Basic Index Number Theory:
Comments on W. E. Diewert

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1. Like all of Erwin Diewert's writings, this chapter is a model of clarity and rigour. It presents much of the theory of index numbers in a concise and precise manner. Some of the material, such as the one on Young and Lowe indices, is very innovative. And as always, Erwin is very meticulous and he provides a great service to the profession by citing early contributions and historical sources. I have enjoyed reading this chapter and learned a great deal from it. I have no criticism to offer, but let me just dwell on a few items that came to my mind as I was working my way through the paper.

2. It seems to me that statisticians tend to place somewhat too much emphasis on demand at the expense of supply. This chapter is no exception as illustrated by the following two quotes (my emphasis):

> "Under *normal economic conditions*, when the price ratios pertaining to two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index index will be larger than the corresponding Paasche index" (paragraph **15.18**)

> "In the *vast majority of situations* covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indices tend systematically to record *greater increases* than Paasche with the gap between them tending to widen with time" (Peter Hill, 1993, quoted in paragraph **15.18**, footnote 14).

I would argue that in economics, supply is just as important as demand. In the context of the national accounts, in the presence of a concave production possibilities frontier, supply considerations tend to prevail. The same would be true in the case of PPI *output* indices., It is thus not generally true that the Laspeyres price index will be larger than the Paasche price index. Instead, the following relationships will hold (see Figures 1 and 2):

(1) $P_L(p^0, p^t, q^0) > P_P(p^0, p^t, q^t)$ (demand-determined quantities)

(2) $P_L(p^0, p^t, q^0) < P_P(p^0, p^t, q^t)$ (supply-determined quantities) .

3. Note that even if we limit our attention to demand systems, it does not follow from (1) that:

(3) $$\frac{P_L(p^0, p^t, q^0)}{P_L(p^0, p^{t-1}, q^0)} > \frac{P_P(p^0, p^t, q^t)}{P_P(p^0, p^{t-1}, q^{t-1})} .$$

This is confirmed by Figure 1. That is, even in the context of demand, it is not necessarily true, contrary to what is suggested by the second quote above, that the Laspeyres price index will record greater increases than the Paasche index.

4. According to Erwin's terminology (see Section F), indices are either of the chained type or of the fixed-based type. I do not find this terminology very enlightening, since, as Erwin readily admits (paragraph **15.80**), both types of indices are expressed relative to a fixed base period (period 0), with the base-period price level being generally set to 1 or to 100. The question rather is whether or not the base period also systematically serves as the reference period. That is, does one make a direct comparison between the current period and the base period, or does one chain indices that make comparisons over consecutive periods? I therefore tend to have a preference for Afriat's (1977) terminology and use the term "direct" to designate "unchained" indices.

5. Note also that the term fixed-basket cannot be used in opposition to chained indices, since in the direct (i.e. unchained) Paasche case the basket changes every period. This observation also runs counter Erwin's comment (paragraph **15.26**) that the Paasche index can be viewed as a special case of a fixed-basket or pure price index, since the basket will change every period. The concept of a fixed-basket only seems to make sense if the number of periods under consideration exceeds two. If there are only two periods, only one bilateral comparison can be made: the basket therefore does not have the opportunity to change, and hence it will be fixed!

6. Many economists tend to favor chained indices over direct (i.e. unchained) ones, as confirmed by the following quote:

> "The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and the Laspeyres indices" (**15.84**).

Is this necessarily true? After all, the compounding of small errors might lead to large errors. What is crucial is whether the spread between the two indices is a convex or a concave function of relative prices. Thus, the argumentation is incomplete. Figures 1 and 2 suggest, however, that the statement is correct, since the spread appears to be a convex function of relative prices.

