Construction and asymptotics of relativistic diffusions on Lorentz manifolds

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Motivations

There are deep links between the short-time and long-time asymptotics of Brownian motion on a Riemannian manifold and its geometry. This makes the heat kernel a powerfull tool in many analytic and geometric problems.

Does there exists similar links in a Lorentzian setting ?

- What is a Brownian motion on a Lorentz manifold ?
- Does its study teach us something on the geometry of the underlying manifold ?

Relativistic diffusion in Minkowski space-time

Minkowski space-time and hyperbolic space

We denote by $\mathbb{R}^{1,d} := \{\xi = (\xi^0, \xi^i) \in \mathbb{R} \times \mathbb{R}^d\}$ the Minkowski space-time of special relativity, endowed with the metric :

$$q(\xi) = \langle \xi, \xi \rangle := -|\xi^0|^2 + \sum_{i=1}^{d} |\xi^i|^2,$$

and by $\mathbb{H}^d$ the positive part of its unit pseudo-sphere :

$$\mathbb{H}^d := \{\xi \in \mathbb{R}^{1,d} \,|\, \xi^0 > 0 \text{ and } \langle \xi, \xi \rangle = -1\}.$$

Basic facts on stochastic process

A continuous stochastic process X, with values in a differentiable manifold $\widetilde{\mathcal{M}}$, can be seen equivalently as :

- a random variable

$$\begin{array}{rcll} X: & (\Omega, \mathcal{F}, \mathbb{P}) & \to & \Big(C(\mathbb{R}^+, \widetilde{\mathcal{M}}), \mathcal{B}\Big) \\ & \omega & \mapsto & X(\omega) = (s \mapsto X(\omega)(s) = X_s(\omega)), \end{array}$$

  and thus a probability measure on $C(\mathbb{R}^+, \widetilde{\mathcal{M}})$;

- a family of probability measures $(\mathbb{P}_z)_{z \in \widetilde{\mathcal{M}}}$, where the support of $\mathbb{P}_z$ is the set $\{f \in C(\mathbb{R}^+, \widetilde{\mathcal{M}}),\ f(0) = z\}$, *i.e.* $\mathbb{P}_z$ is the law of sample paths starting at $X_0 = z$.

Geometric characterization of the Euclidian BM

Proposition

Among the processes with values in $\mathbb{R}^d$, the Brownian motion is the unique process that satisfies the three following properties :

- it is Markovian ;
- its sample paths are continuous ;
- its law is invariant under the action of Euclidian affine isometries, *i.e.* $\forall \, \phi \in \mathrm{Isom}(\mathbb{R}^d), \; A$ measurable :

$$\mathbb{P}_0(A) = \mathbb{P}_z(z + A), \quad \mathbb{P}_0(A) = \mathbb{P}_0(\phi(A)).$$

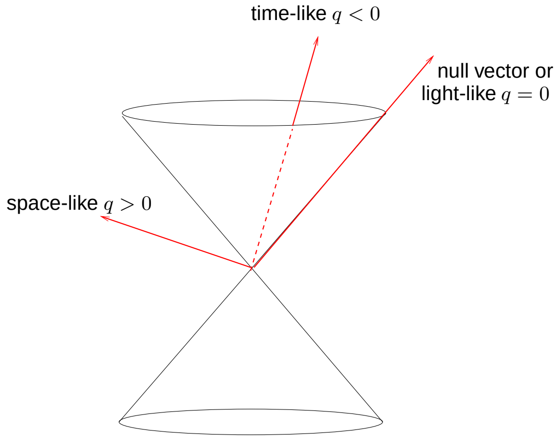
The end of the talk ?

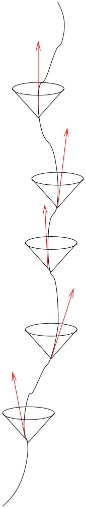
Theorem (Dudley, 1966)

There is no process with values in $\mathbb{R}^{1,d}$, being both

- Markovian ;
- continuous ;
- and whose law is Lorentz-covariant.

Nature of trajectories in Minkowski space-time





Towards a relativistic Brownian motion

Question : does there exist a stochastic process with the following properties ?

- It is Markovian ;
- its sample path are continuous, future-directed and time-like, *i.e.* they are continuous in $\widetilde{\mathcal{M}} = T^1_+\mathbb{R}^{1,d} \simeq \mathbb{R}^{1,d} \times \mathbb{H}^d$;
- its law is Lorentz-covariant.

