From the Ising model star-triangle move to a bicolor loop model

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Les Diablerets

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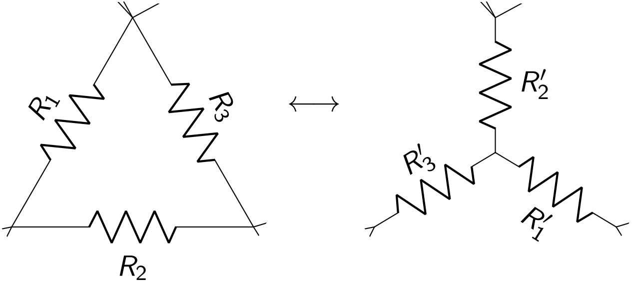
1. The cube recurrence and groves

2. Kashaev's recurrence and bicolor loops

3. Limit shapes

4. Relation Ising-loops

Star-triangle move in resistor networks *[Kennelly 1899]*



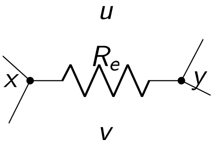
$$R_1' = \frac{R_2 R_3}{R_1+R_2+R_3} \qquad R_2' = \frac{R_1 R_3}{R_1+R_2+R_3} \qquad R_3' = \frac{R_1 R_2}{R_1+R_2+R_3}$$

$$R_1 = \frac{R_1' R_2' + R_2' R_3' + R_3' R_1'}{R_1'} \qquad R_2 = \frac{R_1' R_2' + R_2' R_3' + R_3' R_1'}{R_2'} \qquad R_3 = \frac{R_1' R_2' + R_2' R_3' + R_3' R_1'}{R_3'}$$

Star-triangle move in resistor networks *[Kashaev 1995]*

Let $G = (V, E)$ be a planar graph.
Suppose that

$$R_e = \frac{f_u f_v}{f_x f_y}$$



for some variables $(f_i)_{i \in V \cup F}$ on the vertices and faces of G.

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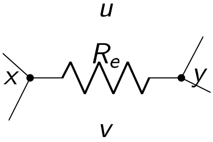

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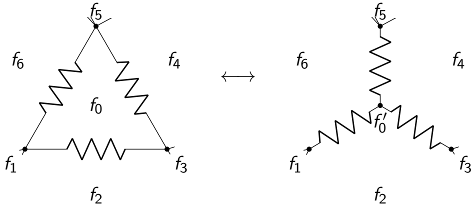
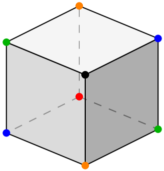
Then



iff $\quad f_0' = \frac{f_1 f_4 + f_2 f_5 + f_3 f_6}{f_0}$

The cube recurrence



$$f_{i,j,k} f_{i+1,j+1,k+1} = f_{i+1,j,k} f_{i,j+1,k+1} + f_{i,j+1,k} f_{i+1,j,k+1} + f_{i,j,k+1} f_{i+1,j+1,k}$$

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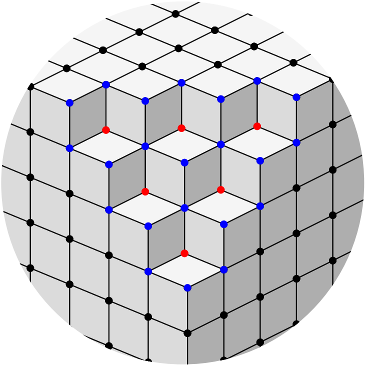

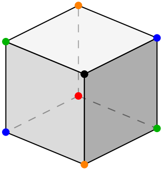

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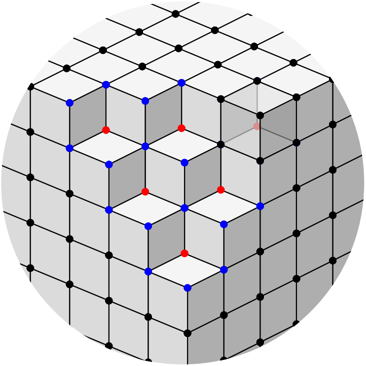

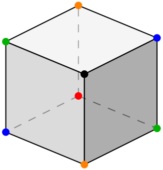

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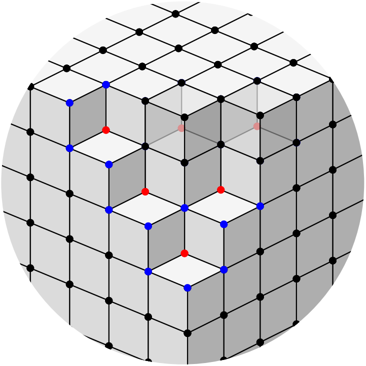

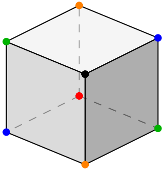

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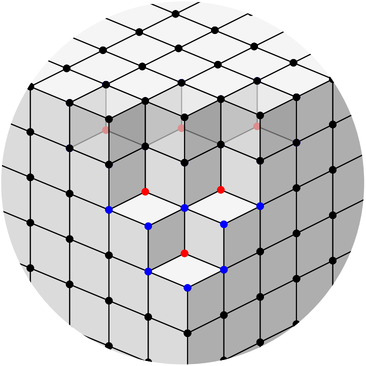

