Long-time homogenization of the wave equation

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Genève
Credits

Inspired by


Independent results by

- Allaire, Rauch (in preparation)

Similar results for the Schrödinger equation with potential

- Duerinckx, Gloria, Shirley (in preparation)
Main result in the periodic setting

For all $\varepsilon > 0$,

- $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$, a periodic + symmetric tensor
- $\square_\varepsilon := \partial_{tt}^2 - \nabla \cdot a_\varepsilon \nabla$ the wave operator

There exists a family $\{\bar{a}_j\}_{j \in \mathbb{N}}$ of $j + 2$-order tensors (with $\bar{a}_{2j+1} = 0$), and for all $\ell \in \mathbb{N}$ we set

$\bar{\square}_{\ell,\varepsilon} := \partial_{tt}^2 - \sum_{j=0}^{\ell-1} \varepsilon^j \bar{a}_j \cdot \nabla^{j+2} - K_\ell(i\varepsilon)^2([\frac{\ell-1}{2}]+1) \text{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)}$.

For all $\ell \in \mathbb{N}$, well-chosen $K_\ell = K_\ell(\bar{a}_0, \ldots \bar{a}_{\ell-1})$, all $u_0 \in S(\mathbb{R}^d)$, the solutions $u_\varepsilon, u_{\ell,\varepsilon} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ of

\[
\begin{align*}
\square_\varepsilon u_\varepsilon &= 0 \\
u_\varepsilon(0, \cdot) &= u_0 \\
\partial_t u_\varepsilon(0, \cdot) &= 0
\end{align*}
\]

\[
\begin{align*}
\bar{\square}_{\ell,\varepsilon} u_{\ell,\varepsilon} &= 0 \\
u_{\ell,\varepsilon}(0, \cdot) &= u_0 \\
\partial_t u_{\ell,\varepsilon}(0, \cdot) &= 0
\end{align*}
\]

satisfy for all $T > 0$:

\[
\sup_{t \leq T} \|u_\varepsilon(t) - u_{\ell,\varepsilon}(t)\|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).
\]
\[
\sup_{t \leq T} \| u_\varepsilon(t) - u_{\ell, \varepsilon}(t) \|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T)
\]

- dispersive effect for \( \ell = 3, 4 \):
  \[
  \partial_{tt}^2 - \overline{a}_0 \cdot \nabla^2 - \varepsilon \overline{a}_1 \cdot \nabla^3 - \varepsilon^2 \overline{a}_2 \cdot \nabla^4 - \varepsilon^3 \overline{a}_3 \cdot \nabla^5 - \text{(yellow term)} = 0
  \]
  \[
  = 0
  \]

Estimate new for \( \ell = 4 \) in periodic case (cf. DLS). Possible to use Boussinesq trick instead of regularization (\( \overline{a}_2 \) has a sign).

- for \( \ell > 4 \): new (but \( \overline{a}_6 \) has no sign: no Boussinesq trick)

- proof is robust: natural norm, no regularity assumption on \( a \), any order \( \ell \), quasi-periodic coefficients OK, results for random coefficients (subtle), related to localization/delocalization and diffusive/ballistic transport
Outline

Rest of the talk:

- Approach à la DLS: exact spectral theory + Fourier analysis
- Alternative approach: approximate spectral theory + Fourier analysis + energy estimates
- A few words on correctors and quantitative (stochastic) homogenization
Part 1: Approach à la DLS

Consider the wave operator\( \Box := \partial_{tt}^2 - \triangle \) and let \( u \) be the solution in \( \mathbb{R}_+ \times \mathbb{R}^d \) of
\[
\begin{cases}
\Box u = 0 \\
u(0, \cdot) = u_0 \in L^2(\mathbb{R}^d) \\
\partial_t u(0, \cdot) = 0.
\end{cases}
\]

Exact spectral theory: Fourier transform diagonalizes \(-\triangle\)

Reformulation of wave equation as a family of ODEs parametrized by frequencies: for all \( k \in \mathbb{R}^d \),
\[
\begin{cases}
\partial_{tt} \hat{u}(t, k) - |k|^2 \hat{u}(t, k) = 0 \\
\hat{u}(0, k) = \hat{u}_0(k) \\
\partial_t \hat{u}(0, k) = 0.
\end{cases}
\]

Yields explicit formula by time integration and inverse Fourier transform
\[
u(t, x) = \int_{\mathbb{R}^d} e^{ik \cdot x} \hat{u}_0(k) \cos(|k|t) d^* k.
\]
Floquet-Bloch analysis

Let a periodic and set $\mathcal{L} := -\nabla \cdot a \nabla$ and $\Box := \partial_{tt}^2 + \mathcal{L}$.