Such a process will be called a *relativistic Brownian motion* or simply a *relativistic diffusion*.

Towards of relativistic diffusion

Theorem (Dudley, 1966)

There exist a unique process $(\xi_s, \dot{\xi}_s)_{s \geq 0}$ with values in $T^1_+\mathbb{R}^{1,d}$ that satisfies the preceeding conditions, it is obtained by taking for $\dot{\xi}_s$ a Brownian motion in $\mathbb{H}^d$ and its primitive

$$\xi_s := \xi_0 + \int_0^s \dot{\xi}_u du.$$

Conclusion of the Minkowskian case

- Relativistic diffusions make sense at the level of the unitary tangent bundle of a Lorentz manifold, not in the base space.

- By construction, the relativistic diffusion $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ is a continuous process in $T^1_+\mathbb{R}^{1,d}$, hence its first projection $(\xi_s)_{s\geq 0}$ with values in $\mathbb{R}^{1,d}$ has a C^1 regularity.

Relativistic diffusions on a general Lorentz manifold

Relativistic diffusions on a general Lorentz manifold

In 2007, Franchi and Le Jan extend Dudley's work by constructing, on a general Lorentz manifold $\mathcal{M}$, a process $(\xi_s, \dot{\xi}_s)_{s\geq 0}$

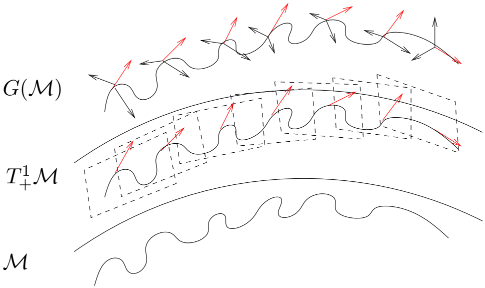
- with values in $T^1_+\mathcal{M}$;
- which is Markovian and continuous ;
- and whose law is Lorentz-covariant.

The process resulting from their construction we be simply called *relativistic diffusion* in the sequel.

Geometric description of the construction

Generalization of Dudley's work via parallel transport

The relativistic diffusion is constructed as the projection of a diffusion on the frame bundle $G(\mathcal{M})$, using a kind of "vertical lift".



Equivalently, it is obtained starting from Dudley's diffusion on a fixed tangent space using stochastic parallel transport.

Geometric description of the relativistic diffusion

Let $\mathcal{M}$ be a Lorentz manifold, $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$, and $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ the process starting from $(\xi_0, \dot{\xi}_0)$ resulting of the Franchi and Le Jan's construction.

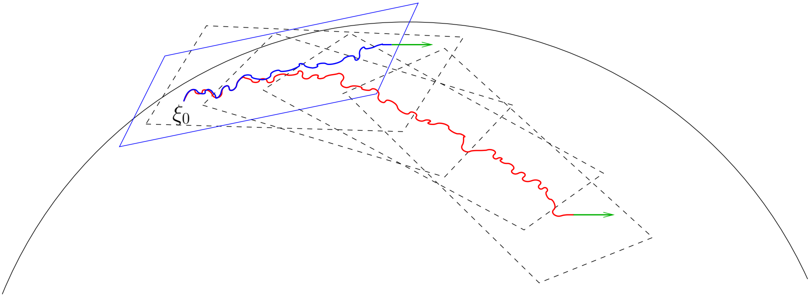
Theorem / Definition (Franchi–Le Jan, 2007)

If $\overleftarrow{\xi}(s) : T_{\xi_s}\mathcal{M} \to T_{\xi_0}\mathcal{M}$ denote the inverse parallel transport along the C^1 curves $(\xi_{s'} \,|\, 0 \leq s' \leq s)$, then $\zeta_s := \overleftarrow{\xi}(s)\,\dot{\xi}_s$ is an hyperbolic Brownian motion in $T^1_{\xi_0}\mathcal{M} \simeq \mathbb{H}^d$.

Stochastic anti-development

Dudley's diffusion in $T^1_{\xi_0}\mathcal{M} \approx \mathbb{H}^d$

Relativistic diffusion



Dynamical description

The notion of infinitesimal generator

Fact : there is a correspondence between diffusion processes $(X_s)_{s\geq 0}$ with values in a manifold $\mathcal{M}$ and differential operators $\mathcal{L}$, of order 2, acting on $C^\infty(\mathcal{M}, \mathbb{R})$.

