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Flowers, Forests and Fields in Physics

Thèse

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par

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Aux trois coqs

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Résumé

Cette thèse est consacrée à l'étude de plusieurs modèles de Mécanique Statistique qui peuvent être vus comme des modèles d'interfaces, i.e. constituant une modélisation des interfaces entre plusieurs phases (e.g. liquide-vapeur, aimantation positive-négative). Elle est articulée en deux parties traitant de problématiques assez différentes.

Dans la première partie, nous étudions l'ensemble des mesures de Gibbs en volume infini correspondant aux modèles d'Ising et de Potts dans le régime de coexistence de phases, i.e. sous la température critique. L'analyse des fluctuations des interfaces joue un rôle important dans ce domaine.

Le premier résultat est un raffinement du célèbre théorème indépendamment établi par Aizenman et Higuchi durant la fin des années 70, affirmant que toutes les mesures de Gibbs en volume infini correspondant au modèle d'Ising plus proche voisins sur $\mathbb{Z}^2$ sont de la forme

$$\alpha \mathbf{P}^+ + (1-\alpha)\mathbf{P}^-, \quad \alpha \in [0,1]$$

où $\mathbf{P}^+$ et $\mathbf{P}^-$ sont les deux phases pures du modèle. Nous présentons une nouvelle approche à ce résultat, qui a un certain nombre d'avantages : nous obtenons un analogue optimal et quantitatif en volume fini (impliquant le résultat classique) ; la philosophie de la preuve est plus naturelle et fournit une meilleure image du phénomène physique sous-jacent.

Le second résultat est la caractérisation de l'ensemble des mesures de Gibbs du modèle de Potts en dimension 2. Nous avons pu étendre l'heuristique générale de l'approche développée pour le modèle d'Ising, pour prouver que tous les états de Gibbs du modèle de Potts plus proche voisins à q états sur $\mathbb{Z}^2$ sous la température critique sont les combinaisons convexes des q phases monochromatiques. En particulier, ils sont tous invariants sous les translations. Ce résultat était conjecturé par les physiciens depuis longtemps, mais n'avait jamais été prouvé rigoureusement.

Nous insistons sur le fait que ces deux résultats sont non-perturbatifs et valides jusqu'à la température critique du modèle (non-incluse).

Dans la seconde partie de la thèse, notre motivation principale est de comprendre les surfaces aléatoires en environnement aléatoire. Leurs analogues en dimension 1, appelés modèles de polymères aléatoires, ont beaucoup été étudiés dans la dernière décennie, grâce à la structure de renouvellement sous-jacente, qui ne survit pas en dimensions supérieures. Par conséquent, toutes les questions importantes sont encore ouvertes en dimensions 2 et plus.

Nous étudions le champ libre Gaussien discret en dimension d, vu comme un modèle d'interface en dimension $d + 1$, en présence d'un potentiel aléatoire dont le support est un hyperplan horizontal fixé. Ce potentiel est soit attractif soit répulsif avec une intensité choisie aléatoirement.

Pour une intensité positive (potentiel attractif), choisie aussi petite que l'on veut, il est connu que l'interface est localisée, i.e. elle est collée sur l'hyperplan pour une densité positive de sites. Les questions que nous posons concernent l'influence du désordre sur le comportement de l'interface : est-ce que la présence du désordre modifie les propriétés macroscopiques du système ?

Le premier résultat est l'inégalité stricte entre les énergies libres "trempée" et "recuite"[1]. Plus précisément nous prouvons, sous des hypothèses minimales sur la loi de l'environnement, que l'énergie libre trempée associée au modèle existe, est positive, déterministe, et strictement inférieure à l'énergie libre recuite lorsque celle-ci est positive.

Le second résultat est une caractérisation partielle du diagramme de phase du modèle, pour un environnement attractif / répulsif suivant une loi de Bernoulli. Nous prouvons que dans le plan de coordonnées données par l'espérance et la variance de l'environnement, la ligne critique trempée (séparant les phases d'énergie libre positive et nulle) se trouve strictement sous la ligne d'espérance nulle. Ceci montre en particulier qu'il existe une région non-triviale où l'interface est localisée bien que repoussée en moyenne par l'environnement.

Je souhaite au lecteur une agréable promenade parmi
les fleurs, les fôrets et les champs...

[1]traduction littérale de "quenched" et "annealed".

Summary

This thesis is devoted to several models of Statistical Mechanics which can be seen as interfaces models, i.e. they constitute a modelization of interfaces between several phases (e.g. liquid-vapor, positive-negative magnetization). It is organized in two parts dealing with quite different problematics.

In the first part of the thesis, we study the set of infinite volume Gibbs measures of the Ising and Potts models in the phase coexistence regime, i.e. below the critical temperature. The analysis of interfaces fluctuations plays a crucial role in this domain.

The first result is a refinement of the celebrated theorem independently established by Aizenman and Higuchi in the late 1970s, which states that all infinite volume Gibbs measures of the ferromagnetic nearest-neighbor Ising model on $\mathbb{Z}^2$ are of the form

$$\alpha \mathbf{P}^+ + (1-\alpha)\mathbf{P}^-, \quad \alpha \in [0,1]$$

where $\mathbf{P}^+$ and $\mathbf{P}^-$ are the two pure phases of the model. We present a new approach to this result, with a number of advantages: we obtain an optimal finite volume, quantitative analogue (implying the classical claim); the scheme of our proof is more natural and provides a better picture of the underlying phenomenon.

The second result is the characterization of the set of Gibbs measures of the two-dimensional Potts model. We have been able to extend the general heuristics behind the approach introduced for the Ising model to prove that all Gibbs states of the q-state nearest neighbor Potts model on $\mathbb{Z}^2$ below the critical temperature are convex combinations of the q monochromatic phases; in particular, they are all translation invariant. This result was conjectured by physicists since a long time but never proved rigorously.

We emphasize that both these results are non-perturbative and valid up to the (non-included) critical temperature of the model.

In the second part of the thesis, our main motivation is to understand random surfaces in random environment. Their one-dimensional counterparts, called random polymers, have been extensively studied in the last decade, thanks to the underlying renewal structure present in the model, which does not survive in higher dimensions. Consequently, all important questions are still open in dimension 2 and more.

We study the discrete d-dimensional Gaussian free field, seen as an interface model in dimension d+1, in the presence of a random potential which support is a fixed horizontal hyperplane. This potential is either attractive or repulsive with a randomly chosen strength.

For a constant positive (attraction) strength, chosen as small as we want, it is known that the interface is localized, i.e. it is sticked on the hyperplane for a positive density of sites. The questions we ask concern the influence of the disorder on the behavior of the interface: do the presence of disorder modify macroscopic properties of the system?

The first result is the strict inequality between the quenched and annealed free energies. More precisely, we prove under minimal assumptions on the law of the environment, that the quenched free energy associated to this model exists in $\mathbb{R}^+$, is deterministic, and strictly smaller than the annealed free energy whenever the latter is strictly positive.

The second result is a partial characterization of the phase diagram of the model, for an attractive/repulsive Bernoulli environment. We prove that in the plane of coordinates given by the expectation and the variance of the environment, the quenched critical line (separating the phases of positive and zero free energy) lies strictly below the line of zero expectation, showing in particular that there exists a non trivial region where the interface is localized though repulsed on average by the environment.

I wish to the reader a nice walk among
flowers, forests and fields...

Introduction

Mathematical physics represents the purest image that the view of nature may generate in the human mind; this image presents all the character of the product of art; it begets some unity, it is true and has the quality of sublimity; this image is to physical nature what music is to the thousand noises of which the air is full...

Théophile de Donder

Statistical physics is the branch of mathematical physics which treats of systems with a very large number of particles, typically of the order 10^{23}. From a microscopic modelization of a system, typically the description of the interaction between neighboring atoms, its goal is to predict the behavior of macroscopic observables, such as the density or the magnetization of a sample. In this thesis, we are particularly interested in lattice models where several phases coexist; for example liquid-vapor or positive-negative magnetization. The description of the interfaces between the phases will be our main object of study.

I Physical phenomena

Consider for example a piece of ferromagnetic metal in thermal equilibrium. The atoms constituting this sample form a crystal lattice, and each of them carries a magnetic moment resulting from the spin of its electrons, which can be represented as a vector on $\mathbb{S}^3$. The interaction properties of the electrons in the crystal are such that nearest neighbor atoms have a tendency to align their magnetic moment. This energetic gain is in competition with thermal fluctuations. The following transition is well known: at low temperature the coupling of magnetic moments is dominant and the sample presents a spontaneous magnetization, the piece of metal is a magnet, whereas above a certain critical temperature, called the Curie temperature, no macroscopic magnetic field is induced in the sample, see Figure 1.

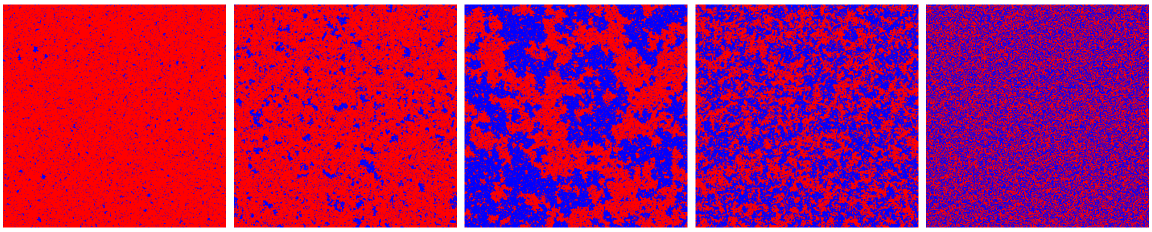

Figure 1 – The paramagnetic-ferromagnetic phase transition. Left: two low temperature samples, the red zone of positive magnetization percolates throughout the sample while blue zones of negative magnetization form very small clusters. Center: a critical sample, red and blue clusters are big and occupy about the same volume. Right: two high temperature samples, red and blue clusters still occupy the same volume but the behavior of each atom becomes more and more independent.

A similar example is the liquid-vapor phase transition. The liquid phase is due to the Van der Waals attraction between the particles which is in competition with the thermal fluctuations. Imagine we heat a glass of water. At low temperature the sample is in a liquid phase characterized by a high density of particles below the liquid-air interface, whereas above the boiling temperature all the sample is in a gaseous phase characterized by a low density of particles, see Figure 2.

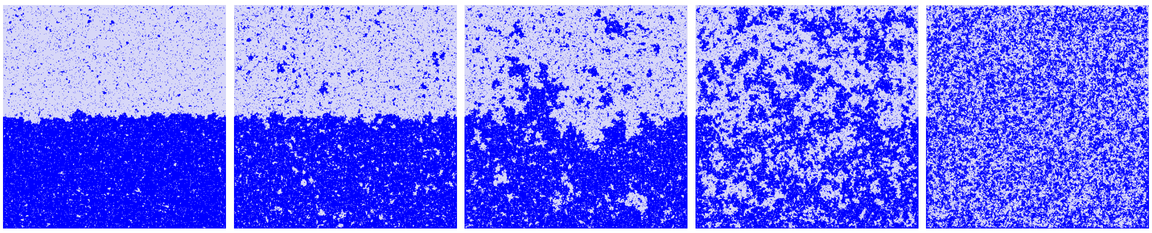

Figure 2 – The liquid-gas phase transition. Left: below the boiling temperature the liquid is present but the liquid-air interface fluctuates more and more. Center: at the boiling temperature the interface fluctuates at the scale of the box. Right: above the boiling temperature the gaseous phase occupies all the glass.

There exist also material with three phases at low temperature: the Strontium Titanate ($SrTiO_3$) is a crystal which has a symmetry called Perovskite structure. The lattice mesh consists of a central atom of Titanium, surrounded by an octahedron of 6 atoms of Oxygen, which are located at the center of the faces of a cube of 8 atoms of Strontium, see Figure 3.

Under a stress, the crystal has three favorite directions of deformation, which are the three directions of the cube $[100], [010]$ and $[001]$. If we stress the crystal in the $[111]$ direction, the crystal tends to locally deform itself in the three above directions. Moreover, below a certain critical temperature, the atom of Titanium undergoes a displacement along the direction of deformation. This induces a local polarization[2]

[2]Note that Perovskite crystals are well known piezoelectric materials thanks to their property of polarizing under a stress.

of each lattice mesh, which can take three values. At microscopic scale, the lattice consists of a number of domains of the three phases. We will be interested in the behavior of these interfaces between domains.

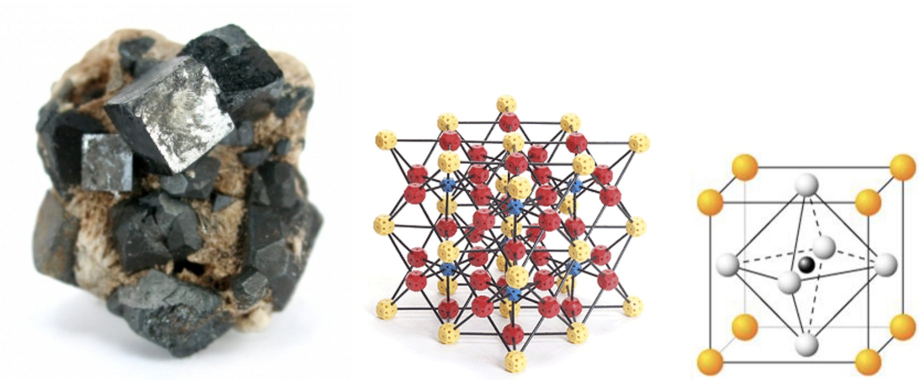

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Figure 3 – Left: Natural Perovskite crystal. Center: schematization of its atomic structure. Right: Zoom on one lattice mesh.

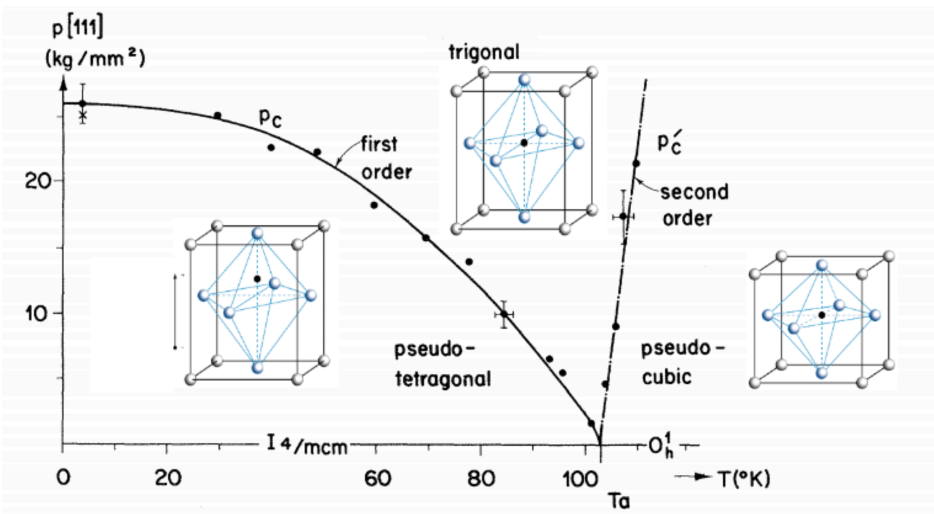


Figure 4 – Experimental phase diagram of the $SrTiO_3$ crystal [73]. The horizontal axis represents the temperature T in Kelvin degrees, and the vertical axis the stress strength p in direction [111] expressed in kg/mm^2.

II Modelization

In 1920, Wilhelm Lenz introduced a modelization of ferromagnetic materials where the magnetic moments of each atom of the lattice L can point either upwards or downwards. An energy is associated to each configuration $\sigma = \{\sigma_x\}_{x\in \mathrm{L}}$ where $\sigma_x \in \{-1, +1\}$ is the magnetic moment, or "spin" of the atom x. It takes the form:

$$\mathcal{H}(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y \tag{1}$$

where the notation $x \sim y$ denotes neighboring atoms in L. Each pair of atoms contribute negatively to the energy if they have the same sign, and positively if they have opposite signs. So the bigger the fraction of aligned spins the smaller the energy. In practice, the number of atoms is so large that it is impossible to measure the state of each of them in the lattice. That is why in Statistical Physics, the equilibrium states of the system are described by probability measures on the space of configurations of the system. In other words, we enumerate all the possible spin configurations and assign them a probability of realization. Of course, every configuration has not the same probability to occur: according to the second principle of Thermodynamics which amounts to minimize the information we have about the micro-states of the system, this probability measure has to maximize the entropy[3]at a given average energy. It is not difficult to show that the probability measure describing the system has thus to take the form:

$$\mathbb{P}(\sigma) = \frac{\exp(-\mathcal{H}(\sigma)/\mathrm{T})}{Z} \tag{2}$$

where T is the temperature and Z is a normalization constant. Such a finite volume measure is called a Gibbs distribution. At low temperature, it favors the configurations which almost minimize the energy, that is where the spins have almost all the same sign, whereas at high temperatures, every atom is more or less free to behave at random; the system's behavior is almost like that of a free system with independent components. We thus imagine why this is a good candidate for a model of phase transition.

The underlying lattice L is supposed to model real materials. The square lattice is a natural example, but we can think about more general periodic lattices, as the Perovskite structure represented on Figure 3. As we will see along this thesis, it turns out that bidimensional systems with enough symmetry have duality and integrability properties which make them much more understood than their higher dimensional counterparts.

Ernst Ising, who was Lenz's PhD student, studied the one-dimensional version of the model, and proved [58] that the latter does not undergo a phase transition: a wire of ferromagnetic atoms interacting with their nearest neighbors is never magnetized at any positive temperature. In 1936, Rudolf Peierls proved [75] the existence of a phase transition in the same model in dimension larger than or equal to two. That

[3]The entropy of a probability measure $\mathbb{P}$ is the number between 0 and 1 given by $S = \sum_\sigma \mathbb{P}(\sigma) \log \mathbb{P}(\sigma)$. One can verify easily that the uniform measure has entropy 1, while the atomic measure has entropy 0.

means that in the limit of an infinitely large sample, there exists a well defined positive temperature below which the system is magnetized and above which it is not. Let us emphasize that such a transition from positive to zero average magnetization cannot occur in finite volume because the magnetization is a weighted sum of analytic functions of the inverse temperature. The absolute value of the magnetization measured in laboratories is very close to zero but positive below T_c, see Figure 10. The now called "Ising model" was the first model of Statistical Physics for which it was possible to prove the existence of a phase transition, which constituted a first sign of success of this theory. This result was followed in 1944 by the work of Onsager [74] who obtained the "exact solution" of the Ising model in two dimensions on a square lattice. He computed in particular the Curie temperature. The analytical expression for the magnetization as a function of temperature was then derived rigorously and published by Yang [90].

The Ising model can also be used to model the liquid-vapor phase transition described above. In a real situation, the particles can be anywhere in a box, in contrast with the magnet where the atoms form a lattice, but we can study a simplified model by dividing the container of the gas into a large number of cells (which form a lattice) and which are of about the same volume as the particles. Hence a particular cell is either occupied or empty (we label it by 1 or 0). This simplified picture is called a "lattice gas". An energy of the form (1) can also be used in this case; the term $\sigma_x\sigma_y$ is only non-zero when the cells x and y are occupied, which means that the particles attract each other. The magnetization in the Ising model corresponds to the difference between the density of the liquid and the gaseous phase. From a formal point of view, the liquid-vapor transition of this lattice gas is similar to the spontaneous magnetization of an Ising ferromagnet as long as the coupling between magnetic moments of the atoms is similar to the effective interaction between the occupation numbers of the cells.

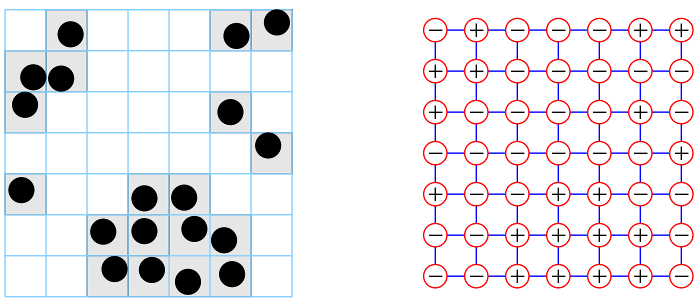


Figure 5 – The mapping between the lattice gas and the Ising ferromagnet.

In 1952, R.B. Potts introduced [79] a generalization of the Ising model where the notion of "spin" is replaced by the one of "color": each atom of the lattice have a certain color labeled between 1 and q. The energetic cost is invariant under permutations of the labels and is only due to nearest neighbor pairs of atoms which do not have the same color. Similarly as in the Ising model, the energy associated to each configuration $\sigma = \{\sigma_x\}_{x\in L}$ with $\sigma_x \in \{1, \dots, q\}$ takes the following form:

$$\mathcal{H}(\sigma) = -\sum_{x\sim y} \mathbb{1}_{[\sigma_x=\sigma_y]}. \tag{3}$$

For $q = 2$ it can be mapped to the Ising model just by replacing $+1$ spins by color 1 and -1 spins by color 2, and noticing that for Ising spins $(1+\sigma_x\sigma_y)/2 = \mathbb{1}_{[\sigma_x=\sigma_y]}$. It is thus as well a model for ferromagnetic materials. For $q = 3$ it models for example the cubic-tetragonal transition of the Perovskite described above. The interesting feature of this model is that the phase transition properties depend non-trivially on q and the dimension d of the lattice. As in the Ising model, there is no phase transition in dimension 1. The existence of a phase transition in dimension 2 and higher can be proved with the analog of Peierl's argument. Another proof is due to Fortuin and Kasteleyn [34] who introduced an ingenious coupling with another model called "random-cluster": below a certain critical temperature T_c, the system spontaneously orders: it has more atoms with a particular color. So there is a well defined order parameter which is the density of this particular color: below T_c it is larger than $1/q$. In planar lattices, it is conjectured that the phase transition is of second order for $q \leqslant 4$ (the order parameter is continuous at the transition) and first order for $q > 4$ (the order parameter makes a jump).

III Phase transition and fluctuation of interfaces

III.1 Ising model

Consider a layer of fixed $+1$ spins all around the sample. If the spin located in the middle of the sample is more probable to be $+1$ than -1 it means that the system is influenced by the boundary condition and is positively magnetized.

It is a very general fact that no phase transition occurs in one-dimensional systems with finite range interactions [81]. Indeed, exact computations are possible in this case which amount to express all relevant quantities as powers of an appropriate "transfer matrix"; its eigenvalues are real analytic functions of the local potentials, and thus of the inverse temperature. We quote Ruelle to summarize the interest of one-dimensional systems: "the statistical mechanics of classical one-dimensional systems is relatively tractable and at the same time relatively uninteresting". In the sequel we will thus be interested in lattices of dimension 2 or larger.

Let us explain the Peierls argument which shows the existence of a phase transition in the two dimensional Ising model. If the central atom has a spin -1 then there must exist an interface (separating spins of opposite signs) surrounding this atom. The "energetic cost" of observing a given interface of length n is equal to $\exp(-2n/T)$, according to (1). On the other hand, the number of possible interfaces of length n surrounding the central atom is upper-bounded by 4^n. Thus, the probability for the central atom to have a spin -1 is upper-bounded by

$$\sum_{n\geqslant 0} \text{Number of interfaces of size } n \text{ surrounding } 0 \quad \times \\ \times \quad \text{Weight of a contour} \quad = \quad \sum_{n\geqslant 0} 4^n \cdot e^{-2n/T}.$$

This sum is strictly smaller than $1/2$ when T is sufficiently small, hence the system is magnetized at small enough temperature.

Peierls argument shows that the phase transition in the Ising model can be understood by looking at the typical configurations of broken lines or surfaces separating the domains with plus and minus spins. At high temperature, the center of a large piece of lattice will typically encounter more or less the same configurations no matter what boundary conditions outside this domain are imposed. This is not the case below the critical temperature, where what happens at the boundary of the sample has an influence deep inside it.

Note that the critical temperature is increasing in the dimension of the lattice. Indeed, each atom has $2d$ nearest neighbors, so at a given temperature the system is more ordered as d increases. In other words, a larger temperature is needed to disorder the system as d increases. As there exists a phase transition in dimension 2, this is also true in any dimension larger than 2.

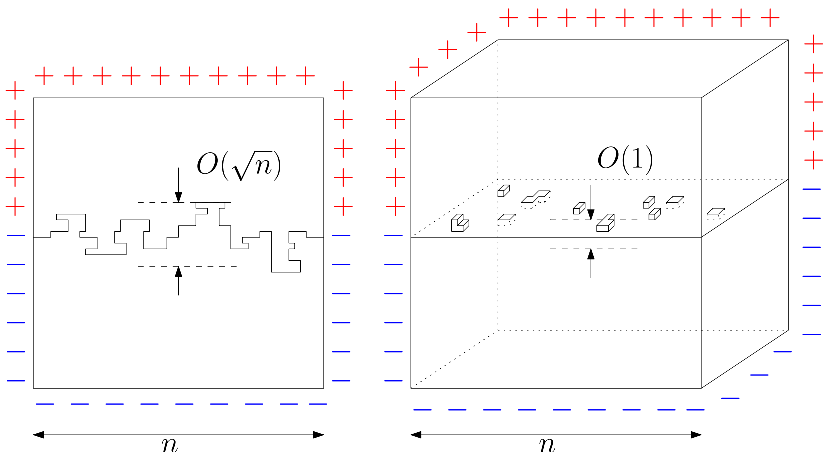


Figure 6 – The Dobrushin interface of the Ising model at low temperature, $d = 2$ and 3.

Imposing a non constant boundary condition amounts to imposing the existence of interfaces inside the system. A natural way to study interfaces is to choose a boundary condition which imposes the existence of at least one interface throughout the system: put $+1$ spins on the upper-half boundary of a domain, and -1 spins on the lower-half; this is called the Dobrushin boundary condition. On the square lattice, it is known [37, 30] that the induced interface is delocalized in dimension 2, whereas it is flat in dimension 3 at small enough temperature, see Figure 6.

More precisely, in dimension 2 the interface is spread over a typical width which is the square root of the side-length of the sample [48]), and in dimension 3 it has small fluctuations of constant size. If we write φ_0 the height of the interface above the origin, we have:

$$\mathrm{Var}(\varphi_0) = \begin{cases} O(\sqrt{n}) & \text{for } d = 2 \\ O(1) & \text{for } d \geqslant 3 \end{cases}$$

The intuition behind these results can be explained as follows. In 2 dimensions, the interface can be typically decomposed into a chain of irreducible pieces whose increments constitute an effective one-dimensional directed random walk. The precise study of the correlations between the increments shows that this random walk has diffusive Gaussian fluctuations.

In 3 dimensions, the situation is different. Note that at zero temperature the interface is flat, it is the horizontal plane passing through the origin. At very low but positive temperature, a Peierls type argument can be implemented to show that only small local fluctuations appear: the “energetic cost” for creating a deformation of height 1 in the plane is proportional to its perimeter, see Figure 7. As before, the weight of a deformation of perimeter n is $\exp(-2n/T)$ and the number of such (non-overhanging) deformations is bounded above by 4^n. So the probability to see any deformation tends to zero as $T \to 0$.

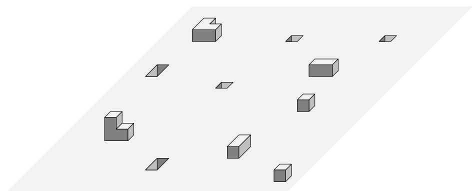


Figure 7 – Deformations of the zero temperature interface in the 3-dimensional Ising model with Dobrushin boundary conditions.

The same kind of argument shows that interfaces in dimension d are localized at low temperature for all $d \geqslant 3$.

III.2 Potts model

For the Potts model, the analog of most of the results described above are still valid. The existence of a phase transition can be proved either via a Peierls argument, or with the auxiliary random-cluster model, which will be introduced later in this thesis.

The fluctuations of the Dobrushin interface (with any two colors on the boundary) in a sample of side-length n are also known [21] to be of order $O(\sqrt{n})$ in 2 dimensions, and of $O(1)$ in dimension 3 and higher at small enough temperature, see Figure 8.

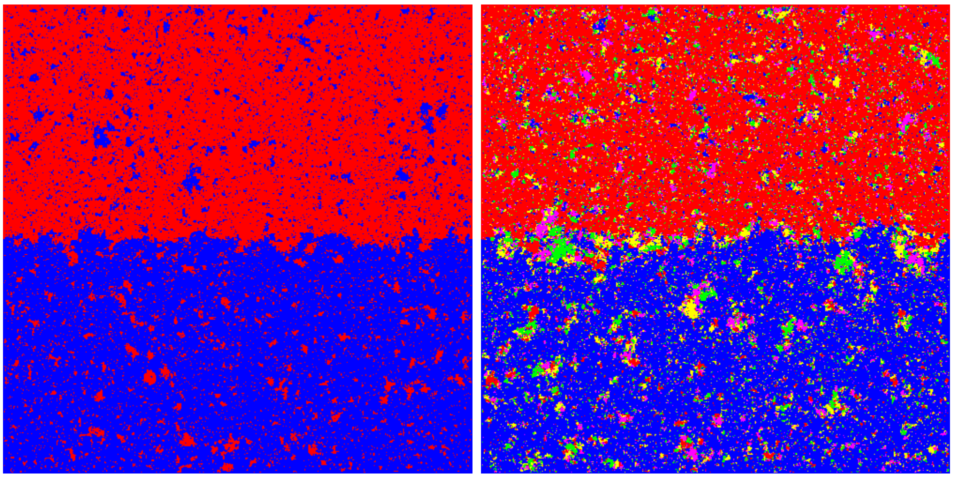


Figure 8 – The Dobrushin interface of the Ising model (left) and Potts model (right, $q = 5$) below T_c in dimension 2 on the square lattice.

III.3 Effective interface models and pinning

As we have seen, the energy of a configuration is proportional to the number of disagreeing pairs of spins. At very low temperature, overhangs of the Dobrushin interface are very much penalized and a reasonable simplified interface model consists in choosing at random a height-function over a $d-1$ dimensional lattice L, with the constraint that big gradients are penalized and the height is zero at the boundary of the sample. Consider for example the following models where a function $\varphi = (\varphi_x)_{x\in L}$ with $\varphi_x \in \mathbb{Z}$ or $\mathbb{R}$ is chosen with a weight:

$$\exp\left(-\frac{1}{4(d-1)}\sum_{x\sim y}(\varphi_x - \varphi_y)^2\right)$$

where $d-1$ is the dimension of the underlying lattice. These models represent an interface between two d-dimensional media; the cases $d = 1, 2$ are relevant in a 3-dimensional world. Both $\mathbb{Z}$ and $\mathbb{R}$-valued models are interesting in themselves, and different techniques allow to study them. The latter is called the Gaussian free field.

The $\mathbb{Z}$-valued model represents a free interface, with no overhangs, whose law is close to the Ising one for a 1-dimensional Dobrushin interface at low temperature. If the variables φ_x are $\mathbb{R}$-valued, the variance of the height over site x can be computed with a useful random walk representation [25]: the covariance between variables φ_x and φ_y under the measure μ coincides with the expected number of visits in y of a simple symmetric random walk starting at x before it exits the finite lattice L. Well known results about the random walk [84] allow us to compute the variance of the interface over the origin (written 0) in a sample of side-length n. Here are the variances for the $\mathbb{Z}$ and $\mathbb{R}$-valued models.

$$\mathrm{Var}_{\mathbb{Z}}(\varphi_0) = \begin{cases} O(\sqrt{n}) & \text{for } d = 1+1 \\ O(1) & \text{for } d \geqslant 2+1 \\ & \text{ and } T \ll 1 \end{cases} \qquad \mathrm{Var}_{\mathbb{R}}(\varphi_0) = \begin{cases} O(\sqrt{n}) & \text{for } d = 1+1 \\ O(\sqrt{\log n}) & \text{for } d = 2+1 \\ O(1) & \text{for } d > 2+1 \end{cases}$$

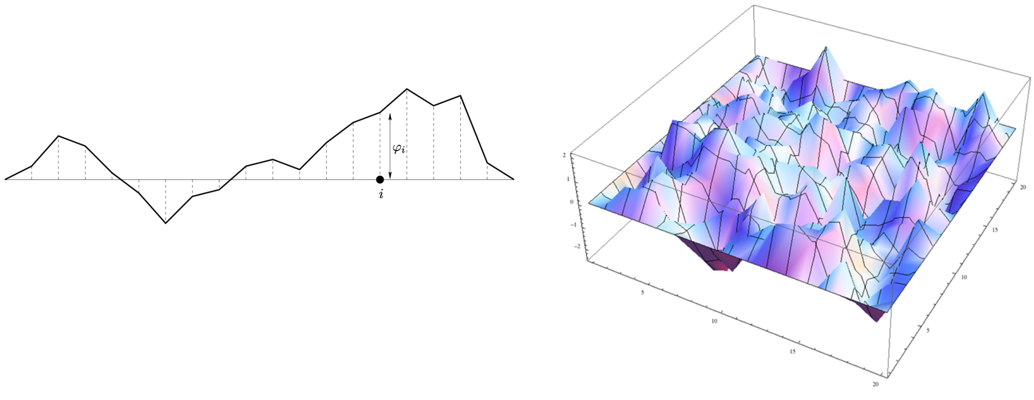


Figure 9 – Interpretation of the discrete Gaussian free field in dimension 1 and 2 as interface models in $\mathbb{R}^2$ and $\mathbb{R}^3$.

As in the Ising model, we observe that the 1-dimensional interface of the $\mathbb{Z}$-valued model is delocalised over a width which is the square root of the side-length of the sample.

An interesting modification of the model consists in adding a potential, for example an attraction or a repulsion at zero height. One can then view the previous interface as a random 1-dimensional polymer, or d-dimensional surface which is attracted or repelled by a deterministic hyperplane.

In Part 2 of this thesis, a new version of this model is studied: we consider the Gaussian free field model in the presence of a potential on the hyperplane, which is either attractive or repulsive with a randomly chosen strength. For a constant positive (attraction) strength, chosen as small as we want, it is known [16] that the (hyper-)surface is localized, i.e. a positive density of its monomers are sticked at zero height. In our work, the question we ask concern the influence of the disorder on the behavior of the interface: does the presence of disorder modify macroscopic quantities in the system?

IV Thermodynamic limit and Gibbs measures

As the number of atoms in a ferromagnet or in a gas is very large, and the phase transition phenomenon (magnetization or density curve as a function of the temperature) is sharper and sharper when the volume of the sample increases (see Figure 10), it is natural to study the models introduced above in the limit of large volume $|L| \to \infty$, also called thermodynamic limit.

Of course, the Hamiltonian (1) is not well defined on an infinite lattice and one must either take suitable limits of finite volume measures, or try to characterize an infinite volume distribution by the specification of its restrictions in finite volume. A probability measure on an infinite lattice whose finite volume conditional probabilities have the form (2) is called a Gibbs measure. It turns out that this characterization is intimately connected with the limiting procedure we describe now.

In a large but finite system, the measurement of local observables in the bulk (i.e.