Then for all $k \in \mathbb{R}^d$ there exist $\Lambda(k) \geq 0$ and $\psi(\cdot, k) : \mathbb{R}^d \to \mathbb{C}$ periodic (and of norm $L^2$ unity on the torus) with

$$\mathcal{L}(e^{ik \cdot x} \psi(x, k)) = \Lambda(k)e^{ik \cdot x}\psi(x, k).$$

Bloch-Floquet theory: for all $g \in L^2(\mathbb{R}^d)$:

$$\tilde{g}(k) := \int_{\mathbb{R}^d} g(x)e^{-ik \cdot x}\psi(x, k)^* dx, \quad g(x) = \int_{\mathbb{R}^d} \tilde{g}(x)e^{ik \cdot x}\psi(x, k)d^*k.$$

Diagonalization of $-\nabla \cdot a \nabla$ and explicit ODE integration in frequencies:

$$\left\{ \begin{array}{l} \Box u = 0 \\ u(0, \cdot) = u_0 \\ \partial_t u(0, \cdot) = 0 \end{array} \right. \implies u(t, x) = \int_{\mathbb{R}^d} e^{ik \cdot x}\psi(x, k)\tilde{u}_0(k)\cos(t\sqrt{\Lambda(k)})d^*k.$$
Strategy of DLS

Rescale $a$ for $\varepsilon > 0$: $\mathcal{L}_\varepsilon := -\nabla \cdot a(\cdot\varepsilon) \nabla$ and $\Box_\varepsilon := \partial_{tt}^2 + \mathcal{L}_\varepsilon$.

Bloch-Floquet analysis yields (after rescaling):

$$\left\{ \begin{array}{l}
\Box_\varepsilon u_\varepsilon = 0 \\
u_\varepsilon(0, \cdot) = u_0 \implies u_\varepsilon(t, x) = \int_{\mathbb{R}^d} e^{i k \cdot x} \psi(\frac{x}{\varepsilon}, \varepsilon k) \tilde{u}_0^\varepsilon(\varepsilon k) \cos\left(\frac{t}{\varepsilon} \sqrt{\Lambda(\varepsilon k)}\right) d^* k.
\end{array} \right.$$ 

Strategy of DLS: quantify the convergences

- $\psi(\frac{x}{\varepsilon}, \varepsilon k) = 1 + O(\varepsilon)$,
- $\tilde{u}_0^\varepsilon(\varepsilon k) = \hat{u}_0(k) + O(\varepsilon)$,
- $\Lambda(\varepsilon k) = \varepsilon^2 \bar{a}_0 \cdot k \otimes^2 + \varepsilon^4 \bar{a}_2 \cdot k \otimes^4 + o(\varepsilon^4)$,

which suggest $u_\varepsilon(t, x) \simeq \int_{\mathbb{R}^d} e^{i k \cdot x} \hat{u}_0(k) \cos(t \sqrt{\bar{a}_0 \cdot k \otimes^2 + \varepsilon^2 \bar{a}_2 \cdot k \otimes^4}) d^* k$. 