The links between processes and operators is the following :

$$\mathcal{L}f(x) := \lim_{s\to 0} \mathbb{E}_x \left[\frac{f(X_s) - f(x)}{s} \right].$$

Besides, $(X_s)_{s\geq 0}$ is a solution of the stochastic differential equations system associated to $\mathcal{L}$.

Generator of the relativistic diffusion

In the case on the relativistic diffusion $(\xi_s, \dot{\xi}_s)_{s\geq 0}$ with values in $T^1_+\mathbb{R}^{1,d}$ introduced by Dudley, the operator $\mathcal{L}$ associated to the process is given by :

$$\mathcal{L}f(\xi,\dot{\xi}) := \underbrace{\dot{\xi}\,\partial_\xi f(\xi,\dot{\xi})}_{\text{geodesic flow}} + \frac{1}{2}\underbrace{\Delta_{\mathbb{H}^d} f(\xi,\dot{\xi})}_{\text{perturbation}} .$$

Dynamical description of the relativistic diffusion

By definition, the infinitesimal generator $\mathcal{L}$ of the relativistic diffusion introduced by Franchi and Le Jan decomposes into a sum :

$$\mathcal{L} := \mathcal{L}_0 + \frac{1}{2}\,\Delta_{\mathcal{V}},$$

where

- $\mathcal{L}_0$ is the generator of the geodesic flow ;
- $\Delta_{\mathcal{V}}$ is the vertical Laplacian.

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A more down-to-earth description

Given a local chart ξ^μ on $(\mathcal{M}, g_{\mu\nu})$, the relativistic diffusion on $T^1_+\mathcal{M}$ is the solution of the stochastic differential equations system :

$$(\star) \quad \begin{cases} d\xi_s^\mu = \dot{\xi}_s^\mu ds, \\ d\dot{\xi}_s^\mu = -\Gamma^\mu_{\nu\rho}(\xi_s)\dot{\xi}_s^\nu \dot{\xi}_s^\rho ds + \dfrac{\dim(\mathcal{M})}{2}\dot{\xi}_s^\mu ds + dM_s^\mu, \end{cases}$$

with

$$d\langle M^\mu,\ M^\nu\rangle_s = \left(\dot{\xi}_s^\mu \dot{\xi}_s^\nu + g^{\mu\nu}\right) ds.$$

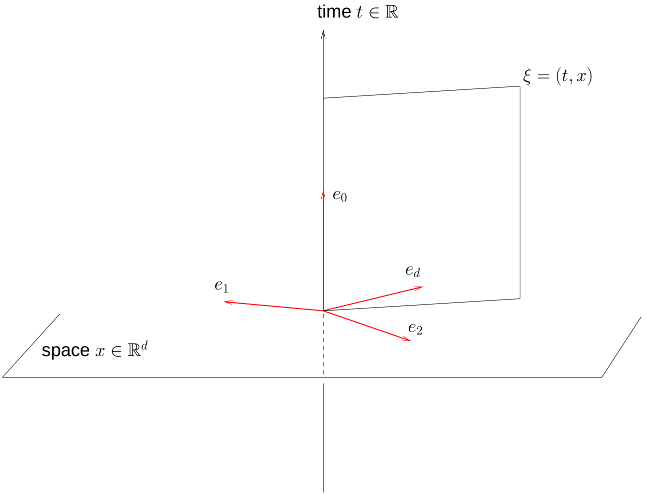
Morality

- The relativistic diffusion on a general Lorentz manifold $\mathcal{M}$ can be seen as the stochastic development of Dudley's diffusion in Minkowski space-time ;
- The flow associated to its generator is a perturbation of the geodesic flow on $\mathcal{M}$ by the vertical Laplacian.

