

# Cardy embedding of random planar maps

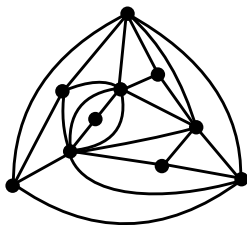
Nina Holden

ETH Zürich, Institute for Theoretical Studies

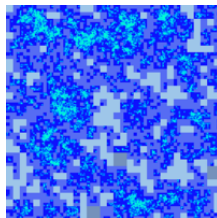
Collaboration with Xin Sun. Based on our joint works with Albenque, Bernardi, Garban, Gwynne, Lawler, Li, and Sepúlveda.

February 9, 2020

# Two random surfaces



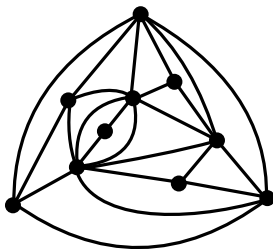
random planar map (RPM)



Liouville quantum gravity (LQG)

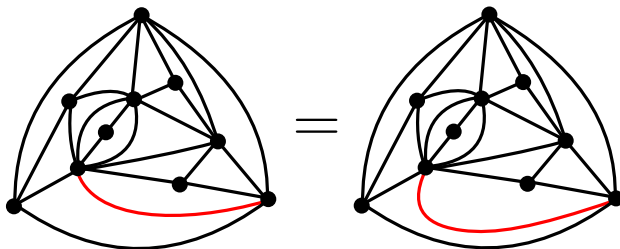
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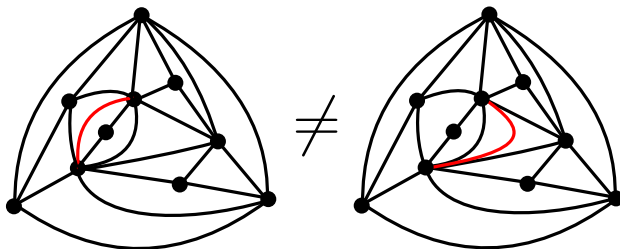
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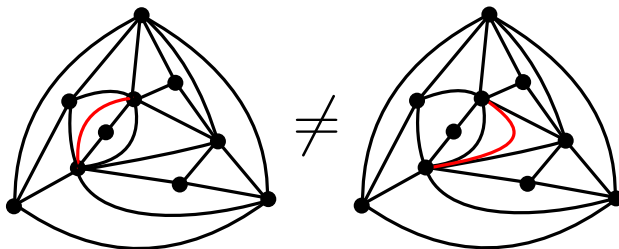
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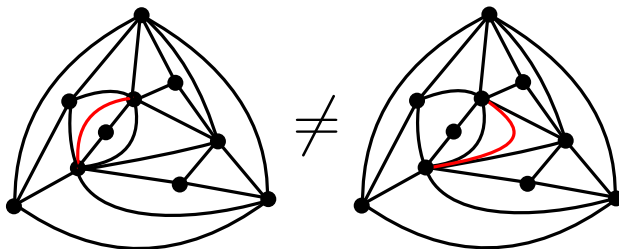
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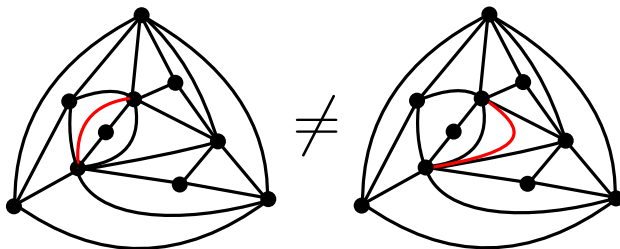
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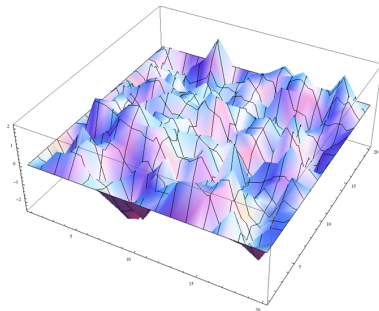
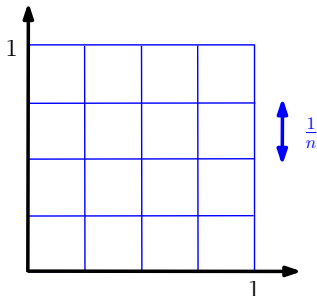
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- Enumeration results by Tutte and Mullin in 60's.



# The Gaussian free field (GFF)

- Hamiltonian  $H(f)$  quantifies how much  $f$  deviates from being harmonic

$$H(f) = \frac{1}{2} \sum_{x \sim y} (f(x) - f(y))^2, \quad f : \frac{1}{n} \mathbb{Z}^2 \cap [0, 1]^2 \rightarrow \mathbb{R}.$$

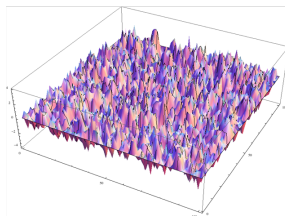
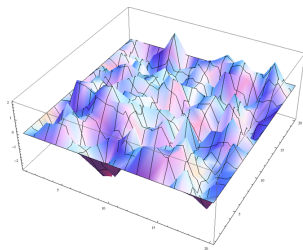