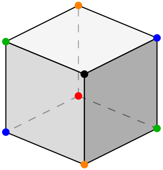

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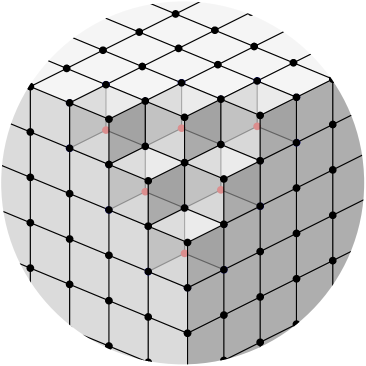

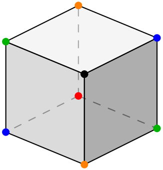

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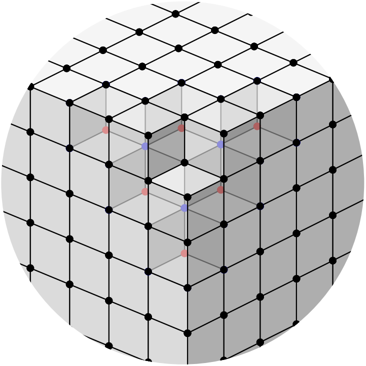

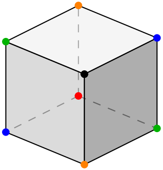

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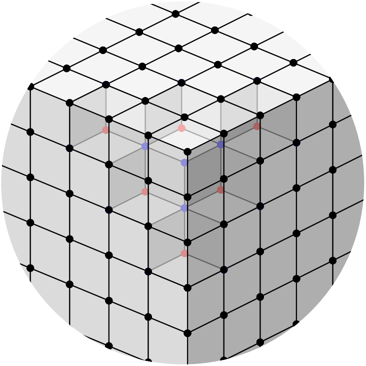

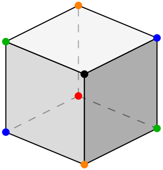

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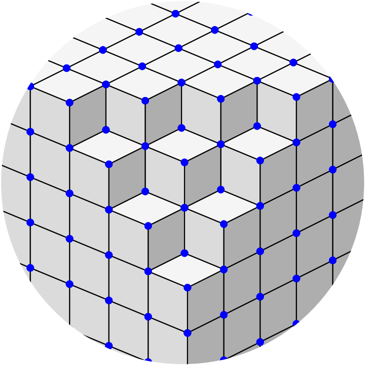
Laurent polynomial in the initial conditions.

Question (Propp, 2001)

Is the solution of the cube recurrence a partition function of some model?

Cube groves *[Carroll - Speyer, 2004]*

A *cube grove* **g**
is a choice of one diagonal per face, such that the union is a **spanning forest** (with some boundary coniditions).



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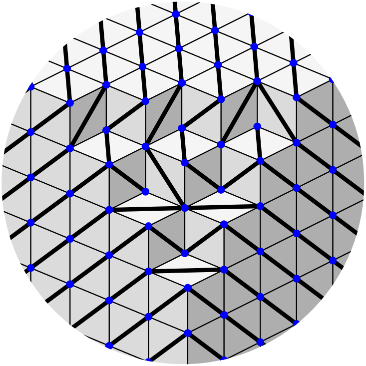
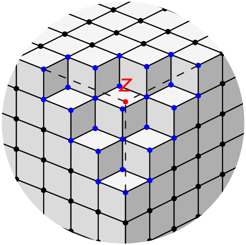
A *cube grove* $\mathbf{g}$
is a choice of one diagonal per face, such that the union is a **spanning forest** (with some boundary coniditions).
Given
weights (f_i) on the vertices,

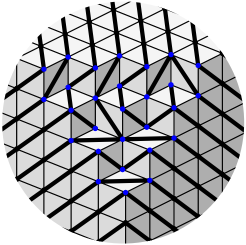
$$w(\mathbf{g}) = \prod_i f_i^{\deg_{\mathbf{g}}(i)-2}$$

$$\mathcal{Z} = \sum_{\mathbf{g}} w(\mathbf{g}).$$



Solution of the cube recurrence *[Carroll - Speyer 2004]*



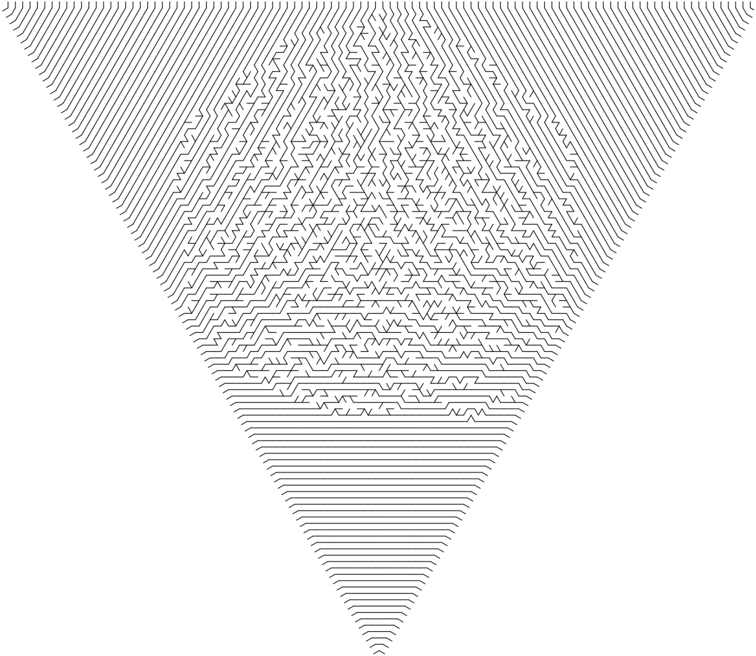


Theorem (*Carroll, Speyer*)