7. In my view, the important question is not so much the question of the choice between direct and chain indices, but rather the question of the choice between *runs* of direct indices and *runs* of chain indices. A *run* (or a pattern, to use Erwin's terminology in paragraphs **15.78** and **15.79**) of direct Paasche price indices is given by the following sequence:

$$(4) \qquad 1, \quad \frac{\sum_i p_i^1 q_i^1}{\sum_i p_i^0 q_i^1}, \quad \frac{\sum_i p_i^2 q_i^2}{\sum_i p_i^0 q_i^2}, \quad \ldots, \quad \frac{\sum_i p_i^{t-1} q_i^{t-1}}{\sum_i p_i^0 q_i^{t-1}}, \quad \frac{\sum_i p_i^t q_i^t}{\sum_i p_i^0 q_i^t},$$

or, in more compact form:

$$(5) \qquad 1, \quad P_P(p^0, p^1, q^1), \quad P_P(p^0, p^2, q^2), \ldots, \quad P_P(p^0, p^{t-1}, q^{t-1}), \quad P_P(p^0, p^t, q^t).$$

It is common practice to use elements of this sequence to make comparisons between arbitrary pairs of periods. For instance, if one wanted to compare period *t* with period *t-1*, one would calculate $\Pi(p^0, p^{t-1}, p^t, q^{t-1}, q^t)$ defined as follows:

$$(6) \qquad \Pi(p^0, p^{t-1}, p^t, q^{t-1}, q^t) \equiv \frac{P_P(p^0, p^t, q^t)}{P_P(p^0, p^{t-1}, q^{t-1})} = \frac{\sum_i p_i^t q_i^t}{\sum_i p_i^{t-1} q_i^{t-1}} \Bigg/ \frac{\sum_i p_i^0 q_i^t}{\sum_i p_i^0 q_i^{t-1}}.$$

It is important to note that $\Pi(p^0, p^{t-1}, p^t, q^{t-1}, q^t)$ is not itself a Paasche index (it could be interpreted as an *implicit* Lowe index, or as a unit value index), and it turns out to have some rather undesirable properties; more about this under point 11 below. A run of chained Paasche price indices, on the other hand, is given by the following sequence:

$$1,\ P_P(p^0,p^1,q^1),\ \prod_{s=1}^{2} P_P(p^{s-1},p^s,q^s),\ \ldots,\ \prod_{s=1}^{t-1} P_P(p^{s-1},p^s,q^s),\ \prod_{s=1}^{t} P_P(p^{s-1},p^s,q^s).$$

It is obvious then that the comparison between two consecutive periods, say period *t-1* and period *t*, yields a true Paasche price index, i.e. $P_P(p^{t-1},p^t,q^t)$, and we are thus on familiar and unproblematic grounds.

8. As mentioned by Erwin, statisticians have long been aware of substitution effects (see paragraph **15.46**, footnote 41, for instance, and point 2 above). That is, in a demand system quantities are typically determined by relative prices. This consideration has led to the economic approach to index numbers. Taking demand relationships into account, considering the fact that the Laspeyres and the Paasche indices are homogeneous of degree zero in quantities, and setting $p^0 = 1$ for simplicity, we can write:

(7) $\quad P_L(p^0,p^t,q^0) = P_L[1,p^t,q(1)] \equiv P^L(p^t)$

(8) $\quad P_P(p^0,p^t,q^t) = P_P[1,p^t,q(p^t)] \equiv P^P(p^t)$,

where $q(p)$ denotes the vector of demand functions.

9. Incidentally, it is interesting to note that Walsh's comment (paragraph **15.46**, footnote 41) is not just a statement about the direction of the substitution effect, but also about its size. If we interpret the expressions "spending more" and "spending less" are referring to the amounts spent, rather than just the prices paid, then Walsh's statement is only true if the elasticity of substitution is less than one.

10. The behavior of the direct Laspeyres and Paasche price indices can be further analyzed if we specify a functional form for the underlying aggregator function. Assume two goods only, and let the unit expenditure (or cost) function have the following *CES* form:[1]

(9) $\quad c^t = \left[(p_1^t)^\rho + (p_2^t)^\rho\right]^{1/\rho}, \quad \rho < 1$.

