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ON A PARTITIONED INVERSION FORMULA HAVING USEFUL APPLICATIONS IN ECONOMETRICS [1]

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Abstract

In this paper a novel partitioned inversion formula is obtained in terms of the orthogonal complements of off-diagonal blocks, with the emblematic matrix of unit-root econometrics springing up as the leading diagonal block of the inverse. On the one hand, the result paves the way to a stimulating reinterpretation of restricted least-squares estimation and, on the other, to a straightforward derivation of a key-result of time-series econometrics.

JEL classification: C13, C32, C69

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§1. THE PARTITIONED-INVERSION THEOREM

Partitioned inversion formulas have been long since (e.g. Goldberger, 1964: Theil, 1971) a basic tool of the standard algebraic equipment of econometricians. In this section we establish a new result in partitioned inversion which proves helpful in classical and unit-root econometrics as well

THEOREM 1.1 - Let $\mathbf{A}$ be a square matrix of order $n \times n$ and $\mathbf{B}$ and $\mathbf{C}$ two rectangular matrices of order $n \times m$ and full column-rank. Further let $\mathbf{B}_\perp$ and $\mathbf{C}_\perp$ denote the orthogonal complements of $\mathbf{B}$ and $\mathbf{C}$.

Consider the partitioned matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}' & \mathbf{O} \end{bmatrix} \tag{1}$$

If $\mathbf{B}'_\perp \mathbf{A} \mathbf{C}_\perp$ is of full rank, then

(a) the matrix $\mathbf{P}$ is nonsingular;

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(b) its partitioned inverse is

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{E} & (\mathbf{I}_n - \mathbf{EA}) \cdot (\mathbf{C}')^{-} \\ \mathbf{B}^{-} \cdot (\mathbf{I}_m - \mathbf{AE}) & \mathbf{B}^{-} \cdot (\mathbf{AEA} - \mathbf{A}) \cdot (\mathbf{C}')^{-} \end{bmatrix} \tag{2}$$

where

$$\mathbf{E} = \mathbf{C}_\perp \cdot (\mathbf{B}'_\perp \mathbf{A}\, \mathbf{C}_\perp)^{-1} \cdot \mathbf{B}'_\perp \tag{3}$$

and $\mathbf{B}^{-}$ and $(\mathbf{C}')^{-}$ are left and right inverses of $\mathbf{B}$ and $\mathbf{C}'$, respectively, namely $\mathbf{B}^{-} = (\mathbf{B}'\mathbf{B})^{-1} \cdot \mathbf{B}'$ and $(\mathbf{C}')^{-} = \mathbf{C} \cdot (\mathbf{C}'\mathbf{C})^{-1}$.

Proof: The result (a) follows from the rank expansion of the partitioned matrix $\mathbf{P}$ (see Marsaglia and Styan, 1974, Theorem 19):

$$\mathrm{r}(\mathbf{P}) = \mathrm{r}(\mathbf{B}) + \mathrm{r}(\mathbf{C}) + \mathrm{r}\{(\mathbf{I} - \mathbf{BB}^{-}) \cdot \mathbf{A} \cdot [\mathbf{I} - (\mathbf{C}')^{-}\mathbf{C}']\} = \mathrm{r}(\mathbf{B}) + \mathrm{r}(\mathbf{C}) + \mathrm{r}(\mathbf{B}'_\perp \mathbf{AC}_\perp) \tag{4}$$

given that

$$\mathbf{I} - \mathbf{D} \cdot (\mathbf{D}'\mathbf{D})^{-1} \cdot \mathbf{D}' = \mathbf{D}_\perp \cdot (\mathbf{D}'_\perp \mathbf{D}_\perp)^{-1} \cdot \mathbf{D}'_\perp \tag{5}$$

for every full column-rank matrix $\mathbf{D}$ (e.g., Johansen, 1995, p. 39).