Comments on the strategy of DLS

Fundamental observation: need to quantify the convergences of

1. $\psi(\frac{x}{\varepsilon}, \varepsilon k) = 1 + O(\varepsilon),$
2. $\tilde{u}_0^\varepsilon(\varepsilon k) = \hat{u}_0(k) + O(\varepsilon),$
3. $\Lambda(\varepsilon k) = \varepsilon^2 \tilde{a}_0 \cdot k \otimes^2 + \varepsilon^4 \tilde{a}_2 \cdot k \otimes^4 + o(\varepsilon^4).$

Limitations:

1. starting point: Floquet-Bloch theorem only holds for periodic coefficients (wrong for quasi-periodic or random coefficients)
2. technique: estimates by hands on Fourier formula are difficult, which yields suboptimal norm ($L^2 + L^\infty$) and estimate in DLS
3. deep algebraic structure yet to be understood
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- technique: estimates by hands on Fourier formula are difficult, which yields suboptimal norm ($L^2 + L^\infty$) and estimate in DLS
- deep algebraic structure yet to be understood
From Bloch waves to homogenization: $|k| \ll 1$

Recall that we have $\mathcal{L}(e^{ik \cdot x} \psi(x, k)) = \Lambda(k)e^{ik \cdot x} \psi(x, k)$ for periodic $\psi(\cdot, k)$. Expand: magnetic eigenvalue problem on the torus

$$-(\nabla + ik) \cdot a(\nabla + ik)\psi(x, k) = \Lambda(k)\psi(x, k) \text{ on } \mathbb{T} = [0, 2\pi)^d.$$  

Observation: for $k = 0$, $\Lambda(0) = 0$ and $\psi(\cdot, 0) \equiv 1$.

Linearization in the regime $k = \kappa e$, $e$ unit vector of $\mathbb{R}^d$ and $0 < \kappa \ll 1$:  

$\psi(x, k) = 1 + i\kappa \phi_e(x) + o(\kappa), \ \Lambda(k) = \kappa^2 \lambda_e + o(\kappa^2)$

$$-\nabla \cdot a(\nabla \phi_e(x) + e) = 0 \text{ on } \mathbb{T}$$

Multiply original equation by $\psi$, integrate over $\mathbb{T}$, use expansion:

$$\lambda_e = \int_{\mathbb{T}} (\nabla \phi_e + e) \cdot a(\nabla \phi_e + e)$$
From Bloch waves to homogenization: $|k| \ll 1$

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**Linearization** in the regime $k = \kappa e$, $e$ unit vector of $\mathbb{R}^d$ and $0 < \kappa \ll 1$: $\psi(x, k) = 1 + i\kappa \phi_e(x) + o(\kappa)$, $\Lambda(k) = \kappa^2 \lambda_e + o(\kappa^2)$

$$-\nabla \cdot a(\nabla \phi_e(x) + e) = 0 \text{ on } \mathbb{T} \rightsquigarrow \text{corrector}$$

Multiply original equation by $\psi$, integrate over $\mathbb{T}$, use expansion:

$$\lambda_e = \int_{\mathbb{T}} (\nabla \phi_e + e) \cdot a(\nabla \phi_e + e) \rightsquigarrow \text{homogenized coefficient}$$
Justification & limitation

The magnetic eigenvalue problem

\[-(\nabla + ik) \cdot a(\nabla + ik)\psi(x, k) = \Lambda(k)\psi(x, k) \text{ on } \mathbb{T} = [0, 2\pi)^d.\]

has compact resolvent (by Rellich) so that \[-(\nabla + ik) \cdot a(\nabla + ik)\] admits (for all \(k \in [0, 2\pi)^d\)) a sequence of eigenvectors and eigenvalues...

In the neighborhood of 0, \(k \mapsto \psi(\cdot, k)\) is analytic so that the linearization can be justified. Higher-order expansions yield \(\phi_2, \phi_3, ..., \bar{a}_1, \bar{a}_2, ...\) And we have quantitative estimates for truncations of the series.

**Justifies the DLS strategy:** \(\psi(x, \varepsilon k) = 1 + \varepsilon \kappa \phi_e(x) + O(\kappa^2).\)

**Main limitation:** \[-(\nabla + ik) \cdot a(\nabla + ik)\] does not have compact resolvent for a quasi-periodic, almost-periodic, or random...
Part 2: Alternative approach

Observation 1: To state the result, we only need “correctors”

Observation 2: To prove the result, we only need to ”diagonalize” the elliptic operator at the bottom of the spectrum since frequencies are rescaled by $\varepsilon$

Strategy: Use correctors to develop an approximate spectral theory

- construct correctors at any order and define approximate Bloch waves close to 0 as a jet using the correctors ($\equiv$Taylor-Bloch waves)