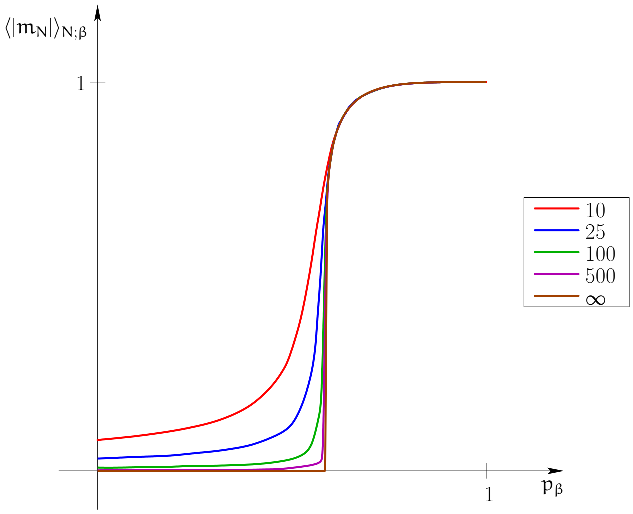


Figure 10 – Expected magnetization as a function of $p = 1 - e^{-1/T}$ for the bidimensional Ising model, for different values of the side-length N of the sample.

observables which depend on a finite number of spins) should not depend on finite-size effects, and thus should be well approximated by an infinite-volume measure. The mathematical framework which corresponds to this requirement is given by the weak convergence of measures: a sequence of probability measures $\mathbb{P}_L$ on a finite lattice L converges to some measure $\mathbb{P}$ on the infinite lattice if the expectation of all local observables under $\mathbb{P}_L$ (these are real numbers) converge towards their counterpart under $\mathbb{P}$.

Hence, it is the local behavior of a given model which allows to characterize the set of its infinite volume Gibbs measures. In particular, the fluctuations of interfaces play a crucial role.

In the Ising and Potts models, we call "pure phases"[4] the probability measures on a d-dimensional lattice which are the limits, as $n \to \infty$, of finite volume measures with constant boundary conditions outside a sample of side-length n. For example, the "plus" phase $\mathbb{P}^+$ (or gaseous phase in the lattice gas picture) of the Ising model consists, below the critical temperature, of an infinite "sea" of plus spins with finite islands of minus spins, while the "minus" (or liquid) phase $\mathbb{P}^-$ consists of the opposite picture. In the Potts model, there are q pure phases, each of them having an infinite "sea" of atoms which have color i, with $i = 1, \ldots, q$. Of course all these measures depend on the temperature.

The set $\mathcal{G}$ of infinite volume Gibbs measures associated to a given model describes the set of equilibrium states of the system. It is a convex set and actually also a simplex, thus characterized by a set of extremal measures. The phenomenon of phase transition corresponds to a transition from uniqueness to non-uniqueness of the equi-

[4]In the Physics literature, the term "pure phase" designs an extremal, translation-invariant measure. As we prove in this thesis, the limits of measures with monochromatic boundary conditions turn out to be the pure phases of the 2d Potts model. That is why we use this abuse of terminology.

librium states of the system. Indeed, below the Curie temperature, a ferromagnet can for example magnetize upwards or downwards, this corresponds to the plus and minus phases of the Ising model, which are extremal. At least two equilibrium states are thus possible below the critical temperature, while thermal fluctuations allow only one equilibrium state above the critical temperature.

To know whether there are more than the pure phases in $\mathcal{G}$, it is useful to consider measures enforcing the presence of macroscopic interfaces throughout the system. We explain what happens with Dobrushin boundary conditions for the Ising model; analogous results holds for the Potts model as well. We call Dobrushin measure $\mathbb{P}^{\pm}$ the limit, as $n \to \infty$, of the finite volume measures with Dobrushin boundary conditions outside a sample of side-length n, see Figure 8.

In dimension 2, the interface is delocalized and has typical fluctuations of order $\sqrt{n}$ so with high probability any local observable will "see" configurations that are typical either of the plus phase or of the minus phase. Indeed, with high probability, any local part of the sample is far from the macroscopic interface, and thus surrounded by (far away) spins with a constant value, which make it close to a typical configuration under the corresponding pure phase[5]. This implies that we can express the measure as a convex combination of the plus phase and the minus phase. By symmetry, we have

$$\mathbb{P}^{\pm}_{d=2} = \frac{1}{2}(\mathbb{P}^{+} + \mathbb{P}^{-}).$$

On the other hand, in dimension 3, at small enough temperature, the interface is localized and has bounded fluctuations, so the local observables whose support is above (resp. below) the interface will see the plus (resp. minus) phase. Moreover, observables probing the region close to the interface (this is a deterministic location) will feel local phase coexistence. Consider indeed the observable

"spin of the atom at $(0,0,10)$ − spin of the atom at $(0,0,-10)$."

Its expectation is close to $m^{\star} - (-m^{\star}) = 2m^{\star}$ under $\mathbb{P}^{\pm}_{d=3}$, where $m^{\star}$ is the magnetization in the plus phase (the interface passes between the two atoms with high probability). Under the plus phase, the minus phase (in any dimension), and $\mathbb{P}^{\pm}_{d=2}$ the expectation of the above observable is equal to 0 with high probability (the interface passes far away from the above two atoms located at $(0,10)$ and $(0,-10)$).

These observations have a very important consequence: the Dobrushin measure is translation invariant[6] in dimension 2 while it is not in dimension 3. In particular, $\mathbb{P}^{\pm}_{d=3}$ cannot be expressed as a convex combination of the plus and minus phases. It is a different equilibrium state of the system.

---

[5]It is indeed known that the law of the spins located in a region surrounded by spins of constant value converges exponentially fast towards the pure phase corresponding to this value.

[6]A measure is called translation invariant if it gives the same expectation to a local function and all its translates.

On $\mathbb{Z}^2$, it is known [1, 55] that the set $\mathcal{G}_T$ of Gibbs measures of the Ising model at temperature T is

$$\mathcal{G}_T = \{\alpha\mathbb{P}_T^+ + (1-\alpha)\mathbb{P}_T^-,\ \alpha \in [0,1]\}.$$

Namely, only the pure phases are the relevant equilibrium states. In the first half of Part 1 of this thesis, we refine this result by showing that, for any $\varepsilon > 0$, the finite volume measures in a sample of side-length n with any boundary conditions are convex combinations of the plus and minus phases up to probabilities of order $O(n^{-1/2+\varepsilon})$. In other words, we prove that, whatever boundary condition we choose (which can enforce a priori up to $O(n)$ interfaces throughout the system) local phase coexistence does not occur: all macroscopic interfaces are located at least at distance $O(n^{1/2-\varepsilon})$ from any fixed local region with high probability.

In dimension 3, it is known that the Dobrushin measure, as well as all its vertical translates, are extremal. The set $\mathcal{G}$ is thus more complicated in this case. Moreover, very little is known yet about other types of boundary conditions.

The second half of Part 1 is devoted to the Potts model, for which much less was known about the set $\mathcal{G}$, even in dimension 2. Only the perturbative regimes of low temperature or large parameter q where studied [78, 71]. In this thesis we prove that on $\mathbb{Z}^2$ every Gibbs measure of the Potts model below the critical temperature is a convex combination of these q pure phases.

$$\mathcal{G}_T = \left\{ \sum_{i=1}^{q} \alpha_i \mathbb{P}_T^i,\ \sum_{i=1}^{q} \alpha_i = 1 \right\}.$$

A corollary of this result is that all Gibbs measures are translation invariant (as the pure phases are), and that the set of extremal measures coincides with the pure phases, neither of which was known before up to the critical temperature. Above the critical temperature there exists a unique Gibbs state, and at the critical temperature it is conjectured that $\mathcal{G}$ contains the q pure phases as extremal elements for $q \leqslant 4$, and those plus the high temperature phase when $q > 4$. This in intimately connected with the expected first order phase transition for $q > 4$ we already mentioned.

The idea behind our result is a finite volume estimate which is the analog of the one for the Ising model: we show that any fixed finite part of a finite sample of side-length n is at a distance $O(n^{1/2-\varepsilon})$ from any macroscopic interface. Note that the structure of the Potts interfaces is more complicated that the Ising one: in the bidimensional case, the Ising interfaces are just lines separating plus and minus spins, while the Potts interfaces are locally trees with several branches. The work consists in proving that only a finite number of interfaces reach the internal half of the sample, and that due to surface tension the macroscopic structure of remaining interfaces concentrates on a forest consisting of trees with internal nodes of degree 3, see Figure 11. Then, the analysis of interfaces fluctuations separating 2 or 3 phases is needed: they are big enough to make any local region most likely to be located deep inside a pure phase. Exponential relaxation into pure phases allows to conclude.

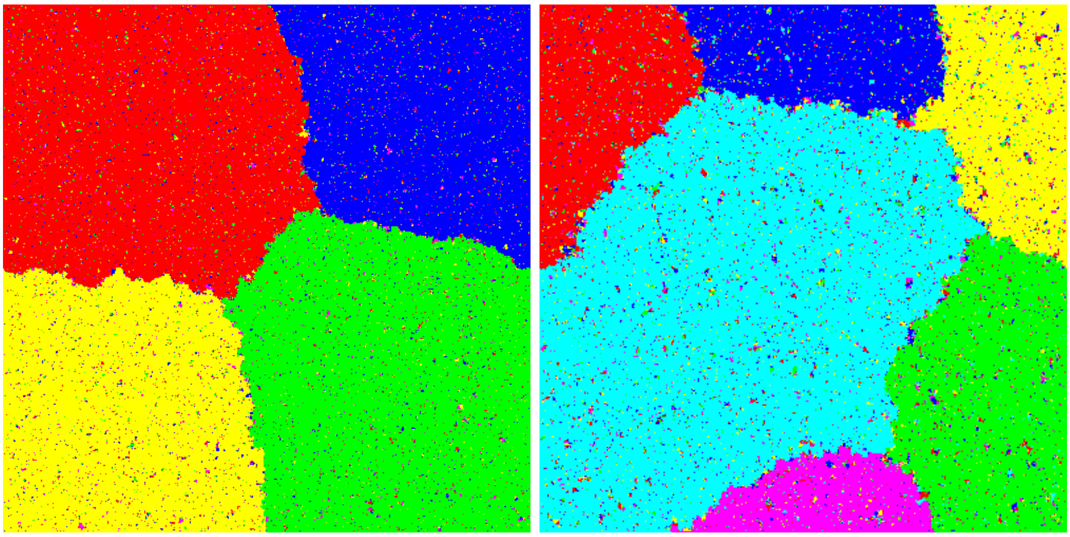


Figure 11 – Macroscopic interfaces in the 2d Potts model with piecewise constant boundary condition. (Left: $q = 4$. Right: $q = 6$.) Interfaces internal nodes of degree more than 3 are prohibited by the surface tension.

Part 1

On the Gibbs states of the noncritical Ising and Potts models on $\mathbb{Z}^2$

Si la vie n'est qu'un passage, sur ce passage au moins semons des fleurs.

Michel de Montaigne, "Les Essais".

Chapter 1

A review of the Ising and Potts models

In this chapter we give a detailed review of the known results about the non-critical Ising and Potts models, which are needed to study the set of their infinite volume measures. Most of the elementary results are proven, and some heuristics is given for the more involved ones. In both cases, we mention the reference to original literature.

1.1 Finite volume Gibbs measures

Let $G = (V, \mathcal{E})$ be a finite connected graph, where V and $\mathcal{E}$ are the vertices and edges of G. We write $V = V_0 \cup \partial V$, where ∂V denotes the distinguished set of boundary vertices of G, see Figure 1.1. We write $e = [i, j]$ to denote the edge $e \in \mathcal{E}$ which connects the vertices i and j. A particular vertex, "the origin", will be also distinguished, we write it 0. We also define the ball of radius n and center x in G, $B_n(x) = \{y \in V : d_G(x, y) \leqslant n\}$ where d_G denotes the graph distance (we will write it d when no confusion is possible).

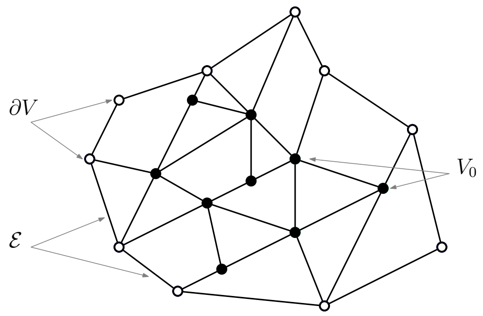


Figure 1.1 – A graph $G = (V, \mathcal{E})$, with $V = \partial V \cup V_0$.

1.1.1 The Ising model

Consider the configuration space $\mathbf{\Sigma}_V = \{-1,+1\}^V$. An Ising configuration in $\mathbf{\Sigma}_V$ is written $\sigma = \{\sigma_i\}_{i\in V}$. It associates to each vertex $i \in V$ a "spin" in $\{-1,+1\}$. Let $\overline{\sigma} \in \mathbf{\Sigma}_{\partial V}$ be a boundary condition. We associate to each configuration σ its energy in G, which depends on the inverse temperature $\beta \in [0,\infty)$, and is described by the following Hamiltonian:

$$\boldsymbol{\mathcal{H}}^{\overline{\sigma}}_{G,\beta}(\sigma) = -\beta \left(\sum_{[i,j]\in\mathcal{E}} \sigma_i\sigma_j + \sum_{\substack{i\in V, j\in\partial V\\ [i,j]\in\mathcal{E}}} \sigma_i\overline{\sigma} \right) \tag{1.1}$$

The Ising model on G at inverse temperature β is the following probability measure on $\mathbf{\Sigma}^{\overline{\sigma}}_V = \{\sigma \in \mathbf{\Sigma}_V : \sigma \equiv \overline{\sigma} \text{ on } \partial V\}$ (with the associated product $\sigma-$algebra):

$$\mathbf{P}^{\overline{\sigma}}_{G,\beta}(\sigma) = \frac{1}{\mathbf{Z}^{\overline{\sigma}}_{G,\beta}} \exp\left(-\boldsymbol{\mathcal{H}}^{\overline{\sigma}}_{G,\beta}(\sigma)\right) \tag{1.2}$$

The normalization constant $\mathbf{Z}^{\overline{\sigma}}_{G,\beta}$ is called the partition function. We denote by $\mathbf{P}^{+}_{G,\beta}$, resp. $\mathbf{P}^{-}_{G,\beta}$, the measures obtained using $\overline{\sigma} \equiv +1$, resp. $\overline{\sigma} \equiv -1$. We denote by $\mathbf{P}^{\mathbf{f}}_{G,\beta}$ the measure obtained without specifying the value of the spins on ∂V.

1.1.2 The Potts model

Consider the configuration space $\Sigma_V = \{1,...,q\}^V$. A Potts configuration in Σ_V is written $\sigma = \{\sigma_i\}_{i\in V}$. It associates to each vertex $i \in V$ a number in $\{1,...,q\}$, usually called "color" or "state". Let $\overline{\sigma} \in \Sigma_{\partial V}$ be a boundary condition. We associate to each configuration σ its energy in G, which depends on the inverse temperature $\beta \in [0,\infty)$, and is described by the following Hamiltonian:

$$\mathcal{H}^{\overline{\sigma}}_{G,\beta}(\sigma) = -\beta \left(\sum_{[i,j]\in\mathcal{E}} \delta_{\sigma_i,\sigma_j} + \sum_{\substack{i\in V, j\in\partial V\\ [i,j]\in\mathcal{E}}} \delta_{\sigma_i,\overline{\sigma}} \right) \tag{1.3}$$

The q-state Potts model on G at inverse temperature β is the following probability measure on $\Sigma^{\overline{\sigma}}_V = \{\sigma \in \Sigma_V : \sigma \equiv \overline{\sigma} \text{ on } \partial V\}$.

$$\mathbb{P}^{\overline{\sigma}}_{G,\beta,q}(\sigma) = \frac{1}{Z^{\overline{\sigma}}_{G,\beta,q}} \exp\left(-\mathcal{H}^{\overline{\sigma}}_{G,\beta}(\sigma)\right) \tag{1.4}$$

Note that:

- For $q = 2$, if we map the colors 1 and 2 onto +1 and -1, since $\delta_{\sigma_i\sigma_j} = \frac{1+\sigma_i\sigma_j}{2}$, we recover the Ising model on G at inverse temperature $\beta/2$.
- When $\overline{\sigma} \equiv i$ for some $i = 1\ldots q$, we will speak about "the finite volume Potts measure with pure boundary condition i". We denote by $\mathbb{P}^{f}_{G,\beta,q}$ the measure obtained without specifying the value of the spins on ∂V.

1.2 A few elementary results for the Ising model

A lot of techniques have been developed in the last decades in order to study the Ising model. We will not be exhaustive here; in the first part of this section, we emphasize the fact that, unlike the Potts model, the natural ordering present in the Ising model allows to prove "easily" important results. More specific techniques are available for planar graphs; in the second part of this section we present the Kramers-Wannier duality and the random-line representation, which will be useful for our new results about the Ising model.

1.2.1 Basic properties on finite graphs

In this section we describe basic properties of the Ising model on a finite connected graph $G = (V, \mathcal{E})$.

1.2.1.1 The spatial Markov property

If σ_1 and σ_2 are the restrictions of a configuration to two disjoint parts of G, we denote by $\sigma_1 \vee \sigma_2$ their concatenation. For a subgraph $G' = (V', \mathcal{E}')$ of G we write $\partial V'$ for the set of vertices sharing an edge with $V \setminus V'$. A simple property of the Ising model is the following:

Proposition 1.1 (Spatial Markov property)
Let $\beta \geqslant 0$. For all subgraphs $G' = (V', \mathcal{E}')$ of the graph $G = (V, \mathcal{E})$, for all $\eta \in \mathbf{\Sigma}_V^{\overline{\sigma}}$ and for all boundary conditions $\overline{\sigma}$ on ∂V, we have

$$\mathbf{P}^{\overline{\sigma}}_{G,\beta}(\sigma \,|\, \sigma \equiv \eta \text{ on } V \setminus V') = \mathbf{P}^{\overline{\sigma}}_{G,\beta}(\sigma \,|\, \sigma \equiv \eta \text{ on } \partial V') = \mathbf{P}^{\eta \vee \overline{\sigma}}_{G',\beta}(\sigma)$$

The proof is elementary and consists in observing the cancellation of the weights due to the nearest neighbor range of the model. Of course the spatial Markov property is also valid for the Potts model.

1.2.1.2 The GKS inequalities

The following very useful inequalities are named after Griffiths, Kelly and Sherman [49, 50, 60]. They are restricted to + and free boundary conditions, and give information about expectation and covariances of products of spins. We will use the following notation throughout this chapter:

Definition 1.1 *For $A \Subset V$, we write*

$$\sigma_A \doteqdot \prod_{x \in A} \sigma_x .$$

Proposition 1.2 (GKS inequalities) *Let $\beta \geqslant 0$, and $A, B \Subset V$. Then,*

$$\mathbf{P}^{+}_{G,\beta}(\sigma_A) \geqslant 0$$
$$\mathbf{P}^{+}_{G,\beta}(\sigma_A \cdot \sigma_B) \geqslant \mathbf{P}^{+}_{G,\beta}(\sigma_A) \cdot \mathbf{P}^{+}_{G,\beta}(\sigma_B)$$

The same is true for the measure $\mathbf{P}^{\mathbf{f}}_{G,\beta}$.

We refer to the original articles cited above for the proof. A first corollary of Proposition 1.2 is that the expectations $\mathbf{P}^{+}_{G,\beta}(\sigma_A)$ are increasing in the coupling constants of the model. Indeed, writing the Hamiltonian (1.1) as

$$\mathcal{H}^{+}_{G,\beta}(\sigma) = -\beta \left(\sum_{\substack{i,j \in V \setminus \partial V \\ [i,j] \in \mathcal{E}}} J_{ij}\sigma_i\sigma_j + \sum_{\substack{i \in V, j \in \partial V \\ [i,j] \in \mathcal{E}}} J_{ij}\sigma_i \right)$$

with $J_{ij} = 1$ is useful to observe that:

$$\frac{d}{dJ_{ij}}\mathbf{P}^{+}_{G,\beta}(\sigma_A) = \mathbf{P}^{+}_{G,\beta}(\sigma_A\sigma_i\sigma_j) - \mathbf{P}^{+}_{G,\beta}(\sigma_A)\mathbf{P}^{+}_{G,\beta}(\sigma_i\sigma_j) \overset{\text{GKS}}{\geqslant} 0. \tag{1.5}$$

This result is valid even if the J_{ij} differ from one site to the other, as long as the model is ferromagnetic ($J_{ij} \geqslant 0, \forall i, j$).

1.2.1.3 The FKG inequality

There is a natural partial order on the set of configurations of the Ising model:

$$\sigma \leqslant \sigma' \iff \sigma_x \leqslant \sigma'_x \quad \forall x \in V.$$

Therefore, we can define a notion of monotonicity: a function $f : \mathbf{\Sigma}_V \to \mathbb{R}$ is called increasing if $\sigma \leqslant \sigma' \implies f(\sigma) \leqslant f(\sigma')$. An event is called increasing if its indicator function is increasing. The following inequality is named after Fortuin, Kasteleyn and Ginibre [35], and states that increasing functions are positively correlated, independently of the boundary condition.

Proposition 1.3 (FKG inequality)
Let $\beta \geqslant 0$. For any boundary condition $\overline{\sigma}$, let $f, g : \mathbf{\Sigma}^{\overline{\sigma}}_V \to \mathbb{R}$ be two increasing functions. Then

$$\mathbf{P}^{\overline{\sigma}}_{G,\beta}(f \cdot g) \geqslant \mathbf{P}^{\overline{\sigma}}_{G,\beta}(f) \cdot \mathbf{P}^{\overline{\sigma}}_{G,\beta}(g).$$

Despite the intuitiveness of the statement, which comes from the ferromagnetic type of the coupling constants, the original proof is somehow technical. We refer to [41] for a nice proof; see also [56]. The FKG inequality has many interesting consequences. One of them is the stochastic comparison between boundary conditions:

Proposition 1.4 *For any increasing function* f*, inverse temperature* $\beta \geqslant 0$*, and boundary conditions* $\overline{\sigma} \leqslant \overline{\sigma}'$*, we have*

$$\mathbf{P}^{-}_{G,\beta}(f) \leqslant \mathbf{P}^{\overline{\sigma}}_{G,\beta}(f) \leqslant \mathbf{P}^{\overline{\sigma}'}_{G,\beta}(f) \leqslant \mathbf{P}^{+}_{G,\beta}(f).$$

We say that the above measures are stochastically ordered, and we write

$$\mathbf{P}^{-}_{G,\beta} \preccurlyeq \mathbf{P}^{\overline{\sigma}}_{G,\beta} \preccurlyeq \mathbf{P}^{\overline{\sigma}'}_{G,\beta} \preccurlyeq \mathbf{P}^{+}_{G,\beta}.$$

Proof We can write

$$\mathcal{H}^{\overline{\sigma}}_{G,\beta}(\sigma) = \mathcal{H}^{\overline{\sigma}'}_{G,\beta}(\sigma) + \beta \sum_{\substack{i\in V, j\in\partial V\\ [i,j]\in\mathcal{E}}} \sigma_i(\overline{\sigma}_j - \overline{\sigma}'_j) \doteqdot \mathcal{H}^{\overline{\sigma}'}_{G,\beta}(\sigma) + I(\sigma)$$

and note that $I(\sigma)$ is a decreasing function. Therefore,

$$\mathbf{P}^{\overline{\sigma}}_{G,\beta}(f) = \frac{\mathbf{P}^{\overline{\sigma}'}_{G,\beta}(f\cdot e^{I})}{\mathbf{P}^{\overline{\sigma}'}_{G,\beta}(e^{I})} \leqslant \frac{\mathbf{P}^{\overline{\sigma}'}_{G,\beta}(f)\mathbf{P}^{\overline{\sigma}'}_{G,\beta}(e^{I})}{\mathbf{P}^{\overline{\sigma}'}_{G,\beta}(e^{I})} = \mathbf{P}^{\overline{\sigma}'}_{G,\beta}(f)$$

The same can be done by replacing $\overline{\sigma}$ by $+$ or $-$. ■

Let us use the following notations:

Definition 1.2 *We write*

$$n_x = \frac{1}{2}(1+\sigma_x) \quad \textit{and} \quad n_A = \prod_{x\in A} n_x.$$

It is easy to prove that the following functions are increasing for any $x \in V$ and $A \Subset V$:

$$\sigma_x, \quad n_x, \quad n_A, \quad \sum_{x\in A} n_x - n_A. \tag{1.6}$$

Definition 1.3 *The support of a function* $f : \Sigma_V \to \mathbb{R}$ *is the smallest set* A *such that* $f(\sigma) = f(\sigma')$ *as soon as* $\sigma_{|A} = \sigma'_{|A}$*, where* $\sigma_{|A}$ *denotes the restriction of* σ *to* A*. It is denoted Support*(f)*.*

The following lemma shows that two classes of local functions, which are amenable to the use of correlation inequalities (either GKS or FKG), form a basis of the set of local functions.

Proposition 1.5 (Decomposition onto local increasing functions) *Let* $f : \Sigma_V \to \mathbb{R}$ *be a function. Then there exist coefficients* $\hat{f}_A$ *and* $\tilde{f}_A$*, for all* $A \subseteq$ *Support*(f) *such that*

$$f = \sum_{A\subseteq \mathit{Support}(f)} \hat{f}_A \sigma_A = \sum_{A\subseteq \mathit{Support}(f)} \tilde{f}_A n_A$$

Proof Observe that the following orthogonality relation holds:

$$2^{-|B|} \sum_{A \subseteq B} \sigma_A \sigma'_A = \mathbb{1}_{[\sigma_x = \sigma'_x \, \forall x \in B]}$$

Indeed, suppose $\sigma_x = \sigma'_x$ for all $x \in B$. Then $\sigma_A \sigma'_A = 1$, and the relation holds. Suppose there exists some $x \in B$ such that $\sigma_x \neq \sigma'_x$, i.e. $\sigma_x \cdot \sigma'_x = -1$. Then,

$$\sum_{A \subseteq B} \sigma_A \sigma'_A = \sum_{A \subseteq B \setminus \{x\}} \sigma_A \sigma'_A + \sigma_{A \cup \{x\}} \sigma'_{A \cup \{x\}} = \sum_{A \subseteq B \setminus \{x\}} \sigma_A \sigma'_A + \sigma_x \sigma'_x \sigma_A \sigma'_A = 0,$$

and the relation holds too. Apply it to $B =$Support(f):

$$\begin{aligned} f(\sigma) &= \sum_{\sigma'} f(\sigma') \mathbb{1}_{[\sigma_{|\text{Support}(f)} = \sigma']} = \sum_{\sigma'} f(\sigma') 2^{-|\text{Support}(f)|} \sum_{A \subseteq \text{Support}(f)} \sigma_A \sigma'_A \\ &= \sum_{A \subseteq \text{Support}(f)} \underbrace{\left(2^{-|\text{Support}(f)|} \sum_{\sigma'} f(\sigma') \sigma'_A \right)}_{\hat{f}_A} \sigma_A \end{aligned}$$

The second decomposition follows from the observation $\sigma_A = \prod_{x \in A} (2n_x - 1)$. ■

1.2.2 Infinite volume measures

As we are interested in large systems, it is interesting to have a notion of infinite volume measures, namely a set of measures on an infinite graph G_∞, which are limits of measures on finite subgraphs of G_∞, for a suitable notion of convergence. To this end, we consider a sequence of graphs $G = (G_n)_n$ such that $G_n \Subset G_\infty = (V_\infty, \mathcal{E}_\infty)$ and $G_n \uparrow G_\infty$ as $n \to \infty$ (which we write $G \uparrow G_\infty$ in the sequel), where G_∞ is an infinite, locally finite, connected graph (i.e. each vertex has a finite number of outcoming edges). We will add hypothesis on G_∞ for some of the results below. We also consider a sequence of boundary conditions $\overline{\sigma}_n$ on G_n, and we will define a notion of limiting measure for sequences $\mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta}$.

First of all, let $\mathbf{\Sigma} = \{-1, +1\}^{V_\infty}$. We say that a function $f : \mathbf{\Sigma} \to \mathbb{R}$ is local if its support is finite, or in other words, if it depends only on a finite number of spins.

Definition 1.4 *A sequence* $(\mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta})_n$ *converges to some limiting measure* $\mathbf{P}$ *on* $(\mathbf{\Sigma}, \mathcal{F})$*, where* $\mathcal{F}$ *is the product* $\sigma-$*algebra, if and only if*

$$\lim_{n \to \infty} \mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta}(f) = \mathbf{P}(f)$$

for all local functions $f : \mathbf{\Sigma} \to \mathbb{R}$.

Let us introduce notations for the set of all infinite-volume measures. The next chapter is devoted to the properties of this set and contains more details.

Definition 1.5 *Let $\beta \geqslant 0$. A probability measure $\mathbf{P}$ on $(\boldsymbol{\Sigma}, \mathcal{F})$ is an infinite volume measure of the Ising model at inverse temperature β if $\mathbf{P}$ is an accumulation point of some sequence of finite volume measures $\{\mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta}\}_n$, for the above notion of convergence. We also call such an accumulation point a "weak limit", and write*

$$\mathcal{G}_\beta = \{\textit{Weak limits of sequences } (\mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta})_n, \textit{ with } G_n \uparrow G_\infty, \textit{ and } \overline{\sigma}_n \in \boldsymbol{\Sigma}\}$$

The physical motivation for this definition is that in a large but finite system, the measurement of local observables in the bulk should not depend on the finite-size effects, and thus should be well approximated by an infinite-volume measure.

On the other hand, the mathematical framework is exactly given by the weak convergence of measures in the compact metric space $(\boldsymbol{\Sigma}, \mathcal{F})$, where $\mathcal{F}$ is the product $\sigma-$algebra, endowed with the metric:

$$d(\sigma, \sigma') = \sum_{x \in V_\infty} 2^{-\|x\|_1} \mathbb{1}_{[\sigma_x \neq \sigma'_x]}$$

Indeed, we can easily check that the local functions are dense in the set of continuous functions for the product topology, which satisfy

$$\forall \varepsilon > 0, \, \exists A \Subset V_\infty : \sup_{\substack{\sigma,\sigma' : \\ \sigma_{|A} = \sigma'_{|A}}} |f(\sigma) - f(\sigma')| \leqslant \varepsilon.$$

We emphasize that the existence of a probability measure $\mathbf{P}$ on $(\boldsymbol{\Sigma}, \mathcal{F})$ such that $\mathbf{P}(f) = \lim_{n\to\infty} \mathbf{P}^{\overline{\sigma}_n}_{G_n,\beta}(f)$ is ensured by the Caratheodory extension theorem and the Kolmogorov theorem [12]. Indeed, the product $\sigma-$algebra $\mathcal{F}$ is generated by the cylinders

$$C_{A,\omega} = \{\sigma \in \boldsymbol{\Sigma} \, : \, \sigma_x = \omega_x, \, \forall x \in A\} \quad \text{with } A \Subset V_\infty \text{ and } \omega \in \boldsymbol{\Sigma}_A,$$

whose indicator function is a local function. Then, the function $\hat{\mathbf{P}}$ characterized by the limiting expectations of indicator of cylinders can be extended to a probability measure $\tilde{\mathbf{P}}$ on the algebra generated by the cylinders, which on its turn can be extended in a unique way to a probability measure $\mathbf{P}$ on the induced $\sigma-$algebra.

Let us still recall some general definitions about infinite graphs:

Definition 1.6

1. *A graph $G = (V, \mathcal{E})$ is of bounded degree if there exists some constant $K \in [1, \infty)$ such that any vertex $x \in V$ has a degree smaller than K.*
2. *An automorphism τ of a graph G is a permutation of the vertex set V such that the pair of vertices (x, y) form an edge if and only if the pair $(\tau(x), \tau(y))$ also form an edge. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of the graph G.*
   A graph G is called transitive if for any two vertices $x, y \in V$ there exists an automorphism $\tau \in \mathrm{Aut}(G)$ such that $\tau(x) = y$.
3. *A graph G is called periodic if $\mathrm{Aut}(G)$ contains a subgroup isomorphic to $\mathbb{Z}^d$, whose elements are called "translations", and which acts freely with a finite number of orbits. In other words, G can be embedded periodically in $\mathbb{R}^d$ in such a way that those*

"translations" correspond to Euclidean translations of $\mathbb{R}^d$*. The value of* d *is well defined and we call it the dimension of the graph. We call "fundamental domain" a set of representatives of each orbit of the vertex set under the translations.*

4. *An infinite graph* G *is said to have sub-exponential balls if*

$$\sup_{x \in V} \lim_{n \to \infty} \frac{1}{n} \log |B_n(x)| = 0.$$

5. *An infinite graph* G_∞ *is said to be Van Hove if there exists sequences of finite graphs* $G \uparrow G_\infty$ *such that,*

$$\lim_{G \uparrow G_\infty} \frac{|\partial V|}{|V|} = 0$$

As we will see, the FKG inequality allows us to prove easily the existence of the infinite volume measures with $+$,$-$ and free boundary conditions.

Proposition 1.6 *Let* G_∞ *be a locally finite graph,* $\beta \geqslant 0$*.*
The weak limits

$$\mathbf{P}^+_\beta = \lim_{G \uparrow G_\infty} \mathbf{P}^+_{G,\beta} \quad , \quad \mathbf{P}^-_\beta = \lim_{G \uparrow G_\infty} \mathbf{P}^-_{G,\beta} \quad \text{and} \quad \mathbf{P}^{\mathbf{f}}_\beta = \lim_{G \uparrow G_\infty} \mathbf{P}^{\mathbf{f}}_{G,\beta}$$

exist and are independent of the choice of the sequence $G \uparrow G_\infty$*. Moreover,*

1. $\mathbf{P}^+_\beta$ *and* $\mathbf{P}^-_\beta$ *are invariant under the automorphisms of* G_∞*,*
2. $\mathbf{P}^-_\beta \preccurlyeq \mathbf{P} \preccurlyeq \mathbf{P}^+_\beta$ *for all* $\mathbf{P} \in \mathcal{G}_\beta$*. (In particular* $\mathbf{P}^-_\beta \preccurlyeq \mathbf{P}^{\mathbf{f}}_\beta \preccurlyeq \mathbf{P}^+_\beta$*.)*

Proof We easily get the following monotonicity property: for every increasing local function f, and $G_1 = (V_1, \mathcal{E}_1) \subset G_2 = (V_2, \mathcal{E}_2) \Subset G_\infty$,

$$\mathbf{P}^+_{G_1,\beta}(f) = \mathbf{P}^+_{G_2,\beta}(f \,|\, \sigma_x \equiv +1 \text{ on } V_2 \setminus V_1) \geqslant \mathbf{P}^+_{G_2,\beta}(f)$$

As any local function is bounded, we deduce the convergence of expectations of increasing local functions, as $G \uparrow G_\infty$. As any local function can be decomposed onto increasing local functions by Proposition 1.5, this implies the weak convergence of $\mathbf{P}^+_{G,\beta}$ towards some measure $\mathbf{P}^+_\beta$. The same argument (with reversed inequality) shows the existence of the infinite measure with $-$ boundary conditions. For free boundary condition we use the GKS inequality: for any $A \subset G_1 \subset G_2 \Subset G_\infty$, using (1.5) we deduce

$$\mathbf{P}^{\mathbf{f}}_{G_1,\beta}(\sigma_A) = \mathbf{P}^{\mathbf{f}}_{G_2,\beta,J\equiv 0 \text{ on } G_2 \setminus G_1}(\sigma_A) \leqslant \mathbf{P}^{\mathbf{f}}_{G_2,\beta}(\sigma_A)$$

Proposition 1.5 must be used as well for the decomposition of local functions onto the functions σ_A. The fact that $\mathbf{P}^+_\beta, \mathbf{P}^-_\beta$ and $\mathbf{P}^{\mathbf{f}}_\beta$ do not depend on the chosen sequence of graphs is standard: suppose it does, then we can construct two growing sequences

of graphs $(G_n^1)_n$ and $(G_n^2)_n$ which lead to the limits $\mathbf{P}_1$ and $\mathbf{P}_2$. Consider an alternate growing sequence of graphs $(G_n)_n$ such that the even graphs come from $(G_n^1)_n$ and the odd ones from $(G_n^2)_n$. This new sequence leads a priori to a third limit $\mathbf{P}$. But $\mathbf{P} = \mathbf{P}_1 = \mathbf{P}_2$ because all subsequences of $(G_n)_n$ must lead to the same limit.
For the automorphism invariance, let $\tau \in \mathrm{Aut}(G_\infty)$. For any local function f,

$$\mathbf{P}^+_{G,\beta}(f) = \mathbf{P}^+_{\tau G,\beta}(\tau^{-1} f).$$

Now passing to the limit $G \uparrow G_\infty$, we get

$$\mathbf{P}^+_\beta(f) = \mathbf{P}^+_\beta(\tau^{-1} f),$$

where the last limit follows from the fact that $\mathbf{P}^+_\beta$ does not depend on the chosen sequence of graphs. The same argument (with reversed inequality) works for $-$ boundary conditions, and for free boundary condition we use the GKS inequality and the functions σ_A.
Finally, the stochastic extremality property follows easily from the FKG inequality in finite volume (Prop. 1.3). ■

1.2.3 Phase transition

The stochastic ordering of Ising measures implies the following uniqueness criterion for infinite volume measures.