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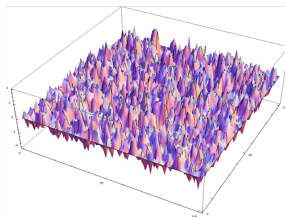
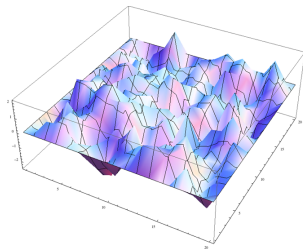
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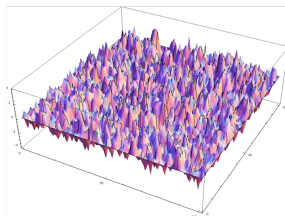
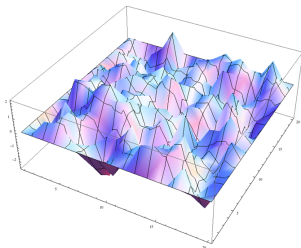
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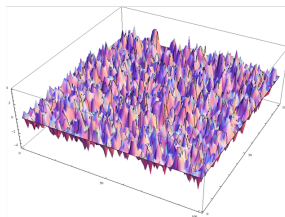
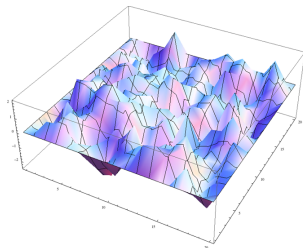
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- The GFF is a **random distribution** (i.e., random generalized function).



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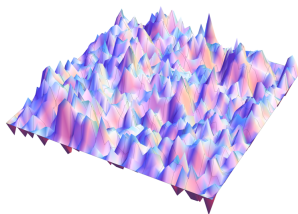
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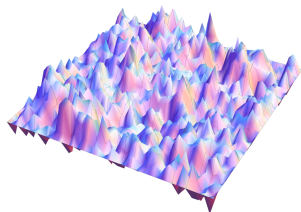
discrete  
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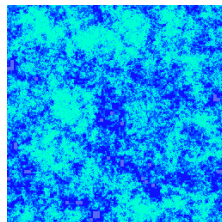
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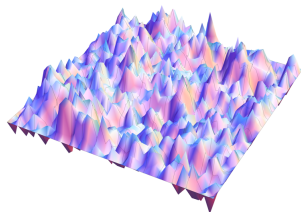
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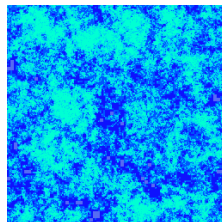
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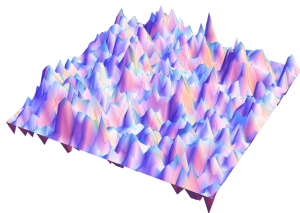
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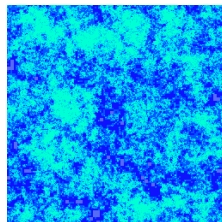
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- The area measure is non-atomic and any open set has positive mass a.s., but the measure is a.s. singular with respect to Lebesgue measure.



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# Random planar maps converge to LQG

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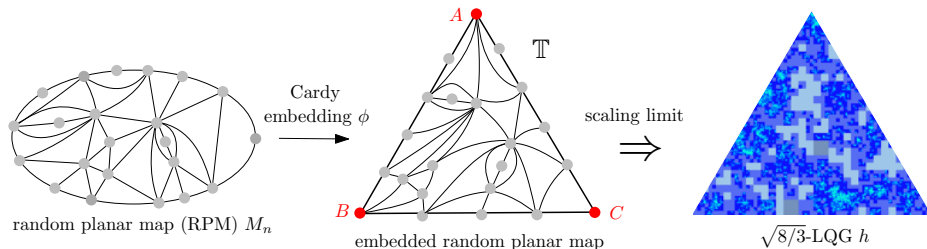
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What does it mean for a RPM to converge?

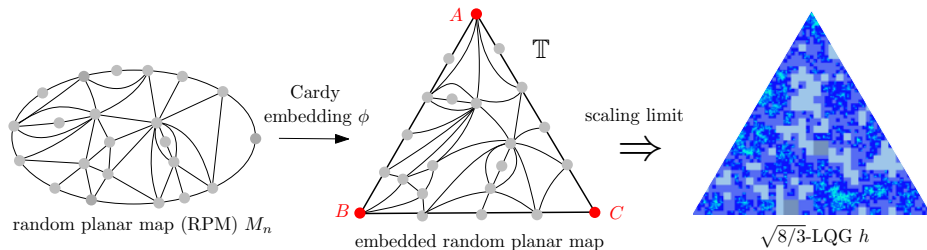
- Metric structure (Le Gall'13, Miermont'13)
- Conformal structure (H.-Sun'19)
- Statistical physics observables (Duplantier-Miller-Sheffield'14, ...)