- Taylor-Bloch waves are approximate extended waves: there is an error in the eigenvector/eigenvalue relation ($\equiv$eigendefect), the structure of which is very special

- control the large-time error due to the eigendefect directly using the wave equation and its special structure ($\equiv$energy estimates)
New family of correctors \((\phi_j, \sigma_j, \chi_j)\)

\[\phi_0 \equiv 1, \text{ and for all } j \geq 1, \ \phi_j \text{ is a scalar field solving } -\nabla \cdot a \nabla \phi_j = \nabla \cdot (-\sigma_{j-1}e + ae\phi_{j-1} + \nabla \chi_{j-1});\]
New family of correctors \((\phi_j, \sigma_j, \chi_j)\)

\[
\Phi_0 \equiv 1, \text{ and for all } j \geq 1, \ \phi_j \text{ is a scalar field solving } -\nabla \cdot a \nabla \phi_j = -\nabla \cdot (a \chi_j - 1 - \lambda_{j-1}) + \nabla \chi_j; \\
\bar{a}_j \cdot e^{\otimes (j+1)} = \int_{\mathbb{T}} \, a(\nabla \phi_{j+1} + e \phi_{j}), \ \lambda_j := \bar{a}_j \cdot e^{\otimes (j+2)}; \\
\chi_0 \equiv 0, \ \chi_1 \equiv 0, \text{ and for all } j \geq 2, \ \chi_j \text{ is a scalar field solving } -\nabla \chi_j = \nabla \chi_{j-1} \cdot e + \sum_{l=1}^{j-1} \lambda_{j-1-l} \phi_l; \\
\]

Very subtle algebraic structure
New family of correctors \((\phi_j, \sigma_j, \chi_j)\)

- for all \(j \geq 1\), \(q_j\) is the vector field
  \[
  q_j := a(\nabla \phi_j + e\phi_{j-1}) - \lambda_{j-1}e + \nabla \chi_{j-1} - \sigma_{j-1}e, \quad \int_{\mathbb{T}} q_j = 0; 
  \]
- \(\sigma_0 \equiv 0\), and for all \(j \geq 1\), \(\sigma_j\) is a skew-symmetric matrix field, i.e. \(\sigma_{jkl} = -\sigma_{jlk}\), that solves
  \[-\Delta \sigma_j = \nabla \times q_j, \quad \nabla \cdot \sigma_j = q_j, \]
  with the three-dimensional notation:
  \[
  [\nabla \times q_j]_{mn} = \nabla_m [q_j]_n - \nabla_n [q_j]_m, 
  \]
New family of correctors \((\phi_j, \sigma_j, \chi_j)\)

- \(\phi_0 \equiv 1\), and for all \(j \geq 1\), \(\phi_j\) is a scalar field solving 
  \[-\nabla \cdot a \nabla \phi_j = \nabla \cdot (-\sigma_{j-1} e + ae\phi_{j-1} + \nabla \chi_{j-1});\]

- for all \(j \geq 0\), \(\bar{a}_j \cdot e^{\otimes(j+1)} = \int_\mathbb{T} a(\nabla \phi_{j+1} + e\phi_j), \lambda_j := \bar{a}_j \cdot e^{\otimes(j+2)};\]

- \(\chi_0 \equiv 0, \chi_1 \equiv 0\), and for all \(j \geq 2\), \(\chi_j\) is a scalar field solving 
  \[-\Delta \chi_j = \nabla \chi_{j-1} \cdot e + \sum_{l=1}^{j-1} \lambda_{j-1-l} \phi_l;\]

- for all \(j \geq 1\), \(q_j\) is the vector field 
  \(q_j := a(\nabla \phi_j + e\phi_{j-1}) - \lambda_{j-1} e + \nabla \chi_{j-1} - \sigma_{j-1} e, \quad \int_\mathbb{T} q_j = 0;\]

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  \(\sigma_{jkl} = -\sigma_{jlk}\), that solves 
  \(-\Delta \sigma_j = \nabla \times q_j, \quad \nabla \cdot \sigma_j = q_j\), with the three-dimensional notation: 
  \([\nabla \times q_j]_{mn} = \nabla_m [q_j]_n - \nabla_n [q_j]_m;\)

Very subtle algebraic structure.

