alpha \neq 0$.
    - Rohde and Zinsmeister (2005) obtained estimates on the dimension of scaling limits for a regularized version of HL(α) when $\alpha > 0$.
    - Sola, T., Viklund (2015) showed scaling limit of regularized HL(α) is a growing disk for all α provided regularization parameter σ is large enough.
- Sola, T., Viklund (2018) showed scaling limit of ALE(α, η, σ) is a single slit if $\alpha \geq 0$ and $\eta > 1$ when using slit particles, provided σ is very small.

Phase transition

Open Problem:

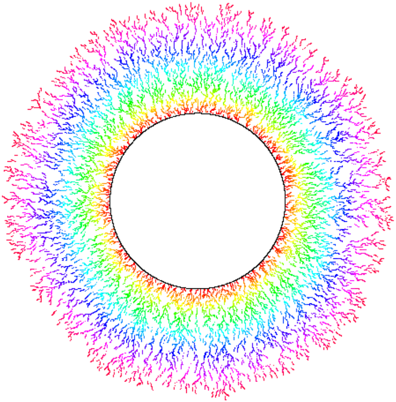
Does ALE(α, η, σ) exhibit a phase transition from disks to non-disks along the line $\alpha + \eta = 1$ (for 'broad' choices of the regularization parameter σ)?

- Longstanding conjectures:
  - HL(α) has a phase transition at $\alpha = 1$.
  - DBM(η) has a phase transition at $\eta = 0$.

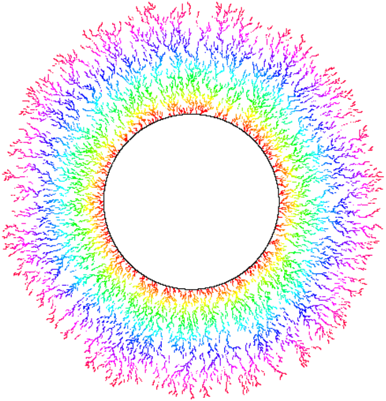
Scaling limits for ALE$(0, \eta, \sigma)$

- Natural to consider particle sizes that are very small compared to the overall size of the cluster and scaling limits where $n \to \infty$ while $c \to 0$.
- Models are difficult to analyse mathematically as all models (except HL(0)) exhibit long-range dependencies.
- Additional difficulty, when $\alpha \neq 0$, is total capacity of cluster is random and cannot, a priori, be bounded above or below, so unclear at what rate to let $n \to \infty$.
- When $\alpha = 0$, K_n has capacity cn, so natural to look for scaling limits when $n = \lfloor T/c \rfloor$.

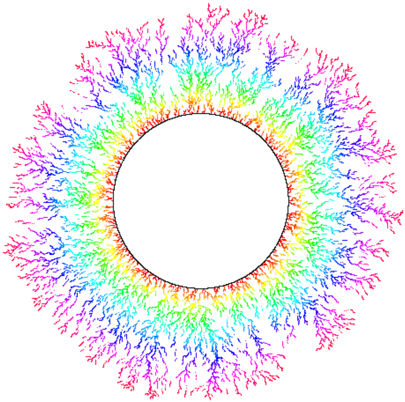
ALE(0,0) cluster with 8,000 particles for $c = 10^{-4}$



ALE(0,0.5,0.02) cluster with 8,000 particles for $c = 10^{-4}$



ALE(0,1,0.02) cluster with 8,000 particles for $c = 10^{-4}$



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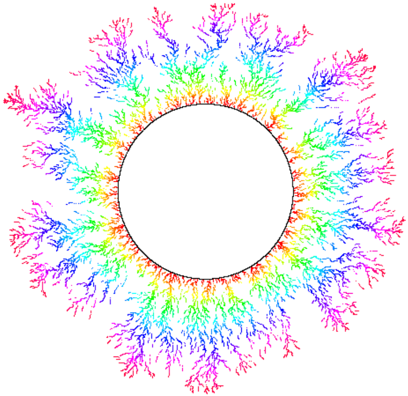
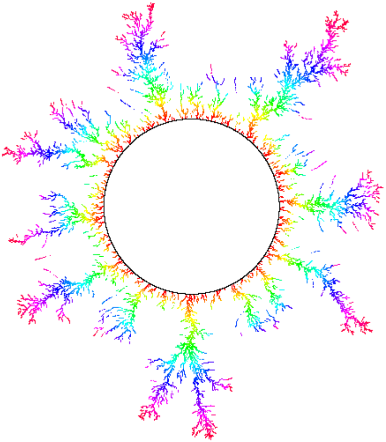


ALE(0,2,0.02) cluster with 8,000 particles for $c = 10^{-4}$



Disk theorem for ALE$(0, \eta, \sigma)$ when $\eta < 1$

Theorem:

For all $\eta < 1$, $T \in [0, \infty)$, $\epsilon \in (0, 1/3)$ and $e^{\sigma} \geq 1 + c^{1/3-\epsilon}$, there exists a constant C such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/3-\epsilon}$,

$$|\Phi_n(z) - e^{cn}z| \leq \frac{Cc^{1/2-\epsilon}}{r}\left(\left(1 + \log\left(\frac{r}{r-1}\right)\right)^{1/2} + \frac{c^{1/2}}{(e^{\sigma}-1)^2}\right).$$

(Almost) Theorem:

The same result holds for ALE(α, η, σ) when $\alpha + \eta < 1$ (with e^{cn} replaced by $\exp(\sum_{k=1}^{n} c_k)$).