For initial conditions $(f_i)_{i\in I}$ on $I \subset \mathbb{Z}^3$, and for a $z \in \mathbb{Z}^3$ whose value f_z is defined by the cube recurrence,

$$f_z = \sum_{\mathbf{g}} \prod_{i} f_i^{\deg_{\mathbf{g}}(i)-2}.$$

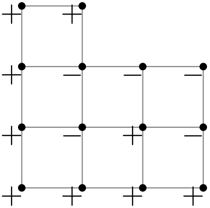
Limit shape



Ising model

Let $G = (V, E)$ be a planar graph, with positive coupling constants $(J_e)_{e \in E}$.
$\sigma : V \to \{-1, +1\}$.

$$\mathbb{P}(\sigma) = \frac{1}{Z_{\text{Ising}}(G, J)} \prod_{e=uv \in E} \exp\left(J_e \sigma_u \sigma_v\right),$$

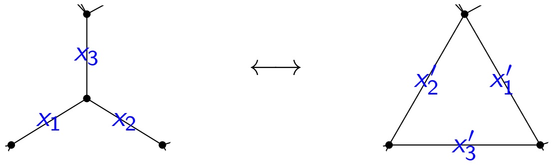


Partition function:

$$Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1,1\}^V} \prod_{e=uv \in E} \exp\left(J_e \sigma_u \sigma_v\right).$$

Let $x_e = \exp(-2J_e)$.

Star-triangle for the Ising model *[Wannier 1945]*



The Ising model is invariant in distribution *iff*

$$x_1' = \sqrt{\frac{(x_2+x_1x_3)(x_3+x_1x_2)}{(x_1x_2x_3+1)(x_1+x_2x_3)}}$$

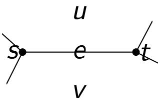
$$x_2' = \sqrt{\frac{(x_1+x_2x_3)(x_3+x_1x_2)}{(x_1x_2x_3+1)(x_2+x_1x_2)}}$$

$$x_3' = \sqrt{\frac{(x_1+x_2x_3)(x_2+x_1x_3)}{(x_1x_2x_3+1)(x_3+x_1x_2)}}$$

Star-triangle for the Ising model *[Kashaev 1995]*

Suppose that

$$\left(\frac{x_e - x_e^{-1}}{2}\right)^2 = \frac{g_s g_t}{g_u g_v}$$



for some potentials $(g_i)_{i \in V \cup F}$ on the vertices and faces of G.

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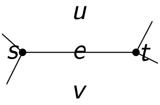

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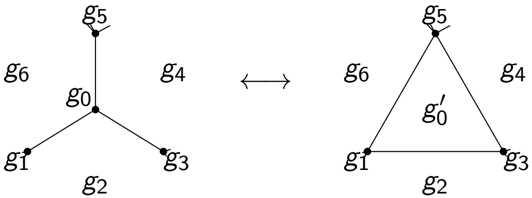

The Ising model is invariant in distribution *iff*

$$g_0^2 g_0'^2 + g_1^2 g_4^2 + g_2^2 g_5^2 + g_3^2 g_6^2 - 2(g_1g_3g_4g_6 + g_2g_3g_5g_6 + g_1g_2g_4g_5) \\ -2g_0g_0'(g_1g_4 + g_2g_5 + g_3g_6) - 4(g_0g_2g_4g_6 + g_0'g_1g_3g_5) = 0$$

Principal minors

Let M be a square matrix of size n.
For $I \subset [n]$, let $a_I = \det\left(M_I^I\right)$ be a **principal minors**.
What are the polynomial relations amongst the a_I? Polynomials in $\mathbb{C}\left[(a_I)_{I\subset[n]}\right]$ that are always null?

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Theorem *[Holtz & Sturmfels 2006; Oeding 2011]*

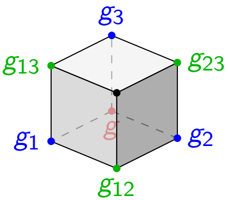
If M is symmetric,

$$\begin{aligned} a_\emptyset^2 a_{123}^2 + a_3^2 a_{12}^2 + a_2^2 a_{13}^2 + a_1^2 a_{23}^2 \\ -2a_\emptyset a_3 a_{12} a_{123} - 2a_\emptyset a_2 a_{13} a_{123} - 2a_\emptyset a_{23} a_1 a_{123} \\ -2a_3 a_2 a_{13} a_{12} - 2a_3 a_{23} a_{12} a_1 - 2a_2 a_{23} a_{13} a_1 \\ +4a_\emptyset a_{23} a_{13} a_{12} + 4a_3 a_2 a_1 a_{123} = 0 \end{aligned}$$

and all the relations of the (a_I) are generated by this.