For periodic coefficients all the correctors exist and are periodic.
Approximate spectral theory

Taylor-Bloch wave $\psi_{k,\ell}$ and Taylor-Bloch eigenvalue $\tilde{\lambda}_{k,\ell}$ of order $\ell$ in direction $k = \kappa e$ are defined by

$$
\psi_{k,\ell} := \sum_{j=0}^{\ell} (i \kappa)^j \phi_j, \quad \tilde{\lambda}_{k,\ell} := \kappa^2 \sum_{j=0}^{\ell-1} (i \kappa)^j \lambda_j \in \mathbb{R}.
$$

Almost diagonalization of magnetic Laplacian for $0 < \kappa \ll 1$:

$$
-(\nabla + ik) \cdot a(\nabla + ik) \psi_{k,\ell} = \tilde{\lambda}_{k,\ell} \psi_{k,\ell} - (i \kappa)^{\ell+1} \vartheta_{k,\ell},
$$

where the Taylor-Bloch eigendefect $\vartheta_{k,\ell}$ is given by

$$
\vartheta_{k,\ell} = \nabla \cdot \left(-\sigma_{\ell} e + ae \varphi_{\ell} + \nabla \chi_{\ell}\right) + i \kappa \left(e \cdot ae \varphi_{\ell} - \sum_{j=1}^{\ell} \sum_{l=\ell-j}^{\ell-1} (i \kappa)^{j+l-\ell} \lambda_l \phi_j\right).
$$

Subtle structure: eigendefect = divergence term + higher order term
Approximate solution of the wave equation

**Strategy:** use the approximate spectral theory to construct an approximate solution of the wave equation

\[
\begin{align*}
\Box_\varepsilon u_\varepsilon &= 0 \\
 u_\varepsilon(0, \cdot) &= u_0 \\
 \partial_t u_\varepsilon(0, \cdot) &= 0
\end{align*}
\]

- Well-prepare initial condition (cannot use that \( u_0 \) can be expanded on Taylor-Bloch waves)
- Use that Taylor-Bloch wave almost diagonalize the elliptic operator and control the error by energy estimates
- Reformulate the almost-solution and write an approximate (high-order) homogenized equation

*[Estimates are first presented in the periodic and quasi-periodic setting]*
Approximate solution of the wave equation

**Step 1:** replace $u_0$ by a well-prepared data

$$u_{0,\ell,\varepsilon} := \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik \cdot x} \psi_{k,\ell,\varepsilon} \frac{x}{\varepsilon} d^* k.$$  

Energy estimate: Solution $v_{\varepsilon,\ell}$ with initial condition $u_{0,\ell,\varepsilon}$

$$\| u_{\varepsilon} - v_{\varepsilon,\ell} \|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^d))} \leq \| u_0 - u_{0,\ell,\varepsilon} \|_{L^2(\mathbb{R}^d)} \leq C(u_0)\varepsilon.$$  

**Step 2:** almost diagonalization of the wave equation

Set $\Lambda_{\ell}(k) := \sqrt{\max\{0, \tilde{\lambda}_{k,\ell}\}}$, and define

$$w_{\varepsilon,\ell}(t,x) = \int_{\mathbb{R}^d} \hat{u}_0(k) e^{ik \cdot x} \psi_{k,\ell,\varepsilon} \frac{x}{\varepsilon} \cos(\varepsilon^{-1} \Lambda_{\ell}(\varepsilon k) t) d^* k$$

Energy estimate: for all $T \geq 0$,

$$\| v_{\varepsilon,\ell} - w_{\varepsilon,\ell} \|_{L^\infty([0,T],L^2(\mathbb{R}^d))} \leq C(u_0)(\varepsilon + \varepsilon^\ell T).$$
Sketch of the argument for step 2

One of the error terms solves

\[
\begin{cases}
\Box_\varepsilon \delta v(t, x) = \varepsilon^\ell \int_{\mathbb{R}^d} G(t, k) \nabla \cdot \left( g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \right) \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk \\
\delta v(0, \cdot) = \partial_t \delta v(0, \cdot) = 0
\end{cases}
\]

**Difficulty**: how not to lose accuracy in \( \varepsilon \)?

- wave equation is not regularizing: need to estimate RHS in \( L^2(\mathbb{R}^d) \)
- \( \nabla \cdot \left( g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \right) \) is only bounded by \( \varepsilon^{-1} \) in \( L^2(\mathbb{R}^d) \)

First need to estimate \( \| \partial_t \delta v(t) \|_{L^2(\mathbb{R}^d)} + \| \nabla \delta v(t) \|_{L^2(\mathbb{R}^d)} \).