# Conformally embedded RPM converge to $\sqrt{8/3}$ -LQG



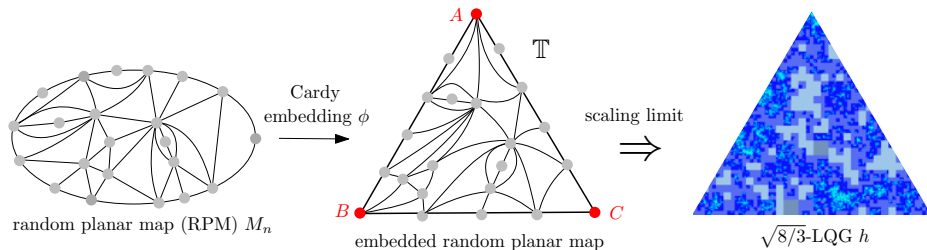
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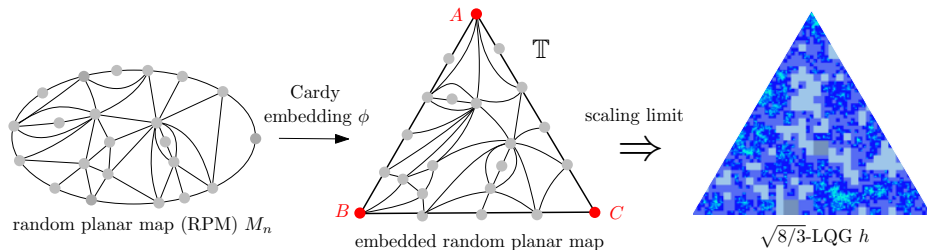
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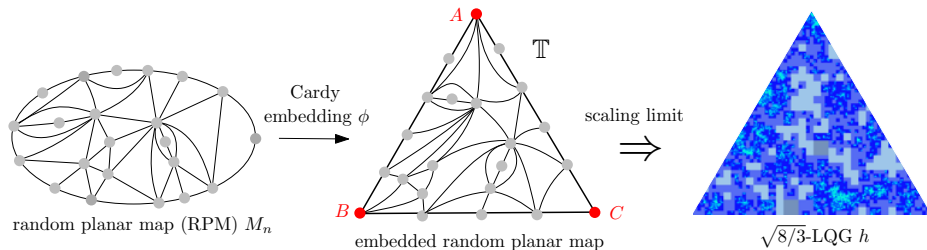
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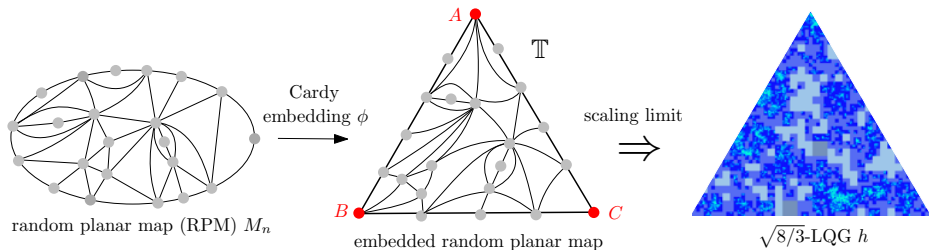
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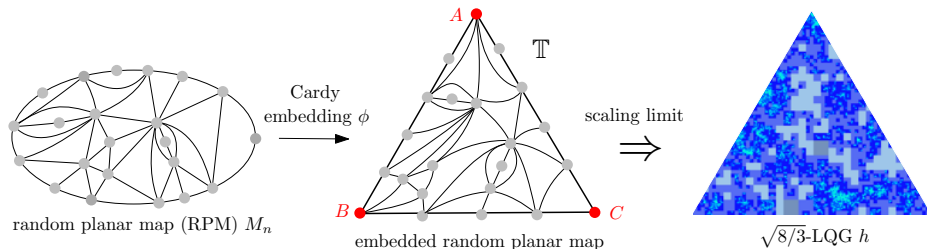


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More precisely,  $\exists$  coupling of  $M_n$  and  $h$  s.t. with probability 1, as  $n \rightarrow \infty$ ,

- $\int f d\mu_n \rightarrow \int f d\mu \quad \forall$  continuous  $f : \mathbb{T} \rightarrow [0, 1]$  (measure convergence)
- $d_n(z, w) \rightarrow d(z, w)$ , uniformly in  $z, w \in \mathbb{T}$  (metric convergence)

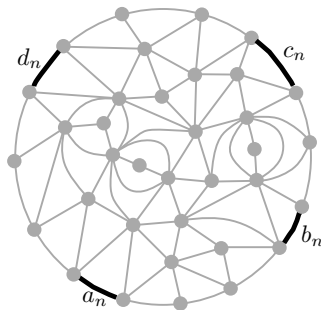
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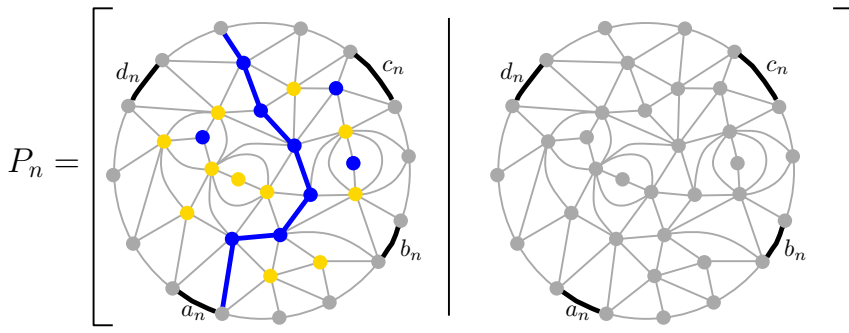
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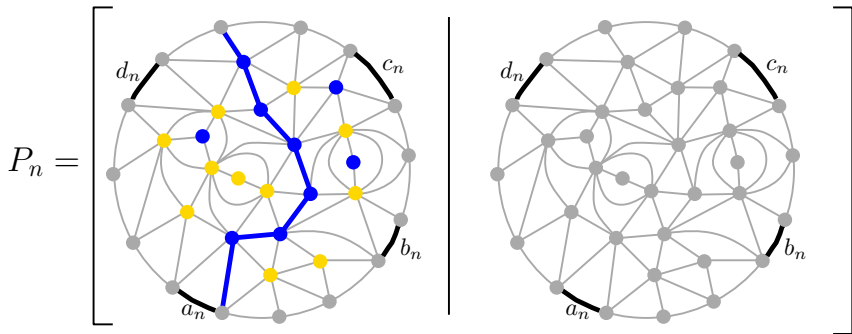
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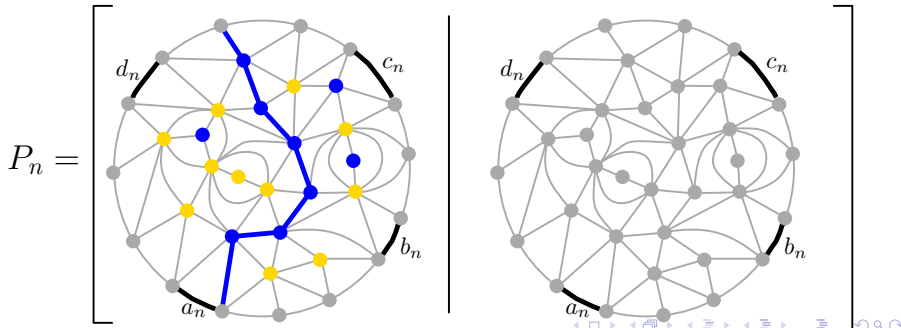
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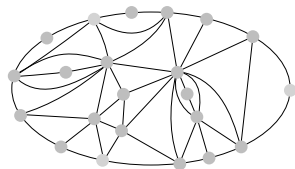