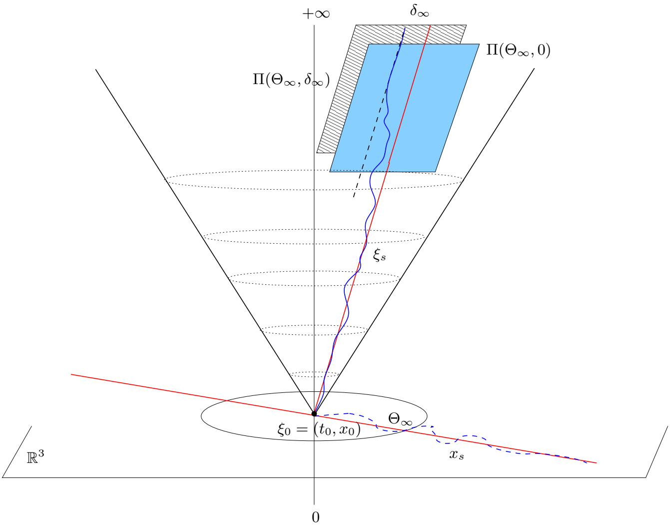
Asymptotics of the relativistic diffusion

The case of Minkowski space-time

Minkowski space-time



Typical sample path of the relativistic diffusion



Theorem (Bailleul 08)

Let $(\xi_0, \dot{\xi}_0)$ be a point in $T^1_+\mathbb{R}^{1,d} \simeq \mathbb{R}^{1,d} \times \mathbb{H}^d$ and $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}$ the law of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ starting from $(\xi_0, \dot{\xi}_0)$.

Then $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely, there exists

- a random limiting angle $\Theta_\infty \in \mathbb{S}^2$,
- a random plane $\Pi(\Theta_\infty, \delta_\infty)$,

such that, as s goes to infinity, the process ξ_s tends to infinity in the direction Θ_∞ along $\Pi(\Theta_\infty, \delta_\infty)$.

Robertson-Walker space-times

Robertson-Walker space-times

These spaces are cartesian products $I \times M$ where

$i)$ $I = (0, T)$ is an interval of $\mathbb{R}$;

$ii)$ M is an homogeneous and isotropic Riemannian manifold, $i.e.$ $M = \mathbb{S}^3, \mathbb{R}^3,$ ou $\mathbb{H}^3$.

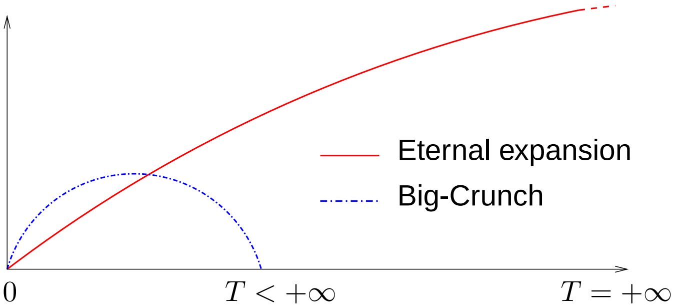
endowed with a metric of the form :

$$ds^2 = -dt^2 + \alpha^2(t) d\ell^2.$$

where α is a positive function on I and $d\ell^2$ is the usual Riemannian metric on M.

These manifolds, denoted by $\mathcal{M} := I \times_\alpha M$, are the natural geometric framework for the theory of Big-Bang.