Multiply by \( \partial_t \delta v \) and integrate over \([0, t] \times \mathbb{R}^d\):

\[
\| \partial_t \delta v(t) \|_{L^2(\mathbb{R}^d)}^2 + \| \nabla \delta v(t) \|_{L^2(\mathbb{R}^d)}^2 \lesssim \varepsilon^\ell I,
\]

\[
I := - \int_{[0, t] \times \mathbb{R}^d} \int_{\mathbb{R}^d} G(s, k) \nabla \cdot \left( g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \right) \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk \partial_t \delta v(s, x) ds dx
\]
Sketch of the argument for step 2

Key observation: integrate by parts in space first, then in time

\[ I = - \int_{[0,t] \times \mathbb{R}^d} G(s, k) \nabla \cdot \left( g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \right) \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk \partial_t \delta v(s, x) ds dx \]

\[ = \int_{[0,t] \times \mathbb{R}^d} G(s, k) g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk \nabla \partial_t \delta v(s, x) ds dx \]

\[ = \int_{[0,t] \times \mathbb{R}^d} \varepsilon^{-1} \Lambda_\ell(\varepsilon k) G(s, k) \left( g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \right) \sin(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk \nabla \delta v(s, x) ds dx \]

\[ + \int_{\mathbb{R}^d} \nabla \delta v(t, x) \cdot \int_{\mathbb{R}^d} g\left( \frac{x}{\varepsilon} \right) e^{ik \cdot x} \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk dx \]

Recall that

\[ \| \partial_t \delta v(t) \|^2_{L^2(\mathbb{R}^d)} + \| \nabla \delta v(t) \|^2_{L^2(\mathbb{R}^d)} \lesssim -\varepsilon^\ell I, \]

so that by Young’s inequality

\[ \| \partial_t \delta v(t) \|^2_{L^2(\mathbb{R}^d)} + \| \nabla \delta v(t) \|^2_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0) \varepsilon^\ell T. \]
Approximate solution of the wave equation

**Step 3:** Throw away the correctors

\[ w_{\varepsilon, \ell}(t, x) = \int_{\mathbb{R}^d} \hat{u}_0(k)e^{ik \cdot x} \psi_{\varepsilon k, \ell}(\frac{x}{\varepsilon}) \cos(\varepsilon^{-1} \Lambda_{\ell}(\varepsilon k) t) d^* k \]

\[ \simeq \int_{\mathbb{R}^d} \hat{u}_0(k)e^{ik \cdot x} \cos(\varepsilon^{-1} \Lambda_{\ell}(\varepsilon k) t) d^* k =: v_{\varepsilon}, \]

With \( \tilde{\Box}_{\ell, \varepsilon} := \partial_{tt}^2 - \sum_{j=0}^{\ell-1} \varepsilon^j \bar{a}_j \cdot \nabla^{j+2} \), we have for all \( T \geq 0 \),

\[
\begin{cases}
\Box_{\varepsilon} u_{\varepsilon} = 0 \\
u_{\varepsilon}(0, \cdot) = u_0 \\
\partial_t u_{\varepsilon}(0, \cdot) = 0
\end{cases},
\begin{cases}
\tilde{\Box}_{\ell, \varepsilon} v_{\varepsilon} = 0 \\
v_{\varepsilon}(0, \cdot) = u_0 \\
\partial_t v_{\varepsilon}(0, \cdot) = 0
\end{cases}
\]

\[
\sup_{t \leq T} \| u_{\varepsilon}(t) - v_{\varepsilon}(t) \|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).
\]

