Objectifs :
- Report → Mean square exponential stability ⇒ almost sure exponential stability (for “very large” class of stochastic systems),
- Report → Almost sure exponential stability (for “very large” class of stochastic systems),
- Relaxation of the stability conditions used in the literature for stochastic systems with multiplicative noises
  → Replace the mean square exponential stability in (literature) by the almost sure exponential stability,
- Application to observers synthesis ,
- Application to control synthesis.

Stochastic systems and notions of stability considered :
\[ \begin{align*}
\dot{x} &= f(x, u) \, dt + g(x, u) \, dw_x \\
\dot{y} &= h(x) \, dt + g(x) \, dw_y 
\end{align*} \]

- \( x \in \mathbb{R}^n \) is the state vector, \( y \in \mathbb{R}^p \) is the output vector and \( u \in \mathbb{R}^m \) is the vector of known inputs,
- \( w_x \in \mathbb{R}^d \) and \( w_y(t) \in \mathbb{R}^q \) are multi-dimensional independent Brownian motions,
- \( f(x, u) \) the drift part of the stochastic differential equation (SDE),
- \( g(x, u) \) the diffusion part of the stochastic differential equation (SDE).

 Almost sure exponential stability : \( \limsup_{t \to +\infty} \ln(\|x(t, t_0, x_0)\|) < -\alpha < 0 \quad \forall x_0 \in \mathbb{R}^n \) almost surely

Application to the observer design

Problem statement
The Observer is given by
\[ \dot{x} = f(\hat{x}, u) \, dt + \psi(u)(y - h(\hat{x})) \, dt \]
The filtering error \( e = x - \hat{x} \)
\[ \begin{align*}
\dot{e} &= (f(x, u) - f(\hat{x}, u)) \, dt + g(x, u) \, dw_x \\
&\quad + g(\hat{x}, u) \, dw_y - \psi(u)(\hat{x} - x) \, dt
\end{align*} \]
\( \psi(u) \) is the matrix gain to determine such that the observation error \( e(t) \) converges exponentially almost surely.

The almost sure exponential stability of \( e \) needs the almost sure exponential stability \( x \).

Problem : The approaches based “Lyapunov” (literature) are reduced to the mean square exponential stability

Approach used : Stability of stochastic triangular systems
We consider a class of stochastic differential equation
\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, u) \, dt + g_1(x_1, u) \, dw \\
\dot{x}_2 &= f_2(x_1, x_2, u) \, dt + g_2(x_1, x_2, u) \, dw \\
\dot{\tau}_1 &= f_1(\tau_1, u) \, dt + g_1(\tau_1, u) \, dw \\
\dot{\tau}_2 &= f_2(\tau_1, \tau_2, u) \, dt + g_2(\tau_1, \tau_2, u) \, dw
\end{align*} \]

Assumption 1 : it exists a real \( k > 0 \) such that, \( \forall t \geq 0 \)
\[ \begin{align*}
&\|f_2(x_1, x_2, u) - f_2(0, 0, u)\| \leq k \|x_1 + x_2 - \tau_2\|, \\
&\text{trace}\left((g_2(x_1, u) - g_2(0, u))(g_2(x_1, u) - g_2(\tau_1, u))\right) \leq k \|x_1 - \tau_1\|^2, \\
&\text{trace}\left((g_2(x_1, u) - g_2(0, u))(g_2(x_1, u) - g_2(\tau_1, u))\right) \leq k \|x_1 - \tau_1\|^2.
\end{align*} \]

Theorem 1 : With assumption 1, the equilibrium point of SDE (1) is almost surely exponentially stable if and only if the equilibrium point of SDE (2) is almost surely exponentially stable.

Application of theorem 1 to the design observer

Theorem 2 : If the assumption 1 is satisfied with SDE \( \dot{x} = f(x, u) \, dt + g(x, u) \, dw \), then the system \( \dot{x} = f(x, u) \, dt + \psi(u)(y - h(x)) \, dt \) is an observer for the considered stochastic system Guaranteeing the almost sure exponential stability of the filtering error if

1. The SDE \( \dot{x} = f(x, u) \, dt + g(x, u) \, dw \) is stable exponentially almost surely,
2. It exists a matrix gain \( \psi(u) \) such that the Ordinary differential equation (ODE)
\[ e = -f(-e, u) + \psi(u)h(-e) \]
is exponentially stable.

Example

\[ \begin{align*}
\text{State of the system } x(t) \text{ and inputs } u_1(t) (\mathbb{R}) \text{ and } u_2(t) (\mathbb{R})
\end{align*} \]

Conclusion

- Decoupling of the stability of the stochastic system from the stability of the filtering error.
- Stability of the SDE of the system → Itô calculus, LMI, etc...
- Calcul de la gain \( \psi(u) \) of the observer → Literature of the observer for the nonlinear ODE → Lyapunov, LMI, great gain, etc...

E-mail : asma.barbata@univ-lorraine.fr, barbata_asma@yahoo.fr