Kashaev's recurrence

$$
\begin{aligned}
&g^2 g_{123}^2 + g_1^2 g_{23}^2 + g_2^2 g_{13}^2 + g_3^2 g_{12}^2 \\
&-2(g_2 g_3 g_{13} g_{12} + g_1 g_3 g_{23} g_{12} + g_1 g_2 g_{23} g_{13}) \\
&-2 g g_{123}(g_1 g_{23} + g_2 g_{13} + g_3 g_{12}) \\
&-4(g g_{23} g_{13} g_{12} + g_{123} g_1 g_2 g_3) = 0
\end{aligned}
$$



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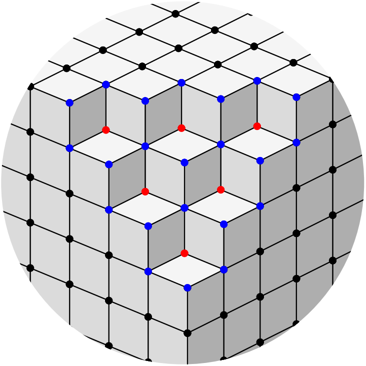

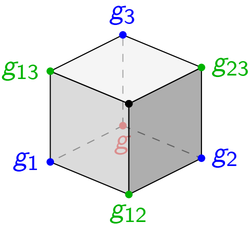

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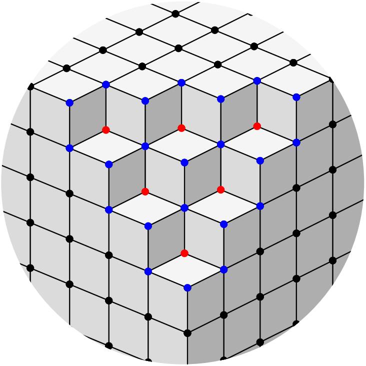

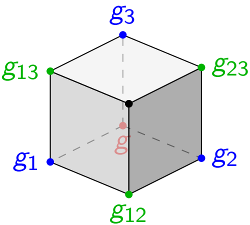

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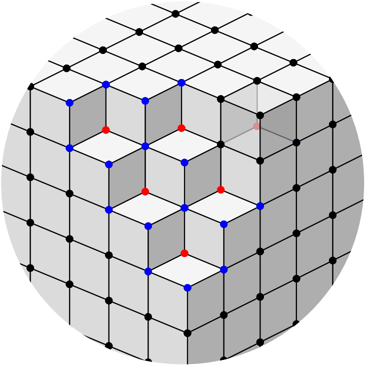

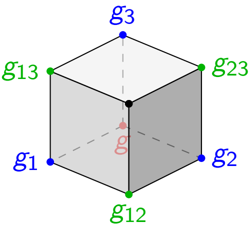

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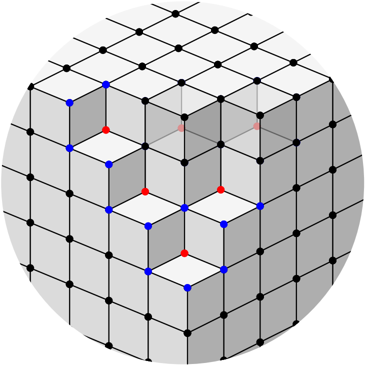

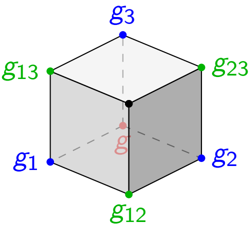

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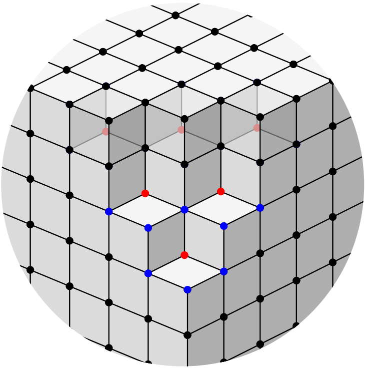

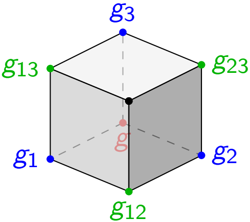

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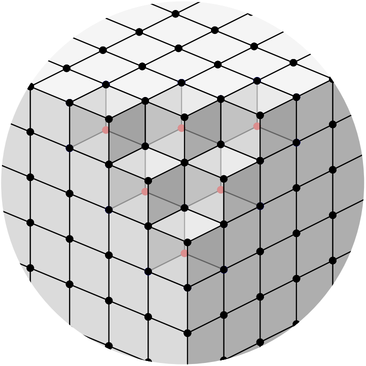

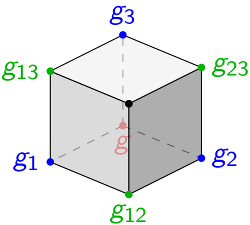

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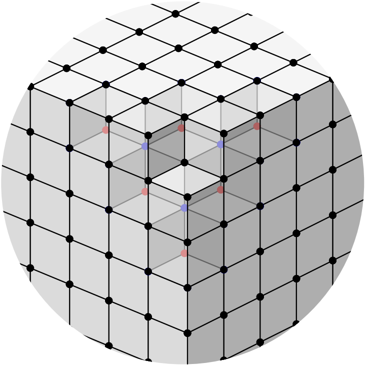