And it remains to add a regularizing term to \( \tilde{\Box}_{\ell, \varepsilon} \) to make it invertible.
Main result in the periodic setting

For all $\varepsilon > 0$, 

- $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$, a periodic + symmetric tensor
- $\Box_\varepsilon := \partial_{tt}^2 - \nabla \cdot a_\varepsilon \nabla$ the wave operator

There exists a family $\{\overline{a}_j\}_{j \in \mathbb{N}}$ of $j + 2$-order tensors (with $\overline{a}_{2j+1} = 0$), and for all $\ell \in \mathbb{N}$ we set

- $\overline{\Box}_{\ell,\varepsilon} := \partial_{tt}^2 - \sum_{j=0}^{\ell-1} \varepsilon^j \overline{a}_j \cdot \nabla^{j+2} - K_\ell(i\varepsilon)^2([\frac{j-1}{2}]+1)I_d \cdot \nabla^2([\frac{\ell-1}{2}]+2)$.

For all $\ell \in \mathbb{N}$, well-chosen $K_\ell = K_\ell(\overline{a}_0, \ldots \overline{a}_{\ell-1})$, all $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the solutions $u_\varepsilon, u_{\ell,\varepsilon} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^d))$ of

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\begin{align*}
\Box_\varepsilon u_\varepsilon &= 0 \\
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\[
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u_{\ell,\varepsilon}(0, \cdot) &= u_0 \\
\partial_t u_{\ell,\varepsilon}(0, \cdot) &= 0
\end{align*}
\]

satisfy for all $T > 0$: 

\[
\sup_{t \leq T} \|u_\varepsilon(t) - u_{\ell,\varepsilon}(t)\|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0)(\varepsilon + \varepsilon^\ell T).
\]
Part 3: Bounds on correctors in the random case

Example of a given by Poisson random inclusions of fixed size we have for the correctors [G-Otto,G-Neukamm-Otto,Armstrong-Kuusi-Mourrat]:

\[
\begin{align*}
|&(\phi_1, \sigma_1, \nabla \chi_1)(x)| \lesssim \omega \\
&\quad \begin{cases}
    d = 1 : & (1 + |x|)^{\frac{1}{2}} \\
    d = 2 : & \log(2 + |x|)^{\frac{1}{2}} \\
    d > 2 : & 1
\end{cases} \\
|&(\phi_2, \sigma_2, \nabla \chi_2)(x)| \lesssim \omega \\
&\quad \begin{cases}
    d = 3 : & (1 + |x|)^{\frac{1}{2}} \\
    d = 4 : & \log(2 + |x|)^{\frac{1}{2}} \\
    d > 4 : & 1
\end{cases} \\
|&(\phi_3, \sigma_3, \nabla \chi_3)(x)| \lesssim \omega \\
&\quad \begin{cases}
    d = 5 : & (1 + |x|)^{\frac{1}{2}} \\
    d = 6 : & \log(2 + |x|)^{\frac{1}{2}} \\
    d > 6 : & 1
\end{cases}
\end{align*}
\]

Apply strategy of Part 2: dispersive effects appear for \( d \geq 5 \).
In smaller dimensions, homogenization breaks down before the occurrence of dispersive effects.
Further comments on the random case

- Sharp bounds on the correctors can be proved for correlated fields as well [Duerinckx-G., G-Neukamm-Otto]
- The main result can be formulated as asymptotic ballistic transport of classical waves at the bottom of the spectrum (for random case, requires $d > 2$)
- Two phenomena could occur when “homogenization breaks down”:
  - the transport remains ballistic (as for wave equation), but the effective equation is different (if any)
  - the transport stops being ballistic, and might become diffuse (as for the random Schrödinger equation), radiative transfer?
Summary of the talk

- Main idea 1: develop an approximate spectral theory at the bottom of the spectrum (a lot of structure), cf. Taylor-Bloch waves

- Main idea 2: combine Fourier space (in the form of estimates of Fourier multipliers) with energy estimates given by the wave equation

- Main result: bounds on extended correctors drive long-time homogenization of the wave equation (periodic, quasi-periodic, almost periodic, random...)

- Main challenging problem: what happens when homogenization breaks down? (For Schrödinger, work in progress with Duerinckx & Shirley)