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- The random variable  $P_n$  converges in law as  $n \rightarrow \infty$ .
- $P_n$  gives some notion of **extremal distance** between  $a_n b_n$  and  $c_n d_n$ .

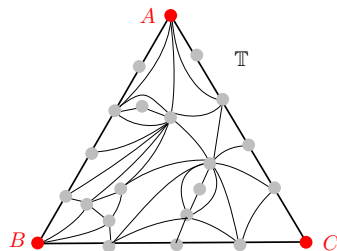


# Cardy embedding: percolation-based embedding



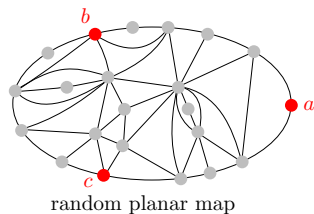
random planar map

Cardy embedding  $\phi$

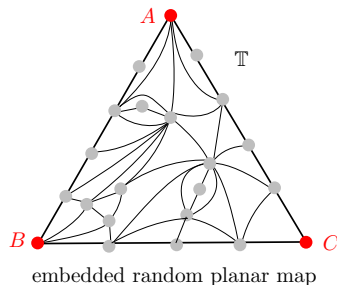


embedded random planar map

# Cardy embedding: percolation-based embedding

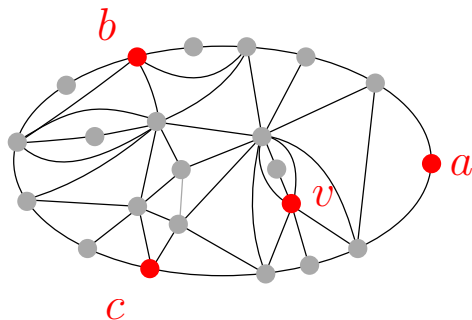


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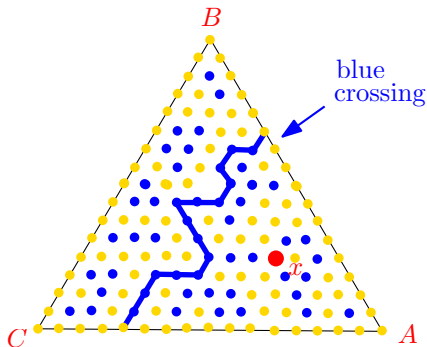
# Cardy embedding: percolation-based embedding

- What is the “correct” position of  $v$  in  $\mathbb{T}$ ?



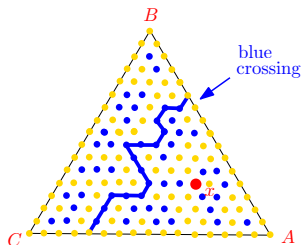
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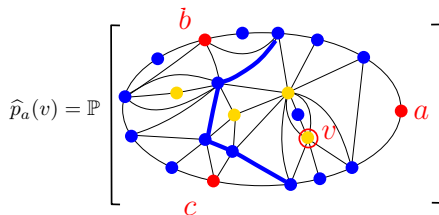
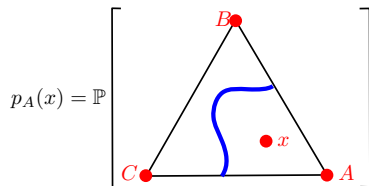
$$p_A(x) = \mathbb{P} \left[ \begin{array}{c} B \\ \text{triangle with crossing} \\ C \quad A \end{array} \right]$$

A diagram of a triangle with vertices labeled  $A$ ,  $B$ , and  $C$  in red. The interior of the triangle is empty. A blue path, representing a crossing, starts from the bottom edge (between  $C$  and  $A$ ), winds through the interior, and exits from the right edge (near  $A$ ). A red dot labeled  $x$  is located in the lower-right region of the triangle.