Expansion functions considered



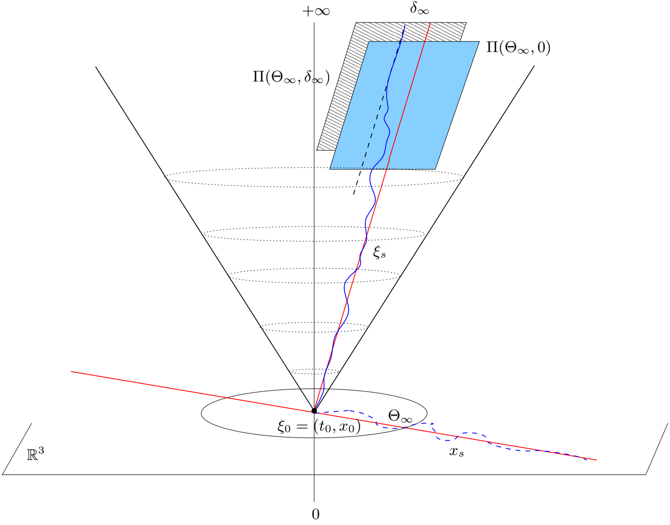
Existence, uniqueness, lifetime

Proposition

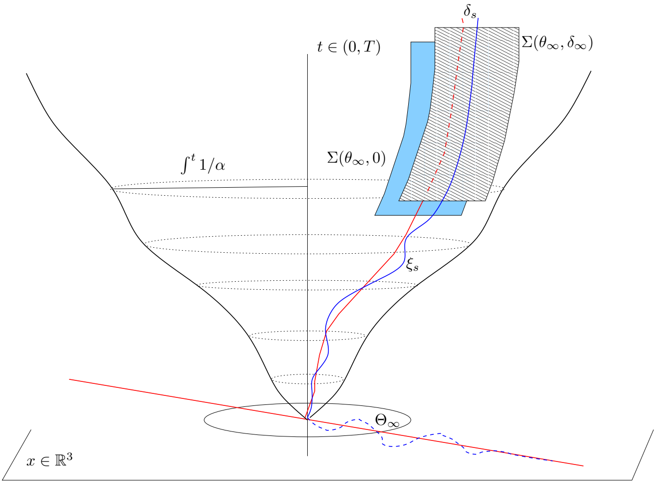
Let $\mathcal{M} = (0,T) \times_\alpha M$ be a Roberton-Walker space-time, and $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$. The system $(\star)$ that defines the relativistic diffusion admits a unique strong solution $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ starting from $(\xi_0, \dot{\xi}_0)$. This solution is defined up to the explosion time $\tau := \inf\{s > 0,\ t_s = T\}$.

Asymptotics of the diffusion in Robertson-Walker space-times

Reminder of the Minkowskian case



The case when $M = \mathbb{R}^3$ and $\int^T \frac{du}{\alpha(u)} = +\infty$


δ_s
$t \in (0, T)$
$\Sigma(\theta_\infty, \delta_\infty)$
$\int^t 1/\alpha$
$\Sigma(\theta_\infty, 0)$
ξ_s
Θ_∞
$x \in \mathbb{R}^3$

Theorem

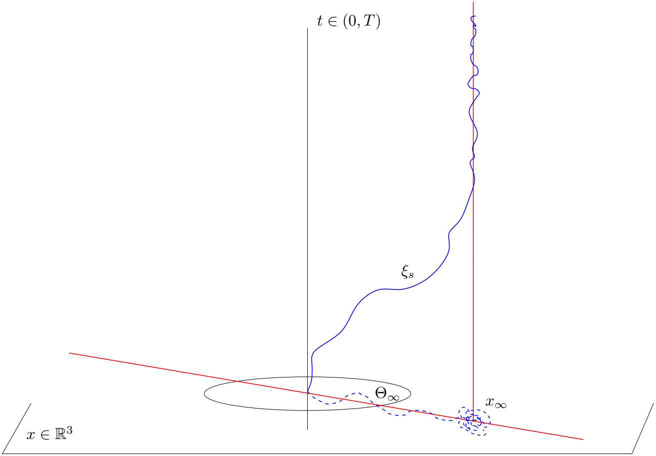
Let $(\xi_0, \dot{\xi}_0)$ be a point of $T^1_+\mathcal{M}$ and $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}$ the law of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ starting from $(\xi_0, \dot{\xi}_0)$.

Then $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely, there exists

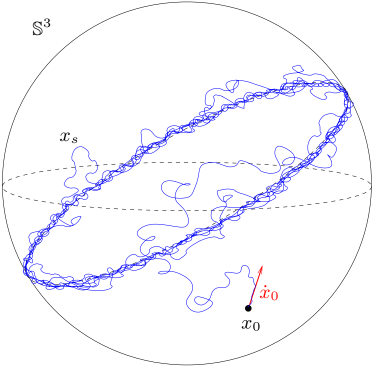
- a random limiting angle $\Theta_\infty \in \mathbb{S}^2$,
- a random hypersurface $\Sigma(\Theta_\infty, \delta_\infty)$,

such that, as s goes to infinity, the process ξ_s goes to infinity in the direction Θ_∞ along $\Sigma(\Theta_\infty, \delta_\infty)$.