To prove (b) let the inverse of $\mathbf{P}$ be

$$\mathbf{Q} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G}' & \mathbf{H} \end{bmatrix} \tag{6}$$

where the blocks in $\mathbf{Q}$ are of the same order as the corresponding blocks in $\mathbf{P}$. Then, in order to express the blocks of the former in terms of the blocks of the latter, write $\mathbf{QP} = \mathbf{I}$ and $\mathbf{PQ} = \mathbf{I}$ in partitioned form as:

$$\begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G}' & \mathbf{H} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}' & \mathbf{O} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_m \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}' & \mathbf{O} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G}' & \mathbf{H} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_m \end{bmatrix} \tag{7'}$$

From (7) and (7′) we can derive the pair of equation systems in the unknown submatrices $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$:

$$\begin{cases} \mathbf{EA} + \mathbf{FC}' = \mathbf{I}_n & (8) \\ \mathbf{EB} = \mathbf{O} & (9) \\ \mathbf{G}'\mathbf{A} + \mathbf{HC}' = \mathbf{O} & (10) \\ \mathbf{G}'\mathbf{B} = \mathbf{I}_m & (11) \end{cases} \qquad \begin{cases} \mathbf{AE} + \mathbf{BG}' = \mathbf{I}_n & (8') \\ \mathbf{AF} + \mathbf{BH} = \mathbf{O} & (9') \\ \mathbf{C}'\mathbf{E} = \mathbf{O} & (10') \\ \mathbf{C}'\mathbf{F} = \mathbf{I}_m & (11') \end{cases}$$

Equations (9) and (10′) have the common solution (see Rao and Mitra, 1971, Lemma 2.3.1):

$$\mathbf{E} = [\mathbf{I} - (\mathbf{C}')^{-}\mathbf{C}'] \cdot \mathbf{Z} \cdot (\mathbf{I} - \mathbf{B}\mathbf{B}^{-}) = [\mathbf{I} - \mathbf{C} \cdot (\mathbf{C}'\mathbf{C})^{-1} \cdot \mathbf{C}'] \cdot \mathbf{Z} \cdot [\mathbf{I} - \mathbf{B} \cdot (\mathbf{B}'\mathbf{B})^{-1} \cdot \mathbf{B}'] \tag{12}$$

for some $\mathbf{Z}$, which is equivalent to:

$$\mathbf{E} = \mathbf{C}_{\perp}\mathbf{W}\mathbf{B}'_{\perp} \tag{12′}$$

for some $\mathbf{W}$, in view of (5) above.

Substituting (12′) into (8), postmultiplying both sides by $\mathbf{C}_{\perp}$ and considering that $\mathbf{C}'\mathbf{C}_{\perp} = \mathbf{O}$, we obtain

$$\mathbf{C}_{\perp}\mathbf{W}\mathbf{B}'_{\perp}\mathbf{A}\mathbf{C}_{\perp} = \mathbf{C}_{\perp} \tag{13}$$

Because $\mathbf{C}_{\perp}$ is of full column-rank, equality (13) holds iff:

$$\mathbf{W}\mathbf{B}'_{\perp}\mathbf{A}\mathbf{C}_{\perp} = \mathbf{I} \;\Leftrightarrow\; \mathbf{W} = (\mathbf{B}'_{\perp}\mathbf{A}\mathbf{C}_{\perp})^{-1} \tag{14}$$

Combining (12′) and (14) we obtain

$$\mathbf{E} = \mathbf{C}_{\perp} \cdot (\mathbf{B}'_{\perp}\mathbf{A}\mathbf{C}_{\perp})^{-1} \cdot \mathbf{B}'_{\perp} \tag{15}$$

Consider now equations (11) and (8′), with $\mathbf{E}$ given by (15). Then simple computations show that

$$\mathbf{B} = (\mathbf{I} - \mathbf{A}\mathbf{E}) \cdot \mathbf{B} \tag{16}$$

Hence a common solution exists, namely (Rao and Mitra, 1971, Th. 2.3.3):

$$\mathbf{G}' = \mathbf{B}^{-} \cdot (\mathbf{I} - \mathbf{A}\mathbf{E}) + \mathbf{B}^{-} - \mathbf{B}^{-}\mathbf{B}\mathbf{B}^{-} + (\mathbf{I} - \mathbf{B}^{-}\mathbf{B}) \cdot \mathbf{Z} \cdot (\mathbf{I} - \mathbf{B}\mathbf{B}^{-}) \tag{17}$$

for some $\mathbf{Z}$, which, noting that $\mathbf{B}^{-}\mathbf{B} = \mathbf{I}$, can be reduced to

$$\mathbf{G}' = \mathbf{B}^{-} \cdot (\mathbf{I} - \mathbf{A}\mathbf{E}) \tag{18}$$

The expression

$$\mathbf{F} = (\mathbf{I} - \mathbf{E}\mathbf{A}) \cdot (\mathbf{C}')^{-} \tag{19}$$

is similarly established, by referring to equations (8) and (11′).