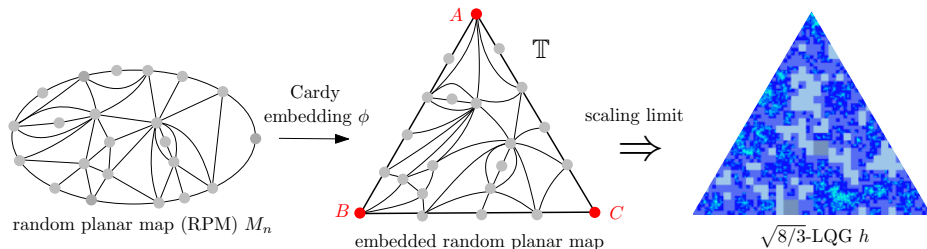
# Cardy embedding: percolation-based embedding

- What is the “correct” position of  $v$  in  $\mathbb{T}$ ?
- Map  $v \in V(M)$  to  $x \in \mathbb{T}$  such that

$$(p_A(x), p_B(x), p_C(x)) = (\hat{p}_a(v), \hat{p}_b(v), \hat{p}_c(v)).$$



# RPM $\Rightarrow$ LQG under conformal embedding

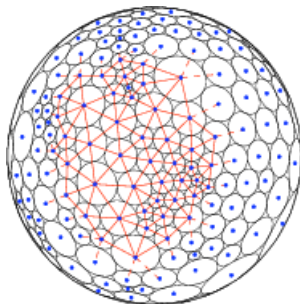


Our result is for **uniform triangulations** and the **Cardy embedding**, but is also believed to hold for other

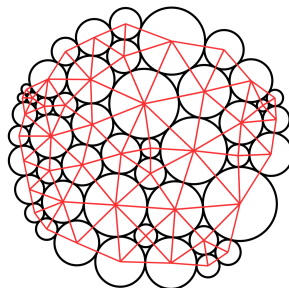
- 1 conformal embeddings,
- 2 local map constraints, and
- 3 universality classes of random planar maps.

# Discrete conformal embeddings

- Circle packing
- Riemann uniformization
- Tutte embedding
- Cardy embedding



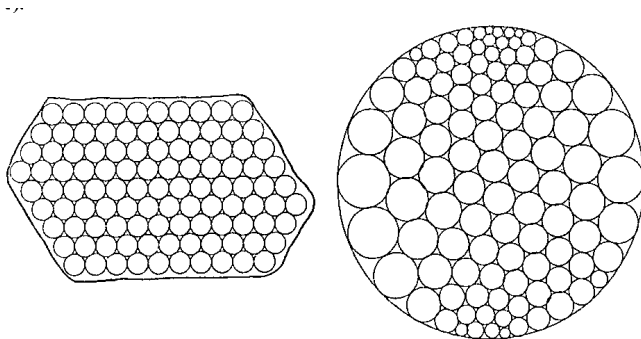
circle packing (sphere topology)



circle packing (disk topology)

# Discrete conformal embeddings

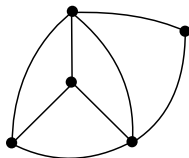
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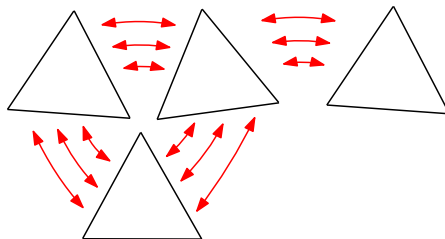
Rodin and Sullivan (1987): *The convergence of circle packings to the Riemann mapping*

# Discrete conformal embeddings

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Random planar map

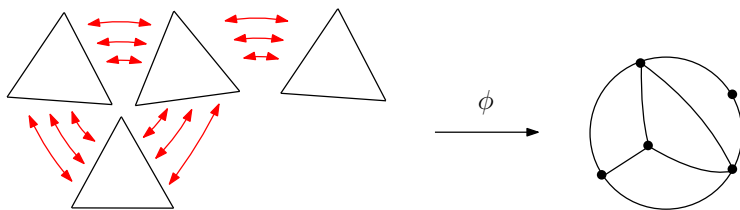


Riemannian manifold

# Discrete conformal embeddings

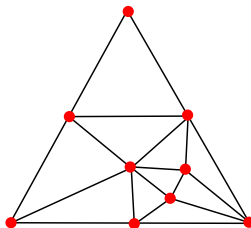
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Uniformization theorem: For any simply connected Riemann surface  $M$  there is a conformal map  $\phi$  from  $M$  to either  $\mathbb{D}$ ,  $\mathbb{C}$  or  $\mathbb{S}^2$ .



# Discrete conformal embeddings

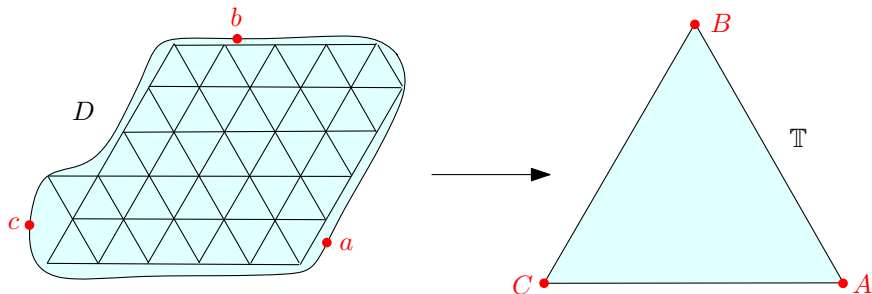
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Tutte embedding

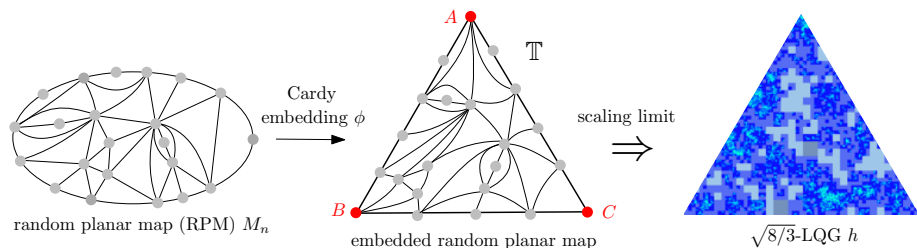
# Discrete conformal embeddings

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Smirnov (2001): The Cardy embedding of the triangular lattice restricted to  $D$  converges to the Riemann mapping  $D \rightarrow \mathbb{T}$ .