The case when $M = \mathbb{R}^3$ and $\int^T \frac{du}{\alpha(u)} < +\infty$


$t \in (0, T)$
ξ_s
Θ_∞
x_∞
$x \in \mathbb{R}^3$

The case when $M = \mathbb{S}^3$ and $\displaystyle\int^T \frac{du}{\alpha(u)} = +\infty$


$\mathbb{S}^3$
x_s
$\dot{x}_0$
x_0

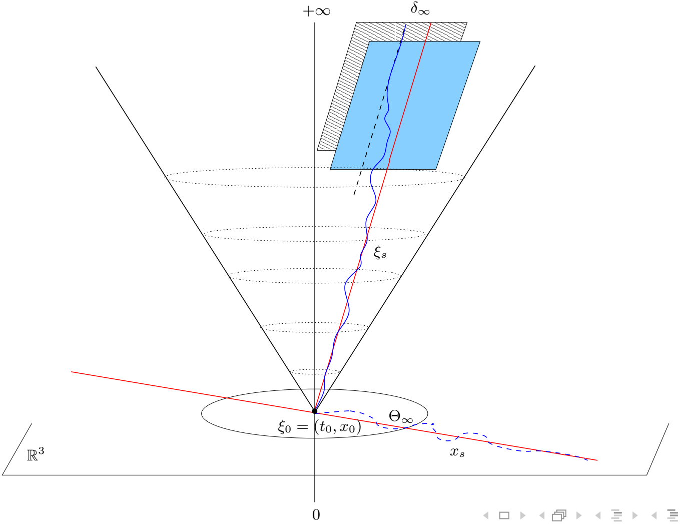
Concise (re)formulation with the help
of the notion of causal boundary

The notion of causal boundary

A strongly causal Lorentz manifold $\mathcal{M}$ admits a natural boundary $\partial\mathcal{M}_c = \partial\mathcal{M}_c^- \cup \partial\mathcal{M}_c^+$, called the *causal boundary*, composed of equivalence classes of causal curves (*i.e.* time-like or light-like curves).

In the case of Robertson-Walker space-times $\mathcal{M} = I \times_\alpha M$, this causal boundary was computed explicitly : it depends naturally on the expansion factor, the base interval I and the fiber M.

In Minkowski space-time, the causal boundary $\partial \mathcal{M}_c^+$ identifies with a cone $\mathbb{R}^+ \times \mathbb{S}^2$.



Theorem (reformulation of Bailleul's result)

Let $(\xi_0, \dot{\xi}_0)$ be a point in $T^1_+\mathbb{R}^{1,d} \simeq \mathbb{R}^{1,d} \times \mathbb{H}^d$ and $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}$ the law of the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ starting from $(\xi_0, \dot{\xi}_0)$.

Then $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely, as s goes to infinity, the process ξ_s converges towards a random point $(\Theta_\infty, \delta_\infty)$ in $\partial \mathcal{M}_c^+$.

Theorem

Let $\mathcal{M} = (0,T) \times_\alpha M$ be a Robertson-Walker space-time. Let $(\xi_0, \dot{\xi}_0)$ be a point in $T^1_+\mathcal{M}$ and $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}$ the law of the relativistic diffusion $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ starting from $(\xi_0, \dot{\xi}_0)$. Then $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely, as s goes to $\tau = \inf\{s > 0,\, t_s = T\}$, the process ξ_s converges towards a random point in $\partial\mathcal{M}_c^+$.

- By proving the last theorem, we confirm a result conjectured by Franchi and Le Jan :

  "the sample paths of the relativistic diffusion asymptotically follows random light-like geodesics".

- The different geometric situations are treated on a case by case basis, the proofs rely on fine stochastic analysis techniques.

Asymptotics of the normalized derivative

when $\displaystyle\int^{T} \frac{du}{\alpha(u)} < +\infty$

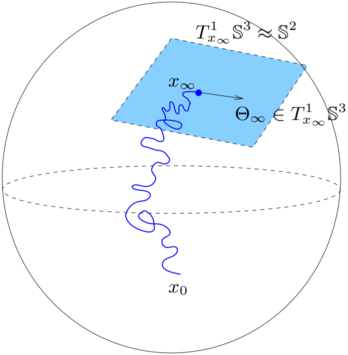
Theorem (Case when $\int^T 1/\alpha < +\infty$)