Proposition 1.7 *Let G_∞ be a locally finite, periodic graph, and $\beta \geqslant 0$.*
The following statements are equivalent:

1. *There exists a unique infinite volume Gibbs measure.*
2. $\mathbf{P}^+_\beta = \mathbf{P}^-_\beta$.
3. $\mathbf{P}^+_\beta(\sigma_x) = \mathbf{P}^-_\beta(\sigma_x)$ *for all x in a fundamental domain.*

Assertions 1 and 2 are equivalent to the following modified assertion in the case of a non-periodic graph: 3′. $\mathbf{P}^+_\beta(\sigma_x) = \mathbf{P}^-_\beta(\sigma_x)$ for all x in G_∞.

Proof The equivalence $1 \Leftrightarrow 2$ follows from the stochastic ordering given by Proposition 1.6. Implication $2 \Rightarrow 3$ is trivial and $3 \Rightarrow 2$ follows from the next Lemma 1.1 together with the automorphism invariance of $\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$: if $\mathbf{P}^+_\beta(\sigma_x) = \mathbf{P}^-_\beta(\sigma_x)$ for all x in a fundamental domain, then $\mathbf{P}^+_\beta(\sigma_x) = \mathbf{P}^-_\beta(\sigma_x)$ for all $x \in G_\infty$. ■

Lemma 1.1 *Let $\mathbf{P}_1$, $\mathbf{P}_2$ be two probability measures on $\{-1,+1\}^V$ (with V some countable set). Let $\sigma = (\sigma_x)_{x\in V} \in \{-1,+1\}^V$.*
If $\mathbf{P}_1 \preccurlyeq \mathbf{P}_2$ and $\mathbf{P}_1(\sigma_x) = \mathbf{P}_2(\sigma_x)$ for all $x \in V$, then $\mathbf{P}_1 = \mathbf{P}_2$.

Proof We have seen (1.6) that the functions $\sum_{x\in A} n_x - n_A$ are increasing for all $A \Subset E$. Hence,

$$\mathbf{P}_1\left(\sum_{x\in A} n_x - n_A\right) \leqslant \mathbf{P}_2\left(\sum_{x\in A} n_x - n_A\right),$$

which implies that

$$\sum_{x\in A} \left(\mathbf{P}_2(n_x) - \mathbf{P}_1(n_x)\right) \geqslant \mathbf{P}_2(n_A) - \mathbf{P}_1(n_A) \geqslant 0.$$

If $\mathbf{P}_1(\sigma_x) = \mathbf{P}_2(\sigma_x)$ for all $x \in E$, then $\mathbf{P}_1(n_x) = \mathbf{P}_2(n_x)$ for all $x \in E$, and hence $\mathbf{P}_1(n_A) = \mathbf{P}_2(n_A)$ for all $A \Subset E$. Proposition 1.5 finishes the proof. ∎

Definition 1.7 *By the FKG inequality the following limit always exists and is called the average magnetization under* $\mathbf{P}^+_\beta$*:*

$$m^\star_\beta \doteqdot \lim_{G\uparrow G_\infty} \mathbf{P}^+_{G,\beta}\left(\frac{1}{|G|}\sum_{x\in V} \sigma_x\right).$$

Note that by symmetry the average magnetization under the measure $\mathbf{P}^-_\beta$ *equals* $-m^\star_\beta$.

We will show later in a more general context (see Proposition 1.26), that if G_∞ is a locally finite, Van Hove, transitive graph, then

$$\mathbf{P}^+_\beta(\sigma_0) = \lim_{G\uparrow G_\infty} \mathbf{P}^+_{G,\beta}\left(\frac{1}{|G|}\sum_{x\in V} \sigma_x\right)$$

whenever G is a Van Hove sequence. The same holds for $\mathbf{P}^-_\beta$. This allows us to use the name “average magnetization” for the quantities $\mathbf{P}^+_\beta(\sigma_0)$ and $\mathbf{P}^-_\beta(\sigma_0)$ as well.

Proposition 1.7 allows to define a critical inverse temperature β_c above which there are more than one unique infinite volume Gibbs measure. Indeed, by symmetry $\mathbf{P}^+_\beta(\sigma_x) = -\mathbf{P}^-_\beta(\sigma_x)$, by the FKG inequality $\mathbf{P}^+_\beta(\sigma_x) \geqslant \mathbf{P}^-_\beta(\sigma_x)$, and the consequence (1.5) of the GKS inequalities ensures that $\mathbf{P}^+_\beta(\sigma_x) - \mathbf{P}^-_\beta(\sigma_x)$ is increasing in β.

Definition 1.8 *The critical inverse temperature of the Ising model is defined as*

$$\beta_c \doteqdot \sup\{\beta \geqslant 0 : \mathbf{P}^+_\beta = \mathbf{P}^-_\beta\} = \sup\{\beta \geqslant 0 : m^\star_\beta = 0\}.$$

There exist several methods to prove that $0 < \beta_c < \infty$, namely non-uniqueness at low temperature for periodic graphs of dimension $d \geqslant 2$. One can mention the Peierls argument [75], or the “disagreement percolation” method [85]. Another one is detailed in Proposition 1.20 and Corollary 1.6, which has the advantage of treating the Potts models as well.

1.2.4 Planar techniques

In this section we treat only the case of finite planar graphs (embedded in $\mathbb{R}^2$). The Ising model presents a duality property, which was first noticed by Kramers and Wannier [62, 63]. It links certain properties of the Ising model at low temperature to properties at high temperature, and is an important tool for the non-perturbative analysis of the Ising model. The presence of such a duality is one of the main reasons why the model is much better understood in dimension 2 than in higher dimensions.

Let $G = (V, \mathcal{E})$ be a (simply connected) planar graph. Its dual $G^\star = (V^\star, \mathcal{E}^\star)$ is constructed as follows: place a dual vertex within each face of G (including the infinite face), forming the set of dual vertices $V^\star$, and place a dual edge $e^\star$ joining two dual vertices whenever they are placed on adjacent primal faces. This forms the set of dual edges $\mathcal{E}^\star$. Note that $V^\star$ is in one-to-one correspondence with the set of faces of G and $\mathcal{E}^\star$ is in one-to-one correspondence with $\mathcal{E}$.

For our purposes here, we will consider the dual vertex on the infinite face as splited into as many vertices as incoming half-edges, see Figure 1.2. We call this set of vertices $\partial V^\star$.

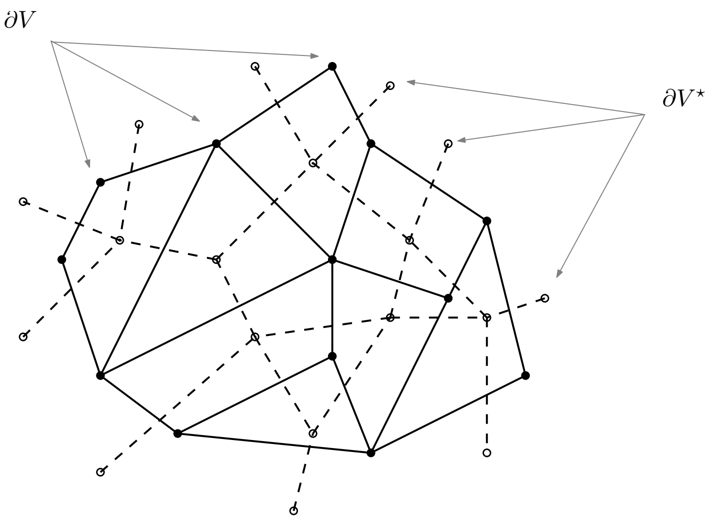
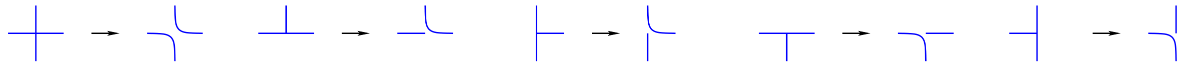


Figure 1.2 – A planar graph and its dual. Primal edges $\mathcal{E}$ are drawn with solid lines, while dual edges $\mathcal{E}^\star$ are drawn with dashed lines.

1.2.4.1 Low temperature representation

Let $\overline{\sigma}$ be some boundary condition. To a configuration σ compatible with this boundary condition, we associate the set $\mathcal{E}^\star(\sigma)$ of all edges of the dual graph separating a pair i, j of nearest-neighbor vertices of G such that and $\sigma_i \neq \sigma_j$. The set of edges $\mathcal{E}^\star(\sigma)$ can be decomposed into a family of self-avoiding lines by applying a fixed deformation rules at each vertex of the dual graph at which more than two edges of $\mathcal{E}^\star$ meet. Here is the case of $\mathbb{Z}^2$:



where the last four deformation rules can only occur on the boundary of the box. Each of these lines is called a *contour* of σ and is denoted γ. Its length is denoted $|\gamma|$. We write $\bar{\Gamma}(\sigma)$ for the set of all *closed contours* of σ, i.e. the closed lines.

Of particular interest will be the *open contours* $\Gamma(\sigma) = (\lambda_1(\sigma), \ldots, \lambda_M(\sigma))$ of the configuration σ, i.e., the open lines. Observe that each of those has its two endpoints on $\partial V^\star$. The set $\mathbf{b}(\overline{\sigma}) \equiv \{b_1, \ldots, b_{2M}\}$ of all endpoints of open contours is completely determined by the boundary condition $\overline{\sigma}$, see Figure 1.3. The notation $\partial\Gamma = \mathbf{b}(\overline{\sigma})$ means that the set of open contours Γ is compatible with $\mathbf{b}(\overline{\sigma})$ (i.e., the set of endpoints of Γ is $\mathbf{b}(\overline{\sigma})$). We sometimes use the notation $\lambda : b \to b'$ in place of $\partial\lambda = \{b, b'\}$.

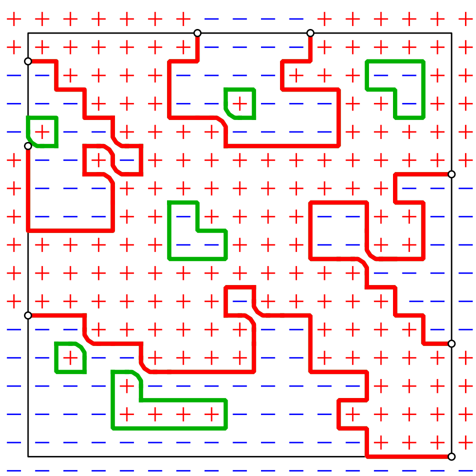


Figure 1.3 – The contour representation of the Ising model on $\mathbb{Z}^2$. The open contours are drawn in red, the closed contours in green. The set of endpoints of open contours (locations of the spin changes) is drawn with void dots.

We also say that a family of contours $\boldsymbol{\gamma}$ is $(\overline{\sigma}, G)$-*compatible*, and write $\boldsymbol{\gamma} \sim (\overline{\sigma}, G)$ if there exists a configuration σ in G, compatible with the boundary condition $\overline{\sigma}$, such that $\boldsymbol{\gamma}$ is the family of contours of σ. The sum of the length of a family of contours $\boldsymbol{\gamma}$ is denoted $|\boldsymbol{\gamma}|$.

Note that by rewriting the Hamiltonian (1.1) as

$$\mathcal{H}^{\overline{\sigma}}_{G,\beta}(\sigma) = -\beta \left(\sum_{\substack{i,j \in V \setminus \partial V \\ [i,j] \in \mathcal{E}}} (\sigma_i \sigma_j - 1) + \sum_{\substack{i \in V, j \in \partial V \\ [i,j] \in \mathcal{E}}} (\sigma_i \overline{\sigma} - 1) \right) - \beta |\mathcal{E}|$$

and by observing that $\sigma_i\sigma_j - 1 \neq 0$ if and only if $\sigma_i \neq \sigma_j$, i.e. iff the vertices i and j are separated by a contour, we deduce:

$$\begin{aligned} \mathbf{Z}^{\overline{\sigma}}_{G,\beta} &= e^{\beta|\mathcal{E}|} \sum_{\sigma\in\boldsymbol{\Sigma}^{\overline{\sigma}}_V} \Bigg(\prod_{\gamma\in\bar{\boldsymbol{\Gamma}}(\sigma)} e^{-2\beta|\gamma|}\Bigg)\Bigg(\prod_{\lambda\in\boldsymbol{\Gamma}(\sigma)} e^{-2\beta|\lambda|}\Bigg) \\ &= e^{\beta|\mathcal{E}|} \underbrace{\sum_{\sigma\in\boldsymbol{\Sigma}^{\overline{\sigma}}_V} \exp(-2\beta|\bar{\boldsymbol{\Gamma}}(\sigma)|)\cdot\exp(-2\beta|\boldsymbol{\Gamma}(\sigma)|)}_{\doteqdot\mathcal{Z}^{\overline{\sigma}}(G)}. \end{aligned} \tag{1.7}$$

More generally, given any $(\overline{\sigma}, G)$-compatible family $\boldsymbol{\lambda}$ of open contours, we set

$$\mathcal{Z}^{\overline{\sigma}}(G|\boldsymbol{\lambda}) = \sum_{\substack{\sigma\in\boldsymbol{\Sigma}^{\overline{\sigma}}_V\\ \boldsymbol{\Gamma}(\sigma)=\boldsymbol{\lambda}}} \exp(-2\beta|\bar{\boldsymbol{\Gamma}}(\sigma)|).$$

We define the weight $q^{\overline{\sigma}}_{G,\beta}(\boldsymbol{\lambda})$ of an arbitrary family of open contours $\boldsymbol{\lambda}$ as follows

$$q^{\overline{\sigma}}_{G,\beta}(\boldsymbol{\lambda}) = \begin{cases} \exp(-2\beta|\boldsymbol{\lambda}|)\cdot\frac{\mathcal{Z}^{\overline{\sigma}}(G|\boldsymbol{\lambda})}{\mathcal{Z}^{+}(G)} & \text{if } \partial\boldsymbol{\lambda} = \mathbf{b}(\overline{\sigma}) \text{ and } \boldsymbol{\lambda}\sim(\overline{\sigma}, G) \\ 0 \quad \text{otherwise.} \end{cases} \tag{1.8}$$

Note that the weights $q^{\overline{\sigma}}_{G,\beta}(\boldsymbol{\lambda})$ do not define a probability measure on the set of $(\overline{\sigma}, G)$-compatible open contours since above we divide by $\mathcal{Z}^{+}(G)$ and not by $\mathcal{Z}^{\overline{\sigma}}(G)$.

1.2.4.2 High temperature representation

Consider free boundary conditions. The elementary identity

$$e^{\beta\sigma_i\sigma_j} = \cosh\beta + \sigma_i\sigma_j\sinh\beta = \cosh\beta(1+\sigma_i\sigma_j\tanh\beta)$$

is used to derive a second useful representation of the partition function.

$$\begin{aligned} \mathbf{Z}^{\mathbf{f}}_{G,\beta} = \sum_{\sigma\in\boldsymbol{\Sigma}^{\mathbf{f}}_V} \prod_{\substack{i,j\in V\\ \{i,j\}\in\mathcal{E}}} e^{\beta\sigma_i\sigma_j} &= (\cosh\beta)^{|\mathcal{E}|} \sum_{\sigma\in\boldsymbol{\Sigma}^{\mathbf{f}}_V} \prod_{\substack{i,j\in V\\ \{i,j\}\in\mathcal{E}}} (1+\sigma_i\sigma_j\tanh\beta) \\ &= 2^{|V|}(\cosh\beta)^{|\mathcal{E}|} \underbrace{\sum_{\partial\bar{\boldsymbol{\Gamma}}=\varnothing} (\tanh\beta)^{|\bar{\boldsymbol{\Gamma}}|}}_{\doteqdot\mathcal{Z}(G)} \end{aligned} \tag{1.9}$$

Indeed, after the summation over σ, only terms labelled by sets of bonds with empty boundary give a non-zero contribution. And any term of the expansion which gives a non-zero contribution can be uniquely labelled by a family $\bar{\boldsymbol{\Gamma}}$ of closed contours which are compatible with the graph G (and the free boundary condition).

More generally, given any $(\mathbf{f}, G)$-compatible family $\boldsymbol{\gamma}$ of (non-necessary closed) contours, we set

$$\mathcal{Z}(G|\boldsymbol{\gamma}) = \sum_{\substack{\bar{\boldsymbol{\Gamma}}:\,\partial\bar{\boldsymbol{\Gamma}}=\varnothing\\ \bar{\boldsymbol{\Gamma}}\vee\boldsymbol{\gamma}\sim(\mathbf{f},G)}} (\tanh\beta)^{|\bar{\boldsymbol{\Gamma}}|}$$

where $\bar{\Gamma} \vee \boldsymbol{\gamma}$ stands for the concatenation of the two families of contours, using the deformation rule explained at the beginning of the section. We define the weight $q_{G,\beta}(\boldsymbol{\gamma})$ of an arbitrary family of contours $\boldsymbol{\gamma}$ as follows

$$q_{G,\beta}(\boldsymbol{\gamma}) = \begin{cases} (\tanh\beta)^{|\boldsymbol{\gamma}|} \cdot \frac{\mathcal{Z}(G|\boldsymbol{\gamma})}{\mathcal{Z}(G)} & \text{if } \boldsymbol{\gamma} \sim (\mathbf{f}, G) \\ 0 \quad \text{otherwise.} \end{cases} \tag{1.10}$$

1.2.4.3 Kramers-Wannier duality and random-line representation of the even correlation functions

The Kramers-Wannier duality relates the high and low temperature expansions by the following observations:

Proposition 1.8 (Kramers-Wannier duality)
Let $G = (V, \mathcal{E})$ *and* $G^\star = (V^\star, \mathcal{E}^\star)$ *be a finite graph and its dual.*
Let $\beta^\star$ *be the solution of*

$$\tanh\beta^\star = e^{-2\beta}.$$

Then,

$$2^{-|V^\star|}(\cosh\beta^\star)^{-|\mathcal{E}^\star|} \cdot \mathbf{Z}^{\mathbf{f}}_{G^\star,\beta^\star} = e^{-\beta|\mathcal{E}|} \cdot \mathbf{Z}^{+}_{G,\beta}$$

Moreover, let $\overline{\sigma}$ *be a boundary condition. Then the set of* $(\overline{\sigma}, G)$*-compatible families of contours coincides with the set of compatible families of contours* $\boldsymbol{\gamma}$ *of the graph* $G^\star$ *such that* $\partial\boldsymbol{\gamma} = \mathbf{b}(\overline{\sigma})$. *Namely, let* $\boldsymbol{\lambda}$ *be a family of open contours such that* $\partial\boldsymbol{\lambda} = \mathbf{b}(\overline{\sigma})$, *then*

$$q^{\overline{\sigma}}_{G,\beta}(\boldsymbol{\lambda}) = q_{G^\star,\beta^\star}(\boldsymbol{\lambda}).$$

Proof The first assertion comes from the comparison between (1.7) and (1.9), and the second one from the comparison between (1.8) and (1.10). ■

The usefulness of the definition of the weights $q_{G,\beta}(\boldsymbol{\lambda})$ comes from the following representation of the correlation function $\mathbf{P}^{\mathbf{f}}_{G,\beta}(\prod_{x\in A}\sigma_x)$. If the cardinality of A is odd, then by symmetry this correlation function vanishes. If $|A| = 2m$, for some $m \geqslant 1$, then by expanding the numerator of this correlation function as in (1.9), we observe that the presence of the variables σ_x, $x \in A$, implies that the only terms which give non-zero contributions, are those labelled by compatible families of contours containing a sub-family $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_m\}$ of m open contours such that $\partial\boldsymbol{\lambda} = A$. By summing over all closed contours for such a given family $\boldsymbol{\lambda}$, we get a contribution to the numerator which is exactly $(\tanh\beta)^{|\boldsymbol{\lambda}|}\mathcal{Z}(G|\boldsymbol{\lambda})$.

This observation implies the following *random-line representation* for the even correlation functions:

$$\mathbf{P}^{\mathbf{f}}_{G,\beta}\left(\prod_{x\in A}\sigma_x\right) = \sum_{\boldsymbol{\lambda}\,:\,\partial\boldsymbol{\lambda}=A} q_{G,\beta}(\boldsymbol{\lambda})$$

And we deduce the following duality, which comes from Proposition 1.8, which we also call "random-line representation" henceforth.

Proposition 1.9 (Random-line representation)
Let $G = (V, \mathcal{E})$ and $G^\star = (V^\star, \mathcal{E}^\star)$ be a finite planar graph and its dual, $\beta^\star$ the solution of $\tanh \beta^\star = e^{-2\beta}$.
Let $\overline{\sigma}$ be a boundary condition on G and $\mathbf{b}(\overline{\sigma}) \subseteq \partial V^\star$ be the set of locations of spin changes along the boundary of G. Then,

$$\frac{Z^{\overline{\sigma}}_{G,\beta}}{Z^{+}_{G,\beta}} = \sum_{\boldsymbol{\lambda}\,:\,\partial\boldsymbol{\lambda}=\mathbf{b}(\overline{\sigma})} q^{\overline{\sigma}}_{G,\beta}(\boldsymbol{\lambda}) = \mathbf{P}^{\mathbf{f}}_{G^\star,\beta^\star}\Bigl(\prod_{x\in\mathbf{b}(\overline{\sigma})} \sigma_x\Bigr)$$

One of the interests of the random-line representation is that the weights $q_{G,\beta}$ have number of useful properties which follow essentially from the GKS inequalities. We give here some examples, with precise references to where a proof can be found.

As before, let $G = (V, \mathcal{E})$ and $G^\star = (V^\star, \mathcal{E}^\star)$ be a finite graph and its dual. Then,

- Let $G_1 \subset G_2$ and $\boldsymbol{\Gamma}$ a family of contours on $\mathcal{E}^\star_1$. Then

$$q_{G_1,\beta}(\boldsymbol{\Gamma}) \geqslant q_{G_2,\beta}(\boldsymbol{\Gamma}). \tag{1.11}$$

  See [77, Lemma 6.3] for the proof.

- Let $\mathbf{b}_1, \mathbf{b}_2$ be two disjoint subsets of even cardinality of $\partial V^\star$. The weights satisfy the following BK-type inequality,

$$\sum_{\substack{\boldsymbol{\Gamma}_1,\boldsymbol{\Gamma}_2:\\ \partial\boldsymbol{\Gamma}_1=\mathbf{b}_1,\partial\boldsymbol{\Gamma}_2=\mathbf{b}_2}} q_{G,\beta}(\boldsymbol{\Gamma}_1 \vee \boldsymbol{\Gamma}_2) \leqslant \sum_{\boldsymbol{\Gamma}_1\,:\,\partial\boldsymbol{\Gamma}_1=\mathbf{b}_1} q_{G,\beta}(\boldsymbol{\Gamma}_1) \sum_{\boldsymbol{\Gamma}_2\,:\,\partial\boldsymbol{\Gamma}_2=\mathbf{b}_2} q_{G,\beta}(\boldsymbol{\Gamma}_2). \tag{1.12}$$

  See [77, Lemma 6.5] for the proof.

- Let us associate to a $(\overline{\sigma}, G)$-compatible family of open contours the set $\mathfrak{F}(\Gamma_1, \dots, \Gamma_n)$ of all vertices of G whose spin value is completely determined by $\overline{\sigma}$ and these open contours, i.e., the maximal set such that, if σ' is another configuration compatible with $\overline{\sigma}$ such that $\Gamma_1, \dots, \Gamma_n \subset \boldsymbol{\Gamma}(\sigma')$, then $\sigma'_i = \sigma_i$, for all $i \in \mathfrak{F}(\Gamma_1, \dots, \Gamma_n)$. We set $G(\Gamma_1, \dots, \Gamma_n) \doteqdot G \setminus \mathfrak{F}(\Gamma_1, \dots, \Gamma_n)$, and say that $\Gamma_1, \dots, \Gamma_n$ *partition the box* G into the connected components of $G(\Gamma_1, \dots, \Gamma_n)$. Then,

$$q_{G,\beta}(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}, \dots, \Gamma_m) = q_{G,\beta}(\Gamma_1, \dots, \Gamma_n)\, q_{G(\Gamma_1,\dots,\Gamma_n),\beta}(\Gamma_{n+1}, \dots, \Gamma_m), \tag{1.13}$$

  for all $(\overline{\sigma}, G)$-compatible family $\Gamma_1, \dots, \Gamma_m \subset \boldsymbol{\Gamma}(\sigma)$ of open contours.
  See [77, Lemma 6.4] for the proof.

More properties of the weights will be described in Section 1.5.7. They are based on estimates of the two point correlation functions. We treat this topic together with the Potts models in the following sections.

1.3 Edwards-Sokal coupling between Potts and Random-Cluster models

There are much less tools in terms of the spins for the Potts model than for the Ising model (the Hamiltonian is invariant under the permutations of the q colors and for example no natural partial order allows to compare them). But an extremely useful coupling, introduced by Edwards and Sokal in 1988 [33], allows to express magnetization properties in the Potts model (on a general graph G) as percolation properties in an appropriate model of random subgraphs of G, called the "random-cluster" model. The latter was introduced by Fortuin and Kasteleyn in the seventies [34], and we define it now.

1.3.1 The random cluster model

Consider a finite graph $G = (V, \mathcal{E})$ as before. The configuration space is $\Omega_{\mathcal{E}} \equiv \{0,1\}^{\mathcal{E}}$. A configuration in $\Omega_{\mathcal{E}}$ is denoted $\omega = \{\omega(e)\}_{e\in\mathcal{E}}$. It associates to each edge e a number in $\{0,1\}$. The edge e is called *open* in the configuration ω if $\omega(e) = 1$, it is called *closed* if $\omega(e) = 0$.
Let $\overline{\omega}$ be a boundary condition, i.e. a wiring of the vertices on ∂V. When G is a subgraph of an infinite graph G_∞, we will consider $\overline{\omega}$ to be a configuration on $G_\infty\backslash G$.
Let $\kappa(\omega, \overline{\omega})$ be the number of connected components (also called *clusters*) of the configuration ω, including isolated sites, and counted according to the connections of the boundary condition $\overline{\omega}$. This notation will be often simplified to $\kappa(\omega)$ when the boundary condition is implicitly fixed.
Let $|\omega|$ denote the cardinality of the set of edges of ω seen as a subgraph of G, i.e. the number of open edges of the configuration ω. We write $|\mathcal{E}|$ to denote the number of edges of G.
We define a probability measure on the set $\Omega_{\mathcal{E}}^{\overline{\omega}}$ of configurations which have additional edges between the vertices of ∂V according to the connections prescribed by $\overline{\omega}$. This measure depends on two real parameters $p \in [0,1]$ and $q \in (0,\infty)$:

$$\begin{aligned}\mu^{\overline{\omega}}_{G,p,q}(\omega) &= \frac{1}{Z^{\overline{\omega}}_{G,p,q}} \prod_{e\in\mathcal{E}} \left[p^{\omega(e)}(1-p)^{1-\omega(e)}\right] q^{\kappa(\omega,\overline{\omega})} \\ &= \frac{1}{Z^{\overline{\omega}}_{G,p,q}}\, p^{|\omega|}\,(1-p)^{|\mathcal{E}|-|\omega|}\, q^{\kappa(\omega,\overline{\omega})}.\end{aligned}$$

Note that:

- For $q = 1$, we recover the well known Bernoulli bond Percolation model (see [52]). It is the only parameter for which the variables $\omega(e)$ are independent. The model with $q \in \mathbb{N}^\star$ is related to the q-state Potts model, as we will see.
- We define two "extremal" boundary conditions: $\overline{\omega} \equiv 0$ is called *free* boundary condition (vertices on ∂V are not connected together outside G), we write $\mu^{f}_{G,p,q}$, whereas $\overline{\omega} \equiv 1$ is called "wired" boundary condition (all vertices in ∂V are in the same connected component), we write $\mu^{w}_{G,p,q}$.

1.3.2 The coupling

We present here the Edwards-Sokal coupling between the q-state Potts model, at inverse temperature β, with boundary condition "1", and the random-cluster model with parameters $p = 1 - e^{-\beta}$ and q, and wired boundary condition. More general boundary conditions will be discussed in Section 2.4.3.
Let $q \in \{2, 3, 4, ...\}$, $p \in [0, 1]$, and $G = (V, \mathcal{E})$ as before. Consider the product space $\Sigma_V^1 \times \Omega_{\mathcal{E}}^w = \{1, 2, 3, ...q\}^{V \setminus \partial V} \times \{0, 1\}^{\mathcal{E}}$, with boundary condition 1 on ∂V whose vertices are all connected to each other. We define a probability measure on this space, which is called Edwards-Sokal coupling:

$$\begin{aligned} \nu(\sigma, \omega) &= \frac{1}{Z} \prod_{e \in \mathcal{E}} (1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma) \\ &= \frac{1}{Z} p^{|\omega|} (1-p)^{|\mathcal{E}|-|\omega|} \prod_{e\,:\,\omega(e)=1} \delta_e(\sigma), \end{aligned} \tag{1.14}$$

where $\delta_e(\sigma) = \delta_{\sigma_i,\sigma_j}$ if $e = [i, j] \in \mathcal{E}$ and Z is defined such that $\sum_{\sigma,\omega} \nu(\sigma, \omega) = 1$. Note that ν can be seen as the product measure

$$\left(\bigotimes_{i \in V \setminus \partial V} \mathrm{Uniform}_{\{1,...q\}} \right) \otimes \left(\bigotimes_{e \in \mathcal{E}} \mathrm{Bernoulli}(p) \right)$$

conditioned on the event $C \equiv \{\delta_e(\sigma) = 1 \text{ for all } e \text{ such that } \omega(e) = 1\}$.

Proposition 1.10 *Let $p = 1 - e^{-\beta} \in [0, 1)$, then:*

1. *The marginal of ν on Σ_V^1 is the Potts measure:*

$$\sum_{\omega} \nu(\sigma, \omega) = \frac{1}{Z^1_{G,\beta,q}} \exp\left(\beta \sum_{e \in \mathcal{E}} \delta_e(\sigma) \right),$$

   where the right-hand side is a simplified notation which takes also into account the "1" boundary condition on ∂V.