# Conformally embedded RPM converge to $\sqrt{8/3}$ -LQG



# Conformally embedded RPM converge to $\sqrt{8/3}$ -LQG

The proof is based on multiple works, including:

- Percolation on triangulations: a bijective path to Liouville quantum gravity (Bernardi-H.-Sun)
- Minkowski content of Brownian cut points (Lawler-Li-H.-Sun)
- Natural parametrization of percolation interface and pivotal points (Li-H.-Sun)
- Uniform triangulations with simple boundary converge to the Brownian disk (Albenque-H.-Sun)
- Joint scaling limit of site percolation on random triangulations in the metric and peanosphere sense (Gwynne-H.-Sun)
- Liouville dynamical percolation (Garban-H.-Sepúlveda-Sun)
- Convergence of uniform triangulations under the Cardy embedding (H.-Sun)

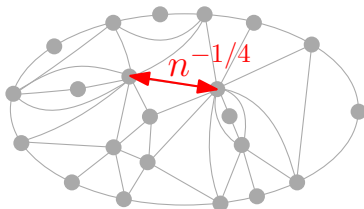


# Convergence as metric measure space

- $M_n$  is a uniform triangulation with  $n$  vertices and bdy length  $\lceil \sqrt{n} \rceil$ .

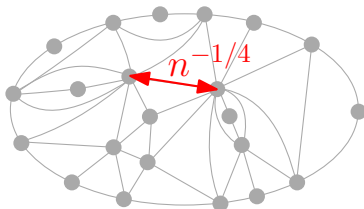
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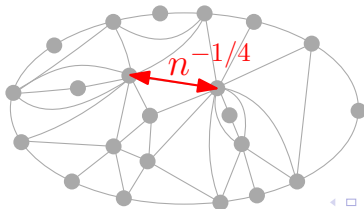
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## Theorem (Albenque-H.-Sun'19)

$M_n \Rightarrow M$  in the GHP topology, where  $M$  is  $\sqrt{8/3}$ -LQG (equivalently, the Brownian disk).

Building on Le Gall'13, Miermont'13, Bettinelli–Miermont'17, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'17, Addario-Berry–Wen'17



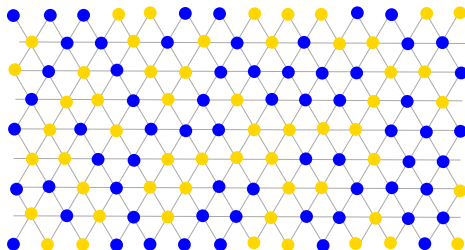
# The Schramm-Loewner evolution (SLE)

- One-parameter family of random fractal curves indexed by  $\kappa \geq 0$ , which describe the scaling limit of statistical physics models
  - loop-erased random walk,  $\kappa = 2$
  - Ising,  $\kappa = 3$ , and FK-Ising,  $\kappa = 16/3$
  - percolation,  $\kappa = 6$
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  - uniform spanning tree,  $\kappa = 8$



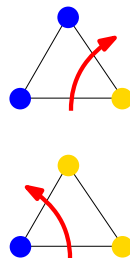
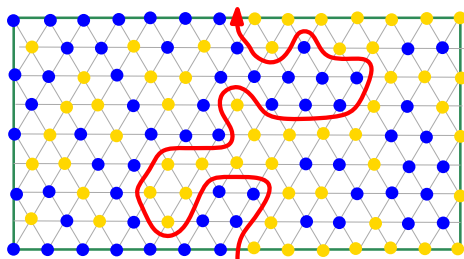
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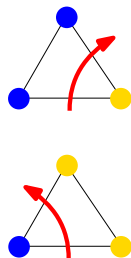
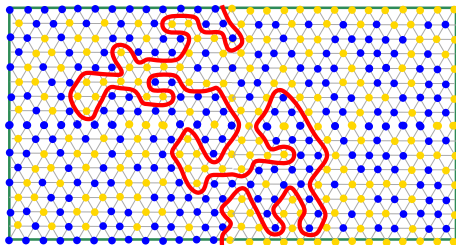
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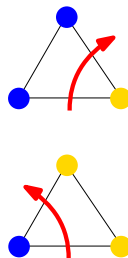
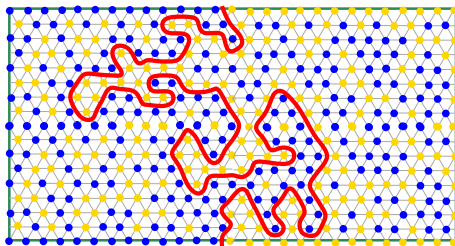
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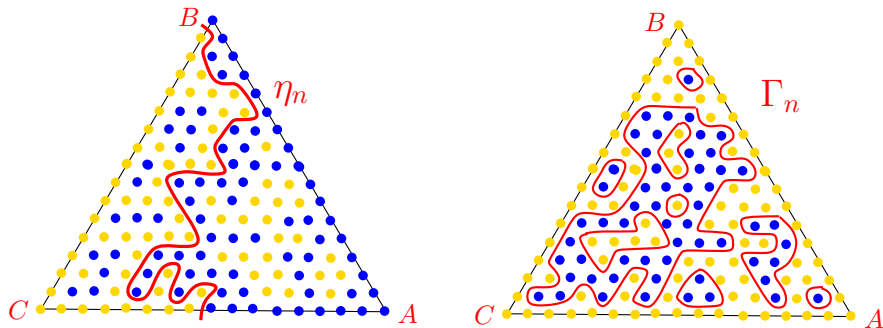


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  - uniform spanning tree,  $\kappa = 8$
- Introduced by Schramm'99: SLE uniquely characterized by **conformal invariance** and **domain Markov property**.