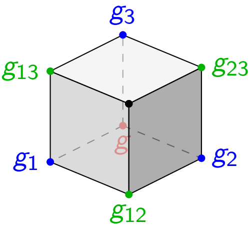

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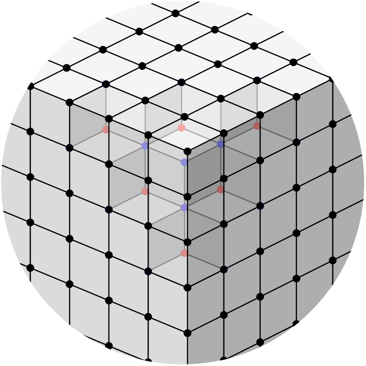
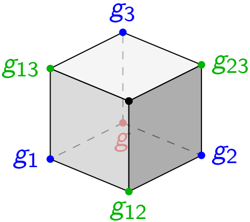




This recurrence also generates **Laurent polynomials**.

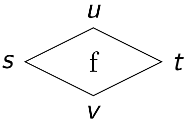
Question *[Kenyon, Pemantle 2016]*

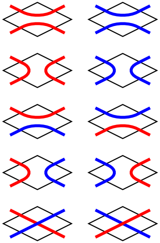
Can one identify the solution of this recurrence with the partition function of a model?

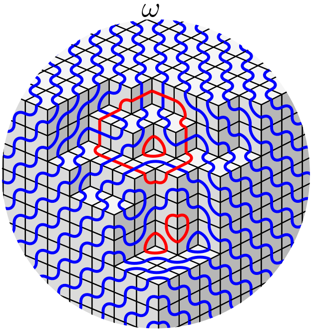
Bicolor loops

For initial conditions $I \subset \mathbb{Z}^3$,
with potentials $(g_i)_{i \in I}$,

allowed local configurations at a face f:





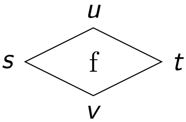


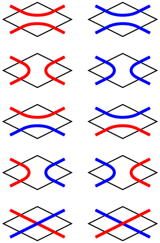
with boundary conditions.

Bicolor loops

For initial conditions $I \subset \mathbb{Z}^3$,
with potentials $(g_i)_{i \in I}$,

allowed local configurations at a face f:
and weight $w_{\mathrm{f}}(\omega)$





$g_s g_t$

$g_u g_v$

$\sqrt{g_s g_t}\sqrt{g_s g_t + g_u g_v}$

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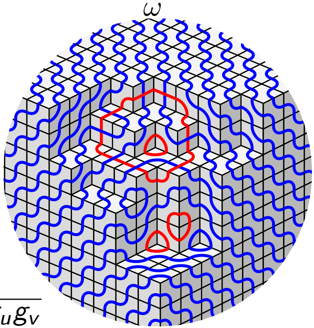

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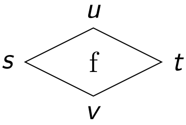

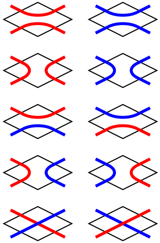

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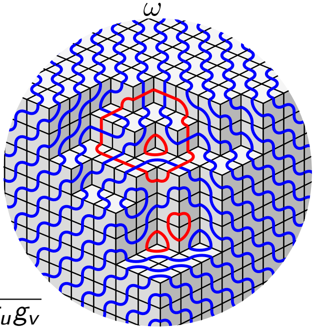

with boundary conditions. Global weight of a loop configuration ω:

$$2^{N(\omega)} \prod_{\mathrm{f}} w_\mathrm{f}(\omega)$$

where $N(\omega)$ is the number of finite loops in ω .

Solution of Kashaev's recurrence

Theorem *[M. 2018]*

The partition function

$$Z_{\text{loops}}(I,g) = \sum_{\text{loop conf. } \omega} \left(2^{N(\omega)} \prod_{\mathrm{f}} w_{\mathrm{f}}(\omega) \prod_{i \in I} g_i^{-2} \right)$$

is the solution of Kashaev's recurrence with initial conditions $(g_i)_{i \in I}$. There is a one-to-one correspondence between loop configurations and monomials of the solution.

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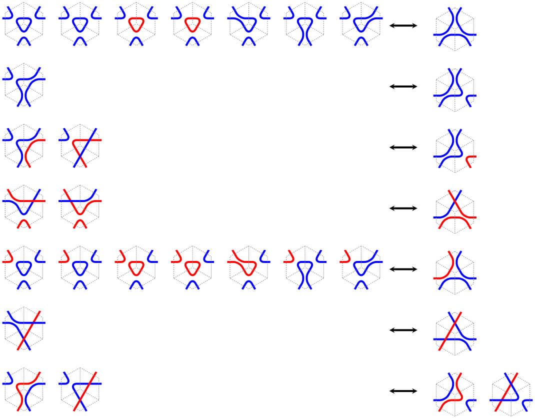
is the solution of Kashaev's recurrence with initial conditions $(g_i)_{i \in I}$. There is a one-to-one correspondence between loop configurations and monomials of the solution.