Finally, consider equation (9′), with $\mathbf{F}$ given by (19). Rearranging terms we *get*

$$\mathbf{B}\mathbf{H} = (\mathbf{A}\mathbf{E}\mathbf{A} - \mathbf{A}) \cdot (\mathbf{C}')^{-} \tag{20}$$

Premultiplication of (20) by $\mathbf{B}^{-}$ thus leads to

$$\mathbf{H} = \mathbf{B}^{-} \cdot (\mathbf{A}\mathbf{E}\mathbf{A} - \mathbf{A}) \cdot (\mathbf{C}')^{-} \tag{21}$$

which completes the proof.

§2. RESTRICTED LEAST-SQUARES REVISITED

In this section some more light is shed on restricted least-squares theory by applying the partitioned inversion formula of Theorem 1.1 in the derivation of the estimator, which leads to a solution that actually is the mirror image of the well-known Theil's (1961) constrained estimator.

Consider the linear model

$$\mathbf{y} = \mathbf{X}\,\boldsymbol{\beta} + \mathbf{e} \tag{22}$$

$$\mathrm{E}\{\mathbf{e}\} = \mathbf{o} \tag{23}$$

$$\mathrm{E}\{\mathbf{e}\mathbf{e}'\} = \sigma^2\,\boldsymbol{\Omega} \tag{24}$$

where $\mathbf{y}$ in the n-vector of values of the regressand, $\mathbf{X}$ is the nxk matrix of values of the regressors and $\mathbf{e}$ is the n-vector of residuals. Moreover σ^2 is an unknown scale factor, $\boldsymbol{\Omega}$ is a known positive-definite matrix normalized so that its trace is equal to n, and $\boldsymbol{\beta}$ is a k-vector of unknown parameters subject to the linear constraints

$$\mathbf{s} = \mathbf{R}'\,\boldsymbol{\beta} \tag{25}$$

where $\mathbf{s}$ is a given m-vector and $\mathbf{R}$ is a given kxm matrix such that

$$\mathrm{r}(\mathbf{R}) = \mathrm{m} < \mathrm{k} \tag{26}$$

$$\mathrm{r}\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{R}' \end{bmatrix}\right) = \mathrm{k} \tag{27}$$

Estimation of $\boldsymbol{\beta}$ can be accomplished by the restricted least-squares method, i.e. by solving the constrained optimization problem

$$\min_{\boldsymbol{\beta}}\{(\mathbf{y} - \mathbf{X}\,\boldsymbol{\beta})' \cdot \boldsymbol{\Omega}^{-1} \cdot (\mathbf{y} - \mathbf{X}\,\boldsymbol{\beta})\}$$

subject to

$$\mathbf{s} = \mathbf{R}'\,\boldsymbol{\beta} \tag{28}$$

Using the method of Lagrange multipliers, the minimizing equations can be compactly written as (e.g., Magnus and Neudecker, 1999, for differentiation rules in compact form)

$$\begin{bmatrix} \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X} & \mathbf{R} \\ \mathbf{R}' & \mathbf{O} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} \\ \mathbf{s} \end{bmatrix} \tag{29}$$

where $\boldsymbol{\lambda}$ is the m-vector of Lagrange multipliers.

The rank qualifications (26) and (27) ensure that the coefficient matrix in (29) is nonsingular. Then the solution for $\mathbf{b}$ is

$$\mathbf{b} = [\mathbf{I},\ \mathbf{O}] \cdot \begin{bmatrix} \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X} & \mathbf{R} \\ \mathbf{R}' & \mathbf{O} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} \\ \mathbf{s} \end{bmatrix} \tag{30}$$

which paves the way to a stimulating application of Theorem 1.1.

To this end we shall now prove:

LEMMA 2.1 - Under the rank qualifications (26) and (27) the rank equality

$$\mathrm{r}(\mathbf{X}\,\mathbf{R}_\perp) = \mathrm{r}(\mathbf{R}_\perp) \tag{31}$$

holds, which in turn entails

$$\mathrm{r}(\mathbf{R}'_\perp \mathbf{X}'\,\boldsymbol{\Omega}^{-1}\mathbf{X}\,\mathbf{R}_\perp) = \mathrm{k} - \mathrm{m} \tag{32}$$