# Conformal invariance of percolation on triangular lattice



- Smirnov'01, Camia-Newman'06:  $\eta_n \Rightarrow \text{SLE}_6$ .
- The conformal loop ensemble ( $\text{CLE}_6$ ) is the loop version of  $\text{SLE}_6$ .
- Smirnov'01, Camia-Newman'06:  $\Gamma_n \Rightarrow \text{CLE}_6$ .

# Convergence as metric measure space with loops

## Theorem (Albenque-H.-Sun'19)

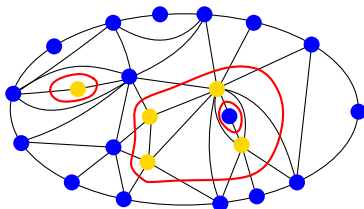
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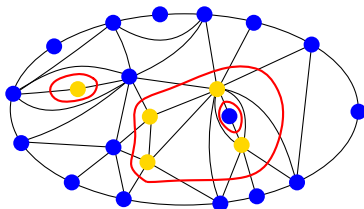


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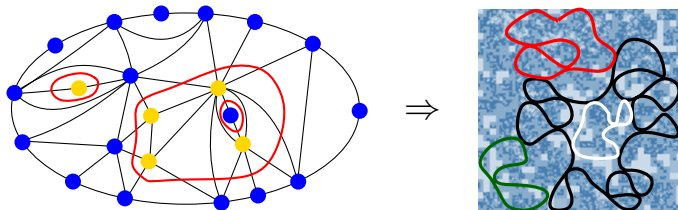
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## Theorem (Gwynne-H.-Sun'19)

$(M_n, P_n) \Rightarrow (M, \Gamma)$  in the GHPL topology, where  $\Gamma$  is the conformal loop ensemble  $CLE_6$ .

Building on Gwynne-Miller'17, Bernardi-H.-Sun'18

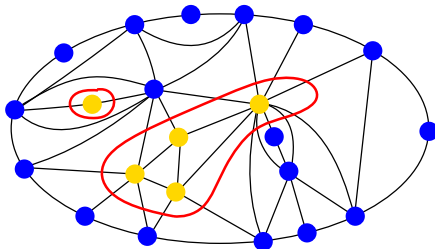


# Liouville dynamical percolation

- **Dynamical percolation**  $(P_t)_{t \geq 0}$  on  $M$ : Each vertex has an exponential clock and its color is resampled when its clock rings.