2. *The marginal of ν on $\Omega_{\mathcal{E}}^w$ is the random-cluster measure, i.e.:*

$$\sum_{\sigma} \nu(\sigma, \omega) = \frac{1}{Z^w_{G,p,q}} \left(\prod_{e \in \mathcal{E}} p^{\omega(e)} (1-p)^{1-\omega(e)} \right) q^{\kappa(\omega)}.$$

3. *The normalization constants satisfy: $Z^w_{G,p,q} = e^{-\beta|\mathcal{E}|} Z^1_{G,\beta,q}$.*

Moreover,

1. *For $\omega \in \Omega^{w}_{\mathcal{E}}$, the conditional measure $\nu(\cdot|\omega)$ on Σ^1_V is obtained by coloring the vertices as follows:*

$$\begin{cases} \text{color "1" on the cluster connected to } \partial V, \\ \text{constant color on each remaining cluster, distributed uniformly on } \{1, \dots q\}, \\ \text{independent between clusters.} \end{cases}$$

2. *For $\sigma \in \Sigma^1_V$, the conditional measure $\nu(\cdot|\sigma)$ on $\Omega^{w}_{\mathcal{E}}$ is obtained as follows: independently for each edge $e = [i, j] \in \mathcal{E}$,*

$$\begin{cases} \text{If } \sigma_i \neq \sigma_j, \text{ then } \omega(e) = 0, \\ \text{If } \sigma_i = \sigma_j, \text{ then } \omega(e) = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p. \end{cases} \end{cases}$$

Proof Let us compute the first marginal:

$$\begin{aligned} \nu(\sigma) = \sum_{\omega} \nu(\sigma, \omega) &= \frac{1}{Z} \sum_{\omega} \prod_{e} \big((1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\big) \\ &= \frac{1}{Z} \prod_{e} ((1-p) + p\delta_e(\sigma)) \\ &= \frac{1}{Z} \prod_{e} \big(e^{-\beta} + (1-e^{-\beta})\delta_e(\sigma)\big) = \frac{1}{Z} \prod_{e} e^{\beta(\delta_e(\sigma)-1)} \\ &= \frac{e^{-\beta|\mathcal{E}|}}{Z} \prod_{e} e^{\beta\delta_e(\sigma)} = \mathbb{P}^1_{G,\beta,q}(\sigma), \end{aligned}$$

where in the last line we used $1 = \frac{e^{-\beta|\mathcal{E}|}}{Z} Z^1_{G,\beta,q}$ (obtained by integrating over σ). For the second marginal we have:

$$\begin{aligned} \nu(\omega) = \sum_{\sigma} \nu(\sigma, \omega) &= \frac{1}{Z} \sum_{\sigma} \prod_{e} \big((1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\big) \\ &= \frac{1}{Z} \sum_{\sigma} \prod_{e} \big((1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\big) \mathbb{1}_{[\delta_e(\sigma)=1 \text{ if } \omega(e)=1]} \\ &= \frac{1}{Z} \prod_{e} (1-p)^{1-\omega(e)} p^{\omega(e)} q^{\kappa(\omega)} = \mu^{w}_{G,p,q}(\omega), \end{aligned}$$

where in the last line we used $1 = \frac{Z^{w}_{G,p,q}}{Z}$ (obtained by integrating over ω). The two last remarks show that $Z^{w}_{G,p,q} = e^{-\beta|\mathcal{E}|} Z^1_{G,\beta,q}$. Let us pick the simplest expressions for:

$$\nu(\sigma, \omega) = \frac{1}{Z} \prod_{e} \big((1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\big) \mathbb{1}_{[\delta_e(\sigma)=1 \text{ if } \omega(e)=1]},$$

$$\nu(\omega) = \frac{1}{Z} \prod_{e} (1-p)^{1-\omega(e)} p^{\omega(e)} q^{\kappa(\omega)} \quad \text{and} \quad \nu(\sigma) = \frac{1}{Z} \prod_{e} ((1-p) + p\delta_e(\sigma)),$$

so that we can divide easily and obtain

$$\nu(\sigma \mid \omega) = \frac{\mathbb{1}_{[\sigma \text{ is constant on the open clusters of } \omega]}}{q^{\kappa(\omega)}}$$

$$\nu(\omega \mid \sigma) = \prod_e \frac{(1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)}{(1-p) + p\delta_e(\sigma)}.$$

which are the announced conditional measures. ■

1.3.3 Potts two-point correlations and magnetization are Random-Cluster connexions and exit probabilities

Let us define the following events

$$\{x \leftrightarrow y\} = \{\text{there exists some path of open edges in } G \text{ connecting } x \text{ to } y\}$$

$$\{x \leftrightarrow \partial V\} = \bigcup_{y \in \partial V} \{x \leftrightarrow y\}.$$

Then, the probability of seeing two vertices having the same color in the Potts model under $\mathbb{P}^1_{G,\beta,q}$ is directly related to the probability of seeing these two vertices connected in the random-cluster model. Similarly, the probability of seeing a vertex having the color "1" in the Potts model is directly related to the probability of seeing this vertex connected to the boundary in the random-cluster model:

Proposition 1.11 *1. For all* $x, y \in V$,

$$\frac{\mathbb{P}^1_{G,\beta,q}(\sigma_x = \sigma_y) - 1/q}{1 - 1/q} = \mu^w_{G,p,q}(x \leftrightarrow y).$$

2. For all $x \in V$,

$$\frac{\mathbb{P}^1_{G,\beta,q}(\sigma_x = 1) - 1/q}{1 - 1/q} = \mu^w_{G,p,q}(x \leftrightarrow \partial G).$$

Proof We have indeed, using the Edwards-Sokal coupling:

$$\begin{aligned}
\mathbb{P}^1_{G,\beta,q}\left(\delta_{\sigma_x,\sigma_y}\right) &= \sum_\sigma \left(\delta_{\sigma_x,\sigma_y}\right) \mathbb{P}^1_{G,\beta,q}(\sigma) = \sum_\sigma \sum_\omega \left(\delta_{\sigma_x,\sigma_y}\right) \nu(\sigma,\omega) \\
&= \sum_\omega \underbrace{\sum_\sigma \left(\delta_{\sigma_x,\sigma_y}\right) \nu(\sigma \mid \omega)}_{\mathbb{1}_{[x\leftrightarrow y]} + \frac{1}{q}\mathbb{1}_{[x\not\leftrightarrow y]}} \mu^w_{G,p,q}(\omega) \\
&= \mu^w_{G,p,q}(x \leftrightarrow y) + \frac{1}{q}\mu^w_{G,p,q}(x \not\leftrightarrow y) \\
&= \left(1 - \frac{1}{q}\right) \mu^w_{G,p,q}(x \leftrightarrow y) + \frac{1}{q}.
\end{aligned}$$

For the second assertion, we notice that

$$\sum_{\sigma} \delta_{\sigma_x,1}\nu(\sigma\,|\,\omega) = \mathbb{1}_{[x\leftrightarrow\partial G]} + \frac{1}{q}\,\mathbb{1}_{[x\not\leftrightarrow\partial G]},$$

which allows the same conclusion with the corresponding event. ■

1.4 Results for the Random-Cluster model

1.4.1 Basic properties on finite graphs

We list here and prove some useful properties of the random-cluster model on a finite graph G. The model is well defined for any positive value of the parameter q. Almost all the properties described below are valid for $q \geqslant 1$ (which is the domain of validity of the FKG inequality, see below). In our work, the random-cluster model constituting a tool for studying the Potts model, we will use it for integer values of q in the next three chapters.

1.4.1.1 The finite energy property

The random-cluster models satisfy a very simple property. The conditional probability for an edge to be open, knowing the states of all the other edges, is bounded away from 0 and 1 uniformly in $p \in (\varepsilon, 1-\varepsilon)$ and in the configuration outside this edge, for all $\varepsilon \in (0, 1/2)$. This property extends to any finite family of edges. Indeed, we have more precisely the following

Proposition 1.12 (Finite energy property)
Let $e = [x, y] \in \mathcal{E}$, $p \in (0, 1)$ and $q \in (0, \infty)$. Then, uniformly in $\omega' \in \Omega^{\varpi}_{\mathcal{E}}$, we have

$$\mu^{\varpi}_{G,p,q}\left(\omega(e) = 1 \middle|\, \omega \equiv \omega' \text{ on } \mathcal{E}\backslash\{e\}\right) = \begin{cases} p & \text{if } x \overset{\mathcal{E}\backslash\{e\}}{\longleftrightarrow} y \text{ in } \omega \\ \frac{p}{p+q(1-p)} & \text{otherwise.} \end{cases}$$

Proof Let us write ω^e (resp. ω_e) for the configuration ω restricted to the edge e, where e is open (resp. closed). Let $\vee$ denote the concatenation of edge configurations. We have by definition,

$$\begin{aligned}\mu^{\varpi}_{G,p,q}\left(\omega(e) = 1 \middle|\, \omega \equiv \omega' \text{ on } \mathcal{E}\backslash\{e\}\right) &= \frac{\mu^{\varpi}_{G,p,q}(\omega^e \vee \omega')}{\mu^{\varpi}_{G,p,q}(\omega^e \vee \omega') + \mu^{\varpi}_{G,p,q}(\omega_e \vee \omega')} \\ &= \frac{\left(\frac{p}{1-p}\right)^{|\omega^e\vee\omega'|} q^{\kappa(\omega^e\vee\omega')}}{\left(\frac{p}{1-p}\right)^{|\omega^e\vee\omega'|} q^{\kappa(\omega^e\vee\omega')} + \left(\frac{p}{1-p}\right)^{|\omega_e\vee\omega'|} q^{\kappa(\omega_e\vee\omega')}}\end{aligned}$$

Obviously we have $|\omega^e| = |\omega_e| + 1$, and

$$\kappa(\omega^e \vee \omega') = \begin{cases} \kappa(\omega_e \vee \omega') & \text{if } x \xleftrightarrow{\mathcal{E}\setminus\{e\}} y \text{ in } \omega \\ \kappa(\omega_e \vee \omega') - 1 & \text{otherwise} \end{cases}$$

Hence,

$$\mu^{\overline{\omega}}_{G,p,q}\left(\omega(e) = 1 |\ \omega \equiv \omega' \text{ on } \mathcal{E}\setminus\{e\}\right) = \begin{cases} \frac{p/(1-p)}{p/(1-p)+1} = p & \text{if } x \xleftrightarrow{\mathcal{E}\setminus\{e\}} y \text{ in } \omega \\ \frac{p/(1-p)}{p/(1-p)+q} = \frac{p}{p+q(1-p)} & \text{otherwise.} \end{cases}$$

■

1.4.1.2 The spatial Markov property

Like the Ising model, the random-cluster measures satisfy the spatial Markov property in the following sense.

Proposition 1.13 (Spatial Markov property)
Let $p \in [0,1]$ and $q > 0$. For all subgraph G' of G and for all boundary condition $\overline{\omega}$ outside G, we have

$$\mu^{\overline{\omega}}_{G,p,q}(\omega \,|\, \omega \equiv \omega'' \text{ on } G\setminus G') = \mu^{\omega''\vee\overline{\omega}}_{G',p,q}(\omega)$$

Proof Let ω' denote the configuration ω restricted to G', so that the left hand side asks for $\omega = \omega' \vee \omega'' \vee \overline{\omega}$. We have

$$\mu^{\overline{\omega}}_{G,p,q}(\omega \,|\, \omega \equiv \omega'' \text{ on } G\setminus G') = \frac{\mu^{\overline{\omega}}_{G,p,q}(\omega' \vee \omega'')}{\sum_{\tilde{\omega}} \mu_{G,p,q}(\tilde{\omega} \vee \omega'')} =$$

$$\frac{\left(\frac{p}{1-p}\right)^{|\omega'|+|\omega''|} q^{\kappa(\omega'\vee\omega'',\overline{\omega})}}{\sum_{\tilde{\omega}} \left(\frac{p}{1-p}\right)^{|\tilde{\omega}|+|\omega''|} q^{\kappa(\tilde{\omega}\vee\omega'',\overline{\omega})}} = \frac{\left(\frac{p}{1-p}\right)^{|\omega'|} q^{\kappa(\omega',\omega''\vee\overline{\omega})}}{\sum_{\tilde{\omega}} \left(\frac{p}{1-p}\right)^{|\tilde{\omega}|} q^{\kappa(\tilde{\omega},\omega''\vee\overline{\omega})}} = \mu^{\omega''\vee\overline{\omega}}_{G',p,q}(\omega)$$

■

Note that this spatial Markov property is of a different type from the one satisfied by the Ising and Potts models. Indeed, the information on $\omega'' \vee \overline{\omega}$ cannot be reduced to the one on edges outcoming from G'. The random-cluster model is not of nearest neighbor range as its Potts counterpart.

1.4.1.3 The FKG inequality

There is a natural partial order on the set of configurations of the random-cluster model:

$$\omega \leqslant \omega' \iff \omega(e) \leqslant \omega'(e) \quad \forall e \in \mathcal{E}.$$

Therefore, a function $f : \Omega_{\mathcal{E}} \to \mathbb{R}$ is called increasing if $\omega \leqslant \omega' \implies f(\omega) \leqslant f(\omega')$. An event is called increasing if its indicator function is increasing.
The FKG inequality, already seen in terms of the spin variables for the Ising model, is valid in the framework of the random-cluster model for any $q \geqslant 1$.

Proposition 1.14 (FKG inequality)
Let $p \in [0,1]$, $q \in [1,\infty)$. For any boundary condition $\overline{\omega}$, let $f, g : \Omega_{\mathcal{E}}^{\overline{\omega}} \to \mathbb{R}$ be two increasing functions. Then

$$\mu^{\overline{\omega}}_{G,p,q}(fg) \geqslant \mu^{\overline{\omega}}_{G,p,q}(f) \cdot \mu^{\overline{\omega}}_{G,p,q}(g)$$

The original proof [35] is somehow technical and not very informative. A nice proof can be found in [41], it is based on [56] and proceeds by constructing a coupling between two Markov processes which have $\mu^{\omega}_{G,p,q}$ and $\mu^{\omega'}_{G,p,q}$, with $\omega \leqslant \omega'$ as stationary distributions.

Note that the FKG inequality is not true for $q < 1$ and therefore very little is known on the corresponding models.

The FKG inequality has number of interesting consequences. One of them is the stochastic comparison between boundary conditions: for any $\omega \leqslant \omega'$,

$$\mu^{f}_{G,p,q} \preccurlyeq \mu^{\omega}_{G,p,q} \preccurlyeq \mu^{\omega'}_{G,p,q} \preccurlyeq \mu^{w}_{G,p,q}$$

where by $\preccurlyeq$ we mean the stochastic domination, namely $\mu \preccurlyeq \nu \Leftrightarrow \mu(f) \leqslant \nu(f)$ for any local increasing function f.

1.4.1.4 Comparison inequalities

A remarkable property of the random-cluster measures is that it is possible to compare expectations of increasing functions under measures with different parameters p or q. The later turns out to be very useful, in particular to show occurrence of a phase transition for every $q \geqslant q_0$ once we prove it for q_0, as we will see later.

Proposition 1.15 (Comparison inequalities)
Let $p_1, p_2 \in [0,1]$ and $q_1, q_2 \in [1,\infty)$ such that either

1. $q_1 \geqslant q_2$ *and* $p_1 \leqslant p_2$, *or*
2. $q_2 \geqslant q_1$ *and* $\frac{p_1}{q_1(1-p_1)} \leqslant \frac{p_2}{q_2(1-p_2)}$

Then for any increasing function f,

$$\mu^{w}_{G,p_1,q_1}(f) \leqslant \mu^{w}_{G,p_2,q_2}(f).$$

Proof We have, under assumption 1:

$$\begin{aligned}
\mu^w_{G,p_2,q_2}(f) &= \frac{1}{Z^w_{G,p_2,q_2}} \sum_\omega f(\omega) \left(\frac{p_2}{1-p_2}\right)^{|\omega|} q_2^{\kappa(\omega)} \\
&= \frac{1}{Z^w_{G,p_2,q_2}} \sum_\omega f(\omega) \underbrace{\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^{|\omega|} \left(\frac{q_2}{q_1}\right)^{\kappa(\omega)}}_{=g(\omega)} \left(\frac{p_1}{1-p_1}\right)^{|\omega|} q_1^{\kappa(\omega)} \\
&= \frac{\mu^w_{G,p_1,q_1}(fg)}{\mu^w_{G,p_1,q_1}(g)} \geqslant \mu^w_{G,p_1,q_1}(f),
\end{aligned}$$

where in the last line we used the FKG inequality, after observing that the function g is increasing. Let us treat assumption 2:

$$\begin{aligned}
\mu^w_{G,p_1,q_1}(f) &= \frac{1}{Z^w_{G,p_1,q_1}} \sum_\omega f(\omega) \left(\frac{p_1}{1-p_1}\right)^{|\omega|} q_1^{\kappa(\omega)} \\
&= \frac{1}{Z^w_{G,p_1,q_1}} \sum_\omega f(\omega) \underbrace{\left(\frac{q_2 p_1(1-p_2)}{q_1 p_2(1-p_1)}\right)^{|\omega|} \left(\frac{q_1}{q_2}\right)^{\kappa(\omega)+|\omega|}}_{=g'(\omega)} \left(\frac{p_2}{1-p_2}\right)^{|\omega|} q_2^{\kappa(\omega)} \\
&= \frac{\mu^w_{G,p_2,q_2}(fg')}{\mu^w_{G,p_2,q_2}(g')} \leqslant \mu^w_{G,p_2,q_2}(f),
\end{aligned}$$

where in the last line we used the FKG inequality, after observing that the function g' is decreasing. ∎

1.4.1.5 Planar duality

Let $G = (V, \mathcal{E})$ be a planar graph. We recall Section 1.2.4 for the definition of its dual $G^\star = (V^\star, \mathcal{E}^\star)$. A random-cluster configuration $\omega \in \{0,1\}^{\mathcal{E}}$ gives then rise to a dual configuration $\omega^\star \in \{0,1\}^{\mathcal{E}^\star}$ defined as $\omega^\star(e^\star) = 1 - \omega(e)$, i.e. $e^\star$ is open if e is closed and conversely. Suppose G is finite. Let $o(\omega) = \{e \in \mathcal{E} : \omega(e) = 1\}$ and $f(\omega)$ be the set of faces of the graph of open edges $(V, o(\omega))$ including the unique infinite face.

Proposition 1.16 (Planar duality)
Let the configuration $\omega \in \Omega^f_{\mathcal{E}}$ be distributed according to $\mu^f_{G,p,q}$.
Then the configuration $\omega^\star \in \Omega^w_{\mathcal{E}^\star}$ is distributed according to $\mu^w_{G^\star,p^\star,q}$, where the dual parameter $p^\star$ is solution of

$$\frac{p^\star}{1-p^\star} = \frac{q(1-p)}{p}.$$

Remark 1.1 *If we put* $q = 2$ *in the above duality relation, and use* $p = 1 - e^{-2\beta}$ *(the factor 2 comes from the mapping from the 2-state Potts model to the Ising model), we easily see that* $p^\star$ *is given by*

$$p^\star = \frac{2(1-p)}{p + 2(1-p)} = \frac{2}{e^{2\beta} + 1}.$$

Hence, defining $\beta^\star$ *by* $p^\star = 1 - e^{-2\beta^\star}$*, we get*

$$e^{-2\beta^\star} = \tanh \beta$$

which is the Kramers-Wannier duality relation, see Proposition 1.8. This duality via the mapping to the random-cluster model is thus another manifestation of the Kramers-Wannier duality, which is more deep, though, because it is valid configuration by configuration.

Proof One sees easily that $f(\omega^\star)$ is in one-to-one correspondence with the connected components of $(V, o(\omega))$, hence:

$$|f(\omega^\star)| = \kappa(\omega) \tag{1.15}$$

By Euler's formula, we have

$$\kappa(\omega) = |V| - |\omega| + |f(\omega)| - 1. \tag{1.16}$$

Note also that

$$|\omega| + |\omega^\star| = |\mathcal{E}|. \tag{1.17}$$

Hence,

$$\begin{aligned}
\mu^f_{G,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\omega)} \quad &\propto \quad \left(\frac{p}{1-p}\right)^{-|\omega^\star|} q^{|f(\omega^\star)|} \quad \text{by (1.15) and (1.17)} \\
&\propto \quad \left(\frac{q(1-p)}{p}\right)^{|\omega^\star|} q^{\kappa(\omega^\star)} \quad \text{by (1.16) applied to } \omega^\star \\
&\propto \quad \mu^w_{G^\star,p^\star,q}(\omega^\star)
\end{aligned}$$

with the announced $p^\star$. Note that the constants of proportionality above are independent of ω and $\omega^\star$. ■

Note that the dual value of $p^\star$ is p, and the unique solution of $p = p^\star$ is called the "self-dual point":

$$p_{sd}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

1.4.2 Infinite volume Random-Cluster measures

As in Section 1.2.2, we now consider a sequence of graphs $G = (G_n)_n$ such that $G_n \Subset G_\infty = (V_\infty, \mathcal{E}_\infty)$ and $G_n \uparrow G_\infty$ as $n \to \infty$, which we write $G \uparrow G_\infty$. We always consider G_∞ to be locally finite and connected, the latter being usually implicit.

1.4.2.1 Definition of infinite volume measures

Analogously as for the Ising model (see Section 1.2.2), we define infinite volume random-cluster measures as weak limits of finite volume measures.

Definition 1.9 *Let* $p \in [0,1]$ *and* $q \in (0,\infty)$. *Let* $\Omega = \{0,1\}^{\mathcal{E}_\infty}$. *A probability measure* μ *on* Ω *is an infinite volume random-cluster measure with parameters* p *and* q *if* μ *is an accumulation point of some sequence of finite volume measures* $\{\mu^{\overline{\omega}_n}_{G_n,p,q}\}_n$, *for the weak topology (see Definition 1.4). We write:*

$$\mathcal{W}_{p,q} = \{\textit{Weak limits of sequences } (\mu^{\overline{\omega}_n}_{G_n,p,q})_n, \textit{ with } G_n \uparrow G_\infty, \textit{ and } \overline{\omega}_n \in \Omega\}$$

Note that every $\mu \in \mathcal{W}_{p,q}$ satisfy the finite energy property (Prop.1.12), the comparison inequalities (Prop.1.15) and the FKG inequality (Prop.1.14) whenever $q \geqslant 1$. We refer to [53] for proofs.

1.4.2.2 Existence of infinite volume measures

As we will see, the FKG inequality allows us to prove easily the existence of the infinite volume measures with free and wired boundary conditions when $q \geqslant 1$.

Proposition 1.17 *Let* G_∞ *be a locally finite graph,* $p \in [0,1]$ *and* $q \in [1,\infty)$. *The weak limits*

$$\mu^w_{p,q} = \lim_{G\uparrow G_\infty} \mu^w_{G,p,q} \quad \textit{and} \quad \mu^f_{p,q} = \lim_{G\uparrow G_\infty} \mu^f_{G,p,q}$$

exist and are independent of the choice of the sequence $G \uparrow G_\infty$. *Moreover,*

1. $\mu^f_{p,q}$ *and* $\mu^w_{p,q}$ *are invariant under the automorphisms of* G_∞,
2. $\mu^f_{p,q} \preccurlyeq \mu \preccurlyeq \mu^w_{p,q}$ *for all* $\mu \in \mathcal{W}_{p,q}$.

Proof As for the Ising model, we easily get the following monotonicity property: for every increasing local function f, and $G_1 = (V_1, \mathcal{E}_1) \subset G_2 = (V_2, \mathcal{E}_2)$,

$$\mu^w_{G_1,p,q}(f) = \mu^w_{G_2,p,q}(f \,|\, \omega(e) \equiv 1 \text{ on } \mathcal{E}_2 \backslash \mathcal{E}_1) \geqslant \mu^w_{G_2,p,q}(f)$$

As any local function is bounded, we deduce the convergence of expectations of increasing local functions, as $G \uparrow G_\infty$. It is a standard fact that any local function can be decomposed onto increasing local functions. This implies the weak convergence of $\mu^w_{G,p,q}$ towards some measure $\mu^w_{p,q}$. The same argument (with reversed inequality) shows the existence of the infinite measure with free boundary conditions. The fact that the limits do not depend on the chosen sequence of graphs, as well as the automorphism invariance, are standard. See the proof of Proposition 1.6. The stochastic extremality property follows easily from the FKG inequality in finite volume (Prop.1.14). ∎

1.4.2.3 Uniqueness of the infinite volume measure

By contrast with the Potts models, the random-cluster model presents a phase transition, as we will see, which is not related to the non-uniqueness of the infinite volume measure. Indeed, for a given q, the set of parameters p such that $|\mathcal{W}_{p,q}| > 1$ is at most countable for a very general class of graphs.

Proposition 1.18 *Let G_∞ be a locally finite, periodic graph. Let $p \in (0,1)$, $q \in [1,\infty)$. Then the set*

$$\mathcal{D}_q = \{p \,:\, |\mathcal{W}_{p,q}| > 1\} \quad \textit{is countable.}$$

For $p \notin \mathcal{D}_q$, we write $\mu_{p,q}$ for the unique infinite-volume random-cluster measure with parameters p and q.

Proof This is a consequence of the convexity of the pressure associated to the random-cluster model. Indeed, up to a constant factor, we have:

$$Z^{\varpi}_{G,p,q} = \sum_{\omega} \left(\frac{p}{1-p}\right)^{|\omega|} q^{\kappa(\varpi,\omega)} = \sum_{\omega} \exp\left(\log\left(\frac{p}{1-p}\right)|\omega| + (\log q)\kappa(\varpi,\omega)\right)$$

It is standard to show the near-multiplicativity of $Z^{\varpi}_{G,p,q}$ as a function of G when the graph is periodic (and thus Van Hove) (see [53, Theorem 4.58] for the proof in the case of $\mathbb{Z}^d$). As a consequence, the limit

$$F(p,q) = \lim_{G\uparrow G_\infty} \frac{1}{|\mathcal{E}|} \log Z^{\varpi}_{G,p,q}$$

exists and is independent of the sequence $G \uparrow G_\infty$ as soon as $|\partial G|/|G| \to 0$ as $G \uparrow G_\infty$. The irrelevance to the limit of the boundary condition ϖ comes also from the Van Hove property. Moreover, it is easy to see that $F(p,q)$ is convex as a function of the two variables $\log(p/(1-p))$ and $\log q$ with $(p,q) \in (0,1) \times \mathbb{R}^+$. The latter fact implies that for each $q \in \mathbb{R}^+$, there exists a (possibly empty) countable subset $\mathcal{D}_q \subset (0,1)$ such that

$$\mathcal{D}_q = \{p \in (0,1) \,:\, F(p,q) \text{ is a non-differentiable function of } p\}$$

Now, let $p \in (0,1)$, $q \in [1,\infty)$, define $\pi = \log\frac{p}{1-p}$ and $F^{\varpi}_G = \frac{1}{|\mathcal{E}|}\log Z^{\varpi}_{G,p,q}$ (so that $F(p,q) = \lim_{G\uparrow G_\infty} F^{\varpi}_G$), and observe that

$$\frac{dF^{\varpi}_G}{d\pi}(p,q) = \frac{\mu^{\varpi}_{G,p,q}(|\omega|)}{|\mathcal{E}|}$$

Since F is convex, if $p \notin \mathcal{D}_q$ then, for all ϖ,

$$\frac{dF^{\varpi}_G}{d\pi}(p,q) \xrightarrow{G\uparrow G_\infty} \frac{dF}{d\pi}.$$

As $|\omega|$ is an increasing function, we have $\mu^{f}_{G,p,q}(|\omega|) \leqslant \mu^{\overline{\omega}}_{G,p,q}(|\omega|) \leqslant \mu^{w}_{G,p,q}(|\omega|)$, and as a consequence of the automorphism invariance of $\mu^{f}_{p,q}$ and $\mu^{w}_{p,q}$, we have,

$$\lim_{G\uparrow G_\infty} \frac{\mu^{b}_{G,p,q}(|\omega|)}{|\mathcal{E}|} = \lim_{G\uparrow G_\infty} \frac{1}{|\mathcal{E}|}\sum_{e\in\mathcal{E}} \mu^{b}_{p,q}(\omega(e)=1) = \frac{1}{|\mathcal{F}|}\sum_{e\in\mathcal{F}} \mu^{b}_{p,q}(\omega(e)=1) \quad \text{for } b = f, w$$

where $\mathcal{F}$ denotes a fundamental domain of the periodic graph. We deduce, when passing to the limit $G \uparrow G_\infty$, that when $p \notin \mathcal{D}_q$,

$$\frac{dF}{d\pi} = \frac{1}{|\mathcal{F}|}\sum_{e\in\mathcal{F}} \mu^{f}_{p,q}(\omega(e)=1) = \frac{1}{|\mathcal{F}|}\sum_{e\in\mathcal{F}} \mu^{w}_{p,q}(\omega(e)=1).$$

As $\mu^{f}_{p,q}(\omega(e) = 1) \leqslant \mu^{w}_{p,q}(\omega(e) = 1)$ for all $e \in \mathcal{E}$, we deduce the equality of the one-site marginals: $\mu^{f}_{p,q}(\omega(e)=1) = \mu^{w}_{p,q}(\omega(e)=1)$ for all $e \in \mathcal{E}$.

Uniqueness of the infinite-volume measure follows from the fact that, since $\mu^{f}_{p,q} \preccurlyeq \mu^{w}_{p,q}$ and the two measures have the same one-site marginals, they are equal (see Lemma 1.1 which trivially holds for measures on $\{0,1\}^E$ by a change of variables). ■

An important consequence of the countability of $\mathcal{D}_q$ together with comparison inequalities is that there exists a well defined value $p_c(q)$ above which the random-cluster model percolates. Let us first define:

Definition 1.10 *Let $\mathcal{C}_0$ denote the open cluster of the origin and define*

$$\{0 \leftrightarrow \infty\} = \{|\mathcal{C}_0| = \infty\}$$

An immediate corollary of Proposition 1.15 is the following:

Corollary 1.1 *Let G_∞ be a locally finite graph, and $q \geqslant 1$.*
Then the function

$$p \mapsto \theta^{w}(p) = \mu^{w}_{p,q}(0 \leftrightarrow \infty) \quad \textit{is increasing}.$$

We can thus unambiguously introduce the following definition:

Definition 1.11 *Let G_∞ be a locally finite, (connected) periodic graph. Then the critical parameter for the percolation of the random-cluster model on G_∞ is*

$$p_c(q) = \sup\{p \,:\, \mu_{p,q}(0 \leftrightarrow \infty) = 0\}.$$

Note that the probability that the cluster of the origin has infinite cardinality may be different under the free or the wired measure, but by monotonicity in p at fixed q (see Corollary 1.1) the supremum of the parameters p for which this probability vanishes is the same.

Remark 1.2 *In the case of a (connected) periodic graph with a fundamental domain $\mathcal{F}$ containing more than one vertex, we should have defined the critical parameter as the supremum* $\sup\{p : \sup_{x\in\mathcal{F}} \mu_{p,q}(x \leftrightarrow \infty) = 0\}$*. However, by the finite energy property, we have* $\mu_{p,q}(x \leftrightarrow \infty) \leqslant c^{d(0,x)}\mu_{p,q}(0 \leftrightarrow \infty)$*, which makes it equivalent to the above definition.*

Let us now prove the easy result that there is uniqueness below $p_c(q)$ which implies $\mathcal{D}_q \subset [p_c(q), 1]$.

Proposition 1.19 *Let G_∞ be a locally finite, periodic graph.*
Let $q \geqslant 1$. Then there is a unique infinite-volume random-cluster measure for all parameters $p < p_c(q)$, i.e.

$$\mu^w_{p,q} = \mu^f_{p,q} \text{ for all } p < p_c(q)$$

(For a graph on which $p_c(q)$ is not uniquely defined, the statement is true for any $p < p^w_c(q)$ with $p^w_c(q) = \sup\{p : \mu^w_{p,q}(0 \leftrightarrow \infty) = 0\}$.)

Proof In view of the above, it is sufficient to prove that $\mu^w_{p,q} \preccurlyeq \mu^f_{p,q}$. As before we write $B_n = \{x \in G_\infty : d(0,x) \leqslant n\}$. Let A be an increasing local event, which depends only on the edges inside B_R, and $E_n = \{B_R \leftrightarrow B^c_n\}$. We have,

$$\mu^w_{p,q}(A) \leqslant \mu^w_{p,q}(E_n) + \mu^w_{p,q}(A \,|\, E^c_n)$$

Since $p < p_c(q)$, we have $\mu^w_{p,q}(0 \leftrightarrow \infty) = 0$, which implies $\mu^w_{p,q}(E_n) \to 0$ as $n \to \infty$. On the other hand, on E^c_n, there exists some subset Δ such that $B_R \subseteq \Delta \subseteq B_n$ and all the edges of $\partial\Delta$ are closed. Let $E^c_{n,\Delta} = E^c_n \cap \{\Delta \text{ is the biggest such subset}\}$ (where biggest is defined with respect to the inclusion). Then, by the Markov property and the FKG inequality we get:

$$\begin{aligned}
\mu^w_{p,q}(A \,|\, E^c_n) &= \sum_{\substack{\Delta:\\ B_R\subseteq\Delta\subseteq B_n}} \mu^w_{p,q}(A \,|\, E^c_{n,\Delta})\, \mu^w_{p,q}(E^c_{n,\Delta} \,|\, E^c_n) \qquad (1.18)\\
&= \sum_{\substack{\Delta:\\ B_R\subseteq\Delta\subseteq B_n}} \mu^f_{\Delta,p,q}(A)\, \mu^w_{p,q}(E^c_{n,\Delta} \,|\, E^c_n)\\
&\leqslant \mu^f_{p,q}(A) \sum_{\substack{\Delta:\\ B_R\subseteq\Delta\subseteq B_n}} \mu^w_{p,q}(E^c_{n,\Delta} \,|\, E^c_n)\\
&= \mu^f_{p,q}(A)
\end{aligned}$$

Taking the limit $n \to \infty$ we get $\mu^w_{p,q}(A) \leqslant \mu^f_{p,q}(A)$ which gives the equality of these two probabilities for local increasing events. We get the statement for any general increasing event by approaching it with local ones. ■

Remark 1.3 *Note that in the planar case, if we know that $p_c(q) = p_{sd}(q)$, which is the case on $\mathbb{Z}^2$ as we will see, then Proposition 1.19 implies that the only possible p for which non-uniqueness could occur is p_c. Indeed, uniqueness for $p < p_{sd}$ implies uniqueness for $p > p_{sd}$ by duality since $\mu^f_{p,q} = \mu^w_{p^\star,q}$. It is conjectured that on $\mathbb{Z}^2$ uniqueness holds at $p_c(q)$ if and only if $q \leqslant 4$. Non-uniqueness at $p_{sd}(q)$ is known only for q large [66].*

1.4.3 Percolation transition

As we just mentioned, on infinite periodic graphs, at fixed q the value of the critical parameter above which the random-cluster model percolates is well defined. The comparison inequalities (Prop. 1.15) have moreover the important yet immediate corollary:

Corollary 1.2 (Comparison between critical parameters)
Let G_∞ be a locally finite, periodic graph. Let $q, q_1, q_2 \geqslant 1$ and $p_1, p_2 \in [0, 1]$. Then the function

$$p_c(q) \text{ is increasing in } q.$$

Moreover, let $1 \leqslant q_1 \leqslant q_2$, then

$$\frac{1}{p_c(q_2)} \leqslant \frac{1}{p_c(q_1)} \leqslant \frac{q_2/q_1}{p_c(q_2)} - \frac{q_2}{q_1} + 1.$$

Hence, the existence of a non-trivial percolation transition in the random-cluster model follows from the corresponding statement for the edge percolation model (i.e. $q = 1$).

A very interesting question is to know whether all phase transitions which can arise in a given model take place at the same critical parameter. In the sub-critical regime, we usually expect exponential decay of connectivities. To this end, we define a second critical parameter:

Definition 1.12 *Let $G_\infty = (V_\infty, \mathcal{E}_\infty)$ be a locally finite graph. The critical parameter for the exponential decay of connectivities on G_∞ is defined by:*

$$\tilde{p}_c(q) = \sup\{p \,:\, \exists\, c > 0 \text{ such that } \forall\, x, y \in V_\infty, \\ \mu_{p,q}(x \leftrightarrow y) \leqslant e^{-c\, d(x-y)}\}$$

Again Proposition 1.15 implies that $\mu^w_{p,q}(x \leftrightarrow y)$ is increasing in p, hence $\tilde{p}_c(q)$ is well defined and increasing in q (by Propositions 1.18 and 1.15).
Note that if $y \in \partial B_n(x)$ then

$$\mu_{p,q}(x \leftrightarrow y) \leqslant \mu_{p,q}(x \leftrightarrow \partial B_n(x)) \leqslant \sum_{y \in \partial B_n(x)} \mu_{p,q}(x \leftrightarrow y) \tag{1.19}$$

hence for any graph such that the size of the ball of radius n grows less than exponentially in n, we have exponential decay of $\mu_{p,q}(x \leftrightarrow \partial B_n(x))$ if and only if we have exponential decay of $\mu_{p,q}(x \leftrightarrow y)$.

Proposition 1.20 *Let G_∞ be a locally finite, periodic graph of dimension $d \geqslant 2$. Let $q \geqslant 1$, then*

$$0 < \tilde{p}_c(q) \leqslant p_c(q) < 1$$

Proof First of all, the inequality $\tilde{p}_c(q) \leqslant p_c(q)$ is trivial since by (1.19), having exponential decay of the two-point function and sub-exponential balls (which holds in the case of periodic graphs) implies that $0 \leftrightarrow \infty$ with probability zero.
By Corollary 1.2, and the analog of its point 2 for $\tilde{p}_c(q)$, the two parameters $p_c(q)$ and $\tilde{p}_c(q)$ are increasing in q. It is then sufficient to prove that $\tilde{p}_c(1) > 0$ to get the first inequality, and that $p_c(1) < 1$ to get the last one for all values of q.

In the latter case ($q = 1$), a very standard "Peierls argument"[1] shows that $\tilde{p}_c(1) > 0$ on any infinite graph with bounded degree. Its relies on the fact that in G_∞, the number of paths of length n starting at $x \in V_\infty$ is upper-bounded by $C(K)^n$ where $C(K)$ is a positive constant and K is the uniform bound on the degree of a vertex. Then,

$$\mu_{p,1}(x \leftrightarrow y) \leqslant (pC(K))^{d(x,y)}$$

and hence $\mu_{p,1}(x \leftrightarrow y)$ decays exponentially in $d(x, y)$ as soon as $p < 1/C(K)$.