Proof: Result (31) follows from (5), (26) and (27) above making use of Result 3.12 of Corollary 6.1 in Marsaglia and Styan, 1974. Formally

$$\mathrm{k} = \mathrm{r}\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{R}' \end{bmatrix}\right) = \mathrm{r}\left(\begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}\cdot(\mathbf{R}'\mathbf{R})^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X} \\ \mathbf{R}' \end{bmatrix}\right) = \mathrm{r}\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{R}\mathbf{R}^{-} \end{bmatrix}\right) =$$

$$= \mathrm{r}\left(\begin{bmatrix} \mathbf{X} \\ \mathbf{I} - \mathbf{R}_\perp\mathbf{R}_\perp^{-} \end{bmatrix}\right) \iff \mathrm{r}(\mathbf{X}\mathbf{R}_\perp) = \mathrm{r}(\mathbf{R}_\perp) = \mathrm{k} - \mathrm{m} \tag{33}$$

Result (32) is straightforward.

With these preliminaries we are ready to demonstrate the following elegant representation theorem.

THEOREM 2.1 - The restricted least-squares estimator of $\boldsymbol{\beta}$ in the linear model specified by the set of hypothesis (22) ÷ (27), is given by

$$\mathbf{b} = \mathbf{R}_\perp \cdot (\mathbf{R}'_\perp \mathbf{X}'\,\boldsymbol{\Omega}^{-1}\mathbf{X}\,\mathbf{R}_\perp)^{-1} \cdot \mathbf{R}'_\perp \mathbf{X}'\,\boldsymbol{\Omega}^{-1}\mathbf{y} +$$

$$+ [\mathbf{I} - \mathbf{R}_\perp \cdot (\mathbf{R}'_\perp \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}\,\mathbf{R}_\perp)^{-1} \cdot \mathbf{R}'_\perp \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}] \cdot (\mathbf{R}')^{-} \cdot \mathbf{s} \tag{34}$$

which, by defining

$$\tilde{\mathbf{b}} = (\mathbf{R}')^{-} \cdot \mathbf{s} \tag{35}$$

$$\boldsymbol{\Phi} = \mathbf{R}_\perp \cdot (\mathbf{R}'_\perp \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}\mathbf{R}_\perp)^{-1} \cdot \mathbf{R}_\perp \mathbf{X}'\boldsymbol{\Omega}^{-1} \tag{36}$$

can be more conveniently rewritten as:

$$\mathbf{b} = \tilde{\mathbf{b}} + \boldsymbol{\Phi} \cdot (\mathbf{y} - \mathbf{X}\,\tilde{\mathbf{b}}) \tag{37}$$

Proof: The result (34) follows from straightforward application - in view of (32) in Lemma 2.1 - of the partitioned inversion formula (2) of Theorem 1.1 to the restricted least-squares estimator (30). Trivial substitutions then leads to (37).

The restricted estimator (37) turns out to be split into two components: the former is a (particular) solution, $\tilde{\mathbf{b}}$, of the a-priori restrictions (25), while the latter mirrors the discrepancies between the observed regressand vector $\mathbf{y}$ and its virtual estimates $\mathbf{X}\,\tilde{\mathbf{b}}$ based on $\tilde{\mathbf{b}}$. The representation (37) is the specular image of the well-known representation (e.g., Goldberger, 1964, Theil, 1961, 1971).

$$\mathbf{b} = \hat{\mathbf{b}} + \boldsymbol{\Psi} \cdot (\mathbf{s} - \mathbf{R}'\,\hat{\mathbf{b}}) \tag{38}$$

where

$$\hat{\mathbf{b}} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} \cdot \mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{y} \tag{39}$$

$$\boldsymbol{\Psi} = (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} \cdot \mathbf{R} \cdot [\mathbf{R}' \cdot (\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} \cdot \mathbf{R}]^{-1} \tag{40}$$

which links the restricted estimator $\mathbf{b}$ to the unrestricted estimator $\hat{\mathbf{b}}$, based on sample data $\mathbf{X}$, $\mathbf{y}$ under a full column-rank hypothesis on the regressor matrix $\mathbf{X}$.

§3. The matrix-residue at the root of VAR econometrics

In this section, after sketching the analytical toolkit relating matrix polinomial inversion to Laurent series expansion, we derive, as a by-product of the partitioned inversion rule of Theorem 1.1, the matrix residue at a simple pole, and we eventually come to a short-cut proof of a crucial representation theorem (e.g., Banerjee et al., 1993, Faliva and Zoia, 1999) of unit-root econometrics.

