Two-step regression quantiles

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Abstract

The α-regression quantile $\left(\hat{\beta}_0(\alpha), \widehat{\boldsymbol{\beta}}(\alpha)^\top\right)^\top$ in the the linear regression model

$$Y_i = \beta_0 + \mathbf{x}_i^\top \boldsymbol{\beta} + e_i, \;\; \mathbf{x}_i, \boldsymbol{\beta} \in \mathbb{R}^p, \;\; i = 1, \ldots, n, \;\; \beta_0 \in \mathbb{R}^1$$

is a solution of the minimization

$$\sum_{i=1}^n \left\{ \alpha[Y_i - b_0 - \mathbf{x}_i^\top \mathbf{b}]^+ + (1-\alpha)[Y_i - b_0 - \mathbf{x}_i^\top \mathbf{b}]^- \right\} = \min \quad \text{with respect to} \;\; b_0 \in \mathbb{R}^1, \mathbf{b} \in \mathbb{R}^p.$$

The two-step α-regression quantile first estimates the slope components $\boldsymbol{\beta}$ with a R-estimate $\widehat{\boldsymbol{\beta}}_R(\alpha)$ as a solution of the minimization

$$\sum_{i=1}^n (Y_i - \mathbf{x}_i^\top \mathbf{b})[a_i(\alpha, \mathbf{b}) - (1-\alpha)] = \min \quad \text{with respect to} \;\; \mathbf{b} \in \mathbb{R}^p$$

where

$$a_i(\alpha, \mathbf{b}) = \begin{cases} 0 & \ldots \quad R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}) < n\alpha \\ R_i - n\alpha & \ldots \quad n\alpha \le R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}) < n\alpha + 1 \\ 1 & \ldots \quad n\alpha + 1 \le R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b}) \end{cases}$$

are Hájek's scores, $R_{ni}(Y_i - \mathbf{x}_i^\top \mathbf{b})$ are the ranks of the residuals, $i = 1, \ldots, n$. In the second step one determines the $[n\alpha]$-quantile $\hat{e}_R(\alpha)$ of the residuals $\{Y_i - \mathbf{x}_i^\top \widehat{\boldsymbol{\beta}}_R(\alpha)\}, \; i = 1, \ldots, n$. Both vectors are asymptotically equivalent,

$$\left\| \begin{pmatrix} \hat{\beta}_0(\alpha) \\ \widehat{\boldsymbol{\beta}}(\alpha) \end{pmatrix} - \begin{pmatrix} \hat{e}_R(\alpha) \\ \widehat{\boldsymbol{\beta}}_R(\alpha) \end{pmatrix} \right\| = o_p(n^{-1/2})$$

as $n \to \infty$, and their population counterpart is $(F^{-1}(\alpha) + \beta_0, \beta_1, \ldots, \beta_p)^\top$. They are very close numerically even for small sample sizes, and the extreme versions ($\alpha \to 0, 1$) of both types coincide exactly. Because $\widehat{\boldsymbol{\beta}}_R(\alpha)$ is invariant to the intercept β_0, it plays only auxiliary role in estimating $F^{-1}(\alpha)$ and thus the quantile inference is based mainly on $\hat{e}_{[n\alpha]}, \; 0 < \alpha < 1$. Some finite-sample properties of both versions will be considered, as well as the possible inference based on them.