Disk theorem for ALE$(0, \eta, \sigma)$ when $\eta = 1$

Theorem:

Suppose $\eta = 1$. For all $T \in [0, \infty)$, $\epsilon \in (0, 1/5)$ and $e^\sigma \geq 1 + c^{1/5-\epsilon}$, there exists a constant C such that, with high probability, for all $n \leq T/c$ and $|z| \geq 1 + c^{1/5-\epsilon}$,

$$|\Phi_n(z) - e^{cn} z| \leq \frac{Cc^{1/2-\epsilon}}{r} \left(\left(\frac{r}{r-1} \right)^{1/2} + \frac{c^{1/2}}{(e^\sigma - 1)^3} \right).$$

(Almost) Theorem:

The same result holds for ALE(α, η, σ) when $\alpha + \eta = 1$ (with e^{cn} replaced by $\exp(\sum_{k=1}^n c_k)$).

1 minute proof ($\eta = 0$)

$$\Phi_n(z) - e^{cn}z = \sum_{k=1}^{n} \Phi_k(e^{c(n-k)}z) - \Phi_{k-1}(e^{c(n-k-1)}z).$$

But

$$\begin{aligned}
\mathbb{E}\left[\Phi_k(z)\,|\mathcal{F}_{k-1}\right] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{k-1}(e^{i\theta} f_c(e^{-i\theta} z))d\theta \\
&= \frac{1}{2\pi i}\int_{|w|=1} \frac{\Phi_{k-1}(wf_c(zw^{-1}))}{w} dw \\
&= \lim_{w\to 0} \Phi_{k-1}\left(wf_c(zw^{-1})\right) \\
&= \Phi_{k-1}(e^c z).
\end{aligned}$$

So $\Phi_n(z) - e^{cn}z$ is a martingale sum and the result follows by your favourite martingale inequality.

Proof idea ($\eta \neq 0$)

Can write

$$\Phi_n(z) = \Phi_{n-1}(e^c z) + L_n(z) + M_n(z) + R_n(z)$$

where $L_n(z)$ is linear in Φ_{n-1}, $M_n(z)$ is a martingale difference and $R_n(z)$ contains higher order error terms.

Therefore there exists an operator P such that

$$\Phi_n(z) - e^{cn} z = \sum_{k=1}^{n} P^{n-k}(M_k(z) + R_k(z)).$$

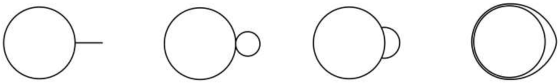
Main work is showing right-hand side is small. We use Marcinkiewicz to control the operator when $\eta \leq 1$ but additional difficulties exist as $M_k(z)$ and $R_k(z)$ depend on Φ'_{k-1}.

Universality of particle shapes

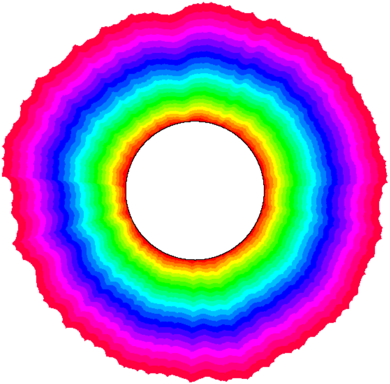
Results apply to any particle shape P with $\gamma \geq 1$ satisfying

$$\log \frac{f_c(z)}{z} = c\frac{\gamma z + 1}{\gamma z - 1} + O\left(\frac{c^{3/2}}{(|z|-1)|z-1|}\right).$$

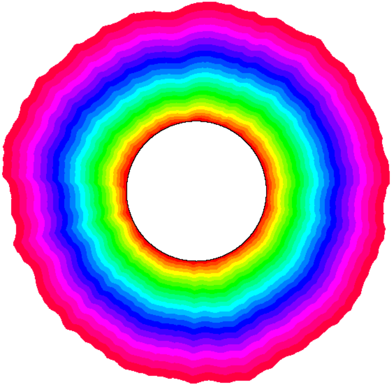
This includes particles that fit within a radius $\sim c^{1/2}$ of 1, but also certain non-local particles.



