Research Center for Statistics and Department of Economics
Time Series Fall semester 2013-2014

ASSIGNMENT 04

Exercise 1

Consider the general form of the ARIMA (AutoRegressive Integrated Moving Average):

$$\nabla^d y_t = \phi_1 \nabla^d y_{t-1} + \ldots + \phi_p \nabla^d y_{t-p} + a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q},$$

where $\{a_t\}$ are iid with expectation 0 and variance σ^2 and $\nabla^d y_t$ represents the differenced series of order d.

We can rewrite the model in terms of the backshift operator B as follows:

$$\underbrace{(I - \phi_1 B - \ldots - \phi_p B^p)}_{AR(p)} \underbrace{(1-B)^d}_{d \quad \text{differences}} y_t = \underbrace{(I - \theta_1 B - \ldots - \theta_q B^q)}_{MA(q)} a_t$$

The R function below allows to simulate trajectories of a certain length for models $ARIMA(p, 0, q)$ for $p \leq 2$ and $q \leq 2$ with a_t being iid observations either drawn from the Gaussian or the t-distribution.

```
ARMAsim <- function(phi,theta,first,serieslength,distribution)
{
  # phi: AR parameters, vector of size 2
  # theta: MA parameters, vector of size 2
  # first: fixed first elements for computation of AR part, vector of size 2
  if(distribution=="t"){a <- rt(serieslength,3)}
  else {a <- rnorm(serieslength,0,1)}
  Y <- first
  for (i in (length(phi)+1):serieslength){
    Y[i] <- phi[1]*Y[i-1]+phi[2]*Y[i-2]+a[i]-theta[1]*a[i-1]-theta[2]*a[i-2]
  }
  return(Y)
}
```

Example of simulation:

```
ts <- ARMAsim(phi=c(0.9,-0.1),theta=c(0.2,0.5),first=c(0,0),
serieslength=1000,distribution="normal")
```

a) Assume $a_t \sim \mathcal{N}(0,1)$ and simulate a trajectory of length 100 for ARIMA(0,0,0). What is this process?

b) Simulate ARIMA(1,0,0) for the following values of the parameter $\phi = \{-0.9, -0.3, 0.5, 0.9, 1\}$. Plot Y_t *vs.* t, Y_t *vs.* Y_{t-1}, and Y_t *vs.* Y_{t-2}. Investigate the plots of ACF and PACF of the obtained series and interpret the results. What is the process corresponding to $\phi = 1$?

c) Simulate ARIMA(0,0,1) for the following values of the parameter $\theta = \{1.9, 0.5, -0.5, -1\}$. Plot Y_t *vs.* t, Y_t *vs.* Y_{t-1}, and Y_t *vs.* Y_{t-2}. Investigate the plots of ACF and PACF of the obtained series and interpret the results.

d) Repeat (b) and (c) when $a_t \sim t_3$, a Student's t distribution with 3 degrees of freedom and compare the results with respect to the Gaussian case.

e) Simulate from an ARIMA(1,0,1) process for $\phi = 0.7$ and $\theta = -0.3$ and obtain the necessary plots as in points (b) or (c).

Exercise 2

Show that the autocorrelation function of a ARMA(1,1) process

$$\text{ARMA(1,1)}: \; y_t = \phi y_{t-1} + a_t - \theta a_{t-1}$$

is given by

$$\begin{aligned} \rho_1 &= \frac{(1-\theta\phi)(\phi-\theta)}{1-2\phi\theta+\theta^2} \\ \rho_k &= \rho_1 \phi^{k-1} \qquad\qquad k \geq 2. \end{aligned}$$

Interpret the result.

Exercise 3

Show that the AR(2) process

$$\text{AR(2)}: \; y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t$$

is stationary under the following conditions:

$$\begin{aligned} -1 < \phi_2 &< 1 \\ \phi_1 + \phi_2 &< 1 \\ \phi_2 - \phi_1 &< 1\,. \end{aligned}$$