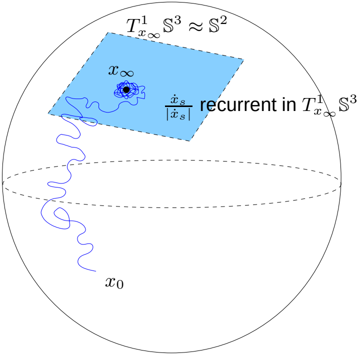
As s goes to $\tau = \inf\{s > 0, t_s = T\}$, the spatial projection x_s converges a.s. toward a random point x_∞ of the fiber M and the normalized derivative $\dot{x}_s/|\dot{x}_s|$ satisfies :

i) if $T < \infty$, then $\dot{x}_s/|\dot{x}_s|$ converges towards $\Theta_\infty \in T^1_{x_\infty} M$;

ii) if $T = +\infty$ and the expansion is polynomial, then $\dot{x}_s/|\dot{x}_s|$ converges towards Θ_∞ in $T^1_{x_\infty} M$;

iii) if $T = +\infty$ and the expansion is exponential, then $\dot{x}_s/|\dot{x}_s|$ asymptotically follows a time-changed spherical Brownian motion in $T^1_{x_\infty} M \approx \mathbb{S}^2$.


$T^1_{x_\infty}\mathbb{S}^3 \approx \mathbb{S}^2$
x_∞
$\Theta_\infty \in T^1_{x_\infty}\mathbb{S}^3$
x_0


$T^1_{x_\infty}\mathbb{S}^3 \approx \mathbb{S}^2$
x_∞
$\frac{\dot{x}_s}{|\dot{x}_s|}$ recurrent in $T^1_{x_\infty}\mathbb{S}^3$
x_0

Poisson boundary of the diffusion

A natural probabilistic question

Fact : on Robertson-Walker space-times, $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely, as s goes to $\tau = \inf\{s > 0,\ t_s = T\}$, the first projection ξ_s of the relativistic diffusion converges towards a random point in $\partial\mathcal{M}_c^+$.

Question : is the whole asymptotic stochastic information encoded in the random point on $\partial\mathcal{M}_c^+$?

The notion of Poisson boundary

The Poisson boundary of a process $X = (X_s)_{s \geq 0}$ can be defined equivalently as :

- the set $\mathrm{Harm}_b(\mathcal{L})$ of bounded $\mathcal{L}-$harmonic functions, where $\mathcal{L}$ is the infinitesimal generator of X ;
- the invariant $\sigma-$field $\mathrm{Inv}(X)$ of the process, composed of the events of the asymptotic $\sigma-$field

$$\bigcap_{t \geq 0} \sigma \left(X_s, \ s > t \right),$$

that are invariant under the shifts $s \mapsto s + s'$, $s' > 0$.

The notion of Poisson boundary

The correspondence between $\mathrm{Inv}(X)$ and $\mathrm{Harm}_b(\mathcal{L})$ is explicit :

Y bounded r.v., measurable w.r.t. $\mathrm{Inv}(X)$

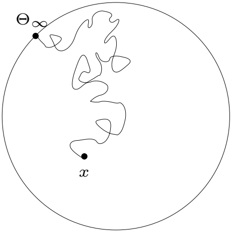
$$\updownarrow$$

$$h \in \mathrm{Harm}_b(\mathcal{L}), \quad h(x) := \mathbb{E}_x(Y)$$

The notion of Poisson boundary

In particular, if $\mathrm{Inv}(X) = \sigma(\ell_\infty)$ with $\ell_\infty \in \partial\mathcal{M}$, then

$$\mathrm{Harm}_b(\mathcal{L}) \simeq \mathbb{L}^\infty(\partial\mathcal{M}), \quad \text{via} \;\; h(x) = \mathbb{E}_x[F(\ell_\infty)] \leftrightarrow F.$$

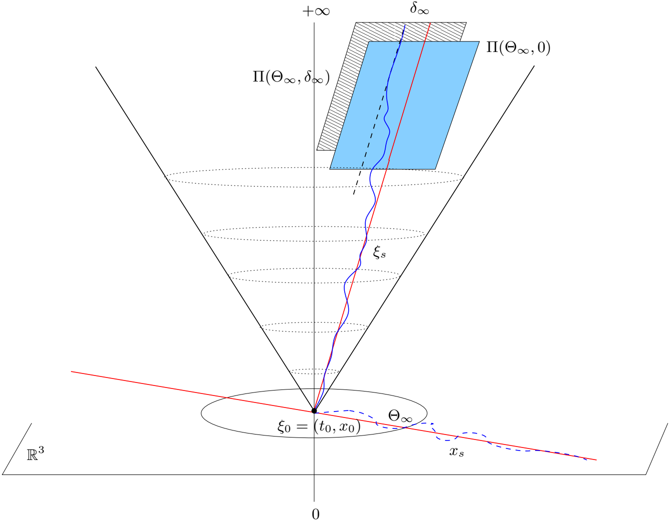


Fundamental example : if X is the killed BM in $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$, $\mathrm{Inv}(X) = \sigma(\Theta_\infty)$ where $\Theta_\infty \in \partial\mathbb{D} = \mathbb{S}^1$ and