Consider the matrix

$$\mathbf{S}(z) = \mathbf{A}(z) \cdot (1 - z) + \mathbf{B}\mathbf{C}' \tag{41}$$

Simple computations shows that $(1 - z)^{-1} \cdot \mathbf{S}(z)$ plays the role of the Schur complement of $(z - 1) \cdot \mathbf{I}$ in the composite matrix:

$$\begin{bmatrix} \mathbf{A}(z) & \mathbf{B} \\ \mathbf{C}' & (z-1) \cdot \mathbf{I} \end{bmatrix}$$

and that, accordingly, the identity

$$(1 - z) \cdot \mathbf{S}^{-1}(z) = [\mathbf{I},\ \mathbf{O}] \cdot \begin{bmatrix} \mathbf{A}(z) & \mathbf{B} \\ \mathbf{C}' & (z-1)\mathbf{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{O} \end{bmatrix} \tag{42}$$

holds true, provided the inverses exist, in view of classical partitioned inversion rules.

Observe next that

$$\lim_{z\to 1}\{(1-z)\cdot \mathbf{S}^{-1}(z)\} = \mathbf{V},\ \ \textit{where } \mathbf{V} \neq \mathbf{O}, \tag{43}$$

provided $z = 1$ is a simple pole of $\mathbf{S}^{-1}(z)$ (e.g., Jeffrey, 1992).

By analogy with the scalar case, the matrix $\mathbf{V}$ is called the matrix residue in the Laurent series expansion

$$[\mathbf{A}(z)\cdot(1-z) + \mathbf{BC}']^{-1} = \mathbf{V}\cdot(1-z)^{-1} + \sum_{i=0}^{\infty} \mathbf{W_i}(1-z)^{\mathbf{i}} \tag{44}$$

of $[\mathbf{A}(z)\cdot(1-z)+\mathbf{BC}']^{-1}$ about the singularity located at $z = 1$.

LEMMA 3.1 - Let $\mathbf{A}(1)$, $\mathbf{B}$ and $\mathbf{C}$ be defined as the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in Theorem 1.1. Then the matrix $[\mathbf{A}(z)\cdot(1-z)+\mathbf{BC}']^{-1}$ has a simple pole at $z = 1$, whose matrix residue is given by;

$$\mathbf{V} = \mathbf{C}_\perp \cdot [\mathbf{B}'_\perp \mathbf{A}(1)\mathbf{C}_\perp]^{-1}\cdot \mathbf{B}'_\perp \tag{45}$$

Proof: Taking limit of both sides of (42) as z tends to 1, we get:

$$\lim_{z\to 1}\{(1-z)\cdot \mathbf{S}^{-1}(z)\} = \lim_{z\to 1}\left\{[\mathbf{I},\mathbf{O}]\cdot \begin{bmatrix} \mathbf{A}(z) & \mathbf{B} \\ \mathbf{C}' & (z-1)\mathbf{I} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{O} \end{bmatrix}\right\}$$

$$\mathbf{V} = [\mathbf{I},\ \mathbf{O}]\cdot \begin{bmatrix} \mathbf{A}(1) & \mathbf{B} \\ \mathbf{C}' & \mathbf{O} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{O} \end{bmatrix} \tag{46}$$

where the right-hand side turns out to be equal to:

$$\mathbf{C}_\perp \cdot [\mathbf{B}'_\perp \mathbf{A}(1)\, \mathbf{C}_\perp]^{-1}\cdot \mathbf{B}'_\perp \tag{47}$$

in view of Theorem 1.1.

Consider now, under the assumption $\mathbf{y_t} \sim CI(1,1)$, a cointegrated autoregressive model (VAR) in differences, namely (e.g., Charemza and Deadman, 1992, Banerjee et al., 1993):

$$\mathbf{A}(\mathcal{L})\nabla \mathbf{y}_t + \mathbf{BC}'\mathbf{y}_t = \boldsymbol{\epsilon}_t \tag{48}$$

where ∇ and $\mathcal{L}$ denote the backward difference and the lag operator, respectively, $\boldsymbol{\epsilon}_t$ is a vector of white-noises, and $\mathbf{A}(\mathcal{L})$ is a matrix polinomial, namely

$$\mathbf{A}(\mathcal{L}) = \mathbf{I} + \sum_{k=1}^{K} \mathbf{A}_k\, \mathcal{L}^k \tag{49}$$