Under cost minimization, the unit (i.e. per unit of output or unit of utility) demand functions are obtained by differentiation:

$$(10) \qquad q_i^t = (p_i^t)^{\rho-1}\left[(p_1^t)^\rho + (p_2^t)^\rho\right]^{(1-\rho)/\rho}, \quad i = 1,2 .$$

Let $p_1^0 = p_2^0 = 1$ once again. It follows that $q_1^0 = q_2^0$. The direct Laspeyres price index can thus be written as:

$$(11) \qquad P^L(p^t) \equiv \frac{p_1^t q_1^0 + p_2^t q_2^0}{q_1^0 + q_2^0} = \frac{1}{2}(p_1^t + p_2^t) ,$$

whereas the direct Paasche price index is given by:

$$(12) \qquad P^P(p^t) \equiv \frac{p_1^t q_1^t + p_2^t q_2^t}{q_1^t + q_2^t} = \frac{(p_1^t)^\rho + (p_2^t)^\rho}{(p_1^t)^{\rho-1} + (p_2^t)^{\rho-1}} .$$

(Note that Figures 1 to 4 are all based on the CES functional form, for alternative values of ρ.)

11. The question then arises whether these indexes satisfy what can be termed an *economic monotonicity test*. That is, are $P^L(p^t)$ and $P^P(p^t)$ monotonically increasing in the components of p^t? It is obvious from (11) that the Laspeyres index satisfies this test, but as shown by Kohli (1986) the Paasche index may fail it. In other words, $\Pi(p^{t-1}, p^t, q^{t-1}, q^t)$ as defined by (6) might register a fall, even though none of the disaggregate prices have fallen, and some have increased. This phenomenon is illustrated in Figures 3 for alternative values of ρ. In Figure 4, the behavior the direct Paasche, Laspeyres and Fisher indices is compared with the true CES index given by (9). The failure of the Paasche price index to pass the economic monotonicity test provides a powerful argument in favor of chaining, one that in my opinion is even more convincing than the one mentioned by Erwin. Similar results hold in the context of supply theory; see Kohli (2004).

12. Erwin asks

> "…if there are index number formulas that give the same answer when either the fixed-base or chain system is used" (**15.89**).

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[1] The Allen-Uzawa elasticity of substitution is constant and it is equal to 1-ρ. In the supply case, ρ > 1 and the function can be interpreted as a unit revenue function.

Erwin then reports that the Cobb-Douglas functional form satisfies this requirement under weak regularity conditions. He might have added, though, that any index number formula would satisfy this requirement as well, as long as it happens to be exact for the true aggregator function. If an index number formula is exact for the underlying aggregator function, the corresponding chain index is necessarily path independent, and hence it is identical to its direct version. This is a strong argument in favor of superlative indices.

13. When dealing with long periods of time during which relative prices tend to trend, and when chaining is not possible (perhaps because quantity data are not available for every period), Erwin suggests the use of mid-year indices (paragraphs **15.50** to **15.54**). I would like to suggest here as an alternative the use of the geometric mean of two Laspeyres (or Lowe) indices, one using the quantities of the first period as reference basket, and the second one using those from the last period. Specifically, consider the two following runs of fixed-basket indices:

$$(13) \quad 1, \; \frac{\sum_i p_i^1 q_i^0}{\sum_i p_i^0 q_i^0}, \; \frac{\sum_i p_i^2 q_i^0}{\sum_i p_i^0 q_i^0}, \; \ldots, \; \frac{\sum_i p_i^{t-1} q_i^0}{\sum_i p_i^0 q_i^0}, \; \frac{\sum_i p_i^t q_i^0}{\sum_i p_i^0 q_i^0}$$

$$(14) \quad \frac{\sum_i p_i^0 q_i^t}{\sum_i p_i^t q_i^t}, \; \frac{\sum_i p_i^1 q_i^t}{\sum_i p_i^t q_i^t}, \; \frac{\sum_i p_i^2 q_i^t}{\sum_i p_i^t q_i^t}, \; \ldots, \; \frac{\sum_i p_i^{t-1} q_i^t}{\sum_i p_i^t q_i^t}, \; 1 .$$