$$\mathrm{Harm}_b(\Delta_{\mathbb{D}}) \simeq \mathbb{L}^\infty(\mathbb{S}^1),$$

via $h(x) = \mathbb{E}_x[F(\Theta_\infty)] \leftrightarrow F$.

The case of Minkowski space-time


$+\infty$
δ_∞
$\Pi(\Theta_\infty, 0)$
$\Pi(\Theta_\infty, \delta_\infty)$
ξ_s
Θ_∞
$\xi_0 = (t_0, x_0)$
$\mathbb{R}^3$
x_s
0

Theorem (Bailleul, 2008)

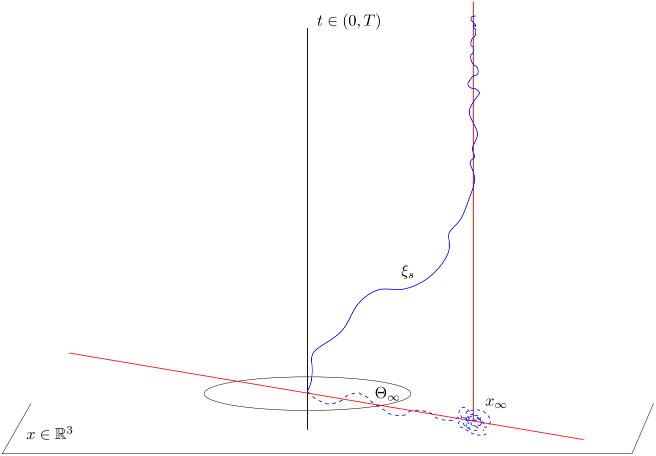
The invariant $\sigma-$field of the process $(\xi_s, \dot{\xi}_s)$ coincides $\mathbb{P}_{(\xi_0,\dot{\xi}_0)}-$almost surely with $\sigma(\Theta_\infty, \delta_\infty)$, the $\sigma-$field generated by the single variable $\ell_\infty = (\delta_\infty, \Theta_\infty) \in \partial\mathcal{M}_c^+ \simeq \mathbb{R}^+ \times \mathbb{S}^2$. Equivalently, one has

$$\mathrm{Harm}_b(\mathcal{L}) \simeq \mathbb{L}^\infty(\partial\mathcal{M}_c^+).$$

Robertson-Walker space-times

$$\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$$

where α has exponential growth.


$t \in (0, T)$
ξ_s
Θ_∞
x_∞
$x \in \mathbb{R}^3$

Let $\mathcal{M} = (0, +\infty) \times_\alpha \mathbb{R}^3$ be a Robertson-Walker space-time where α has exponential growth.

Theorem

Let $(\xi_0, \dot{\xi}_0) \in T^1_+\mathcal{M}$ et let $(\xi_s, \dot{\xi}_s) = (t_s, x_s, \dot{t}_s, \dot{x}_s)$ be the relativistic diffusion starting from $(\xi_0, \dot{\xi}_0)$. Then $\mathbb{P}_{(\xi_0, \dot{\xi}_0)}-$almost surely, the invariant $\sigma-$field of the process $(\xi_s, \dot{\xi}_s)$ coincides with $\sigma(x_\infty)$, the $\sigma-$field generated by the single variable $x_\infty \in \partial\mathcal{M}_c^+ \simeq \mathbb{R}^3$. Equivalently, one has

$$\mathrm{Harm}_b(\mathcal{L}) \simeq \mathbb{L}^\infty(\partial\mathcal{M}_c^+).$$

A challenging question

If $\mathcal{L}$ is the infinitesimal generator of the relativistic diffusion on a general Lorentz manifold, do we have

$$\mathrm{Harm}_b(\mathcal{L}) \simeq \mathbb{L}^\infty(\partial \mathcal{M}_c^+) \quad ?$$