whose matrix coefficients $\mathbf{A}_k$ along with the matrices $\mathbf{B}$ and $\mathbf{C}$ – even if unconstrained by economic theory a-priori's – fulfill the following specification requirements:

i) the matrices $\mathbf{B}$ and $\mathbf{C}$ have full column-rank;

ii) the square matrix $\mathbf{B}'_{\perp}\mathbf{A}(1)\mathbf{C}_{\perp}$ – where $\mathbf{B}_{\perp}$ and $\mathbf{C}_{\perp}$ are the orthogonal complements of $\mathbf{B}$ and $\mathbf{C}$, respectively, and $\mathbf{A}(1) = \mathbf{I} + \sum_{k=1}^{K} \mathbf{A}_k$ – has full rank;

iii) the roots of the characteristic polinomial

$$\pi(z) = \det\left[(1-z)\,\mathbf{A}(z) + \mathbf{B}\mathbf{C}'\right] \tag{50}$$

in the complex argument z, are either equal to one or, in modulus, greater than one.

Then the following representation theorem holds:

THEOREM 3.1: The model (48), under the assumptions i), ii) and iii) stated above, has a solution of the form:

$$\mathbf{y}_t = \mathbf{C}_{\perp} \cdot \left[\mathbf{B}'_{\perp}\mathbf{A}(1)\mathbf{C}_{\perp}\right]^{-1} \cdot \mathbf{B}'_{\perp} \sum_{\tau \le t} \boldsymbol{\epsilon}_{\tau} + \sum_{i=0}^{\infty} \boldsymbol{\Xi}_i \cdot \boldsymbol{\epsilon}_{t-i} \tag{51}$$

which is precisely a CI(1,1) process, given that:

$$\mathbf{y}_t \sim I(1) \tag{52}$$

$$\mathbf{C}'\mathbf{y}_t \sim I(0) \tag{53}$$

Proof: Looking at (48) as a linear operator equation, the closed form expression for $\mathbf{y}_t$:

$$\mathbf{y}_t = \left[\mathbf{A}(\mathcal{L})\nabla + \mathbf{B}\mathbf{C}'\right]^{-1} \cdot \boldsymbol{\epsilon}_t \tag{54}$$

is a particular solution of the nonhomogeneous equation at stake, provided the inverse operator $\left[\mathbf{A}(\mathcal{L})\nabla + \mathbf{B}\mathbf{C}'\right]^{-1}$ exists.

Applying the Laurent series expansion (44) to the inversion of the operator $\mathbf{A}(\mathcal{L})\nabla + \mathbf{B}\mathbf{C}'$ – with $\mathcal{L}$ and ∇ formally replacing z and $1-z$, respectively – eventually leads, after some elementary manipulations, to the expansion (c.g., Engsted and Johansen, 1997; Faliva and Zoia, 1999):

$$\left[\mathbf{A}(\mathcal{L})\nabla + \mathbf{B}\mathbf{C}'\right]^{-1} = \mathbf{V}\nabla^{-1} + \sum_{i=0}^{\infty} \boldsymbol{\Xi}_i\, \mathcal{L}^i \tag{55}$$

where $\mathbf{V}$ is the matrix residue of Lemma 3.1 and the $\boldsymbol{\Xi}_i$'s turn out to be, under the above stated assumptions on the roots of the characteristic polinomial (50), matrices of exponentially decreasing coefficients.

In view of (55) the solution (54) can be rewritten in the form (51), using the formal identity between the antidifference ∇^{-1} and the indefinite sum operator (e.g., Spiegel, 1971), namely:

$$\nabla^{-1} \equiv \sum_{\tau \leq t} \tag{56}$$

The solution (51) is the sum of two components: the former is an I(1) process, namely a linear transformation – through the matrix residue $\mathbf{C}_{\perp} \cdot [\mathbf{B}'_{\perp}\mathbf{A}(1)\mathbf{C}_{\perp}]^{-1} \cdot \mathbf{B}'_{\perp}$ – of the multivariate random-walk $\sum_{\tau \leq t} \boldsymbol{\epsilon}_{\tau}$, while the latter is an I(0) process, namely a causal infinite vector moving average, VMA(∞), of the multivariate white-noise $\boldsymbol{\epsilon}_t$.

Hence the result (52) holds trivially, and the result (53) ensues from the annihilation of the I(1) component of the solution once it is premultiplied by the matrix $\mathbf{C}'$.

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