Consequences

- Laurent polynomial in the g_i and the $X_{\mathrm{f}} = \sqrt{g_s g_t + g_u g_v}$; interpretation of coefficients and exponents.
- Limit shapes.

Idea of proof:

- $Z_{\text{loops}}(I, g)$ is invariant when a cube is added to I, updating g with Kashaev's recurrence.



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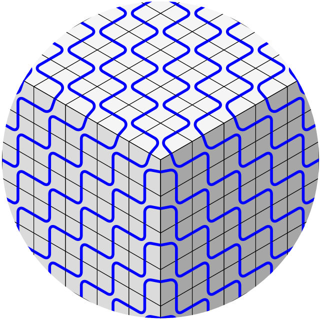

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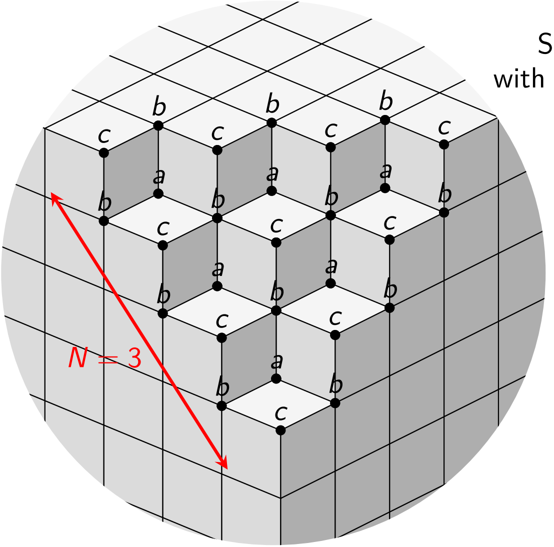
- When we get to the target vertex z, Z_{loops} is equal to g_z.
- Uniqueness: each monomial corresponds to a single configuration.

  Reconstruction algorithm of a configuration from its weight.

Limit shapes

Periodic initial conditions:



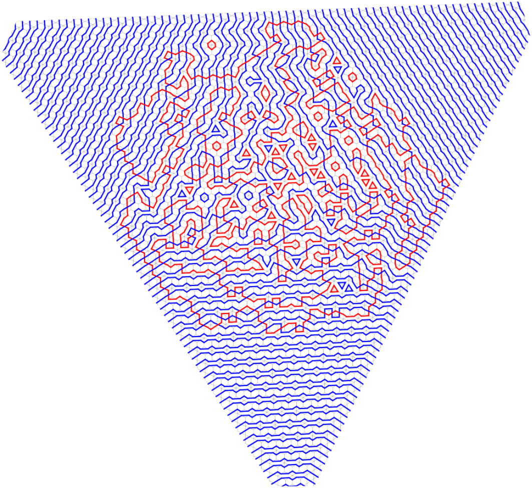
Sample a configuration with a probability proportional to its global weight.

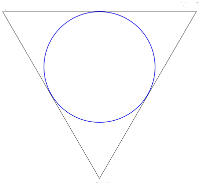
$$R = \frac{ac}{b^2},$$

$$N \to \infty,$$

Limit shapes

Periodic initial conditions: $R = 3, \quad N \to \infty$





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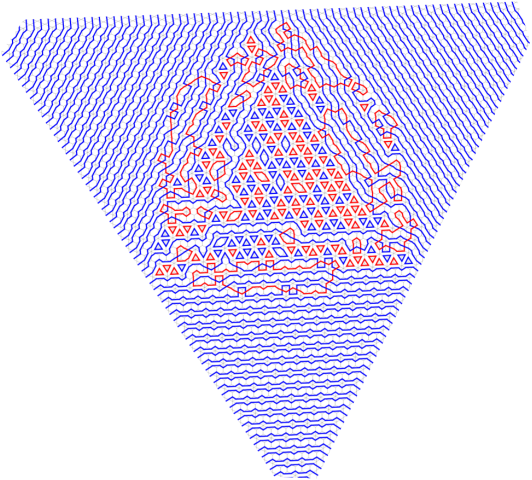

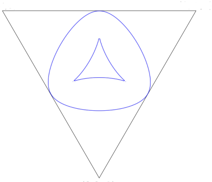

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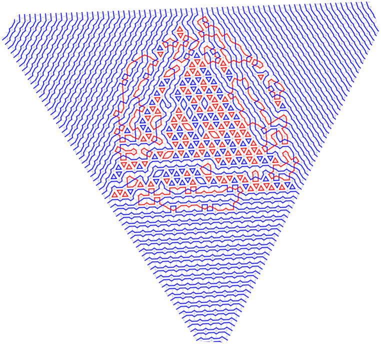
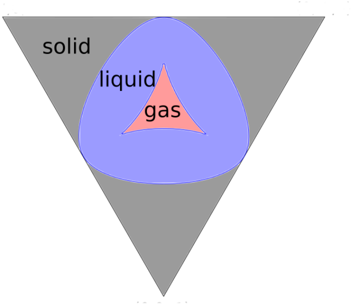




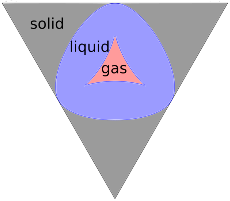
Limit shape: an explicit algebraic curve of degree 8.