(13) is simply a run of direct Laspeyres indices. Each element in sequence (14), on the other hand, can be interpreted as (the inverse of) Paasche price index. Taken as a whole, however, sequence (14) is not a run of Paasche price indices, since in each element the quantities remain those of period t, rather than those of the current period. Next, we can normalize run (14) by dividing all its elements by the first one:

$$(15) \quad 1, \; \frac{\sum_i p_i^1 q_i^t}{\sum_i p_i^0 q_i^t}, \; \frac{\sum_i p_i^2 q_i^t}{\sum_i p_i^0 q_i^t}, \; \ldots, \; \frac{\sum_i p_i^{t-1} q_i^t}{\sum_i p_i^0 q_i^t}, \; \frac{\sum_i p_i^t q_i^t}{\sum_i p_i^0 q_i^t} .$$

Next, one can take the geometric mean of (13) and (15) to get:

$$(16) \quad 1, \; \sqrt{\frac{\sum_i p_i^1 q_i^0}{\sum_i p_i^0 q_i^0} \frac{\sum_i p_i^1 q_i^t}{\sum_i p_i^0 q_i^t}}, \; \ldots, \; \sqrt{\frac{\sum_i p_i^{t-1} q_i^0}{\sum_i p_i^0 q_i^0} \frac{\sum_i p_i^{t-1} q_i^t}{\sum_i p_i^0 q_i^t}}, \; \sqrt{\frac{\sum_i p_i^t q_i^0}{\sum_i p_i^0 q_i^0} \frac{\sum_i p_i^t q_i^t}{\sum_i p_i^0 q_i^t}}$$

I see two main advantages in using (16) rather than a midyear index. First, one sees that the last element in (16) has the Fisher form. Thus, run (16) will give a superlative measure of the cumulated increase in the price level over the entire period. Although the other elements of (16) do not have the Fisher form, they can be viewed as elements of a linear interpolation. Second, quantity data are often available for the "first" period only. Yet, baskets do have to be – and indeed are – updated from time to time, so that end-of-period quantities will eventually become available. Mid-year (i.e. middle of sample) quantitities, on the other hand, might never become available. I would thus suggest that, in those cases where a Laspeyres price index is being used, (16) could be calculated retroactively at the time when a new basket is introduced and splicing has to take place.

14. As documented by Erwin, the literature on index numbers goes back at least two centuries. It seems likely that after the writings of Irving Fisher in the first part of the 20th century, most economists and statisticians felt that everything there was to say about index numbers had been said. Yet, the field went through a second youth over the past thirty years, with many new and exciting results. It is to a large part thanks to Erwin Diewert that this "Renaissance" was made possible.