A dual version of the Peierls argument can be used in the case of periodic graphs to show that, for a graph embedded in $\mathbb{R}^d$, there exist two positive constants $c(d)$ and ν such that

$$\mu_{p,1}(|\mathcal{C}_0| = n) \leqslant N_n(c(d)(1-p))^{\nu n^{(d-1)/d}}$$

where $N_n = O(n^d)$ is the number of edges of $\mathcal{E}_\infty$ having an end-vertex within distance n of the origin. If p is sufficiently close to 1, then $(1-p)c(d) < 1$ and the result follows by the union bound. It is rather easy to make this argument rigorous in dimension 2 for graphs with bounded degree such that the dual graph is also of bounded degree (see [52][Theorem 1.10]). Topological complications occur when $d \geqslant 3$, but the result still holds. ∎

1.4.4 Exponential relaxation

It is very informative to know when a finite volume measure is close to its infinite volume counterpart. We will show that in the case of a graph G_∞ with sub-exponential balls, under the hypothesis of finite volume exponential decay of connectivities with wired boundary condition, the finite volume wired measure (in a ball of radius n in G_∞) is exponentially close (for the weak topology) to its infinite volume counterpart.

Definition 1.13 *We say that the hypothesis $\mathbb{H}_{p,q}$ is verified for the graph G_∞ if there exists a constant $c = c(p, q) > 0$ such that for all $n \geqslant 0$ and all $x \in V_\infty$,*

$$\mu^w_{B_n(x),p,q}(x \leftrightarrow \partial B_n(x)^c) \leqslant e^{-cn} \tag{1.20}$$

[1] See [75] for the original paper by Peierls who proved the existence of a phase transition for the Ising model. The argument for percolation is very similar.

The hypothesis $\mathbb{H}_{p,q}$ is conjectured to hold for all $q \geqslant 1$ and all $p < p_c(q)$. It is proved to hold in any dimension for $q = 1$ (it follows from [2]), for q sufficiently large [66], and for $q = 2$ (the so-called sharpness of the phase transition in the Ising model, i.e. the fact that $p_c(2) = \tilde{p}_c(2)$ was proved in [3] and it follows from the GKS (Proposition 1.2) and the GHS inequality [51] that the magnetization relaxes exponentially fast in a finite box). As we will see in the next section, the hypothesis $\mathbb{H}_{p,q}$ is known to hold on $\mathbb{Z}^2$ for all $q \geqslant 1$ and $p < p_c(q)$.

Proposition 1.21
Let G_∞ be a locally finite graph with sub-exponential balls. Let $q \geqslant 1$ and p such that the hypothesis $\mathbb{H}_{p,q}$ holds.
Then there exist $C_1, C_2 \in (0, \infty)$ depending on p and q such that

$$0 \leqslant \mu^w_{B_n,p,q}(A) - \mu^w_{p,q}(A) \leqslant C_1 e^{-C_2(n-n_0)} \quad \forall n \geqslant n_0,$$

for all events A such that Support(A)$\subseteq B_{n_0}$.
The same statement is valid for free boundary conditions for $p < p_c$.

Proof Suppose A is an increasing event. The result for general functions follows from the standard decomposition of local functions onto increasing ones.

The lower bound follows from the FKG inequality. For the upper bound, let us define $E = \{B_{n_0} \leftrightarrow B_n^c\}$. We have,

$$\mu^w_{B_n,p,q}(A) \leqslant \mu^w_{B_n,p,q}(E) + \mu^w_{B_n,p,q}(A \,|\, E^c)$$

On the one hand, by the hypothesis $\mathbb{H}_{p,q}$ and the FKG inequality we have

$$\begin{aligned}
\mu^w_{B_n,p,q}(E) &\leqslant \sum_{i \in \partial B_{n_0}} \mu^w_{B_n,p,q}(i \leftrightarrow B_n^c) \\
&\leqslant |\partial B_{n_0}| \sup_{i \in \partial B_{n_0}} \sup_{j \in \partial B_{\frac{n-n_0}{2}}} \mu^w_{B_{\frac{n-n_0}{2}}(j),p,q}(j \leftrightarrow B^c_{\frac{n-n_0}{2}}(j)) \\
&\leqslant e^{-c(n-n_0)}.
\end{aligned}$$

On the other hand, on E^c, there exists some subset Δ such that $B_{n_0} \subseteq \Delta \subseteq B_n$ and all the edges of $\partial\Delta$ are closed. Let

$$E^c_\Delta = E^c \cap \{\Delta \text{ is the biggest such subset}\}$$

(where biggest is defined with respect to the inclusion). Then, by the same reasoning as (1.18)

$$\mu^w_{B_n,p,q}(A \,|\, E^c) \leqslant \mu^f_{p,q}(A) \leqslant \mu^w_{p,q}(A)$$

which concludes the proof. ■

1.4.5 Inverse correlation length and surface tension

1.4.5.1 Existence and properties

In this section we consider $G_\infty = \mathbb{Z}^d$ to simplify the notations. Possible generalizations will be mentioned.

The study of (a priori part of) the sub-critical regime of the random-cluster model can be done via the rate of exponential decay of the two-point function, also called inverse correlation length.

Definition 1.14 *Let $G_\infty = \mathbb{Z}^d$, $q \geqslant 1$, $p \in [0,1]$, and $x \in \mathbb{R}^d$, we define*

$$\xi_p(x) = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor) \tag{1.21}$$

where $\mu_{p,q}$ is the unique infinite volume random-cluster measure with parameters p and q (which is well defined for almost every p at fixed q). We will often write $\xi(x)$ when p is implicitly fixed. The dependence on q will always be implicit, to lighten the notation.
This definition can be extended to periodic graphs embedded in $\mathbb{R}^d$. In this case $\lfloor kx \rfloor$ is a (fixed) choice of rounding inside the fundamental domain containing $kx \in \mathbb{R}^d$.

Note that the existence of the limit follows from the super-multiplicativity of $\mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor)$ as a function of k, which is due to the FKG inequality and translation invariance of $\mu_{p,q}$. Indeed, first consider $x \in \mathbb{Z}^d$. Then,

$$\begin{aligned}\mu_{p,q}(0 \leftrightarrow \lfloor (k+\ell)x \rfloor) &\geqslant \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor\,,\ \lfloor kx \rfloor \leftrightarrow \lfloor (k+\ell)x \rfloor) \\ &\geqslant \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor)\mu_{p,q}(\lfloor kx \rfloor \leftrightarrow \lfloor (k+\ell)x \rfloor) \\ &\geqslant \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor)\mu_{p,q}(0 \leftrightarrow \lfloor \ell x \rfloor)\end{aligned}$$

which implies the sub-additivity of $\frac{1}{k}\log \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor)$ and hence existence of the limit. For $x \in \mathbb{R}^d \backslash \mathbb{Z}^d$, some error term can come from rounding error, namely we can have

$$\mu_{p,q}(\lfloor kx \rfloor \leftrightarrow \lfloor (k+\ell)x \rfloor) = \mu_{p,q}\left(0 \leftrightarrow \lfloor \ell x \rfloor \pm \sum_{i=1}^{d} \hat{e}_i\right)$$

where $\hat{e}_i$ denotes the unit vector in direction i. In this case we can use the finite energy property, which implies, after a splitting as before,

$$\mu_{p,q}\left(0 \leftrightarrow \lfloor \ell x \rfloor \pm \sum_{i=1}^{d} \hat{e}_i\right) \geqslant C\mu_{p,q}(0 \leftrightarrow \lfloor \ell x \rfloor) \text{ for some } C > 0$$

We deduce that $C\mu_{p,q}(0 \leftrightarrow \lfloor (k+\ell)x \rfloor) \geqslant (C\mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor))(C\mu_{p,q}(0 \leftrightarrow \lfloor \ell x \rfloor))$ from which follows the existence of the above limit as well.

The surface tension has interesting geometrical properties which are crucial to understand. Here are those we will use the most in the sequel.

Proposition 1.22 *Let $G_\infty = \mathbb{Z}^d$ and $p < \tilde{p}_c(q)$. Then the surface tension*

$$\xi_p \text{ is a norm on } \mathbb{R}^d.$$

Proof

1. $\xi(x) \geqslant 0$ with equality iff $x = 0$. This comes from $p < \tilde{p}_c(q)$.

2. ξ is positive homogenous. Let $\lambda > 0$, then

$$\begin{aligned}\xi(\lambda x) &= -\lim_{k\to\infty} \frac{1}{k} \log \mu_{p,q}(0 \leftrightarrow \lfloor k\lambda x \rfloor) \\ &= -\lim_{k\to\infty} \frac{\lambda}{k\lambda} \log \mu_{p,q}(0 \leftrightarrow \lfloor k\lambda x \rfloor) \\ &= \lambda \xi(x)\end{aligned}$$

   Moreover, by invariance of $\mu_{p,q}$ under the symmetries of $\mathbb{Z}^d$, we have $\xi(x) = \xi(-x)$, hence

$$\xi(\lambda x) = |\lambda| \xi(x) \quad \forall \lambda \in \mathbb{R}$$

3. ξ satisfies the triangle inequality. Indeed, we can proceed as before to show the sub-multiplicativity, and obtain for x and y in $\mathbb{Z}^d$,

$$\mu_{p,q}(0 \leftrightarrow \lfloor k(x+y) \rfloor) \geqslant \mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor) \mu_{p,q}(0 \leftrightarrow \lfloor ky \rfloor)$$

   which implies $\xi(x+y) \leqslant \xi(x) + \xi(y)$. For x or y in $\mathbb{R}^d \backslash \mathbb{Z}^d$, there is as above an error term of at most one unit in each lattice direction, which disappears in the limit $k \to \infty$.

4. ξ is finite. Indeed, by the finite energy property, for a fixed path γ connecting 0 to x, we have $\mu_{p,q}(\gamma \text{ is open } \mid \omega \equiv \omega' \text{ in } \mathbb{Z}^d \backslash \gamma) \geqslant p^{|\gamma|} = e^{-c|x|}$ uniformly in ω' for some finite $c > 0$. Hence,

$$\mu_{p,q}(0 \leftrightarrow \lfloor kx \rfloor) \geqslant p^{\|\lfloor kx \rfloor\|_1} = e^{-c' k|x|} \text{ for some finite } c' > 0$$

   which implies that $\xi(x) \leqslant c'|x| < \infty$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

■

Two convex bodies play an important role in the analysis of the fluctuations of interfaces. We introduce the following definitions:

Definition 1.15 *Let G_∞ be a locally finite, periodic graph. Let $q \geqslant 1$ and $p < \tilde{p}_c(q)$, and let ξ be the inverse correlation length of the subcritical random-cluster model with parameters p and q. Denote by $(\cdot,\cdot)$ the Euclidean scalar product in $\mathbb{R}^d$.*

The equi-decay set *is the unit ball for the* ξ*-norm:*

$$U_\xi = \{x \in \mathbb{R}^d \,:\, \xi(x) \leqslant 1\},$$

The Wulff shape *is the polar transform of* U_ξ*:*

$$W_\xi = \bigcap_{\hat{n} \in \mathbb{S}^{d-1}} \{t \in \mathbb{R}^d \,:\, (t, \hat{n}) \leqslant \xi(\hat{n})\}$$

Pairs $(x, t) \in \mathbb{R}^d \times \partial W_\xi$ *are called* dual *if* $(t, x) = \xi(x)$.

1.4.5.2 Subcritical surface tension coincide with supercritical inverse correlation length on planar graphs

In this section, we define the surface tension for the random-cluster model starting from the corresponding quantity in the Potts model. We will come back to the Ising and Potts surface tensions in Section 1.5.5.

In view of the Edwards-Sokal coupling and the planar duality presented in Sections 1.3 and 1.4.1.5, in two dimensions there is a natural notion of surface tension associated to a super-critical random-cluster model with parameters $p^\star$ and q, with $p^\star$ chosen such that $p < \tilde{p}_c(q)$.
Indeed, let $\hat{x} \in \mathbb{S}^1$. Consider the Potts model in the box $\Lambda_{L,M} = \{-M, \dots M\} \times \{-L, \dots, L\}$, with $\hat{x}$-Dobrushin boundary condition, that is

$$\overline{\sigma}_i = \begin{cases} 1 \text{ if } (\hat{x}, i) \geqslant 0 \\ 2 \text{ if } (\hat{x}, i) < 0 \end{cases}.$$

We write $\mathbb{P}^{\hat{x}}_{\Lambda_{L,M},\beta^\star,q}$ for the corresponding Potts measure, and $x^\star_\ell = x^\star_\ell(L, M)$, resp. $x^\star_r = x^\star_r(L, M)$ for the locations on $(\mathbb{Z}^2)^\star \cap \partial\Lambda_n$ of the color changes. It is not difficult to see that the measure $\mathbb{P}^{\hat{x}}_{\Lambda_{L,M},\beta^\star,q}$ can be coupled (via the Edwards-Sokal coupling) to the random-cluster measure $\mu^{w,\hat{x}}_{\Lambda_{L,M},p^\star,q}(\,\cdot \mid \mathcal{C}_{up} \nleftrightarrow \mathcal{C}_{down})$ where the boundary condition $w, \hat{x}$ consists in wiring the edges around $\partial\Lambda_{L,M}$ except at the location of the Potts color changes, giving two boundary clusters $\mathcal{C}_{up}$ and $\mathcal{C}_{down}$. The dual of this random-cluster measure is $\mu^f_{\Lambda^\star_{L,M},p,q}(\,\cdot \mid x^\star_\ell \leftrightarrow x^\star_r)$. See Figure 1.4.

The surface tension at $p^\star$ in a direction $\hat{x} \in \mathbb{S}^1$ is thus defined as follows:

$$\begin{aligned}
\tau_{p^\star}(\hat{x}) &= -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|x^\star_\ell - x^\star_r|} \log \frac{Z^{\hat{x}}_{\Lambda_{L,M},\beta^\star,q}}{Z^1_{\Lambda_{L,M},\beta^\star,q}} \\
&= -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|x^\star_\ell - x^\star_r|} \log \frac{Z^{w,\hat{x}}_{\Lambda_{L,M},p^\star,q}(\mathcal{C}_{up} \nleftrightarrow \mathcal{C}_{down})}{Z^{w}_{\Lambda_{L,M},p^\star,q}} \\
&= -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|x^\star_\ell - x^\star_r|} \log \mu^f_{\Lambda^\star_{L,M},p,q}(x^\star_\ell \leftrightarrow x^\star_r)
\end{aligned}$$

where the first line involves a ratio of Potts partition functions, which can be rewritten as a ratio of super-critical random-cluster partition functions in the second line

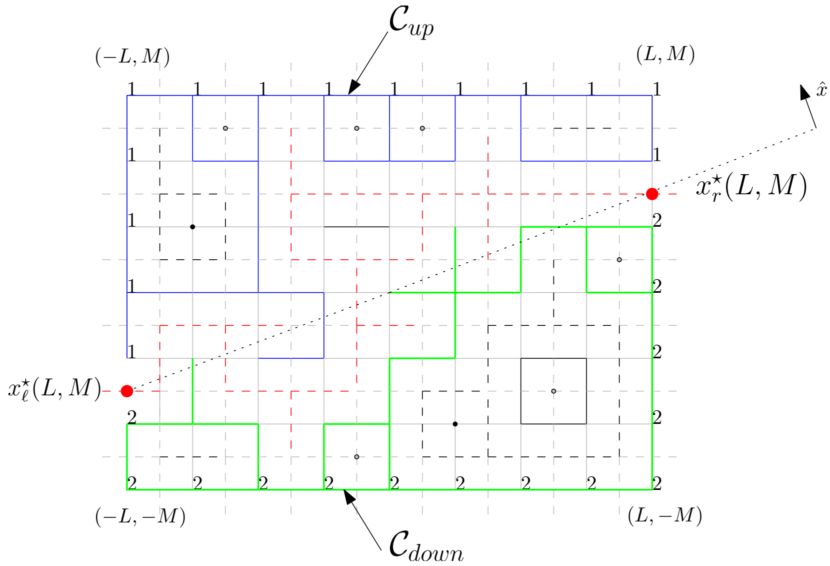


Figure 1.4 – A random-cluster configuration and its dual, corresponding to $\hat{x}$-Dobrushin Potts boundary condition. Open (resp. closed) edges of $\mathbb{Z}^2$ are drawn with dark (resp. light) solid lines, open (resp. closed) edges of $(\mathbb{Z}^2)^\star$ are drawn with dark (resp. light) dashed lines. The dual cluster connecting $x_\ell^\star$ to $x_r^\star$ is drawn in red, while primal clusters $\mathcal{C}_{up}$ and $\mathcal{C}_{down}$ are drawn in blue, resp. green.

thanks to Edwards-Sokal coupling, and then as a sub-critical random-cluster probability via duality.

Note that as soon as we show that the above limit is the same as the one taken in infinite volume, we show that the surface tension at $p^\star$ coincides with the inverse correlation length at p.

Proposition 1.23 *Let $G_\infty = \mathbb{Z}^2$, and $\hat{x} \in \mathbb{S}^1$, $p < \tilde{p}_c(q)$. Let $p^\star$ dual to p as defined in Prop. 1.16, then*

$$\xi_p(\hat{x}) = \tau_{p^\star}(\hat{x})$$

Proof As announced, it is sufficient to show that

$$-\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{|x_\ell^\star - x_r^\star|} \log \mu^f_{\Lambda^\star_{L,M},p,q}(x_\ell^\star \leftrightarrow x_r^\star) = -\lim_{\substack{L\to\infty\\M\to\infty}} \frac{1}{|x_\ell^\star - x_r^\star|} \log \mu^f_{p,q}(x_\ell^\star \leftrightarrow x_r^\star)$$

We usually call the right-hand the "short" correlation length, and the left-hand side the "long" correlation length or the mass-gap, see [83]. Below we follow an argument by [18].

First, note that by the FKG inequality, $\mu^f_{\Lambda^\star_{L,M},p,q}(x_\ell^\star \leftrightarrow x_r^\star) \leqslant \mu^f_{p,q}(x_\ell^\star \leftrightarrow x_r^\star)$, which implies the inequality $\xi_p(\hat{x}) \leqslant \tau_{p^\star}(\hat{x})$.

On the other hand, let us write $\Lambda_n = \Lambda_{nL,nM}$ and $x_i^\star = (\frac{1}{2}, \frac{1}{2}) + n(x_r^\star - (\frac{1}{2}, \frac{1}{2}))$ (in particular $x^\star_\ell = x^\star_{-1}$ and $x^\star_r = x^\star_1$). Again by the FKG inequality we have

$$\mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{-nk} \leftrightarrow x^\star_{nk}) \geqslant \prod_{i=-n}^{n-1} \mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{ik} \leftrightarrow x^\star_{(i+1)k})$$

Now it follows from Proposition 1.21 that for any $\varepsilon > 0$, if the distance of the points to the boundary satisfies $d(x_{ik}, \partial\Lambda_n) \wedge d(x_{(i+1)k}, \partial\Lambda_n) > \delta$, there exists a $\delta > 0$ such that

$$\begin{aligned}\mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{ik} \leftrightarrow x^\star_{(i+1)k}) &> \mu^f_{p,q}(x^\star_{ik} \leftrightarrow x^\star_{(i+1)k}) - \varepsilon \\ &= \mu^f_{p,q}(x^\star_0 \leftrightarrow x^\star_k) - \varepsilon\end{aligned}$$

where in the last line we used translation invariance of the infinite volume measure $\mu^f_{p,q}$. This implies

$$\begin{aligned}\frac{1}{|x^\star_{-nk} - x^\star_{nk}|} \log \mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{-nk} \leftrightarrow x^\star_{nk}) &\geqslant \frac{1}{2n|x^\star_k - x^\star_0|} \sum_{i=-n}^{n-1} \log \mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{ik} \leftrightarrow x^\star_{(i+1)k}) \\ &\geqslant \frac{1}{2n|x^\star_k - x^\star_0|} \Big(O(\delta/k) + (2n - O(\delta/k)) \log \mu^f_{p,q}(x^\star_0 \leftrightarrow x^\star_k) - O(n\varepsilon)\Big)\end{aligned}$$

where for the boundary terms we used the fact that $\mu^f_{\Lambda^\star_{nk},p,q}(x^\star_{ik} \leftrightarrow x^\star_{(i+1)k}) > 0$ (which follows easily from the finite energy property).
We prove $\xi_p(\hat{x}) \geqslant \tau_{p^\star}(\hat{x})$ by taking the limit (sup) $n \to \infty$, followed by $k \to \infty$ and finally $\varepsilon \to 0$. ■

1.4.6 Ornstein-Zernike asymptotics of the two-point function on $\mathbb{Z}^d$

We have seen that exponential decay holds in (a priori part of) the subcritical regime. It is a much more difficult task to prove that it holds all the way to the critical point. This result is known only in the cases $q = 2$, and q large, as we will explain later.

In the case of $\mathbb{Z}^d$, under the hypothesis $\mathbb{H}_{p,q}$, the correction to exponential decay is an algebraic factor. The behavior of the two-point function is called Ornstein-Zernike asymptotics. The article [21] proves this result by introducing crucial coarse-graining methods which we will use in our main result, which is Theorem 2.2.

Below we emphasize the dependencies on p and q of all the relevant quantities.

Theorem 1.1 (Campanino, Ioffe, Velenik, 2008)
Let $G_\infty = \mathbb{Z}^d$, and $d \geqslant 2$. For all $q \geqslant 1$ and $p < p_c(q)$ such that hypothesis $\mathbb{H}_{p,q}$ defined in (1.20) *holds,*

1. *The two-point function satisfies the Ornstein-Zernike asymptotics, i.e.*

$$\mu_{p,q}(0 \leftrightarrow x) = \frac{\Psi_{p,q}(\hat{x})}{|x|^{\frac{d-1}{2}}} e^{-\xi_{p,q}(\hat{x})|x|}(1 + o_{|x|}(1))$$

uniformly as $|x| \to \infty$, where $\hat{x} = x/|x|$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Moreover, the functions $\Psi_{p,q}$ and $\xi_{p,q}$ are positive and locally analytic on $\mathbb{S}^{d-1}$.

2. *The Wulff shape W_ξ has a locally analytic, strictly convex boundary. Moreover, the Gaussian curvature χ_p of W_ξ is uniformly positive,*

$$\chi_p = \min_{t \in \partial W_\xi} \prod_{i=1}^{d-1} \chi_{p,i}(t) > 0$$

where $\chi_{p,i}(t)$, $i = 1, \ldots, d-1$ are the principal curvatures of ∂W_ξ at t. By duality, ∂U_ξ is also locally analytic and uniformly convex.

The general heuristics behind this result is that on the event $0 \leftrightarrow x$, for large $|x|$, the cluster $\mathcal{C}_{0,x}$ containing 0 and x can be typically decomposed into a chain of irreducible sub-clusters, whose increments constitute an effective 1-dimensional directed random walk. More precisely, let t be dual to $\hat{x}$ according to Definition 1.15. The authors show that there exist two constants $c > 0$ and $\delta \in (0, 1)$, depending on p and q, such that up to probabilities of order $\exp(-\xi(x) - c|x|)$, the cluster $\mathcal{C}_{0,x}$ can be represented as a concatenation

$$\mathcal{C}_{0,x} = \gamma^b \vee \gamma_1 \vee \ldots \vee \gamma_{\mathcal{N}} \vee \gamma^f \tag{1.22}$$

where $\mathcal{N} \geqslant 1$ and γ_i, $i = 1, \ldots, \mathcal{N}$ are irreducible clusters in the sense that they are confined into the intersection of shifted backward and forward cones, $\pm\mathcal{Y}$, where $\mathcal{Y} = \{y \in \mathbb{Z}^d : (t, y) > \delta\xi(y)\}$. One can then think about this decomposition in terms of an effective random walk with boundary conditions γ^b and γ^f and steps $V(\gamma_i)$, $i = 1, \ldots, \mathcal{N}$ where $V(\gamma)$ is the displacement along the irreducible cluster γ, see Figure 1.5.

The precise study of the correlations between the increments turn out to fall in the framework of 1d full shifts over countable alphabets, for which a classical local limit theorem holds, giving the factor $|x|^{-(d-1)/2}$, which is called Ornstein-Zernike

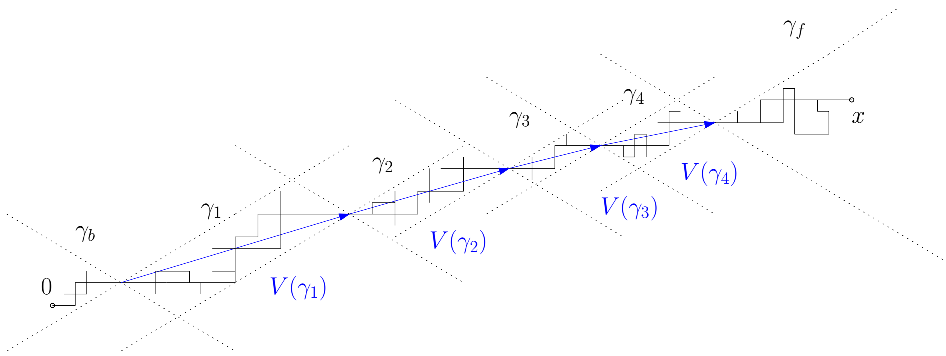


Figure 1.5 – Effective random walk representation.

type asymptotics. As the coarse-graining techniques of [21] will be explained in the proof of Theorem 2.2, we do not enter into more details here.

Note that the uniform convexity of the Wulff shape W_ξ is actually equivalent to a sharper version of the triangle inequality for the τ-norm (cf. [77][Theorem 2.1]), which allow us to compare the curvature of the ξ-ball with the one of the Euclidean ball:

Corollary 1.3 (Sharp triangle inequality) *Let $G_\infty = \mathbb{Z}^d$, and $d \geqslant 2$. For all $q \geqslant 1$ and $p < p_c(q)$ such that hypothesis $\mathbb{H}_{p,q}$ defined in (1.20) holds, then there exists some $\rho = \rho(p) > 0$ such that*

$$\xi(x) + \xi(y) - \xi(x+y) \geqslant \rho(|x| + |y| - |x+y|),$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. The same inequality is true for the surface tension τ of the dual random-cluster model.

The local limit result can be extended to a full invariance principle for the cluster $\mathcal{C}_{0,x_n}$ as $x_n \to \infty$ in the direction $\hat{x}$, i.e. subcritical clusters have a Brownian scaling, with a variance which depends on the direction $\hat{x}$ and is related to the curvature of $\partial W_\xi(t)$, with t dual to $\hat{x}$. Indeed, consider a sequence of vertices $x_n = \lfloor nx \rfloor$ and the corresponding sequence of conditional measures $\mu_{p,q}(\cdot \,|\, 0 \leftrightarrow x_n)$. The authors show that the cluster $\mathcal{C}_{0,x_n}$ can be decomposed as (1.22) up to $\mu_{p,q}(\cdot \,|\, 0 \leftrightarrow x_n)$-probability of order $\exp(-cn)$. Let $X = \{0, u_0, \dots, u_{\mathcal{N}}, x_n\}$ be the trajectory of the corresponding effective random walk, and $\mathcal{L}_n(X)$ be the linear interpolation between the vertices u_i. Let H_x be the $(d-1)$-dimensional hyperplane orthogonal to x. Note that this shifted hyperplane intersects $\mathcal{L}_n(X)$ only in one point because the clusters γ_i are confined in cones. There is thus a natural parametrization of $\mathcal{L}_n(X)$ as a function $\Phi_n : [0,1] \to H_x$. If we define ϕ_n to be the diffusive scaling of Φ_n, i.e.

$$\phi_n(\cdot) = \frac{1}{\sqrt{n}} \Phi_n(\cdot)$$

then it follows from estimates in the proof of Theorem 1.1 (see [21] and [48]) that the function ϕ_n weakly converges to the distribution of a $(d-1)$-dimensional Brownian bridge:

Corollary 1.4 *Let $G_\infty = \mathbb{Z}^d$, and $d \geqslant 2$. For all $q \geqslant 1$ and $p < p_c(q)$ such that hypothesis $\mathbb{H}_{p,q}$ defined in (1.20) holds, then $\{\phi_n(\cdot)\}_n$ weakly converges under $\{\mu_{p,q}(\cdot \,|\, 0 \leftrightarrow x_n)\}_n$ to the distribution of*

$$\left(\sqrt{\chi_{p,1}} B_1(\cdot), \dots, \sqrt{\chi_{p,d-1}} B_{d-1}(\cdot)\right)$$

where $B_i(\cdot)$, $i = 1, \dots, d-1$ are independent Brownian bridges on $[0,1]$, and $\chi_{p,i}(t)$, $i = 1, \dots, d-1$ are the principal curvatures of ∂W_ξ at t.

1.4.7 Exponential decay in finite volume with wired boundary condition and cluster size estimates on $\mathbb{Z}^2$

As already mentioned, it is a hard question for general graphs to prove that the percolation transition takes place at the same critical parameter as the exponential decay of connectivities.

The proof of $p_c(q) = \tilde{p}_c(q)$ on $\mathbb{Z}^2$, was known only for $q = 1, 2$ and q large until very recently (see [52, 74, 66]). It is now proved [11] that $p_c(q) = \tilde{p}_c(q) = p_{sd}(q)$ for all values of $q \geqslant 1$. Note that for the case of large values of q, the paper [66] proves that non-uniqueness holds at $p_{sd}(q)$, which was then identified to be the critical value of the random-cluster model, in view of Remark (1.3).

Theorem 1.2 (Beffara, Duminil-Copin, 2010) *Let $G_\infty = \mathbb{Z}^2$, and $q \geqslant 1$. Then the critical value $p_c(q)$ for the random-cluster model with parameter q satisfies*

$$p_c(q) = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{1.23}$$

Moreover, for all $q \geqslant 1$ and for any $p < p_c(q)$ there exist two constants $c, C \in (0, \infty)$ depending on p and q such that for any $x, y \in \mathbb{Z}^2$,

$$\mu_{p,q}(x \leftrightarrow y) \leqslant C e^{-c|x-y|} \tag{1.24}$$

were $|\cdot|$ denotes the Euclidean norm.

The description of the proof of this result goes beyond the scope of this manuscript. We mention the rough heuristics, which involves two main ingredients. The first one consists in estimating crossing probabilities at the self-dual point $p_{sd}(q)$, i.e. showing that the probability of crossing a rectangle with length n and height αn in the horizontal direction is bounded away from 0 and 1 uniformly in n. It is the analog of the Russo-Seymour-Welsh theorem for percolation (cf. [52][Theorem 11.70]). The second ingredient is called sharp threshold theorem, analog of [36] for percolation, which is used to show that the probability of crossings rectangles goes rapidly to 1 when $p > p_{sd}(q)$. The approach up to here allows the determination of the critical value, but it provides only weak estimate on the speed of convergence towards 0 for crossing probabilities in the sub-critical regime. Another threshold theorem is needed to prove that the cluster-size at the origin has finite moments of any order, which implies exponential decay of the two-point function by [53, Theorem 5.86].

An important consequence of Theorem 1.2, which will be useful for our work, is that on $\mathbb{Z}^2$ the hypothesis $\mathbb{H}_{p,q}$ is satisfied for $p < p_c(q)$ for all $q \geqslant 1$.

Indeed, exponential decay of connectivities in the bulk *together with* uniqueness of the infinite volume measure imply (1.20) for a very general class of graphs. We explain the argument now in the case of $\mathbb{Z}^2$ but a generalization to $\mathbb{Z}^d$ or even non nearest neighbor graphs works the same.

Proposition 1.24 *Let* $G_\infty = \mathbb{Z}^2$, $p \in [0,1]$ *and* $q \geqslant 1$ *such that there exists a unique infinite random-cluster measure* $\mu_{p,q}$, *which, in addition, has exponential decay of connectivities*

$$\mu_{p,q}(0 \leftrightarrow x) \leqslant e^{-c'|x|}$$

Then hypothesis $\mathbb{H}_{p,q}$ *is satisfied, i.e. there exists a constant* $c = c(p,q) > 0$ *such that for all* $n \geqslant 0$,

$$\mu^w_{\Lambda_n,p,q}(0 \leftrightarrow \Lambda_n^c) \leqslant e^{-cn}$$

Proof We follow the lines of [21, Appendix]. First let us prove that

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mu^w_{\Lambda_n,p,q}\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} \mathbb{1}_{[x\leftrightarrow\Lambda_n^c]} \geqslant \delta\right) < 0 \tag{1.25}$$

By uniqueness and exponential decay property of the infinite-volume measure,

$$\lim_{m\to\infty} \mu^w_{\Lambda_m,p,q}(0 \leftrightarrow \Lambda_m^c) = \mu_{p,q}(0 \leftrightarrow \infty) = 0$$

Hence, for any $\delta > 0$ there exists some $m = m(\delta) < \infty$ such that

$$\mu^w_{\Lambda_m,p,q}\left(\frac{1}{|\Lambda_m|}\sum_{x\in\Lambda_m} \mathbb{1}_{[x\leftrightarrow\Lambda_m^c]}\right) \leqslant \delta/2 \tag{1.26}$$

Now divide Λ_n in cells of side-length m such that $n = (2k+1)m$. We have,

$$\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} \mathbb{1}_{[x\leftrightarrow\Lambda_n^c]} \leqslant \frac{1}{|\Lambda_k|}\sum_{x\in k\mathbb{Z}^2\cap\Lambda_n}\left(\frac{1}{|\Lambda_m|}\sum_{y\in\Lambda_m(x)} \mathbb{1}_{[y\leftrightarrow\Lambda_m(x)^c]}\right)$$

Let us write $f(x) = \sum_{y\in\Lambda_m(x)} \mathbb{1}_{[y\leftrightarrow\Lambda_m(x)^c]}$. By inclusion of events, the Markov inequality, and finally the FKG inequality, we have for any $t > 0$,

$$\begin{aligned}
\mu^w_{\Lambda_n,p,q}\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} \mathbb{1}_{[x\leftrightarrow\Lambda_n^c]} \geqslant \delta\right) &\leqslant \mu^w_{\Lambda_n,p,q}\left(\frac{1}{|\Lambda_k|}\sum_{x\in k\mathbb{Z}^2\cap\Lambda_n} f(x) \geqslant \delta\right)\\
&\leqslant e^{-t\delta k^2}\mu^w_{\Lambda_n,p,q}\left(\prod_{x\in\mathbb{Z}^2\cap\Lambda_n} e^{tf(x)}\right)\\
&\leqslant e^{-t\delta k^2}\prod_{x\in\mathbb{Z}^2\cap\Lambda_n}\mu^w_{\Lambda_m(x),p,q}\left(e^{tf(x)}\right)
\end{aligned}$$

Now we estimate the right-most term by using the general result that, for a random variable X such that $\mathbb{E}(X) = 0$ et $a \leqslant X \leqslant b$, we have

$$\mathbb{E}(e^{tX}) \leqslant e^{t^2(b-a)^2/8}. \tag{1.27}$$

This follows from writing $X = \alpha b + (1-\alpha)a$ for an explicit (random) $\alpha \in (0,1)$ and using the convexity of $y \mapsto e^{ty}$ together with the Taylor formula. Using (1.26),(1.27) and the fact that $f(x) \in \{0,1,\dots,m^2\}$, it is easy to see that

$$\mu^w_{\Lambda_m(x),p,q}\left(e^{tf(x)}\right) \leqslant \exp(tm^2\delta/2 + t^2m^4/8).$$

Plugging this result into the last expression we get

$$\begin{aligned}\mu^w_{\Lambda_n,p,q}\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\mathbb{1}_{[x\leftrightarrow\Lambda_n^c]} \geqslant \delta\right) &\leqslant \exp(-t\delta k^2 + k^2(tm^2\delta/2 + t^2m^4/8))\\ &\leqslant \exp\left(k^2\frac{-\delta^2(m^2-2)^2}{2m^4}\right)\end{aligned}$$

where in the last line we minimized over the value of t. As $k = n/m$ we obtain (1.25). Let $A_{\ell,n} = \Lambda_n\backslash\Lambda_\ell$. By (1.25), we can restrict our attention to the configurations where

$$\sum_{x\in A_{\frac{n}{2},n}}\mathbb{1}_{[x\leftrightarrow\Lambda_n^c]} < \delta n^2/2.$$

Thus in these configurations there is at least one square layer $L_\ell = \{x : |x| = n-\ell\}$ which contains less than δn sites connected to the complement of Λ_n. Explore the cluster of the boundary from the exterior and declare you are at layer $L_{\ell^\star}$ if it is the first layer which contains less than δn sites connected to Λ_n^c in $A_{n-\ell^\star,n}$. The events $E_\ell = \{\ell = \ell^\star\}$ partition the event $\{0 \leftrightarrow \Lambda^c_{n/2}\}$:

$$\{0\leftrightarrow\Lambda^c_{n/2}\} = \bigvee_{\ell=0}^{n/2}\{0\leftrightarrow\Lambda^c_{n/2}\,;\,E_\ell\}$$

Let ω^ℓ_{in} denote the random-cluster configuration inside $\Lambda_{n-\ell}$ and ω^ℓ_{out} the one in $A_{n-\ell,n}$. On the event E_ℓ, we have to close less than δn edges on $\partial\Lambda_{n-\ell}$ in order to disconnect $\Lambda_{n-\ell}$ from Λ_n^c. Hence, by the finite energy property, for each $\omega = \omega^\ell_{in}\vee\omega^\ell_{out}$ fulfilling the event E_ℓ,

$$\mu^w_{\Lambda_n,p,q}(\omega^\ell_{in}\vee\omega^\ell_{out}) \leqslant e^{c\delta n}\mu^w_{\Lambda_n,p,q}(\omega^\ell_{in}\vee\omega^\ell_{out};\partial\Lambda_{n/2}\nleftrightarrow\partial\Lambda_n)$$

for some $c<\infty$ uniform in $\omega^\ell_{in},\omega^\ell_{out}$. Hence, for each ℓ we have:

$$\begin{aligned}\mu^w_{\Lambda_n,p,q}(0\leftrightarrow\partial\Lambda_{n/2};E_\ell) &\leqslant e^{c\delta n}\mu^w_{\Lambda_n,p,q}(0\leftrightarrow\partial\Lambda_{n/2};\partial\Lambda_{n/2}\nleftrightarrow\partial\Lambda_n;E_\ell)\\ &\leqslant e^{c\delta n}\mu^w_{\Lambda_n,p,q}(0\leftrightarrow\partial\Lambda_{n/2};\partial\Lambda_{n/2}\nleftrightarrow\partial\Lambda_n)\\ &\leqslant e^{c\delta n}\mu_{p,q}(0\leftrightarrow\partial\Lambda_{n/2}) \leqslant e^{-c'n}\end{aligned}$$

for some $c'>0$ where in the last line we omitted the standard decoupling argument explained in (1.18). ■

We will use the following corollary as a key initial step in the proof of our main result, which is Theorem 2.2 below.