Limit shapes

In the periodic initial conditions,
for any vertex $z \in I$, consider the **observable**

$$\rho(z) = \mathbb{E}\left[n_z(\omega) + \frac{1}{2(1+R)} \sum_{\mathrm{f} \sim z} \epsilon_{\mathrm{f}}(\omega) \right]$$

where $n_z(\omega)$ is the power of g_z in the total weight of ω, and $\epsilon_{\mathrm{f}}(\omega)$ the power of X_{f}.



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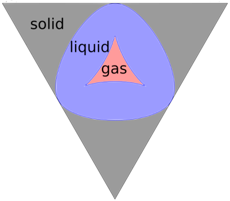

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Theorem *[M. 2018]*

For periodic initial conditions of order N, and $z = (\lfloor Nu \rfloor, \lfloor Nv \rfloor, \lfloor Nw \rfloor)$, when $N \to \infty$,

- if $[u : v : w]$ is in the solid region, $\rho(z) \sim \mathrm{cste}\ e^{-\kappa_{uvw}N}$,
- if $[u : v : w]$ is in the liquid region, $\rho(z) \sim \mathrm{cste}\ N^{-\theta_{uvw}}$,
- if $[u : v : w]$ is in the gas region, $\rho(z) \to \frac{1}{3}$.

[Petersen, Speyer 2004; Di Francesco, Soto-Garrido 2014; Kenyon, Pemantle 2016;...]

Spatial recurrences induce models with limit shapes.

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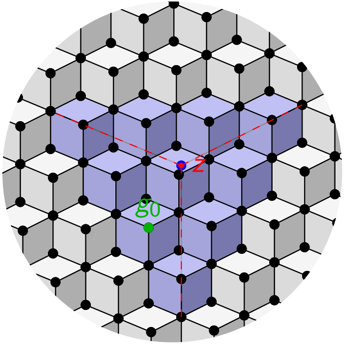

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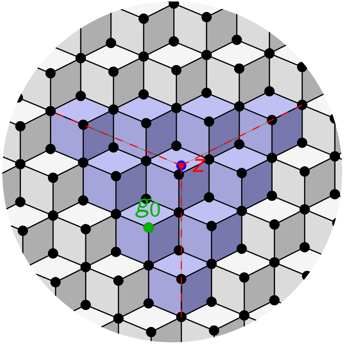
$\rightarrow$ the observables $\rho_z := g_0 \frac{\partial \ln(g_z)}{\partial g_0}$ satisfy a **linear** relation.



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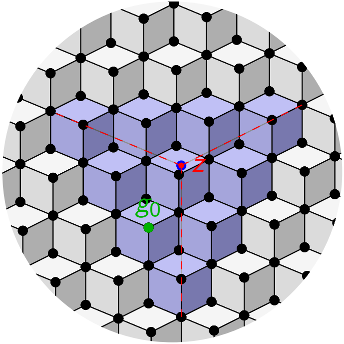
$\to F(x,y,z) = \sum_{i,j,k} \rho_{i,j,k} x^i y^j z^k$: $F(x,y,z) = \frac{P(x,y,z)}{H(x,y,z)}$.



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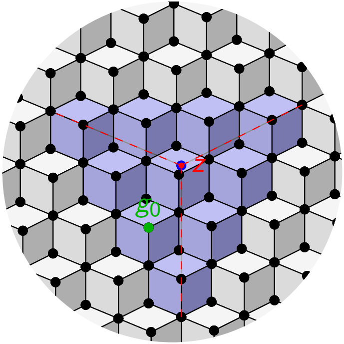


Analytic combinatorics in several
variables *[Pemantle, Wilson]* (analysis of singularities of F) yield
the asymptotic behaviour of its coefficients.

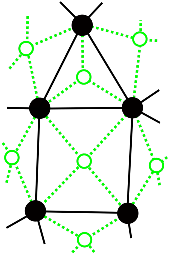
Link Ising - loops

Question

Is there a direct relation between Ising configurations and bicolor loops?

Link Ising - loops

Let $G = (V, E)$ be a finite planar graph with Ising compling constants $(J_e)_{e \in E}$. Let $G^\diamond$ be its *diamond* quadrangulation.



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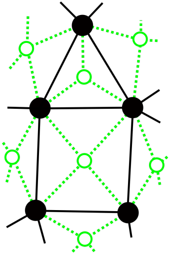
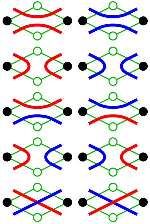
Let $G = (V, E)$ be a finite planar graph with Ising compling constants $(J_e)_{e \in E}$. Let $G^\diamond$ be its *diamond* quadrangulation.