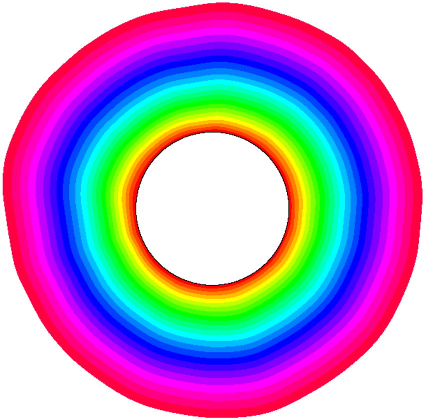
ALE(0,0) cluster with 10,000 non-local particles for $c = 10^{-4}$



Level lines of form $\Phi_n(re^{i\theta})$ in ALE(0,0) cluster with 10,000 non-local particles for $c = 10^{-4}$ and $r - 1 = c^{1/2}$



Level lines of form $\Phi_n(re^{i\theta})$ in ALE(0,0) cluster with 10,000 non-local particles for $c = 10^{-4}$ and $r - 1 = c^{1/4}$



Pointwise fluctuations for ALE$(0, \eta, \sigma)$ when $\eta \leq 1$

Set

$$\mathcal{F}_n(z) = c^{-1/2}(e^{-cn}\Phi_n(z) - z)$$

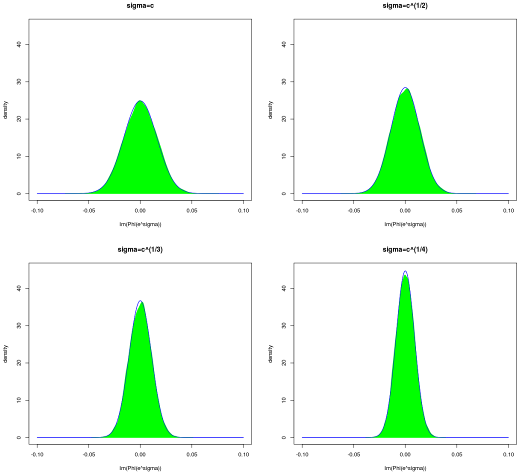
and let $n(t) = \lfloor t/c \rfloor$.

Under the assumptions above, but with $e^\sigma \geq 1 + c^{1/4-\epsilon}$ when $\eta < 1$ and $e^\sigma \geq 1 + c^{1/6-\epsilon}$ when $\eta = 1$,

$$\mathcal{F}_{n(t)}(z) \to \mathcal{N}\left(0, \sum_{m=0}^{\infty} \frac{1 - e^{-2(m(1-\eta)+1)t}}{m(1-\eta)+1} |z|^{-2m}\right).$$

(Note that if $\eta > 1$ would need $|z| > e^{(\eta-1)t}$ for this sum to converge – beginnings of a phase transition?)

Fluctuation distributions in ALE(0,0)



Global fluctuations for ALE$(0, \eta, \sigma)$ when $\eta \leq 1$

Under the assumptions above, $\mathcal{F}_{n(t)}(z) \to \mathcal{W}_t(z)$ where

$$\dot{\mathcal{W}}_t(z) = (1-\eta) z \mathcal{W}_t'(z) - \mathcal{W}_t(z) + \sqrt{2}\xi_t(z).$$

Here $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

Global fluctuations for ALE$(0, \eta, \sigma)$ when $\eta \leq 1$

Specifically

$$\mathcal{W}_t(z) = \sum_{m=0}^{\infty} (A_t^m + iB_t^m) z^{-m}$$

where

$$dA_t^m = -\left(m(1-\eta)+1\right) A_t^m dt + \sqrt{2} d\beta_t^m$$
$$dB_t^m = -\left(m(1-\eta)+1\right) B_t^m dt + \sqrt{2} d\beta_t'^m.$$

Here $\beta_t^m, \beta_t'^m$ are i.i.d. Brownian motions for $m = 0, 1, \ldots$.

Remarks

- The map $z \mapsto \mathcal{W}_t(z)$ is determined (by analytic extension) by the boundary process $\theta \mapsto \mathcal{W}_t(e^{i\theta})$.
- When $\eta = 0$, these boundary fluctations are the same as for internal diffusion limited aggregation (IDLA).
- As $t \to \infty$, $\mathcal{W}_t(e^{i\theta})$ converges to a Gaussian field.
  - When $\eta = 0$, $\mathcal{W}_\infty(e^{i\theta})$ is known as the augmented Gaussian Free Field.
  - When $\eta < 1$, $\mathrm{Cov}\left(\mathcal{W}_\infty(e^{ix}), \mathcal{W}_\infty(e^{iy})\right) \asymp \log|x - y|$.
  - When $\eta = 1$, $\mathcal{W}_\infty(e^{i\theta})$ is complex white noise.

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