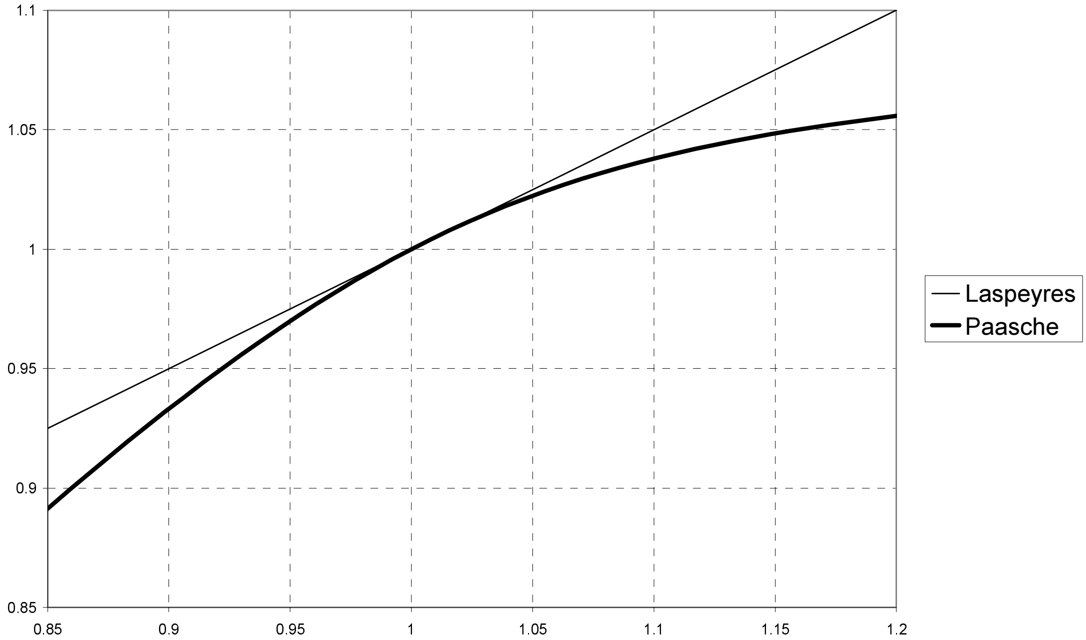

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Figure 1

$P^L(p^t)$ and $P^P(p^t)$ ($\rho = -4.00$) as functions of p_1^t for $p_2^t = 1$ in the demand context. Note that $P^L(p^t) \geq P^P(p^t)$ throughout, but that $\partial P^L(p^t)/\partial p_1^t <$ $\partial P^P(p^t)/\partial p_1^t$ for $p_1^t < 1$.

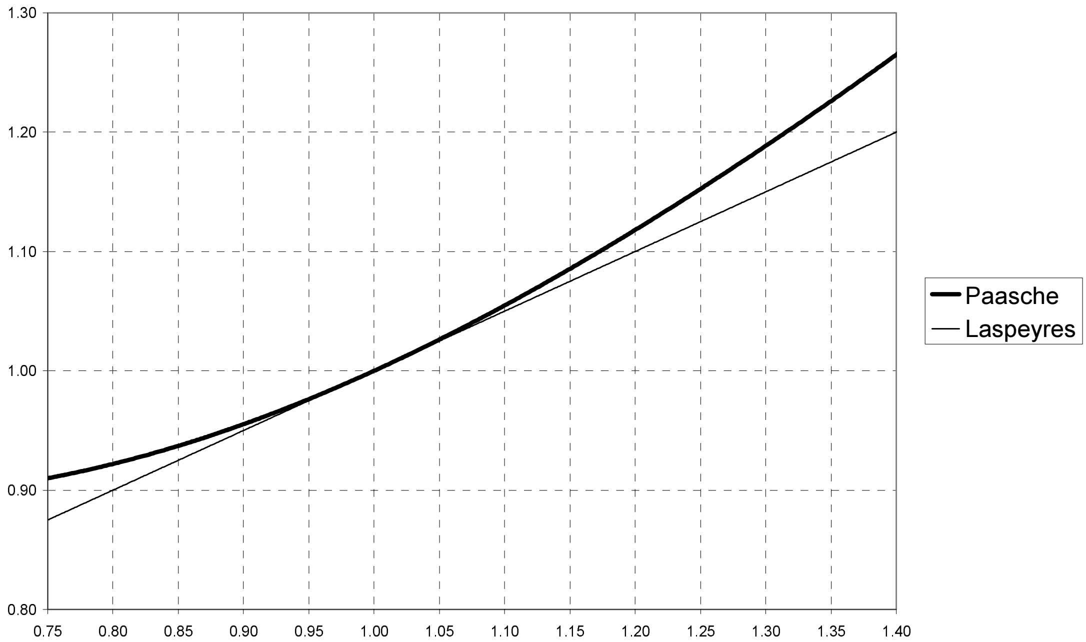

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Figure 2

$P^L(p^t)$ and $P^P(p^t)$ ($\rho = 1.50$) as functions of p_1^t for $p_2^t = 1$ in the supply context. Note that $P^L(p^t) \le P^P(p^t)$ throughout, but that $\partial P^L(p^t)/\partial p_1^t > \partial P^P(p^t)/\partial p_1^t$ for $p_1^t < 1$.