Corollary 1.5 *Let* $G_\infty = \mathbb{Z}^2$*. For all* $q \geqslant 1$ *and* $p < p_c(q)$*,*

1. *there exists a constant* $c = c(p, q) > 0$ *such that for all* $n \geqslant 0$*,*

$$\mu^w_{\Lambda_n,p,q}(0 \leftrightarrow \Lambda_n^c) \leqslant e^{-cn}$$

2. *for* n *large enough and* $m \geqslant n$*,*

$$\mu^w_{\Lambda_n,p,q}(\textit{there exists a cluster of cardinality } m \textit{ in } \Lambda_{n/2}) \leqslant e^{-cm}.$$

Proof The assertion 1 follows from Proposition 1.19 and Theorem 1.2. Now we use assertion 1 to prove assertion 2.

Let us construct a coarse graining at scale K of an open cluster $\mathcal{C}$. Namely, cover $\mathcal{C}$ iteratively with balls for the ξ-norm of radius $K + c\log K$, written $\bar{\mathbf{B}}_K$, while balls of radius K are written $\mathbf{B}_K$: center the first ball at some $u_i = x_0 \in \Lambda_{n/2}$, and continue centering the $(n+1)$-th ball at the site where $\mathcal{C}$ exits $\cup_{i=1}^n \bar{\mathbf{B}}_K(x_i)$, see Figure 1.6. This produces a graph $\mathcal{F}_K$ with vertices $\{x_i\}$ and such that $[x_i, x_j]$ (with $i > j$) is an edge if $\mathcal{C}$ exits $\cup_{k=1}^{i-1} \bar{\mathbf{B}}_K(x_k)$ through $\bar{\mathbf{B}}_K(x_j)$. Writing $\bar{A}_i = \bar{\mathbf{B}}_K(x_i) \setminus \cup_{j=1}^{i-1} \mathbf{B}_K(x_j)$, and $A_i = \mathbf{B}_K(x_i) \setminus \cup_{j=1}^{i-1} \bar{\mathbf{B}}_K(x_j)$, and $N_{\mathcal{F}} = \sharp\{i : x_i \in \mathcal{F}\}$ we have the upper-bound:

$$\begin{aligned}\mu^w_{\Lambda_n,p,q}(\mathcal{F}_K = \mathcal{F}) &\leqslant \mu^w_{\Lambda_n,p,q}\left(\bigcap_{x_i \in \mathcal{F}} x_i \xleftrightarrow{A_i} \partial\mathbf{B}_K(x_i)\right) \\ &\leqslant \prod_{x_i \in \mathcal{F}} \mu^w_{\bar{A}_i}\left(x_i \xleftrightarrow{A_i} \partial\mathbf{B}_K(x_i)\right) \leqslant e^{-KN_{\mathcal{F}}(1-o_K(1))} \qquad (1.28)\end{aligned}$$

where in the second inequality we expand the probability of the intersection as a product of conditional probabilities and then use the FKG inequality to compare these conditional probabilities with the probability with wired boundary conditions.

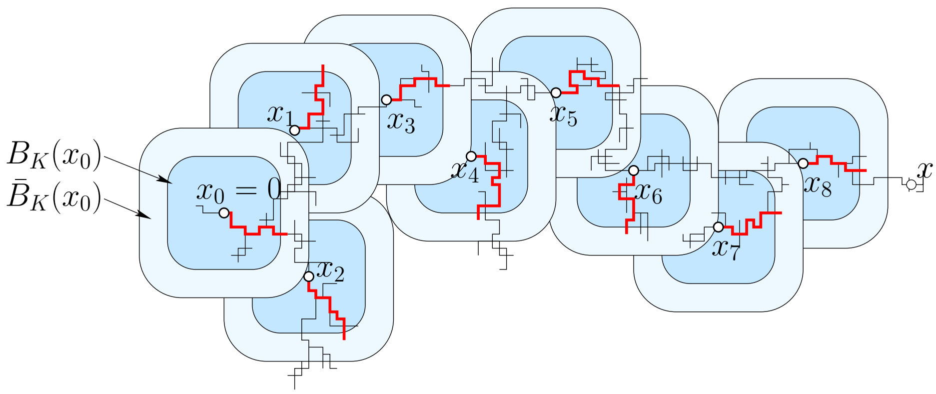


Figure 1.6 – Coarse-graining procedure.

We finally used that,

$$\mu^{w}_{\bar{A}_i,p,q}(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)) = e^{-K}(1 + o_K(1))$$

Indeed, by the FKG inequality,

$$\mu^{w}_{\bar{A}_i,p,q}(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)) \leqslant \mu^{w}_{\bar{A}_i,p,q}(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)) \cdot \mu^{w}_{\bar{A}_i \setminus A_i,p,q}(\partial A_i \leftrightarrow \partial \bar{A}_i) + \mu^{f}_{\bar{A}_i,p,q}(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)).$$

where by assertion 1,

$$\mu^{w}_{\bar{A}_i \setminus A_i,p,q}(\partial A_i \leftrightarrow \partial \bar{A}_i) = o_K(1),$$

and again by FKG and exponential decay in infinite volume,

$$\mu^{f}_{\bar{A}_i,p,q}(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)) \leqslant \mu(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i)) \leqslant e^{-K}.$$

To prove assertion 2, assume in the configuration ω there exists a cluster of cardinality m in $\Lambda_{n/2}$, with $m \geqslant n$. Write $\Lambda_{n/2} = \{y_1, \dots, y_{n^2}\}$. Construct the coarse graining at scale K of the open cluster of y_i and delete it if less than $m/2K^2$ balls are used; continue with y_{i+1} and so on. Given the assumption, we necessarily end up with the (non-empty) coarse-graining $\mathcal{F}_K(\omega)$ of the first cluster of cardinality at least m in ω according to our enumeration of $\Lambda_{n/2}$. Hence, using (1.28), and the fact that $m \geqslant n$, we get

$$\begin{aligned}
&\mu^{w}_{\Lambda_n,p,q}(\text{there exists a cluster of cardinality } m \text{ in } \Lambda_{n/2}) \\
&\quad = \sum_{i=1}^{n^2} \sum_{\substack{\mathcal{F} \text{ starting at } y_i: \\ N_{\mathcal{F}} \geqslant m/2K^2}} \mu^{w}_{\Lambda_n,p,q}(\mathcal{F}_K = \mathcal{F}) \\
&\quad \leqslant n^2 (cK)^{m/2K^2} e^{-K\frac{m}{2K^2}(1-o_K(1))} \leqslant e^{-m[o_K(1)+o_n(1)]} \leqslant e^{-cm}
\end{aligned}$$

for K (and n) sufficiently large, where we upper bound crudely the number of possible coarse-grainings starting at y_i with $m/2K^2$ steps by $(cK)^{m/2K^2}$. ■

1.5 Results for the Ising and Potts models

1.5.1 Infinite volume Potts measures

In Section 1.2.2, we introduced the set $\mathcal{G}_\beta$ of all infinite-volume measures for the Ising model, and proved the existence of the so-called $+$ and $-$ phases of the Ising model, i.e. $\mathbf{P}^{+}_{\beta}$ and $\mathbf{P}^{-}_{\beta}$. The analogous set $\mathcal{G}_{\beta,q}$ of infinite volume Potts measures will be of interest in the sequel.

As before we consider a sequence of graphs $G = (G_n)_n$ such that $G_n \Subset G_\infty = (V_\infty, \mathcal{E}_\infty)$ and $G_n \uparrow G_\infty$ as $n \to \infty$ (which we write $G \uparrow G_\infty$), where G_∞ is an infinite, locally finite, connected graph. We also consider a sequence of boundary conditions $\overline{\sigma}_n$ on G_n. Let $\Sigma = \{1, \dots, q\}^{V_\infty}$, and define:

Definition 1.16 *Let $\beta \geqslant 0$. A probability measure $\mathbb{P}$ on Σ is an infinite volume measure of the Potts model at inverse temperature β if $\mathbb{P}$ is an accumulation point of some sequence of finite volume measures $\{\mathbb{P}^{\overline{\sigma}_n}_{G_n,\beta}\}_n$, for the weak topology (see Definition 1.4). We write*

$$\mathcal{G}_{\beta,q} = \{\text{Weak limits of sequences } (\mathbb{P}^{\overline{\sigma}_n}_{G_n,\beta,q})_n, \text{ with } G_n \uparrow G_\infty, \text{ and } \overline{\sigma}_n \in \Sigma\}$$

In the last section, we have proved the existence of the infinite volume random-cluster measures $\mu^f_{p,q}$ and $\mu^w_{p,q}$. The corresponding Potts measures coupled to them are the free and "1" measures, the latter is also called the pure phase "1". In general we define the following weak limits:

$$\mathbb{P}^f_{\beta,q} = \lim_{G\uparrow G_\infty} \mathbb{P}^f_{G,\beta,q} \qquad \text{(free phase)} \tag{1.29}$$

$$\mathbb{P}^i_{\beta,q} = \lim_{G\uparrow G_\infty} \mathbb{P}^i_{G,\beta,q}, \quad i = 1,\dots,q \qquad \text{(pure phases)} \tag{1.30}$$

where $\mathbb{P}^f_{G,\beta,q}$ is the Potts measure on G with no prescribed color on the boundary of G. We have proved the existence of the analogous $+, -, f$ limits in the case of the Ising model in Proposition 1.6. The easiest way to prove the existence of these Potts infinite volume measures is to use the Edwards-Sokal coupling with a random-cluster on G and use the stochastic monotonicity in the volume in the latter. The coupling survives in infinite volume; we have indeed the following conditional measures.

Proposition 1.25 *Let $G_\infty = (V_\infty, \mathcal{E}_\infty)$ be an infinite, locally finite graph.*

1. *Let ω be distributed according to $\mu^w_{p,q}$. Assign a random color $\sigma_x \in \{1,\dots,q\}$ to every vertex $x \in V_\infty$ in such a way to have*

$$\begin{cases} \text{color 1 on infinite cluster(s)} \\ \text{constant color on each finite cluster, distributed uniformly on } \{1,\dots q\} \\ \text{independent colors between clusters} \end{cases}$$

   Then $\sigma = (\sigma_x)_{x\in V_\infty}$ is distributed according to $\mathbb{P}^1_{\beta,q}$ which satisfy (1.30).

2. *Let σ be distributed according to $\mathbb{P}^1_{\beta,q}$. Assign a random number $\omega(e) \in \{0,1\}$ to every edge $e \in \mathcal{E}_\infty$ in such a way that independently for each edge $e = [i,j]$,*

$$\begin{cases} \text{If } \sigma_i \neq \sigma_j, \text{ then } \omega(e) = 0, \\ \text{If } \sigma_i = \sigma_j, \text{ then } \omega(e) = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases} \end{cases}$$

   Then $\omega = (\omega(e))_{e\in\mathcal{E}_\infty}$ is distributed according to $\mu^w_{p,q}$.

3. *The same is valid for the measures $\mu^f_{p,q}$ and $\mathbb{P}^f_{\beta,q}$ except that infinite open clusters must be colored independently uniformly in $\{1,\dots,q\}$ in the analog of assertion 1.*

Remark 1.4 *The measures $\mathbb{P}^f_{\beta,q}$ and $\mathbb{P}^i_{\beta,q}$ satisfy as well the automorphism invariance properties we proved for $\mu^f_{p,q}$ and $\mu^w_{p,q}$ in Proposition 1.17. We refer to [53] for the proofs.*

1.5.2 Phase transition

As in the Ising case, we can define a notion of magnetization in the Potts model.

Definition 1.17 *For a finite graph* G*, the magnetization of the vertex* $x \in V$ *under the measure* $\mathbb{P}^1_{G,\beta,q}$ *is*

$$m^1_{G,\beta}(x) = \frac{\mathbb{P}^1_{\beta,q}(\sigma_x = 1) - 1/q}{1 - 1/q}.$$

The total magnetization of the finite system is

$$m^1_{G,\beta} = \frac{1}{|G|} \sum_{x \in V} m^1_{G,\beta}(x).$$

Let G_∞ *be an infinite, locally finite graph. The associated infinite volume magnetization under the "1" phase is the following limit, if it exists:*

$$m^1_\beta = \lim_{G \uparrow G_\infty} m^1_{G,\beta}$$

Note that $m^1_{G,\beta} = 1$ if all the vertices of G have the color 1, and $m^1_{G,\beta} = 0$ if each vertex has a uniform color in $\{1, \ldots q\}$ (it is the case for $\beta = 0$). We say that the system is in an ordered phase if the infinite volume magnetization (exists and) is positive, i.e. $m^1_\beta > 0$. On the other hand, as we have seen, we say that percolation occurs in the (infinite volume) random-cluster model if $\mu^w_{p,q}(0 \leftrightarrow \infty) > 0$. We will prove that the two limits coincide for transitive graphs.

Remark 1.5 *Note that a transitive Van Hove graph has necessarily sub-exponential balls.*

Proposition 1.26 *Let* G_∞ *be a locally finite graph. Then the Potts vertex-magnetization is positive if and only if the corresponding random-cluster model percolates:*

$$m^1_\beta(x) = \frac{\mathbb{P}^1_{\beta,q}(\sigma_x = 1) - 1/q}{1 - 1/q} = \mu^w_{p,q}(x \leftrightarrow \infty).$$

In the case $q = 2$*, this can be rewritten in terms of the Ising model as:*

$$\mathbf{P}^+_\beta(\sigma_x) = \mu^w_{p,2}(x \leftrightarrow \infty).$$

Moreover, if G_∞ *is also transitive and Van Hove, the magnetization coincides with the vertex-magnetization at the origin, and*

$$m^1_\beta = m^1_\beta(0) = \mu^w_{p,q}(0 \leftrightarrow \infty).$$

For the Ising model, recalling Definition (1.7)*, we have:*

$$m^\star_\beta = \mathbf{P}^+_\beta(\sigma_0) = \mu^w_{p,2}(0 \leftrightarrow \infty).$$

Proof We will proceed in three steps. First we recall Proposition 1.11: the component 1 of the Potts magnetization at site i is the random-cluster probability that i is connected to the boundary of the graph:

$$\frac{\mathbb{P}^1_{G,\beta,q}(\sigma_x = 1) - 1/q}{1 - 1/q} = \mu^w_{G,p,q}(x \leftrightarrow \partial G)$$

And then we show that the expressions with x averaged over G on both sides converge to respectively $m^1(\beta)$ (in the case of transitive graphs) and $\mu^w_{p,q}(0 \leftrightarrow \infty)$ as $G \uparrow G_\infty$. As before, we have the relation $p = 1 - e^{-\beta}$. Let us first prove that

$$\mu^w_{p,q}(0 \leftrightarrow \infty) = \lim_{G \uparrow G_\infty} \mu^w_{G,p,q}(0 \leftrightarrow G^c)$$

By the FKG inequality we have, for $0 \in \Delta \subset G$:

$$\mu^w_{p,q}(0 \leftrightarrow G^c) \leqslant \mu^w_{G,p,q}(0 \leftrightarrow G^c) \leqslant \mu^w_{G,p,q}(0 \leftrightarrow \Delta^c).$$

Taking the limit $G \uparrow G_\infty$, we obtain:

$$\mu^w_{p,q}(0 \leftrightarrow \infty) \leqslant \lim_{G \uparrow G_\infty} \mu^w_{G,p,q}(0 \leftrightarrow G^c) \leqslant \mu^w_{p,q}(0 \leftrightarrow \Delta^c),$$

and finally taking the limit $\Delta \uparrow \ G_\infty$ leads to:

$$\mu^w_{p,q}(0 \leftrightarrow \infty) \leqslant \lim_{G \uparrow G_\infty} \mu^w_{G,p,q}(0 \leftrightarrow G^c) \leqslant \mu^w_{p,q}(0 \leftrightarrow \infty).$$

as claimed. Now, when the graph G_∞ is supposed to be Van Hove and transitive, we take $|\partial G|/|G| \to 0$ as $G \uparrow G_\infty$, this will allow us to prove that

$$\mathbb{P}^1_{\beta,q}(\sigma_0 = 1) = \lim_{G \uparrow G_\infty} \mathbb{P}^1_{G,\beta,q}\left(\frac{1}{|G|}\sum_{i \in V} \delta_{\sigma_i,1}\right).$$

Write as before $\mathbb{P}^1_{G,\beta,q}(\delta_{\sigma_i,1}) = \mu^w_{G,p,q}(i \leftrightarrow G^c)\left(1 - \frac{1}{q}\right) + \frac{1}{q}$. Note that by the existence of the infinite volume Potts measure $\mathbb{P}^1_{\beta,q}$ (justified above) we can write $\mathbb{P}^1_{\beta,q}(\delta_{\sigma_0,1}) = \lim_{G \uparrow G_\infty} \mathbb{P}^1_{G,\beta,q}(\delta_{\sigma_i,1})$. It remains to prove that

$$\lim_{G \uparrow G_\infty} \mu^w_{G,p,q}(0 \leftrightarrow G^c) = \lim_{G \uparrow G_\infty} \frac{1}{|G|}\sum_{i \in V} \mu^w_{G,p,q}(i \leftrightarrow G^c).$$

On the one hand, by the FKG inequality, for all i we have,

$$\mu^w_{G,p,q}(i \leftrightarrow G^c) \geqslant \mu^w_{p,q}(i \leftrightarrow G^c) \geqslant \mu^w_{p,q}(i \leftrightarrow \infty) = \mu^w_{p,q}(0 \leftrightarrow \infty) = \lim_{G \uparrow G_\infty} \mu^w_{p,q}(0 \leftrightarrow G^c),$$

which implies the upper bound. On the other hand, let $R > 0$. Let us define the ball of radius R centered at i in the graph G by $B_R(i)$ (for the graph distance). For any $i \in V$ located at a distance at least R from ∂G, we have, by the FKG inequality:

$$\mu^w_{G,p,q}(i \leftrightarrow G^c) \leqslant \mu^w_{B_R(i),p,q}(i \leftrightarrow (B_R(i))^c)$$

Hence,

$$\begin{aligned}
&\limsup_{G\uparrow G_\infty} \frac{1}{|G|}\sum_{i\in V} \mu^{w}_{G,p,q}(i\leftrightarrow G^c)\\
&\qquad\leqslant \limsup_{G\uparrow G_\infty}\left(\frac{1}{|G|}\sum_{i\in V}\mu^{w}_{B_R(i),p,q}(i\leftrightarrow (B_R(i))^c) + \frac{|\partial G|\sup_{i\in V}|B_R(i)|}{|G|}\right)\\
&\qquad= \mu^{w}_{B_R(0),p,q}(0\leftrightarrow (B_R(0))^c)\quad \text{by transitivity.}
\end{aligned}$$

Now as $B_R(0)\uparrow G_\infty$ as $R\to\infty$, we have:

$$\begin{aligned}
\limsup_{G\uparrow G_\infty} \frac{1}{|G|}\sum_{i\in V} \mu^{w}_{G,p,q}(i\leftrightarrow G^c) &\leqslant \lim_{R\to\infty} \mu^{w}_{B_R(0),p,q}(0\leftrightarrow (B_R(0))^c)\\
&= \mu^{w}_{p,q}(0\leftrightarrow\infty) = \lim_{G\uparrow G_\infty} \mu^{w}_{G,p,q}(0\leftrightarrow G^c)
\end{aligned}$$

■

Propositions 1.26, 1.18 and 1.11 allow us to define the critical parameter for the magnetization transition and for exponential decay of the correlations in the Potts model. They satisfy $p_c(q) = 1 - e^{-\beta_c(q)}$ (and similarly for $\tilde{p}_c(q)$).

Definition 1.18 *Let G_∞ be a locally finite, periodic graph, and $q \geqslant 1$. Let $\mathfrak{F}(0)$ be the fundamental cell of the graph around the origin. Then the critical parameters for the magnetization and the exponential decay in the Potts model are defined by $\beta_c(q) = -\log(1-p_c(q))$ and $\tilde{\beta}_c(q) = -\log(1-\tilde{p}_c(q))$, namely*

$$\beta_c(q) = \sup\{\beta \geqslant 0 : \inf_{x\in\mathfrak{F}(0)} m^1_\beta(x) > 0\},$$

$$\tilde{\beta}_c(q) = \sup\left\{\beta \geqslant 0 : \exists c>0 \text{ s.t. } \frac{\mathbb{P}^1_{\beta,q}(\sigma_x=\sigma_y) - 1/q}{1-1/q} \leqslant e^{-c|x-y|} \text{ for all } x,y\in V_\infty\right\}$$

We recall Definition 1.8 for the Ising model. Note as well that

$$\tilde{\beta}_c(2) = \sup\left\{\beta \geqslant 0 : \exists c>0 \text{ such that } \mathbf{P}^+_\beta(\sigma_0\sigma_x) \leqslant e^{-c|x-y|} \text{ for all } x,y\in V_\infty\right\}$$

An immediate consequence of Proposition 1.20 and Theorem 1.2 is the following

Corollary 1.6 *Let $q \geqslant 1$.*

1. *Let G_∞ be a locally finite, periodic graph of dimension $d \geqslant 2$, then*

$$0 < \tilde{\beta}_c(q) \leqslant \beta_c(q) < \infty$$

2. *If $G_\infty = \mathbb{Z}^2$,*

$$\tilde{\beta}_c(q) = \beta_c(q) = \log(1+\sqrt{q})$$

1.5.3 Uniqueness of the infinite volume measure

Let $\mathcal{G}_{\beta,q}$ be the set of infinite volume Gibbs measures of the Potts model at inverse temperature β and parameter q as in Definition 1.16. We will show that $\mathcal{G}_{\beta,q}$ is a singleton for any $\beta < \beta_c(q)$. The interesting regime where several Gibbs states can coexist is thus $[\beta_c(q), \infty)$. In the regime $(\beta_c(q), \infty)$, it is easy to show that there are at least q different mutually singular Gibbs states: the q pure phases defined in (1.30). We refer to Chapter 2 for a review of known and new results on $\mathcal{G}_{\beta,q}$.

Proposition 1.27 *Let G_∞ be a locally finite, transitive, Van Hove graph. Let $q \geqslant 1$. Then,*

1. *for all $\beta < \beta_c(q)$, $|\mathcal{G}_{\beta,q}| = 1$,*
2. *for all $\beta > \beta_c(q)$, there exists at least q mutually singular measures in $\mathcal{G}_{\beta,q}$.*

Proof

1. Let $\beta < \beta_c(q)$. The argument is close to the proof Proposition 1.19 which states uniqueness of the random-cluster measure for $p < p_c(q)$. Let $\mathbb{P}$ be the infinite volume Gibbs measure for the Potts model given by the weak limit $\mathbb{P} = \lim_{G\uparrow G_\infty} \mathbb{P}^{\overline{\sigma}}_{G,\beta,q}$, with some arbitrary boundary condition $\overline{\sigma}$. Generalizing the Edwards-Sokal coupling for non-pure boundary condition (as we did in a special case in Section 1.4.5.2), we see that the finite volume measure $\mathbb{P}^{\overline{\sigma}}_{G,\beta,q}$ can be coupled to the random-cluster measure $\mu^{w,\overline{\sigma}}_{G,p,q}(\cdot\,|\,\mathrm{Cond}(\overline{\sigma}))$ where $p = 1-e^{-\beta} < p_c(q)$, the boundary condition $w,\overline{\sigma}$ denotes wired except some holes at edges corresponding to Potts color changes, and $\mathrm{Cond}(\overline{\sigma}) = \{\mathfrak{Col}_i \nleftrightarrow \mathfrak{Col}_j, \forall i \neq j\}$ with $\mathfrak{Col}_i = \{x \in \partial G : \sigma_x = i\}$, and $i = 1 \ldots q$.
We will prove that $\mathbb{P} = \mathbb{P}^f_{\beta,q}$, which intuitively comes from the fact that under the (random-cluster) event $\{0 \nleftrightarrow \infty\}$, there exists a "surface" of closed edges with high probability, making any subcritical measure close to the free measure. The rigorous proof of this fact is not difficult but we explain it deliberately in details to emphasize tricky points.
We must prove that for every Potts local function f, we have
$$\lim_{G\uparrow G_\infty} \mathbb{P}^{\overline{\sigma}}_{G,\beta,q}(f) = \mathbb{P}^f_{\beta,q}(f).$$
Let $B_R \Subset G$ be a ball of radius R such that $\mathrm{Support}(f) \subset B_R$, and define $E_n = \{B_R \leftrightarrow B_n^c\}$. Since $p < p_c(q)$, we have $\mu^w_{p,q}(0 \leftrightarrow \infty) = 0$, which implies $\mu^w_{p,q}(E_n) \to 0$ as $n \to \infty$. By FKG, this is also true under $\mu^{w,\overline{\sigma}}_{G,p,q}(\cdot\,|\,\mathrm{Cond}(\overline{\sigma}))$, since the conditioning is decreasing:
$$\mu^{w,\overline{\sigma}}_{G,p,q}(E_n\,|\,\mathrm{Cond}(\overline{\sigma})) \leqslant \mu^w_{p,q}(E_n) \to 0$$

we can then suppose that E_n^c is realized, since for all functions g:

$$\begin{aligned}\mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})) &= \mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n)\mu^{w,\overline{\sigma}}_{G,p,q}(E_n\,|\,\mathrm{Cond}(\overline{\sigma}))\\ &\quad+\mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n^c)\mu^{w,\overline{\sigma}}_{G,p,q}(E_n^c\,|\,\mathrm{Cond}(\overline{\sigma}))\\ &= o_n(1)+(1-o_n(1))\mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n^c)\end{aligned}$$

On E_n^c, there exists some subset Δ such that $B_R \subseteq \Delta \subseteq B_n$ and all the edges of $\partial\Delta$ are closed. Let $E^c_{n,\Delta} = E_n^c \cap \{\Delta \text{ is the biggest such subset}\}$ (where biggest is defined with respect to the inclusion).

Now observe two crucial facts. First, the Potts local function f gets rephrased as a random-cluster function g depending a priori on every edge, but with the property that on the event $E^c_{n,\Delta}$, g is measurable with respect to edges of Δ. Indeed, the non-local term $\kappa(\omega)$ in the random-cluster measure is splited once edges are closed on a "surface" which surrounds B_R. Second, on the event $E^c_{n,\Delta}$ the event $\mathrm{Cond}(\overline{\sigma})$ is measurable with respect to the edges outside Δ. Indeed, the disconnection between the different parts of the boundary can be achieved partially by the "surface" $\partial\Delta$.

Now, as in (1.18), by the Markov property we get:

$$\begin{aligned}\mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n^c) &= \sum_{\substack{\Delta:\\ B_R\subseteq\Delta\subseteq B_n}} \mu^{w,\overline{\sigma}}_{G,p,q}(g\,|\,\mathrm{Cond}(\overline{\sigma})\cap E^c_{n,\Delta})\times\\ &\qquad\qquad\times\mu^{w,\overline{\sigma}}_{G,p,q}(E^c_{n,\Delta}\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n^c)\\ &= \sum_{\substack{\Delta:\\ B_R\subseteq\Delta\subseteq B_n}} \mu^f_{\Delta,p,q}(g)\,\mu^{w,\overline{\sigma}}_{G,p,q}(E^c_{n,\Delta}\,|\,\mathrm{Cond}(\overline{\sigma})\cap E_n^c)\end{aligned}$$

we emphasize that in the second line we used the last two observations. Using Proposition 1.21 we have,

$$\mu^f_{\Delta,p,q}(g) = \mu^f_{p,q}(g) + o_R(1)$$

which concludes the proof by taking the limit $n \to \infty$ and then $R \to \infty$.

2. When $\beta > \beta_c$, by definition $\mathbb{P}^i_{\beta,q}(\sigma_0 = i) > 1/q$ for all $i = 1, \dots, q$ (as the Hamiltonian is invariant under the permutations of the colors). Moreover, the Edwards-Sokal coupling implies

$$\mathbb{P}^i_{\beta,q}(\sigma_0 = j) = (1-1/q)\mu^w_{p,q}(0 \nleftrightarrow \infty) + 1/q < 1/q \quad \text{for all } j \neq i.$$

Hence the q pure phases are different mutually singular measures in this regime.

■

1.5.4 Exponential relaxation into pure phases

On planar graphs, the exponential relaxation of the finite volume free and wired random-cluster measures towards corresponding infinite volume measures (see Proposition 1.21), imply an analog statement for the pure phases in the Potts model.

Proposition 1.28 *Let G_∞ be a planar transitive graph. Let $\beta > \beta_c(q)$ be such that the dual parameter $p^\star$ corresponding to $p = 1-e^{-\beta}$ satisfy hypothesis $\mathbb{H}_{p^\star,q}$. Then there exists $c > 0$ such that, for any $n > 0$,*

$$|\mathbb{P}^f_{B_{2n},\beta,q}(f) - \mathbb{P}^f_{\beta,q}(f)| \leqslant O(\|f\|_\infty e^{-cn}),$$
$$|\mathbb{P}^i_{B_{2n},\beta,q}(f) - \mathbb{P}^i_{\beta,q}(f)| \leqslant O(\|f\|_\infty e^{-cn}) \quad \forall i = 1, \dots, q$$

for any f such that $\mathrm{Support}(f) \subseteq B_n$.
This can be rewritten for the Ising model as:

$$|\mathbf{P}^+_{B_{2n},\beta}(f) - \mathbf{P}^+_\beta(f)| \leqslant O(\|f\|_\infty e^{-cn})$$

the same holds for $\mathbf{P}^-_\beta$.

Proof We treat the case of the free boundary condition. The other cases follow from the same proof. Since $p^\star$ satisfy hypothesis $\mathbb{H}_{p^\star,q}$, see (1.20), it follows from Proposition 1.21 that

$$|\mu^w_{B_{2n},p^\star,q}(g) - \mu^w_{p^\star,q}(g)| \leqslant O(\|g\|_\infty e^{-cn})$$

for any function g which has support inside B_n. This in turn implies by duality

$$|\mu^f_{B_{2n},p,q}(g) - \mu^f_{p,q}(g)| \leqslant O(\|g\|_\infty e^{-cn}) \tag{1.31}$$

with the same restriction on g. More generally, let F be the event that there exists an open crossing of the annulus $A_{n,3n/2} = B_{3n/2} \backslash B_n$. The event F has exponentially small $\mu^w_{B_{2n},p^\star,q}$ probability by hypothesis $\mathbb{H}_{p^\star,q}$. Consider a function g' depending a priori on every dual edges, but with the property that $g'\mathbb{1}_{[F^c]}$ is measurable with respect to edges of $B_{3n/2}$. We immediately find that

$$\begin{aligned}
\mu^f_{B_{2n},p,q}(g') &= \mu^f_{B_{2n},p,q}(g'\mathbb{1}_{[F]}) + \mu^f_{B_{2n},p,q}(g'\mathbb{1}_{[F^c]}) \\
&= O(\|g'\|_\infty e^{-cn}) + \mu^f_{B_{2n},p,q}(g'\mathbb{1}_{[F^c]}) \\
&= O(\|g'\|_\infty e^{-c'n}) + \mu^f_{p,q}(g'\mathbb{1}_{[F^c]}) \\
&= O(\|g'\|_\infty e^{-c'n}) + \mu^f_{p,q}(g')
\end{aligned}$$

where in the third line we used (1.31) (which uses hypothesis $\mathbb{H}_{p^\star,q}$), and in the last line we used "soft" infinite volume exponential decay. Hence (1.31) is still valid for the function g'.

Now, consider f depending only on spins in B_n. Via the Edwards-Sokal coupling, $\mathbb{P}^f_{B_{2n},\beta,q}(f)$ and $\mathbb{P}^f_{\beta,q}(f)$ can be seen as $\mu^f_{B_{2n},p,q}(g')$ and $\mu^f_{p,q}(g')$ for a certain function g', depending a priori on every edge, but for which (as above) $g'\mathbb{1}_{[F^c]}$ depends on edges in $B_{3n/2}$ only. We conclude that

$$|\mathbb{P}^f_{B_{2n},\beta,q}(f) - \mathbb{P}^f_{\beta,q}(f)| = |\mu^f_{B_{2n},p,q}(g') - \mu^f_{p,q}(g')| \leqslant O(\|g'\|_\infty e^{-cn})$$

■

Let us mention that whenever there is uniqueness of the infinite volume random-cluster measure as well as (almost sure) uniqueness of the infinite cluster (if any) for the parameters $p = 1 - e^{-\beta}$ and q, then point 3 of Proposition 1.25 implies that

$$\mathbb{P}^{f}_{\beta,q} = \frac{1}{q}\sum_{i=1}^{q} \mathbb{P}^{i}_{\beta,q}. \tag{1.32}$$

1.5.5 Ornstein-Zernike asymptotics of the two-point function on $\mathbb{Z}^d$

It is possible to reformulate Theorem 1.1 for the Potts model in the following way, taking into account Theorem 1.2 and Proposition 1.24. Recall as well Proposition 1.26 for the relation between the two-point functions, and Definition 1.15 for the Wulff shape.