Consider the bicolor loop model on $G^\diamond$ with local weights $\lambda = (a_e^2, b_e^2, a_e, b_e, a_e b_e)_{e \in E}$, where $a_e = \tanh 2J_e$ and $b_e = (\cosh 2J_e)^{-1}$.





a_e^2

b_e^2

a_e

b_e

$a_e b_e$

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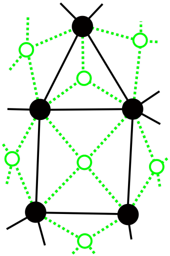
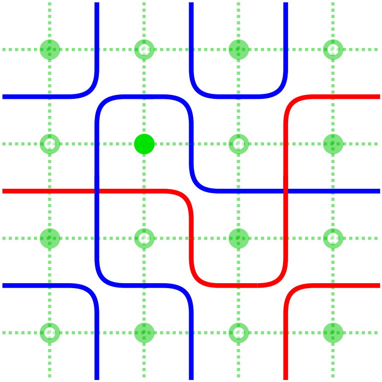


Theorem *[M. 2018]*

$$(Z_{\mathrm{Ising}}(G, J))^4 = \left(2^{2|V|} \prod_{e\in E} \cosh^2 2J_e \right) Z_{\mathrm{loops}}(G^\diamond, \lambda).$$

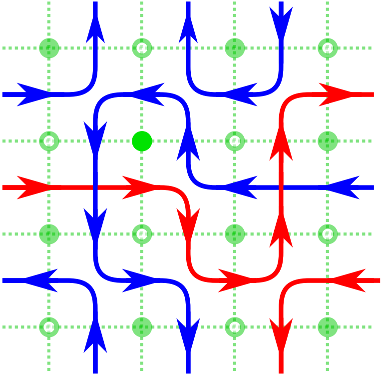
Idea of proof:



$$w(\omega) = 2^{N(\omega)} \prod_{\mathrm{f}} w_{\mathrm{f}}(\omega).$$

10 local configurations.

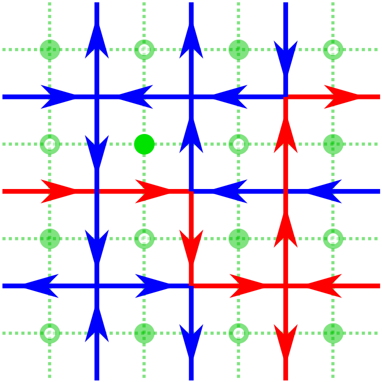
Idea of proof:



$$w(\overrightarrow{\omega}) = \prod_{\mathrm{f}} w_{\mathrm{f}}(\overrightarrow{\omega}).$$

40 local configurations.

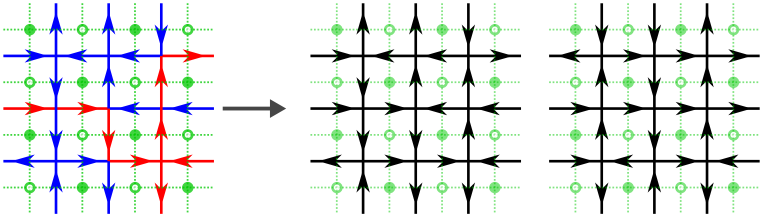
Idea of proof:



$$w(\hat{\overrightarrow{\omega}}) = \prod_{\mathrm{f}} \hat{w}_{\mathrm{f}}(\hat{\overrightarrow{\omega}}).$$

36 local configurations.

Idea of proof:



$$w(\hat{\overrightarrow{\omega}}) = \prod_{\mathrm{f}} \hat{w}_{\mathrm{f}}(\hat{\overrightarrow{\omega}}) = w_{6V}(\tau_1) w_{6V}(\tau_2).$$

where τ_1, τ_2 are two six-vertex configurations with local weights $a_e, b_e, 1$. They satisfy $a_e^2 + b_e^2 = 1$ (free-fermion).

Hence

$$Z_{\mathrm{loops}}(G^{\diamond}, \lambda) = Z_{6V}(G^{\diamond}, (a, b, 1))^2.$$

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Moreover, by **bosonization** *[Dubédat 2011]*,

$$Z_{6V}(G^{\diamond}, (a, b, 1)) = C\ Z_{\text{Ising}}(G, J)^2$$

for $C = 2^{-|V|} \prod_{e \in E} \cosh^{-1} 2J_e$.

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Therefore $Z_{\text{loops}}(G^\diamond, \lambda) = C^2 Z_{\text{Ising}}(G, J)^4$.

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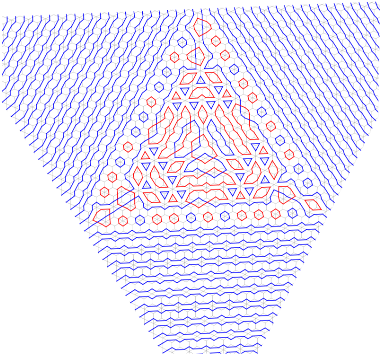
Therefore $Z_{\text{loops}}(G^\diamond, \lambda) = C^2 Z_{\text{Ising}}(G, J)^4$.

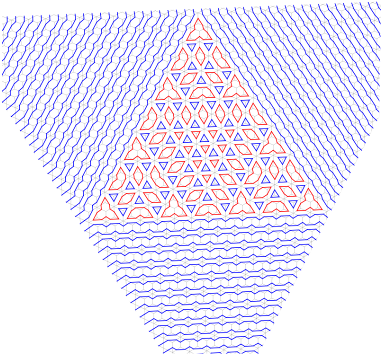
Remark

The blue loops are the interface of the product of 4 Ising models.

Other limit shapes?

For periodic initial conditions, let $R \to 0$ as $N \to \infty$.





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