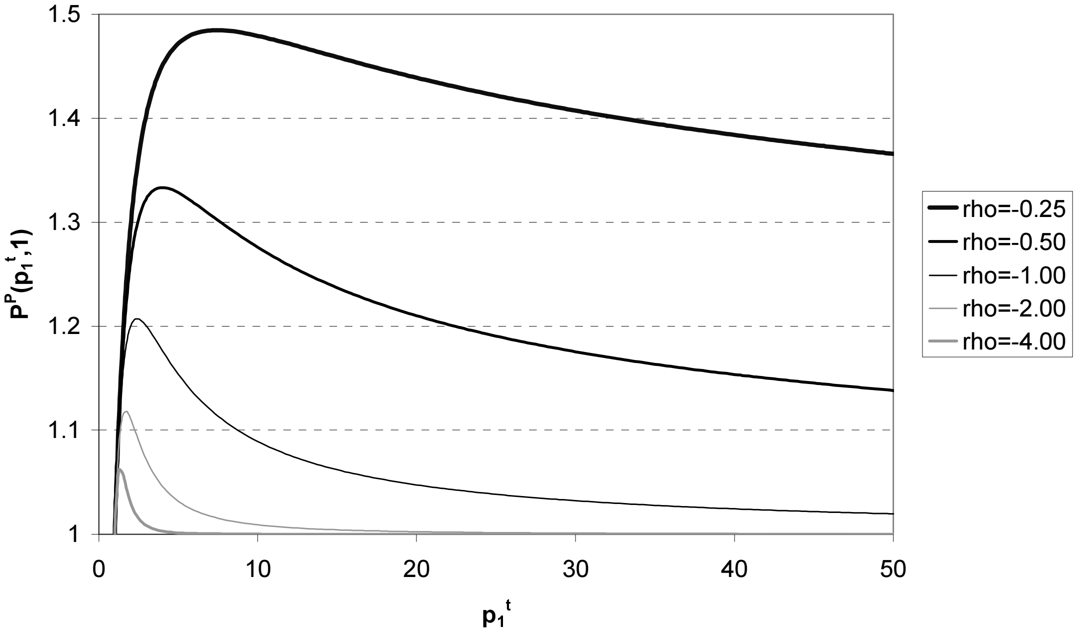

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Figure 3

In the demand theory context, the direct Paasche index fails the economic monotonicity test if the elasticity of substitution is greater than one.

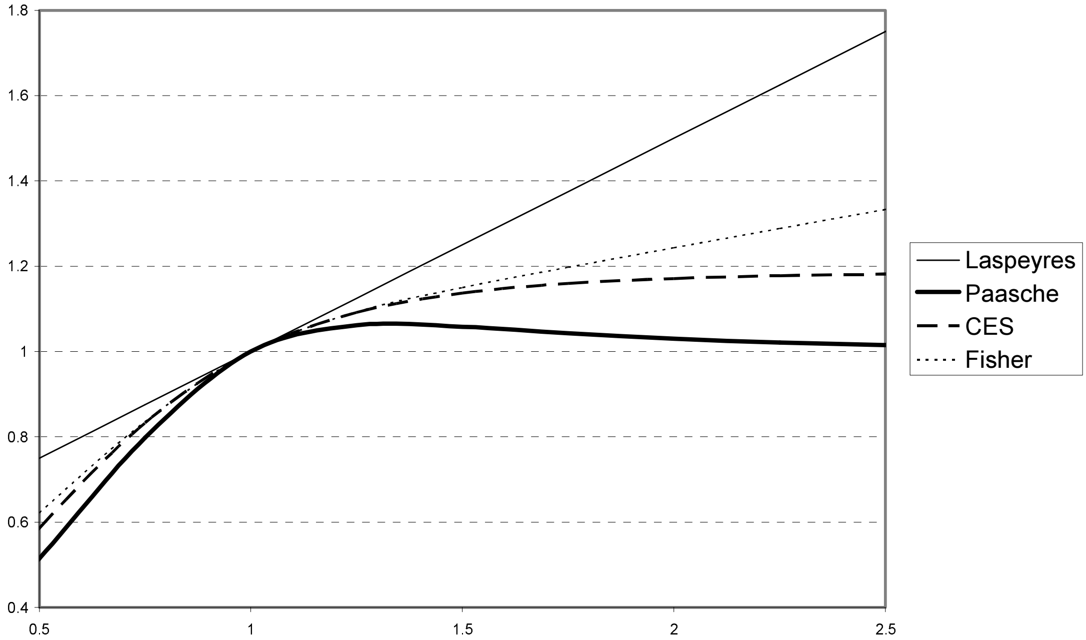

Figure 4

Relationship between the direct Paasche (for $\rho = -4.00$), the direct Fisher, the direct Laspeyres and the true CES price indices.

References

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