Theorem 1.3
Let $G_\infty = \mathbb{Z}^d$, and $d \geqslant 2$. Let $q \geqslant 1$ and $\beta < \beta_c(q)$. We denote by $\mathbb{P}_{\beta,q}$ the unique infinite volume measure of the q-state Potts model at inverse temperature β.

1. *The two-point function satisfies the Ornstein-Zernike asymptotics:*

$$\frac{\mathbb{P}_{\beta,q}(\sigma_0 = \sigma_x) - 1/q}{1 - 1/q} = \frac{\Psi_{\beta,q}(\hat{x})}{|x|^{\frac{d-1}{2}}} e^{-\xi_{\beta,q}(\hat{x})|x|}(1 + o_{|x|}(1))$$

   uniformly as $|x| \to \infty$, where $\hat{x} = x/|x|$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Moreover, the functions Ψ and ξ are positive and locally analytic on $\mathbb{S}^{d-1}$. In the case of the Ising model, denoting the unique infinite volume measure as $\mathbf{P}_\beta$, we have

$$\mathbf{P}_\beta(\sigma_0\sigma_x) = \frac{\Psi_{\beta,2}(\hat{x})}{|x|^{\frac{d-1}{2}}} e^{-\xi_{\beta,2}(\hat{x})|x|}(1 + o_{|x|}(1))$$

2. *The Wulff shape W_ξ has a locally analytic, strictly convex boundary. Moreover, the Gaussian curvature χ_β of W_ξ is uniformly positive,*

$$\chi_\beta \doteqdot \min_{t\in\partial W_\xi} \prod_{i=1}^{d-1} \chi_{\beta,i}(t) > 0$$

   where $\chi_{\beta,i}(t)$, $i = 1, \dots, d-1$ are the principal curvatures of ∂W_ξ at t. By duality, ∂U_ξ is also locally analytic and uniformly convex.

Remark 1.6 *The result for the Ising model has been proved before the general Potts case by the same authors in the paper [20], using the random-line representation defined in Section 1.2.4.3. The BK inequality 1.12 that satisfy the weights simplifies notably the proof.*

Of course the inverse correlation length of the Potts model coincides with the one of the corresponding random-cluster model, i.e. $\xi_{\beta,q} = \xi_{p,q}$ (with $p = 1 - e^{-\beta}$). We do not rewrite in details the fact that $\xi_{\beta,q}$ satisfies the sharp triangle inequality, see Corollary 1.3.

1.5.6 Absence of roughening transition and Brownian scaling of the interfaces on $\mathbb{Z}^2$

In the special case of two-dimensional random cluster model, duality allows us to reinterpret the results of Theorem 1.3 as results about interfaces in the q-state Potts model in the phase coexistence regime ($\beta > \beta_c$).

We have proved in Section 1.4.5.2 that the inverse correlation length of the sub-critical random-cluster model at $p < p_c$ coincides with the surface tension of the Potts model at the corresponding dual inverse temperature $\beta > \beta_c$ (satisfying $p^\star = 1 - e^{-\beta}$).

An important corollary of Theorem 1.3 for the sub-critical Potts model on $\mathbb{Z}^2$ is the absence of roughening transition, due to the Brownian scaling of the Dobrushin interfaces. Before defining these expressions and stating the results, we recall the definition of the surface tension of the Potts and Ising models.

1.5.6.1 Surface tension for the Potts model

Let $\hat{x} \in \mathbb{S}^1$. Consider the Potts model in the box $\Lambda_{L,M} = \{-M, \dots M\} \times \{-L, \dots, L\}$, with $\hat{x}$-Dobrushin boundary condition, that is

$$\overline{\sigma}_i = \begin{cases} 1 \text{ if } (\hat{x}, i) \geqslant 0 \\ 2 \text{ if } (\hat{x}, i) < 0 \end{cases} .$$

As in Section 1.4.5.2, we write $\mathbb{P}^{\hat{x}}_{\Lambda_{L,M},\beta,q}$ for the corresponding Potts measure, and $x^\star_\ell, x^\star_r$ for the locations on $(\mathbb{Z}^2)^\star \cap \partial\Lambda_n$ of the color changes, see Figure 1.4. The surface tension at inverse temperature β in a direction $\hat{x} \in \mathbb{S}^1$ is defined as follows:

$$\tau_{\beta,q}(\hat{x}) = -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|x^\star_\ell - x^\star_r|} \log \frac{Z^{\hat{x}}_{\Lambda_{L,M},\beta,q}}{Z^{1}_{\Lambda_{L,M},\beta,q}}$$

As we saw, this limit is known to exist for all values of β. It is useful to extend τ_β to a function on $\mathbb{R}^2$ by positive homogeneity, setting $\tau_\beta(x) \doteqdot \tau_\beta(\hat{x})|x|$, where $\hat{x} = x/|x|$. When $\beta > \beta_c$, the extended function is a norm on $\mathbb{R}^2$. The proof is the same as the one of Proposition 1.22, via duality and the Edwards-Sokal coupling. (The statement for the whole regime $\beta > \beta_c$ takes into account Theorem 1.2.)

1.5.6.2 Surface tension for the Ising model

In the case of the Ising model, this corresponds to take the $\hat{x}$-Dobrushin boundary condition

$$\overline{\sigma}_i = \begin{cases} +1 \text{ if } (\hat{x}, i) \geqslant 0 \\ -1 \text{ if } (\hat{x}, i) < 0 \end{cases} ,$$

and, denoting by $\mathbf{P}^{(\pm,\hat{x})}_{\Lambda_{L,M}\beta}$ the corresponding Ising measure in $\Lambda_{L,M}$, the surface tension in direction $\hat{x}$ is defined as

$$\tau_{\beta,2}(\hat{x}) = -\lim_{\substack{L\to\infty \\ M\to\infty}} \frac{1}{|x^\star_\ell - x^\star_r|} \log \frac{Z^{(\pm,\hat{x})}_{\Lambda_{L,M},\beta}}{Z^{+}_{\Lambda_{L,M},\beta}}$$

The function $\tau_{\beta,2}(x) \doteqdot \tau_{\beta,2}(\hat{x})|x|$ was known to be positive (and thus a norm on $\mathbb{R}^2$) for all $\beta > \beta_c$ since the work of Lebowitz and Pfister [70].

1.5.6.3 Corollary of the Ornstein-Zernike asymptotics of the two-point function

We say that a system undergoes a roughening transition when the interface between two phases has bounded fluctuations below a certain temperature (uniformly in the size of the sample) and fluctuations which diverge with the size of the sample above this temperature. A corollary of Theorem 1.3 is the absence of roughening transition for the q-state Potts model on $\mathbb{Z}^2$. Indeed, the underlying proof, as briefly described in Section 1.4.6, shows that the interfaces have a diffusive scaling for all $\beta > \beta_c$.

Indeed, let $\hat{x} \in \mathbb{S}^1$, and consider the box $\Lambda_n = \{-n, \dots n\}^2$, with $\hat{x}$-Dobrushin boundary condition, as defined above. We define the interface as the connected component of all dual edges separating disagreeing spins, which is attached to $x^\star_\ell$ and $x^\star_r$ (see Figure 1.7).

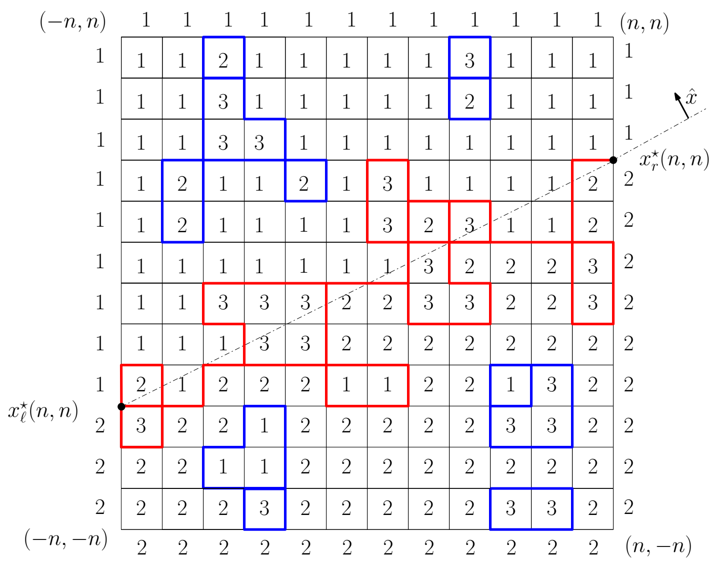


Figure 1.7 – The interface in direction $\hat{x}$ of the 3-state Potts model.

As we saw in Section 1.4.5.2, the measure $\mathbb{P}^{\hat{x}}_{\Lambda_n,\beta,q}$ is coupled to the dual random-cluster measure of $\mu^f_{\Lambda^\star_n,p,q}(\cdot\,|\,x^\star_\ell \leftrightarrow x^\star_r)$. In particular the interface as defined here is a subset of the open cluster connecting x_ℓ to x_r in the latter random cluster model. It can thus can be approximated by the same effective random walk we described in Section 1.4.6.

Denoting as before by Φ_n its linear interpolation, and by ϕ_n diffusive scaling of Φ_n, we get the following corollary:

Corollary 1.7 *Let $G_\infty = \mathbb{Z}^2$, $\beta > \beta_c$ and $q \in \mathbb{N}$. Consider the q-state Potts model with $\hat{x}$-Dobrushin boundary condition in the box Λ_n. Then $\{\phi_n(\cdot)\}_n$ weakly converges under $\{\mathbb{P}^{\hat{x}}_{\Lambda_n,\beta,q}\}_n$ to the distribution of*

$$\{\sqrt{\chi_\beta} B(\cdot)\}$$

where B is the standard Brownian bridge on $[0,1]$ and χ_β is the curvature of ∂W_τ computed at the point $t \in \partial W_\tau$ dual to $\hat{x}$. This curvature is uniformly positive.
In particular, the q-state Potts model on $\mathbb{Z}^2$ do not undergo a roughening transition.

The existence of a Brownian bridge diffusive limit in the Ising model on $\mathbb{Z}^2$ has been established [48] for a single interface, resulting from the Dobrushin boundary condition. Similar results restricted to large β include [37] and [54].

Note that the unboundedness of the Ising interfaces is specific to the two-dimensional model. Indeed, in dimensions $d \geqslant 3$, it was proved by Dobrushin that at sufficiently low temperature, the interface corresponding to Dobrushin boundary condition along coordinate axis has bounded fluctuations. As we will see later this shows that non translation-invariant infinite-volume Gibbs measures exist in dimension 3 and higher.

1.5.7 More results for the random-line representation of the planar Ising model

The previous section gives useful estimates on the decay of correlations. This can be rewritten in the case of the Ising model as estimates on the weights of the random-line representation introduced in Section 1.2.4.3. As we will use these results on $\mathbb{Z}^2$ (were duality plays an important role), we introduce some specific notations.

A subset $A \subset \mathbb{Z}^2$ is said to be (simply) connected if $\bigcup_{i\in A}\left(i + [-\frac{1}{2}, \frac{1}{2}]^2\right)$ is (simply) connected. Let $\Lambda \Subset \mathbb{Z}^2$ be simply connected. Let $\overline{\sigma} \in \{-1,1\}^{\mathbb{Z}^2}$ be some boundary condition. To a configuration σ compatible with this boundary condition, we associate the set $E(\sigma)$ of all edges of the dual lattice $(\mathbb{Z}^2)^\star \doteqdot (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ separating a pair i, j of nearest-neighbor vertices such that $\{i, j\} \cap \Lambda \neq \varnothing$ and $\sigma_i \neq \sigma_j$. We recall the notations and compatibility relations described at the beginning of Section 1.2.4.

The weights $q_{\Lambda,\beta}$ have a number of remarkable properties we began to describe in (1.11), (1.12) and (1.13). Here we add some other useful properties of the weights which are based on estimates on the two-point function of the Ising model on $\mathbb{Z}^2$.

- Let $i, j \in \partial^\star\Lambda$. Then,

$$\sum_{\Gamma: i\to j} q_{\Lambda,\beta}(\Gamma) \leqslant e^{-\tau_\beta(j-i)}. \tag{1.33}$$

  See [77, Lemma 6.6 and Prop. 2.4] for the proof.

- Let $z \in \Lambda^\star$; we write $\Gamma : b \to z \to b'$ when $\Gamma : b \to b'$ and $\Gamma \ni z$. Then,

$$\sum_{\Gamma : b \to z \to b'} q_{\Lambda,\beta}(\Gamma) \leqslant \frac{C(\beta)}{\sqrt{\|z-b\|_2 \|z-b'\|_2}} e^{-\tau_\beta(z-b) - \tau_\beta(z-b')}. \tag{1.34}$$

(this comes essentially from the BK-type inequality (1.12) and from the Ornstein-Zernike behavior of the two-points correlation function in infinite volume, i.e. Theorem 1.3.)

- Let

$$\mathfrak{E}(x, y, \rho) \doteqdot \{t \in (\mathbb{Z}^2)^\star \text{ s.t. } \|x-t\| + \|y-t\| \leqslant \|x-y\| + \rho\}$$

be the ellipse in $\mathbb{R}^2$ with focuses x and y and big axis $2\rho + \|x-y\|$.

The following lemma states that for $\beta^\star < \beta_c$ the set of contours contributing to the infinite volume two points function $\langle \sigma_x \sigma_y \rangle_{\beta^\star}$ is exponentially concentrated into the ellipse $\mathfrak{E}(x, y, \rho)$ if ρ is of order n^ϵ, with $\epsilon > 0$. More precisely, under the infinite volume measure at inverse temperature $\beta^\star < \beta_c$ we have

$$\frac{\sum_{\substack{\lambda : x \to y \\ \lambda \not\subset \mathfrak{E}(x,y,\rho)}} q_{\beta^\star}(\lambda)}{\langle \sigma_x \sigma_y \rangle} = \frac{\sum_{\substack{\lambda : x \to y \\ \lambda \not\subset \mathfrak{E}(x,y,\rho)}} q_{\beta^\star}(\lambda)}{\sum_{\lambda : x \to y} q_{\beta^\star}(\lambda)} \leqslant \mathfrak{c} \cdot |\partial \mathfrak{E}(x, y, \rho)| \cdot \|x-y\|^{1/2} e^{-\kappa_{\beta^\star} \rho} \tag{1.35}$$

for a certain $\mathfrak{c} > 0$, and some $\kappa_{\beta^\star} > 0$. See [77, Lemma 6.10] for a proof.

Chapter 2

About the Gibbs states of the Ising and Potts models

In this chapter, we first present some general results about the set $\mathcal{G}$ of infinite volume Gibbs measures of models with a finite spin space. The results are written in the case of the Potts model to simplify the exposition; the general definition of a Gibbs measure is given thereafter. We continue with a review of the known results toward the characterization of $\mathcal{G}$ for the Ising and the Potts models. We present our new results, the main one being the full characterization of $\mathcal{G}$ for the Potts model on $\mathbb{Z}^2$. Finally, we give a general heuristics, followed by more detailed sketches of the proofs.

2.1 General results

In the last chapter (see Section 1.2.2), we justified physically the definition of the set of infinite-volume measures of a given model as the set of weak limits of finite volume measures. We recall the latter definitions.

Consider a sequence of graphs $G = (G_n)_n$ such that $G_n \Subset G_\infty = (V_\infty, \mathcal{E}_\infty)$ and $G_n \uparrow G_\infty$ as $n \to \infty$ (which we write $G \uparrow G_\infty$), where G_∞ is an infinite, locally finite, connected graph. Let $\mathbf{\Sigma} = \{-1, +1\}^{V_\infty}$, and $\Sigma = \{1, \dots, q\}^{V_\infty}$. We defined, according to Definition 1.4 of weak convergence,

$$\begin{aligned}
\mathcal{G}_\beta &= \{\text{Weak limits of sequences } (\mathbf{P}^{\tilde{\sigma}_n}_{G_n,\beta})_n, \text{ with } G_n \uparrow G_\infty, \text{ and } \tilde{\sigma}_n \in \mathbf{\Sigma}\} \\
\mathcal{G}_{\beta,q} &= \{\text{Weak limits of sequences } (\mathbb{P}^{\overline{\sigma}_n}_{G_n,\beta,q})_n, \text{ with } G_n \uparrow G_\infty, \text{ and } \overline{\sigma}_n \in \Sigma\}
\end{aligned}$$

In this section, we will identify the 2-state Potts model with the Ising model, as well as $\mathcal{G}_\beta$ with $\mathcal{G}_{\beta,2}$, despite a slight abuse of notation.

2.1.1 Markov property and DLR equations

A general feature of the Ising and Potts models is the spatial Markov property (see Proposition 1.1). For an arbitrary subset A of V, let $\mathcal{F}_A$ be the σ-algebra generated by spins in A. The spatial Markov property can also be written as

$$\mathbb{P}^{\overline{\sigma}}_{G,\beta,q}(\cdot\,|\mathcal{F}_{A^c})(\eta) = \mathbb{P}^{\eta\vee\overline{\sigma}}_{A,\beta,q} = \mathbb{P}^{\eta}_{A,\beta,q},$$

where we write A for the subset of vertices and for the corresponding subgraph of G. This remark implies the following "compatibility relation":

Proposition 2.1 *Let $G = (V, \mathcal{E})$ be a finite graph, and $A \subset G$. For all boundary condition $\overline{\sigma}$ and all functions f, we have*

$$\mathbb{P}^{\overline{\sigma}}_{G,\beta,q}(f) = \mathbb{P}^{\overline{\sigma}}_{G,\beta,q}\left(\mathbb{P}^{\cdot\vee\overline{\sigma}}_{A,\beta,q}(f)\right).$$

Let us pass to the infinite volume limit, i.e. let $G \uparrow G_\infty$ and keep A fixed, and consider a limit $\mathbb{P} \in \mathcal{G}_{\beta,q}$. Then the so-called Dobrushin-Lanford-Ruelle equations are satisfied, namely,

Proposition 2.2 (DLR equations)
Let $G_\infty = (V_\infty, \mathcal{E}_\infty)$ be an infinite, locally finite graph and $\mathbb{P} \in \mathcal{G}_{\beta,q}$, then

$$\mathbb{P}(f) = \mathbb{P}\left(\mathbb{P}^{\cdot}_{A,\beta,q}(f)\right).$$

In other words, for $\mathbb{P}$-almost every $\overline{\sigma} \in \Omega$, and all $A \Subset V_\infty$,

$$\mathbb{P}(\cdot|\mathcal{F}_{A^c})(\overline{\sigma}) = \mathbb{P}^{\overline{\sigma}}_{A,\beta,q}. \tag{2.1}$$

Proof Take $G \uparrow G_\infty$ and keep $A \Subset V_\infty$ fixed in Proposition 2.1, then for each (fixed) local function f,

$$\mathbb{P}^{\cdot\vee\overline{\sigma}}_{A,\beta,q}(f) : \Omega_{V_\infty\setminus A} \to \mathbb{R} \quad \text{is a local function of support } \partial A.$$

Indeed, the nearest-neighbor range of the model implies that the above function depends only on the spins located on the boundary of A. The Definition 1.4 of weak convergence finishes the proof. ■

2.1.2 Properties of the set of DLR measures

In the beginning of the seventies, Dobrushin [29], Lanford and Ruelle [68] have introduced a new way to construct probability measures on infinite product probability spaces (i.e. directly in infinite volume) which allow to model phase transitions in statistical mechanics. The key-point of their approach is to replace the definition via accumulation points of finite-volume measures, by the system (2.1) of conditional probabilities with respect to the outside of any finite set. We will explain this approach by taking the Potts model as an example, although the DLR theory deals with the very general framework of "specifications" [80].

Definition 2.1 (DLR Potts measures)
The set of DLR measures for the q*-state Potts model at inverse temperature* β *is the set of probability measures on* $(\Omega, \mathcal{F})$ *(which we denote by* $\mathcal{M}_1^+(\Omega)$*) defined as*

$$\tilde{\mathcal{G}}_{\beta,q} = \{\mathbb{P} \in \mathcal{M}_1^+(\Omega) \,:\, \mathbb{P}(\cdot|\mathcal{F}_{A^c})(\overline{\sigma}) = \mathbb{P}^{\overline{\sigma}}_{A,\beta,q} \textit{ for } \mathbb{P}-a.e.\, \overline{\sigma} \in \Omega \textit{ and all } A \Subset V_\infty\}.$$

The main advantage of this approach is that functional analysis tools allow to prove interesting properties of this set. We have seen in Proposition 2.2 that $\mathcal{G}_{\beta,q} \subset \tilde{\mathcal{G}}_{\beta,q}$. As our motivation for studying infinite volume measures comes from the study of large finite systems, we would also like to have the reversed inclusion. We will see that this is "almost true"; indeed, $\tilde{\mathcal{G}}_{\beta,q}$ turns out to be a simplex, and all its extremal measures (i.e. the relevant measures, which are not convex combinations of others) are weak limits of finite-volume measures. Hence, $\tilde{\mathcal{G}}_{\beta,q}$ contains all the "physically relevant measures".

It is worth noticing here that the analog of Proposition 2.2 is not trivial at all in the case of the random-cluster model. Indeed, the function $\mu^{\cdot}_{A,p,q}(f)$ is not local. A boundary condition on $\mathcal{E}_\infty \backslash \mathcal{E}_A$ induces a set of connections between the vertices of ∂A which depends on the whole configuration outside A. Nothing forbids us, though, to define a set $\tilde{\mathcal{W}}_{p,q}$ of DLR random-cluster measures analogously as in Definition 2.1; this can be inspired by the spatial Markov property (see Proposition 1.13) which is valid for this model. However, very little is known about the relationship between the sets $\mathcal{W}_{p,q}$ and $\tilde{\mathcal{W}}_{p,q}$. It is not known, for example, whether or not $\mathcal{W}_{p,q} \subset \tilde{\mathcal{W}}_{p,q}$, and even the proof of non-emptiness of $\tilde{\mathcal{W}}_{p,q}$ is not trivial. We refer to [53] for more details.

Let us introduce an important σ-algebra.

$$\mathcal{F}_\infty = \bigcap_{A \Subset V_\infty} \mathcal{F}_{A^c}$$

is called the tail σ-algebra; it contains the events whose realization does not depend on the values of the spins inside any finite region.

Definition 2.2 (Choquet simplex) *A Choquet simplex is a metrisable compact convex set* $\mathfrak{G}$ *such that all elements of* $\mathfrak{G}$ *have a unique decomposition onto the set of extremal elements* $\mathbf{ex}\mathfrak{G}$*, i.e. for all* $\mathbb{P}_0 \in \mathfrak{G}$*, there exists a unique measure* $\rho_0 \in \mathcal{M}(\mathbf{ex}\mathfrak{G})$ *such that*

$$\mathbb{P}_0 = \int_{\mathbf{ex}\mathfrak{G}} \mathbb{P}\, \rho_0(d\mathbb{P})$$

where we say that an element $\mathbb{P}$ is extremal, and write $\mathbb{P} \in \mathbf{ex}\mathfrak{G}$, if $\mathbb{P} = \alpha\mathbb{P}_1 + (1-\alpha)\mathbb{P}_2$ with $\alpha \in (0,1)$ and $\mathbb{P}_1, \mathbb{P}_2 \in \mathfrak{G}$ implies $\mathbb{P}_1 = \mathbb{P}_2$.

Proposition 2.3 *The set $\tilde{\mathcal{G}}_{\beta,q}$ is a convex subset of $\mathcal{M}_1^+(\Omega, \mathcal{F})$, which satisfies the following properties*

1. *The elements of $\tilde{\mathcal{G}}_{\beta,q}$ are completely determined by their macroscopic behavior: let $\mathbb{P}_1, \mathbb{P}_2 \in \tilde{\mathcal{G}}_{\beta,q}$,*

$$\text{If } \mathbb{P}_1(E) = \mathbb{P}_2(E) \text{ for all } E \in \mathcal{F}_\infty, \text{ then } \mathbb{P}_1 = \mathbb{P}_2.$$

2. *The extreme elements of $\tilde{\mathcal{G}}_{\beta,q}$ are the probability measures $\mathbb{P} \in \tilde{\mathcal{G}}_{\beta,q}$ that are trivial on the tail σ-algebra $\mathcal{F}_\infty$:*

$$\mathsf{ex}\tilde{\mathcal{G}}_{\beta,q} = \left\{ \mathbb{P} \in \tilde{\mathcal{G}}_{\beta,q} \,:\, \mathbb{P}(E) = \{0,1\}, \,\forall\, E \in \mathcal{F}_\infty \right\}.$$

3. *$\tilde{\mathcal{G}}_{\beta,q}$ is a Choquet simplex.*

The convexity of $\tilde{\mathcal{G}}_{\beta,q}$ is obvious. The proof of the fact that it is a simplex relies on technical functional analysis which does not really correspond to any physical intuition. The points 1 and 2 are not difficult to prove. We refer to [69] for the proofs.

This proposition can be interpreted as follows: the "true" macroscopic equilibrium states of the system are the "physically relevant" measures, and are described by the extremal measures, which give a deterministic value to any macroscopic observable; while non-extremal measures describe an uncertainty about the preparation of the system.

The following proposition shows that $\tilde{\mathcal{G}}_{\beta,q}$ and $\mathcal{G}_{\beta,q}$ coincide on the set of "physically relevant" measures: an infinite-volume extremal measure can be selected by a sequence of finite-volume measures with boundary conditions that are typical for it.

Proposition 2.4 *Let $\mathbb{P} \in \mathsf{ex}\tilde{\mathcal{G}}_{\beta,q}$. Then*

$$\lim_{G \uparrow G_\infty} \mathbb{P}^{\sigma}_{G,\beta,q} = \mathbb{P} \quad \textit{for } \mathbb{P} - \textit{almost every } \sigma \in \Sigma.$$

Proof Let f be a local function and write $G_n = (V_n, \mathcal{E}_n)$. Observe that the random variables $\{\mathbb{P}^{\cdot}_{G,\beta,q}(f) = \mathbb{P}(f \,|\, \mathcal{F}_{V_n^c})(\cdot)\}_n$ constitute a backward martingale for the filtration $\{\mathcal{F}_{V_n^c}\}_n$. The convergence theorem for backward martingales [89] implies

$$\mathbb{P}(f \,|\, \mathcal{F}_{V_n^c})(\cdot) \to \mathbb{P}(f \,|\, \mathcal{F}_\infty)(\cdot) = \mathbb{P}(f) \quad \mathbb{P} - \text{a.s.} \quad \text{as} \quad n \to \infty,$$

where we used the triviality of $\mathbb{P}$ on the tail $\sigma-$algebra (see Proposition 2.3). ∎

Note that in the case of a regular lattice G_∞, an analog of Proposition 2.3 is true when we replace the tail-triviality by the translation invariance. The ergodic theorem plays the role of the backward martingale limit theorem in this framework. However, let us emphasize that the set of extremal translation-invariant measures is far from coinciding with the set of extremal measures itself. The latter form a rather small and sometimes empty set, whereas the former consists of ergodic measures, that are in particular extreme because $\mathcal{F}_{inv} \subset \mathcal{F}_\infty$, where $\mathcal{F}_{inv}$ is the σ-algebra generated by translation invariant events. If one does not focus on translation-invariance, the structure of extremal states could be very rich, and the analysis of the set of Gibbs states $\mathcal{G}$ of a given model is in general a very hard problem which remains essentially completely open in dimension 3 and higher, for any non trivial model, even in perturbative regimes. The known results about the Ising and Potts models are described below. Before, we briefly describe the general notion of Gibbs measure.

2.1.3 General Gibbs measures

The DLR approach of defining the set of infinite volume measure through Definition 2.1 can be applied to very general random fields defined in finite volume. Indeed, the set of probability measures

$$\{\mathbb{P}(\,\cdot\,|\mathcal{F}_{A^c})(\overline{\sigma}) \,:\, \overline{\sigma} \in \Omega,\, A \Subset V_\infty\} \tag{2.2}$$

can be given as a "specification"[1] for the conditional expectations of a random field. In our case, these quantities are the finite-volume Gibbs measures corresponding to some precise Hamiltonian (1.3), describing the Potts model.

For a modelization in finite volume V, the Gibbs measures

$$\mu(\sigma) = \frac{1}{Z}\exp(-\beta H_V(\sigma))$$

are particularly interesting in statistical mechanics because they are easily shown to maximize the entropy $S(\mu) = \sum_{\sigma\in\Omega}\mu(\sigma)\log\mu(\sigma)$ having the given energy $\mu(H_V) = \sum_{\sigma\in\Omega} H_V(\sigma)\mu(\sigma)$. They thus satisfy the "second principle" of thermodynamics, and can reasonably be considered as equilibrium states of a system.

It is hence interesting to know under which condition a given specification (2.2) can be written as a finite volume Gibbs measure. To this end, we define a general notion of Hamiltonian, and we can think about $\{1,\dots,q\}$ being replaced by any finite set called "spin space".

Definition 2.3 (General Hamiltonian and Gibbsian specification)
Let $G = (V, \mathcal{E}) \subset G_\infty = (V_\infty, \mathcal{E}_\infty)$, *and*

$$H_G^{\overline{\sigma}}(\sigma) = \sum_{\substack{A\Subset V_\infty \\ A\cap V\neq\varnothing}} \phi_A(\sigma \vee \overline{\sigma})$$

[1] We voluntarily do not enter the formal definition of the notion of specification, which would be outside the scope of this thesis. We refer to the book [40] and the review [69].

with some interaction potential $\Phi = \{\phi_A\}_{A \Subset V_\infty}$ *such that* $\phi_A : \Omega \to \mathbb{R}$ *is* $\mathcal{F}_A$ *measurable for all* $A \Subset V_\infty$*, such that* $\sum_{A: x \in A} \|\phi_A\|_\infty < \infty$ *for all* $x \in V_\infty$.
The specification of conditional expectations

$$\left\{ \mathbb{P}(\,\cdot\,|\mathcal{F}_{A^c})(\overline{\sigma}) = \frac{1}{Z^{\overline{\sigma}}_{G,\beta}} \exp(-\beta H^{\overline{\sigma}}_G(\cdot)) \,:\, \overline{\sigma} \in \Omega,\, A \Subset V_\infty \right\}$$

is called the Gibbsian specification (associated to Φ*).*

We now describe two important properties which can be fulfilled the specification (2.2), and that will turn out to be equivalent to its Gibbsianity.

Definition 2.4 *We say that a specification* (2.2) *is quasi-local if for all cylindrical event* C *and all* $A \Subset V_\infty$

$$\lim_{V \uparrow V_\infty} \sup_{\substack{\sigma, \sigma' \in \Omega: \\ \sigma_{|V} = \sigma'_{|V}}} |\mathbb{E}(C\,|\,\mathcal{F}_{A^c})(\sigma) - \mathbb{E}(C\,|\,\mathcal{F}_{A^c})(\sigma')| = 0,$$

We say that it is uniformly non-null if for all $A \Subset V_\infty$

$$\inf_{\sigma, \overline{\sigma} \in \Omega} \mathbb{P}(\sigma\,|\,\mathcal{F}_{A^c})(\overline{\sigma}) > 0.$$

Proposition 2.5 *A specification* (2.2) *is Gibbsian (i.e. there exists a potential* Φ *such that Definition 2.3 holds) if and only if it is quasi-local and uniformly non-null.*

This proposition shows that the Gibbsian specifications constitute a quite natural class of specifications. We refer to [61] for a proof.

2.2 The Ising case

To determine the set $\mathcal{G}$ of infinite volume Gibbs measures of a model, it is necessary to understand all the possible local behaviors of the system. Indeed, the expectations of local functions characterize an infinite volume measure, by definition of the weak convergence of finite-volume measures.

Pirogov-Sinaĭ [78] theory gives a way to determine the extremal translation-invariant Gibbs states at low temperature as perturbation of the corresponding ground states. But as soon as local phase coexistence is possible, namely when a macroscopic interface between two phases has bounded fluctutations in the bulk, which breaks translation invariance, the present knowledge is very limited.

Some rare cases for which $\mathcal{G}_\beta$ can be characterized include the two dimensional Ising model (with the result of Aizenman and Higuchi, see below), and now the two-dimensional Potts model, which constitute the main result of this manuscript.

We give now the history of the known results for the Ising model on $\mathbb{Z}^d$.

2.2.1 Previously known results

We proved in Proposition 1.27 that $\mathcal{G}_{\beta,2}$ contains a unique element for all $\beta < \beta_c$. It is known in dimension $d \neq 3$ (and expected in dimension 3) that uniqueness holds also at $\beta = \beta_c$ (see [74] for $d = 2$, [5] for $d \geqslant 4$).

It follows from Proposition 1.6 that $\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$ are always extremal, translation-invariant elements of $\mathcal{G}_{\beta,2}$. Hence, in the non-uniqueness regime $\beta > \beta_c$, the set $\mathcal{G}_{\beta,2}$ contains at least the two distinct extremal measures $\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$.

In 1975, Messager and Miracle-Sole [72] proved that all *translation invariant* infinite-volume Gibbs measures of the Ising model on $\mathbb{Z}^2$ are convex combinations of $\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$, i.e. they are of the form

$$\alpha \mathbf{P}^+_\beta + (1-\alpha)\mathbf{P}^-_\beta, \quad 0 \leqslant \alpha \leqslant 1.$$

An earlier result on that problem was obtained by Gallavotti and Miracle-Sole for large enough β [38].

At this stage, the problem was thus reduced to figuring out whether there exist non-translation invariant infinite-volume Gibbs measures in this model.
Important progress was made in 1979 by Russo [82], who proved that an infinite-volume Gibbs measure for the 2d Ising model which is invariant under translations along one direction is necessarily invariant under all translations.
Building up on these earlier results, Aizenman [1] and Higuchi [55] (see also [42] for a more recent variant) independently established, in the late 1970s, that all infinite-volume Gibbs measures of the 2d Ising model are translation invariant, thus providing a complete description of the set $\mathcal{G}_\beta$.

Note that the absence of non-translation invariant infinite-volume Gibbs measures is specific to the two-dimensional model: In higher dimensions, as was mentioned in other terms in Section 1.5.6, it was proved by Dobrushin [30] that such measures exist at sufficiently large values of β. Indeed, the boundedness of the fluctuations of interfaces uniformly in the volume of the system imply local phase coexistence, and thus existence of non-translation-invariant measures.

However, all translation invariant measures of the Ising model in all dimensions are still convex combinations of $\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$, as was proved by Bodineau in [13].

We emphasize that the main difference between the 2d case and its higher-dimensional counterparts is that interfaces in 2d are one-dimensional objects and as such undergo unbounded fluctuations (with diffusive scaling) at any $\beta < \infty$, while horizontal interfaces in higher dimensions are rigid at large enough values of β.