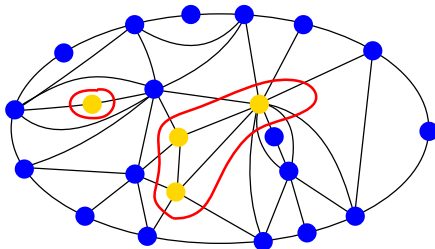
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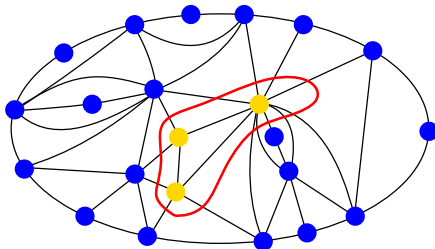
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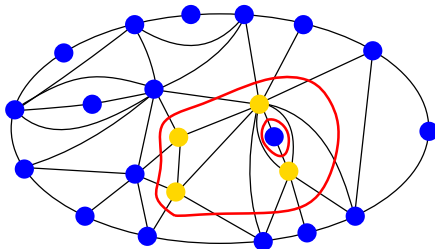
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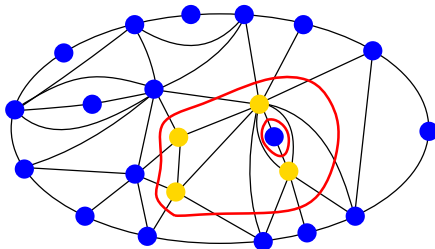
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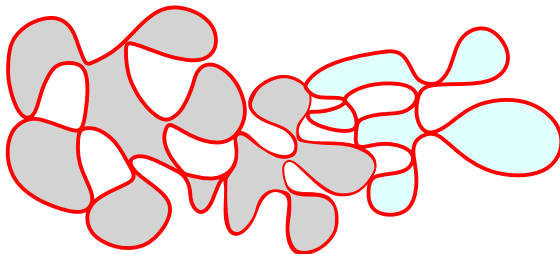
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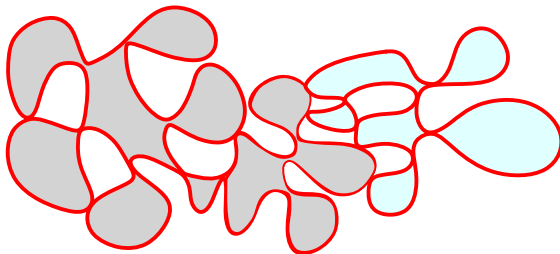
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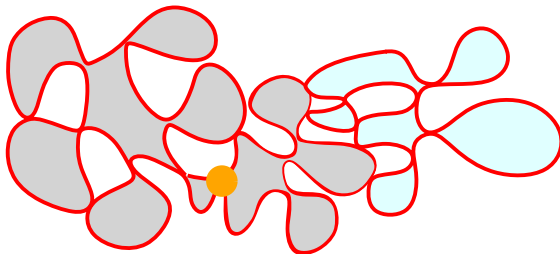
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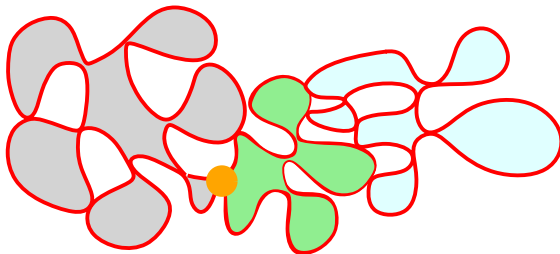
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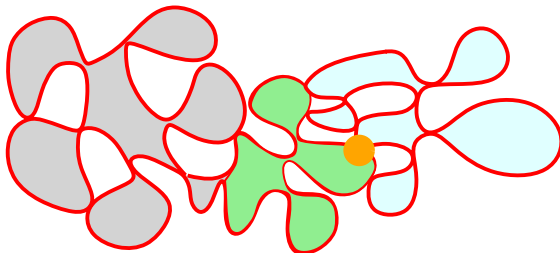
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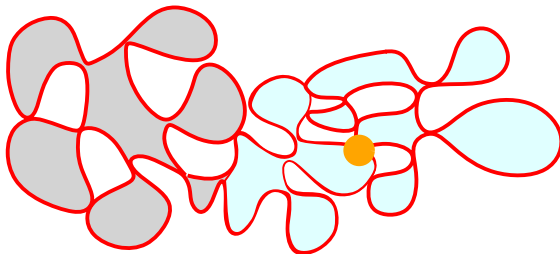
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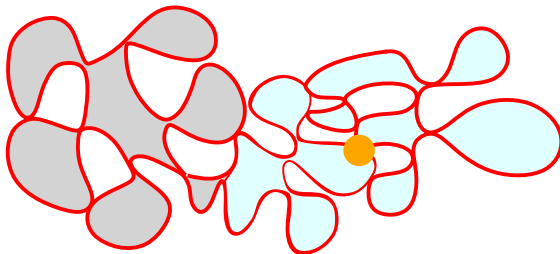
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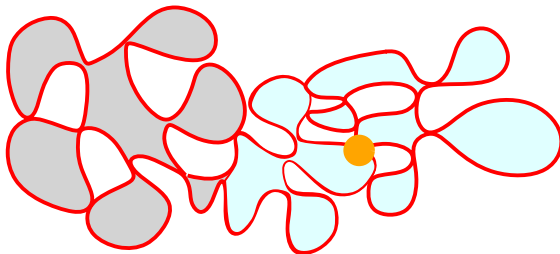
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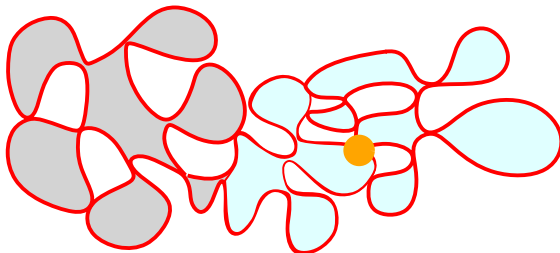
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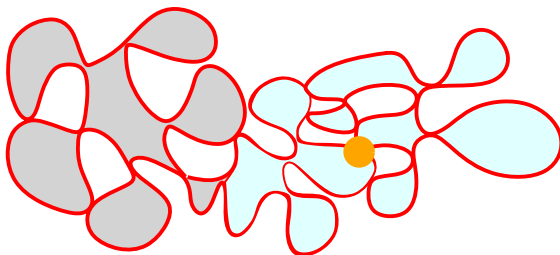
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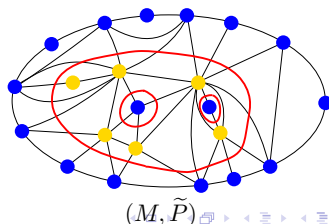
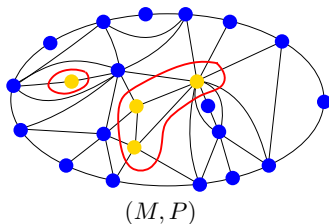
# Liouville dynamical percolation

- **Dynamical percolation**  $(P_t)_{t \geq 0}$  **on**  $M$ : Each vertex has an exponential clock and its color is resampled when its clock rings.
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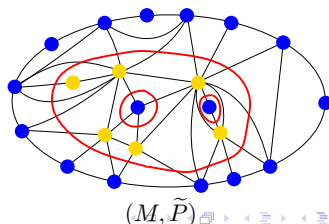
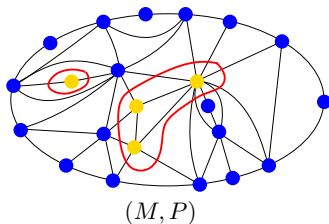
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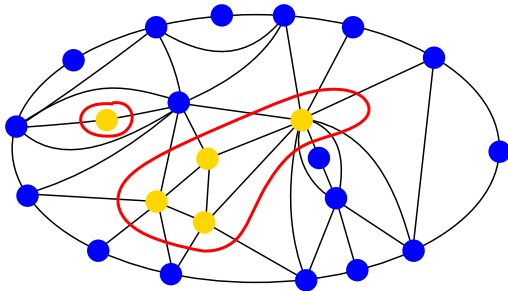
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  - implies **convergence of Cardy embedding** of  $M$  via LLN argument



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Thanks for your attention!