Parareal computation of SDEs with time-scale separation

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In a nutshell

**Goal** Simulate slow-fast SDEs over long time, quickly

**Model** Slow-fast system of SDEs, and a macroscopic model taken from the “fast” limit

**Method** Parallel-in-time algorithm that iteratively improves the macroscopic result

**Result** Reduction in wall clock time

**Bonus** Lower variance than full microscopic model

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**Microscopic model**

- Slow-fast system of coupled SDEs
  
  \[ dX = (-X^2 + X_t(t) + Y^2) dt + \frac{\sqrt{2}}{\sqrt{\beta}} dB^{(x)} \]
  
  \[ dY = \frac{1}{\varepsilon} (X_t(t) - Y) dt + \frac{\sqrt{2}}{\sqrt{\varepsilon}} dB^{(y)}. \]

- Modeled as an ensemble \( \{ X_i \} \) of particles with positions \((X^i_t(t), Y^i)\) and weight \( W^i \)

- Time integrator: a Lie-Trotter splitting, updating \( X_i \) first, then \( Y_i \)

- Validation: deterministic solution given by the Fokker-Planck equation, akin to the macroscopic model

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**Macroscopic model**

- Only slow variable, assume the fast \( Y_i(t) \) is equilibrated and use only the expected value of the term \( Y^2 \)
  
  \[ dZ = (-Z^2 + Z_t + \mu^2) dt + \frac{\sqrt{2}}{\sqrt{\beta}} dB^{(z)}, \]

  or in potential form
  
  \[ dZ = -\partial_z V_{eff}(z) dt + \frac{\sqrt{2}}{\sqrt{\beta}} dB^{(z)}. \]

- The associated Fokker-Planck equation reads
  
  \[ \partial_t p(z) = \partial_z (\mu(z) \partial_z V_{eff} + \frac{1}{\beta} \partial_z p(z)), \]

- The macroscopic state is represented by integral quantities over a regular grid

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**The parareal algorithm**

- **Iteratively improves** the macroscopic propagator by computing the discrepancies between the macroscopic and the microscopic models in parallel

- **Parallel** use of the microscopic propagator gives a reduction in wall clock time if there are fewer iterations needed than time steps

- **Variance** of the stochastic microscopic propagator dominates the error (see figures →)

- **Reduction in variance** by using a particle propagator with correlated noise for the macroscopic model in computing the discrepancies between the models

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**Convergence in iterations**

- **Iteration converges in a few steps, need only \( K/N \) of serial wall clock time**

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**Convergence in number of particles**

- **Parameters used below:** \( \beta = 5, \varepsilon = 0.1, K = N = 20, \Delta t = 0.025, \Delta t = 1.0 \times 10^{-4} \)