As we already mentioned, Greenberg and Ioffe established the existence of a Brownian bridge diffusive limit in 2d [48] for a single interface, resulting from the Dobrushin boundary condition. Earlier results restricted to large β include [37] and [54].

2.2.2 New results and open problems

In collaboration with Yvan Velenik, we introduced a new approach to the Aizenman-Higuchi result, which states that all the infinite volume Gibbs measures of the 2d

Ising model are of the form $\alpha \mathbf{P}_\beta^+ + (1-\alpha)\mathbf{P}_\beta^-$. Our approach has a number of distinctive advantages:

- We obtain a finite-volume, quantitative analogue (of course, implying the classical claim), and our error estimate is of the correct order.
- The scheme of our proof seems more natural, and provides a clear picture of the underlying phenomenon.
- This new approach is more robust, as the underlying general heuristics has been extendable to the Potts model (see Section 2.3), for which the classical approach does not apply.

Here is the statement of our theorem.

Theorem 2.1 *Let $\beta > \beta_c$, and $\omega \in \Omega$.*
Then, for any $\varepsilon \in (0, \frac{1}{2})$ and $b < \frac{1-\varepsilon}{2}$, there exists $n_0 = n_0(\beta, b, \varepsilon)$ and $C_\beta \geqslant 0$ such that, for all $n > n_0$, there exists a constant $\alpha^{n,\omega}(\beta) \in [0,1]$ such that,

$$\left|\mathbf{P}^{\omega}_{\Lambda_n,\beta}(f) - \left(\alpha^{n,\omega}\mathbf{P}^+_\beta(f) + (1-\alpha^{n,\omega})\mathbf{P}^-_\beta(f)\right)\right| \leqslant C_\beta \|f\|_\infty n^{-\frac{1-\varepsilon}{2}+b}.$$

uniformly in all functions f which have support inside Λ_{n^b}.

It is not difficult to deduce the Aizenman-Higuchi theorem from Theorem 2.1, see Chapter 3. Let $\mathcal{G}_\beta$ be the set of all infinite-volume Gibbs measures of the Ising model on $\mathbb{Z}^2$, at inverse temperature β, then,

Corollary 2.1 *For any $\beta > \beta_c$, $\mathcal{G}_\beta = \{\alpha\mathbf{P}^+_\beta + (1-\alpha)\mathbf{P}^-_\beta \ : \ 0 \leqslant \alpha \leqslant 1\}$.*

Moreover, the estimate we have on the error term in Theorem 2.1 is essentially optimal:

Proposition 2.6 *Let $\beta > \beta_c$. There exist a local function f and a constant $c = c(\beta) > 0$ such that, for all n large enough, one can find $\overline{\sigma} \in \Omega$ with*

$$\inf_{\alpha\in[0,1]}\left|\mathbf{P}^{\overline{\sigma}}_{\Lambda_n,\beta}(f) - \alpha\mathbf{P}^+_\beta(f) - (1-\alpha)\mathbf{P}^-_\beta(f)\right| \geqslant cn^{-1/2}.$$

The proof of Theorem 2.1 comprises two main steps:

- Proving that, with high probability, at most one interface approaches the center of the box Λ_n,

- proving that this interface, when present, undergoes unbounded fluctuations (actually of order $\sqrt{n}$).

It then follows that any local observable, with support close to the center of the box lie, with high probability, deep inside the + or − phase. A more detailed sketch of the proof can be found below, while Chapter 3 contains the detailed proof.

As we already mentioned, Greenberg and Ioffe [48] established the existence of a Brownian bridge diffusive limit in 2d for a single interface, resulting from the Dobrushin boundary condition. The behavior of the system under a general boundary condition is the main topic of our work here.

Note that there is one drawback in our approach: It does not imply uniqueness at the critical temperature, while this can be extracted from the classical Aizenman-Higuchi result, e.g., using [17]. However, this should not be surprising, since the general philosophy apply to the q-state Potts model for any $q \geqslant 2$ for which the transition is known to be of first-order for q large.

2.2.2.1 Open problems

2.2.2.1.1 "Generic" boundary condition. As discussed above, the estimate we have on the error term in Theorem 2.1 is essentially optimal. However, it seems very likely that a "generic" boundary condition should yield, with high probability, configurations with no crossing interfaces, which should improve the error term to e^{-cn}. One of the difficulties is to give a precise meaning to the word "generic" in this context. One possible choice would be to sample the boundary condition according to some natural probability measure. Unfortunately, very little is known about the Ising model with a strongly inhomogeneous boundary condition. The only work we are aware of that is related to this question is [87], in which the following result is proved: Let the spins of the boundary condition $\overline{\sigma}$ be independent Bernoulli random variables with parameter $1/2$. Then, for almost all $\overline{\sigma}$, the probability of appearance of an interface goes to zero as the system size goes to infinity, provided that β be large enough. This shows that, for a generic boundary condition, typical configurations of the low-temperature Ising model do not possess macroscopic interfaces.

2.2.2.1.2 Wetting with inhomogenous boundary fields. A related issue, whose solution would probably be helpful in making progress in the previously mentioned problem, is that of wetting above an inhomogeneous substrate. Consider a 2 dimensional Ising model at inverse temperature $\beta > \beta_c$, in a box Λ_n with + boundary condition along the vertical and top sides of the box, and − boundary condition along the bottom side. If the interaction $\sigma_i\sigma_j$ between the spins in the bottom row of Λ_n and those outside the box is modified to $h\sigma_i\sigma_j$, with $h > 0$, then an interface is present along the bottom wall. As long as $h < h_w(\beta)$, for some explicitly known value $0 < h_w(\beta) < 1$, the interface sticks to the bottom wall, its Hausdorff distance to the wall being $O(\log n)$; this is the so-called partial wetting regime. When $h \geqslant h_w(\beta)$, the interface is repelled away from the bottom wall, and the Hausdorff distance becomes $O(\sqrt{n})$; this is the complete wetting regime. The transition between these two regimes is called the wetting transition. All this is rather well understood, see [76] for a review. Understanding the corresponding problem when the homogeneous boundary field h is replaced by site-dependent boundary fields h_i is much more difficult and still mostly open [32].

2.2.2.1.3 Robustness of the Dobrushin boundary condition. A final open problem that might be of interest is to understand how robust the Dobrushin boundary conditions are: Start with such a boundary condition, and randomly flip a density $\rho > 0$ of spins; does the macroscopic interface survive? What can be said about the critical ρ at which the macroscopic interface disappears?

2.3 The Potts case

2.3.1 Previously known results

As we will see in the next chapter, in the case of the Ising model ($q = 2$), the presence of only two phases simplifies the investigation (which is not trivial, though). In the Potts model with $q = 3$ or more states, there are more than two phases and consequently interfaces are more complicated objects, elementary macroscopic interfaces being trees rather than lines.

Proposition 1.27 implies that there exists at least q extremal phases for $\beta > \beta_c$. We will prove in Proposition 2.7 that the q pure phases are actually extremal.

At very low temperature, Dobrushin and Shlosman proved [31], in a general context including the Potts model, that all Gibbs states are translation invariant, and in particular are convex combinations of the q pure phases corresponding to perturbations of the ground states of the model.

In the case of $\mathbb{Z}^d$ and q large enough (depending on the dimension d of the lattice), Martirosian [71] extended the result given by Pirogov-Sinaĭ's theory up to the critical temperature and treated also criticality. He proved that for $\beta > \beta_c(q)$ all translation invariant Gibbs states are convex combination of the q pure states, while for $\beta = \beta_c(q)$, the decomposition is onto $q + 1$ pure phases: the q low-temperature ordered pure phases and the high temperature disordered phase. The same result was proved before in dimension 2 by Lanaait, Messager and Ruiz [67].

2.3.2 Extremality of the q pure phases for $\beta > \beta_c$

The infinite volume Edwards-Sokal coupling allows to prove extremality of the q pure phases of the Potts model below the critical temperature.

Proposition 2.7 *Let $\beta > \beta_c(q)$. Then the pure phases $(\mathbb{P}^i_{\beta,q})_{i=1,\dots,q}$, of the q-state Potts model are extremal.*

Proof Let $i \in \{1, \dots, q\}$ be fixed. As $\mathbb{P}^i_{\beta,q}$ is a Gibbs measure, proving extremality is equivalent to proving tail triviality by Proposition 2.3.

Let $A \in \mathcal{F}_\infty$. Using the infinite volume Edwards-Sokal coupling presented in Proposition 1.25 we can write

$$\mathbb{P}^{i}_{\beta,q}(A) = \int \mu^{w}_{p,q}(d\omega)\nu(A|\omega). \tag{2.3}$$

One can show that $\mu^{w}_{p,q}$ is trivial on the random-cluster tail σ−algebra $\mathcal{F}^{RC}_\infty$ [53, Theorem 4.19]. Therefore, it is sufficient to show that $\nu(A|\cdot)$ is $\mathcal{F}^{RC}_\infty$-measurable.
Let ω, ω' be two random-cluster configurations which differ only on one edge $e = [x, y]$. By iteration, proving that $\nu(A|\omega) = \nu(A|\omega')$ implies the claim. We now construct a coupling $\mathcal{P}$ between the measures $\nu(\cdot|\omega)$ and $\nu(\cdot|\omega')$.
Let $\{c_{-1}, c_0, c_1, c_2, \dots\}$ be a sequence of independent random variables distributed uniformly on $\{1, \dots, q\}$. For a given random-cluster configuration η, generate a coloring of the vertices as follows:

- Let $\Sigma^\eta_z = i$ if $z \leftrightarrow \infty$.
- If $x \not\leftrightarrow \infty$, paint the cluster containing x with the color c_0, i.e. $\Sigma^\eta_z = c_0$ for all $z \leftrightarrow x$.
- If $y \not\leftrightarrow \infty$ and $y \not\leftrightarrow x$ paint the cluster containing y with the color c_{-1}, i.e. $\Sigma^\eta_z = c_{-1}$ for all $z \leftrightarrow y$.
- Number the remaining finite clusters in the lexicographic order and paint the cluster i with the color c_i.

The Edwads-Sokal coupling ensures that if η is distributed according to $\mu^{w}_{p,q}$, then $\Sigma^\eta = (\Sigma^\eta_z)_{z\in V}$ is distributed according to $\mathbb{P}^{i}_{\beta,q}$. We thus have:

$$\nu(A|\omega) - \nu(A|\omega') = \mathcal{P}(A \text{ is realized in } \Sigma^\omega) - \mathcal{P}(A \text{ is realized in } \Sigma^{\omega'}). \tag{2.4}$$

Without loss of generality we can assume that the edge e is open in ω and closed in ω'. Two cases can occur outside the edge e (recall that ω and ω' differ only on e):

1. $y \leftrightarrow \infty$: in this case $\Sigma^\omega_z = \Sigma^{\omega'}_z$ for all $z \in V$ and the difference (2.4) vanishes.
2. $y \not\leftrightarrow \infty$: in this case Σ^ω_z differs from $\Sigma^{\omega'}_z$ with probability $1 - \frac{1}{q}$ for $z \leftrightarrow y$. But the cluster of y is finite and $A \in \mathcal{F}_\infty$. Therefore (2.4) vanishes as well.

We have proved that $\nu(A|\cdot)$ is tail measurable and thus $\mathbb{P}^{i}_{\beta,q}$ is extremal. ■

2.3.3 New results and open problems

We recently proved together with Hugo Duminil-Copin, Dmitry Ioffe and Yvan Velenik that, in the case of $\mathbb{Z}^2$ and $\beta > \beta_c(q)$, all Gibbs states of the Potts model (without the restriction to translation invariant states) are convex combinations of the q pure phases.

Particularizing some of the notations we used before to the case of $\mathbb{Z}^2$, and writing $\mathcal{G}_{\beta,q}$ for the set of all infinite volume Gibbs measures of the two-dimensional q-state

Potts model on $\mathbb{Z}^2$ at inverse temperature β, and $\mathbb{P}^i_\beta$, $i = 1 \dots q$ for the pure phases of the model (the dependence on q is omitted since we work with fixed $q \geqslant 2$), our theorem is the following:

Theorem 2.2 (Coquille, Duminil-Copin, Ioffe, Velenik, 2012)
Let $G_\infty = \mathbb{Z}^2$, $q \geqslant 2$ and $\beta > \beta_c(q)$. Then every Gibbs measure $\mathbb{P} \in \mathcal{G}_{\beta,q}$ is a convex combination of the q pure states $\mathbb{P}^i_\beta$, $i = 1 \dots q$.
Namely there exists a (unique) family of numbers $\alpha_1, \dots, \alpha_q \in [0,1]$ such that

$$\mathbb{P} = \sum_{i=1}^{q} \alpha_i \mathbb{P}^i_\beta, \quad \textit{with} \sum_{i=1}^{q} \alpha_i = 1.$$

Corollary 2.2 *For any $q \geqslant 2$ and for any inverse temperature $\beta > \beta_c(q)$,*

1. *Every Gibbs measure $\mathbb{P} \in \mathcal{G}_{q,\beta}$ is translation invariant.*
2. *The pure phases $\mathbb{P}^i_\beta$, $i = 1 \dots q$ are (the only) extremal measures of the simplex $\mathcal{G}_{\beta,q}$.*

The latter corollary is a consequence of elementary results proved in Section 1.5. Indeed, the pure states are translation invariant (which implies point (1)), and we can observe that in the decomposition of Theorem (2.2), the coefficient α_i is directly related to the i-th component of the magnetization of the system. We have

$$\alpha_i = \frac{\mathbb{P}(\sigma_0 = i) - \Delta(\beta)}{\mathbb{P}^i_\beta(\sigma_0 = i) - \Delta(\beta)}, \quad \text{with} \quad \Delta(\beta) = \frac{1}{q-1}\mathbb{P}^i_\beta(\sigma_0 \neq i). \tag{2.5}$$

This shows that we cannot decompose any $\mathbb{P}^i_\beta$ on the other $\mathbb{P}^j_\beta$ with $j \neq i$, implying point (2).

A sketch of the proof can be found below, and Chapter 4 contains the detailed proof. Note that our proof does not work for $\beta = \beta_c$, but this is not surprising since the result is false at criticality for large values of q. Indeed, the phase transition is known to be of first order in this regime, and conjectured to be so down to $q = 5$.

2.3.3.1 Open problems

2.3.3.1.1 Critical 2d Potts models. The behavior of two-dimensional q-state Potts models in the critical regime $\beta = \beta_c(q)$ is still open. It is conjectured that there is a unique Gibbs state when $q = 3$ and 4, but that, for $q \geqslant 5$, there is coexistence at β_c of $q + 1$ pure phases: the q low-temperature ordered pure phases and the high temperature disordered phase.

2.3.3.1.2 Finite-range 2d models. The extension of Theorem 2.2, even in the Ising case $q = 2$, to general finite-range interactions still seems out of reach today. There are, at least, two main difficulties when dealing with such models: On the one hand, it is difficult to find a suitable non-perturbative definition of interfaces (the classical definitions used, e.g., in Pirogov-Sinaĭ theory become meaningless once the temperature is not very low); on the other hand, interfaces will not partition the system into (random) subsystems with pure boundary conditions anymore, which implies that it will be necessary to understand relaxation to pure phases from impure boundary conditions. The general *philosophy* of the approach we use should still apply, though.

2.3.3.1.3 The question of quasi-periodicity. There is a general conjecture that two-dimensional models with finite spin space should always possess a finite number of extremal Gibbs states, all of which are periodic. In particular, this would imply that all Gibbs states are periodic, and thus that a two-dimensional quasicrystal cannot exist (as an equilibrium state).

2.3.3.1.4 Models in higher dimensions. As already mentioned, the situation in higher dimensions is very different, due to the existence of translation non-invariant states. Even in the very low-temperature 3-dimensional n.n.f. Ising model, the set of extremal Gibbs states is not known. We recall however, that it has been proved, in the case of a d-dimensional Ising model for any $d \geqslant 3$, that all *translation invariant* Gibbs states are convex combinations of the two pure phases at all temperatures [13].

2.4 Heuristics of the proofs

2.4.1 General heuristics

In this section we give the general heuristics which is behind the result that, for both the Ising and the Potts models, the set of extremal infinite volume Gibbs measures $\mathrm{ex}\tilde{\mathcal{G}}_{\beta,q}$ is the set of pure phases ($\mathbf{P}^+_\beta$ and $\mathbf{P}^-_\beta$ for Ising and $\mathbb{P}^1_\beta, \ldots \mathbb{P}^q_\beta$ for Potts).

The way we proceed to prove this result gives at the same time an upper bound on the rate of convergence of any sequence of finite volume measures towards the corresponding convex combination onto pure phases. Moreover, this bound is almost of the optimal order, namely it is almost sharp for a well chosen sequence of boundary conditions together with an appropriate local function.

Note that Proposition 2.4, which states that the extremal measures are weak limits of finite volume measures, ensures that we really can characterize the abstract set $\mathrm{ex}\tilde{\mathcal{G}}_{\beta,q}$ by analyzing the asymptotic behavior of finite volume measures, in the large volume limit.

Proving that any infinite volume measure can be decomposed as a convex combination onto the pure phases amounts to showing, in view of the weak topology we are using, that no local function can "feel" phase coexistence. This statement is closely related to translation invariance: as the pure phases are translation invariant, so is any of their convex combinations, and what we mean by saying that a local function "feels" phase coexistence is that its support is located on an interface between two phases with positive probability, uniformly in the volume of the box.

We must thus prove that, in a finite box of side-length n with boundary condition ω_n, with probability tending to 1 as $n \to \infty$, the macroscopic interfaces fluctuate enough so that every local function has its support far away from any of them (where by far away we mean at a distance which diverges, as $n \to \infty$. In other words we must prove that the support of any local function is located "deep inside a pure phase", up to $o_n(1)$ probability.

Once the latter is proved, spatial relaxation into pure phases (see Proposition 1.28) allows to conclude: if the support $S(f)$ of a given local function f is far away from macroscopic interfaces, then there exists a path surrounding $S(f)$ where the spins take a constant value i (we say "the phase i surrounds $S(f)$"). Conditionally on this event, the expectation of f in the finite box is close to its expectation in the pure phase i, up to a term which is exponentially small in the distance from the support of f to the interface.

We describe now the three steps which show that, with probability $1-O(n^{-1/2+\varepsilon})$ none of the macroscopic interfaces of a box Λ_n of side-length n intersect a (deterministic) sub-box of side-length $O(n^{1/2-\varepsilon})$.

First, observe that an arbitrary boundary condition ω_n can enforce the presence of $O(n)$ interfaces. We claim that, uniformly in ω_n, only a finite number of them reach a sub-box of side-length $n/2$ with high probability. Indeed, below the critical temperature, the surface tension is positive. Hence each interface crossing $\Lambda_n \setminus \Lambda_{n/2}$ from i to j "costs" $e^{-\tau_\beta(i-j)}$ where $i \in \partial\Lambda_n$ and $j \in \partial\Lambda_{n/2}$, the cost of seeing r interfaces crossing is thus upper-bounded by e^{-Crn}, for some $C = C(\beta) > 0$. On the other hand, imposing the spins on $\partial\Lambda_{n-1}$ to have the same value costs $O(e^{-C'8n})$ by the finite energy property, for some $C' = C'(\beta) > 0$. By taking r sufficiently large (but finite), we ensure the probability of seeing r crossing interfaces to be at most exponentially small in n.

Now, we can focus on the box $\Lambda_{n/2}$, and look at the typical macroscopic interfaces configurations with these r endpoints. With a large deviation analysis, we can prove that there exists some $\eta > 0$ such that the latter are located in a ηn neighborhood of the graphs which are solution of the so-called Steiner problem: link the r endpoints in a way which is compatible with the boundary condition and which minimizes the total τ_β-length.

At this stage the Ising and Potts cases are quite different: the above Steiner graphs are (union of) lines for $q = 2$, while they are (union of) trees for $q > 2$. Hence, the Ising model is easier to handle, since only two cases can occur: for δ sufficiently small, the box $\Lambda_{\delta n}$ can intersect either none of the Steiner lines, or one of them. For the Potts model, complication comes from the fact that $\Lambda_{\delta n}$ can also intersect a node of a Steiner tree. But due to the uniform convexity of the τ_β norm, it is known that a node of a Steiner graph has degree three. Only one more case is thus added to the Ising situation: $\Lambda_{\delta n}$ intersects a "Steiner tripod".

Conditionally on the event that $\Lambda_{\delta n}$ does not intersect the Steiner forest, the support of any local function being eventually deep inside $\Lambda_{\delta n/2}$, it is necessary deep inside a pure phase.

Conditionally on the event that $\Lambda_{\delta n}$ intersects a line of the Steiner forest, there is a unique interface going through the box, and the Brownian scaling of interfaces (or equivalently the Ornstein-Zernike asymptotics of the two-point function) ensures that the box $\Lambda_{(\delta n)^{1/2-\varepsilon}}$ is "missed" with high probability by the interface. The same reasoning as before can be made inside $\Lambda_{(\delta n)^{1/2-\varepsilon}/2}$, and this finishes the proof in the case of the Ising model.

Conditionally on the event that $\Lambda_{\delta n}$ intersects a tripod of the Steiner forest, there are three phases present inside the box, and one must analyze the fluctuations of the triple "point". A coarse-graining at scale n^{ε} (for some ε small enough) allows us to get rid of the microscopic details of the interfaces. It is the scale at which the interfaces "look like" the Steiner forest. By the mass gap arguments of [21], which come essentially from strict convexity of the τ_β norm, we can then prove that the triple "point" has typically a diameter of order n^{ε} and that it fluctuates in a Gaussian way. The same reasoning as before can be made inside $\Lambda_{(\delta n)^{\varepsilon}/2}$, and this concludes the proof in the case of the Potts model.

2.4.2 Sketch of the proof for the Ising model

The proof of Theorem 2.1 can be explained in three natural steps, along the lines of the above heuristics, but taking account of the simplifications which come from the Ising model. We now sketch the proof without giving all the technical details, and refer to Chapter 3 for the detailed proof.

Throughout this section the notations $C, C', C'', \mathfrak{c}, \mathfrak{c}', \mathfrak{c}''$ stand for positive constants which can change from line to line.

2.4.2.1 Step 1: Typical configurations have at most one crossing interface

Let ω be a boundary condition around $\Lambda_n = \{-n, \dots, n\}^2$, and $\mathbf{b}(\omega) \equiv \{b_1, \dots, b_{2M}\}$ be the set of endpoints of the open contours induced by ω, see the notations of Section 1.2.4.

Let $a = 1 - \varepsilon$, with $\varepsilon \in (0, 1/2)$. The first step is to prove that a contour $\Gamma_{b,b'}$ such that the segment $\overline{bb'}$ is too far from the center of Λ_n does not reach Λ_{n^a} with high probability. More precisely,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\exists (b,b') \in \mathbf{b}(\omega) \,:\, \Gamma_{b,b'} \cap \Lambda_{n^a} \neq \varnothing, \ \text{and} \ \overline{bb'} \cap \bar{\Lambda}_{2n^a} = \varnothing\big) \leqslant e^{-Cn^{2a-1}}. \quad (2.6)$$

Indeed, let $(b, b') \in \mathbf{b}(\omega)$, such that $\overline{bb'} \cap \Lambda_{2n^a} = \varnothing$, suppose there exists a compatible $\Gamma_{b,b'} : b \to b'$. Observe that on the event $\{\Gamma_{b,b'} \cap \Lambda_{n^a} \neq \varnothing\}$ there exists a path of (w.l.o.g.) $+$ spins going from b to Λ_{n^a}.

The latter event is increasing and, by monotonicity and conditionning on the contours $\mathbf{\Gamma}_1$ which are below $\Gamma_{b,b'}$, its probability is bigger than in a box with Dobrushin

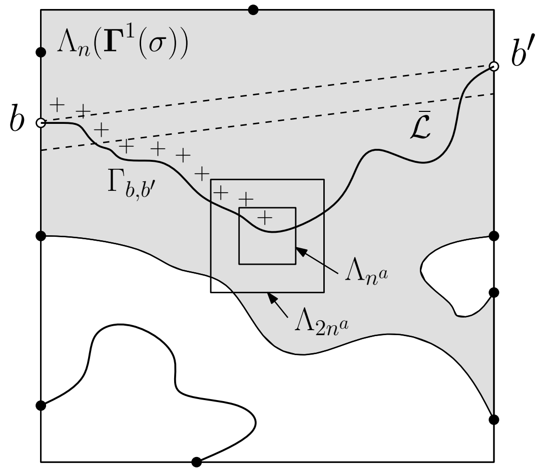


Figure 2.1 – Proof of (2.6). The box $\Lambda_n(\mathbf{\Gamma}^1)$ is shaded. The contour $\Gamma_{b,b'}$ reaches Λ_{n^a} while the segment $\overline{bb'}$ does not intersect Λ_{2n^a}.

boundary condition $\pm(b,b')$ (i.e. $+1$ above and -1 below $\overline{bb'}$):

$$\mathbf{P}^{\omega(\mathbf{\Gamma}^1)}_{\Lambda(\mathbf{\Gamma}^1),\beta}\big(\Gamma_{b,b'}\cap\Lambda_{n^a}\neq\varnothing\big)\leqslant\mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\Gamma_{b,b'}\cap\Lambda_{n^a}\neq\varnothing\big),$$

where the (random) box $\Lambda(\mathbf{\Gamma}^1)$ and the boundary condition $\omega(\mathbf{\Gamma}^1)$ can be infered from Figure 2.1. Now, let $z\in\partial\Lambda_{n^a}$ be the first hitting point of $\Gamma_{bb'}$ on Λ_{n^a} starting from b. The cost of the interface $\Gamma_{bb'}$ is

$$\frac{\mathbf{Z}^{\pm(b,b')}_{\Lambda_n,\beta}(\Gamma_{bb'}:b\to z\to b')}{\mathbf{Z}^{+}_{\Lambda_n,\beta}}\approx e^{-\tau_\beta(z-b)-\tau_\beta(z-b')}.$$

On the other hand, the cost of the unconstrained interface is:

$$\frac{\mathbf{Z}^{\pm(b,b')}_{\Lambda_n,\beta}}{\mathbf{Z}^{+}_{\Lambda_n,\beta}}\approx e^{-\tau_\beta(b-b')}.$$

The constraint to reach Λ_{n^a} imposes a vertical deviation from the segment $\overline{bb'}$ of at least $O(n^a)$, which has small probability due to surface tension: by the sharp triangle inequality (1.3) and the Pythagoras theorem, we have, uniformly in $z\in\partial\Lambda_{n^a}$ and in b,b' such that $\overline{bb'}\cap\Lambda_{2n^a}=\varnothing$,

$$\tau_\beta(z-b)+\tau_\beta(z-b')-\tau_\beta(b'-b)\geqslant\kappa_\beta\big(\|z-b\|_2+\|z-b'\|_2-\|b'-b\|_2\big)\geqslant Cn^{2a-1}.$$

Since there are at most $4n^a$ vertices $z\in\partial\Lambda_{n^a}$, we conclude that, for n large enough,

$$\mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\Gamma_{b,b'}\cap\Lambda_{n^a}\neq\varnothing\big)\leqslant 4n^a\cdot e^{-C\,n^{2a-1}}\leqslant e^{-C'\cdot n^{2a-1}},$$

and the conclusion follows, since there are at most $64n^2$ pairs b,b'. ■

This first result shows that the open contours which cross the box by reaching Λ_{n^a} have their endpoints quasi-diametrally opposed. This will allows us to prove that the number of such crossing contours is either 0 or 1 with high probability. Let us denote by N_{cr} the number of open contours intersecting Λ_{n^a} (which we call crossing contours). Then,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{cr} \geqslant 2\big) \leqslant e^{-Cn^{2a-1}}. \tag{2.7}$$

Indeed, suppose there are at least two of them, and denote the two first ones by $\Gamma_{b_1 b_1'}$ and $\Gamma_{b_2 b_2'}$. By conditioning on the other open contours which are linking the points of $\mathbf{b}(\omega)\setminus\{b_1, b_1', b_2, b_2'\}$ (we call them $\boldsymbol{\Gamma}_1$ above $\Gamma_{b_1 b_1'}$ and $\boldsymbol{\Gamma}_2$ below $\Gamma_{b_2,b_2'}$) we end up with a (random) box with boundary conditions (w.l.o.g.) $+1$ above $\Gamma_{b_1 b_1'}$, and below $\Gamma_{b_2 b_2'}$, and the initial ω inbetween b_1, b_2 and b_1', b_2', see Figure 2.2.

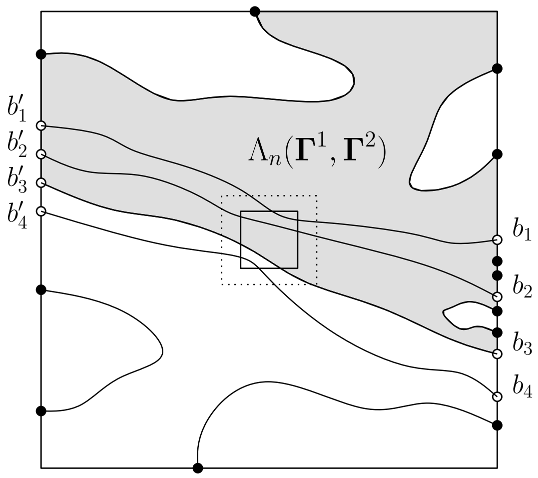


Figure 2.2 – Proof of (2.7). The box $\Lambda_n(\boldsymbol{\Gamma}^1, \boldsymbol{\Gamma}^2)$ is shaded. The contours $\Gamma_{b_1,b_1'}$ and $\Gamma_{b_2,b_2'}$ reach Λ_{n^a} and the segments $\overline{b_1 b_1'}$ and $\overline{b_2 b_2'}$ intersect Λ_{2n^a}.

Observing that if there is more than one crossing contour, by (2.6), their endpoints are located in a $O(n^a)$ neighborhood from each other. The unconstrained contours have thus a cost:

$$\frac{Z^{\omega(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2)}_{\Lambda_n(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2),\beta}}{Z^{+}_{\Lambda_n(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2),\beta}} \approx e^{-C\,n^a},$$

where the box $\Lambda_n(\boldsymbol{\Gamma}^1, \boldsymbol{\Gamma}^2)$ and the boundary condition $\omega(\boldsymbol{\Gamma}^1, \boldsymbol{\Gamma}^2)$ can be inferred from Figure 2.2. Indeed we just have to change boundary conditions from $-$ to $+$ on a total length less of order $O(n^a)$. On the other hand, observing two crossing contours has a cost:

$$\frac{Z^{\omega(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2)}_{\Lambda_n(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2),\beta}(\Gamma_{b_1,b_1'} \text{ and } \Gamma_{b_2,b_2'} \text{ are crossing})}{Z^{+}_{\Lambda_n(\boldsymbol{\Gamma}^1,\boldsymbol{\Gamma}^2),\beta}} \approx e^{-\tau_\beta(b_1'-b_1)-\tau_\beta(b_2'-b_2)} \leqslant e^{-C\cdot n}.$$

since $\min\{\|b_1' - b_1\|_1, \|b_2' - b_2\|_1\} \geqslant Cn$. Combining these estimates, we deduce that

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{cr} \geqslant 2, \mathcal{D}\big) \leqslant C'' n^{2+2a} \cdot e^{C\,n^a} e^{-C'n} \leqslant e^{-c\,n}$$

where $\mathcal{D}$ denotes the event that all crossing contours have endpoints b, b' satisfying $\overline{bb'} \cap \Lambda_{2n^a} \neq \varnothing$. We then have, in view of (2.6) and since $2a - 1 < 1$,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{cr} \geqslant 2\big) \leqslant e^{-Cn^{2a-1}}$$

which finishes the proof. ■

There are now only two cases remaining. In the first case, no open contour reaches the box Λ_{n^a}, which surrounds eventually any local function; in this case a local function f has its support at distance $O(n^a)$ of the nearest open contour, and is thus deep inside a pure phase. In the second case, one open contour reaches Λ_{n^a}. To ensure that no local function can feel the phase coexistence, we must prove that this open contour has fluctuations which diverge with n. We treat this situation in the following step.

2.4.2.2 Step 2: When present, this interface has large fluctuations

We denote by $\mathcal{I}_0$ the event that there is no crossing contour, and by $\mathcal{I}_1$ the event that there is a unique crossing contour. Let $b < a/2$, and Γ be the unique crossing contour on the event $\mathcal{I}_1$. Then there exists some constant $C = C(\beta)$ such that, for all n large enough,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}(\Gamma \cap \Lambda_{2n^b} \neq \varnothing,\, \mathcal{I}_1) \leqslant C n^{b-a/2}, \tag{2.8}$$

Indeed, let us denote by b and b' the endpoints of the unique crossing contour Γ. Let also $\boldsymbol{\Gamma}$ denote the set of all open contours of the configuration inside $\Lambda_n \backslash \Lambda_{n^a}$. Let d, d' be the endpoints of Γ on $\partial\Lambda_{n^a}$, we thus have Dobrushin boundary conditions outside the (random) box $\Lambda_n(\boldsymbol{\Gamma})$ see Figure 2.3.

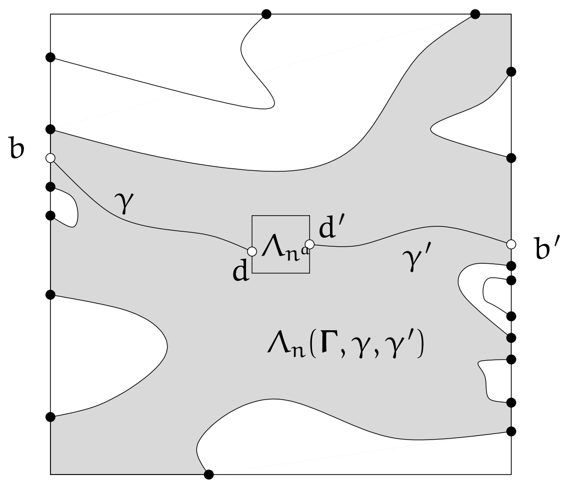


Figure 2.3 – Proof of (2.8). The box $\Lambda_n(\boldsymbol{\Gamma})$ is shaded. The contour $\Gamma_{bb'}$ intersects Λ_{n^a} at the points d and d'.

We can begin to argue as for (2.6), but the sharp triangle inequality gives nothing then, since we do not impose some macroscopic fluctuation to the interface. We have to take care of the sub-exponential behavior of the two-point function.

Inside $\Lambda_n(\boldsymbol{\Gamma})$, the two point function is morally the same as in infinite volume (finite size effects affect only the constant in the Ornstein-Zernike asymptotics), and Theorem 1.3 yields:

$$\frac{Z^{\pm(d,d')}_{\Lambda_n(\boldsymbol{\Gamma}),\beta}}{Z^{+}_{\Lambda_n(\boldsymbol{\Gamma}),\beta}} \approx \frac{C}{n^{a/2}} e^{-\tau_\beta(d'-d)}.$$

Now, let $z \in \partial\Lambda_{2n^b}$ be the first hitting point of $\Gamma_{dd'}$ on Λ_{2n^b} starting from d. The cost of the interface $\Gamma_{dd'}$ is

$$\frac{Z^{\pm(d,d')}_{\Lambda_n(\Gamma),\beta}(\Gamma_{dd'} : d \to z \to d')}{Z^{+}_{\Lambda_n(\Gamma),\beta}} \approx \frac{C}{n^{a/2}} e^{-\tau_\beta(z-d)} \frac{C'}{n^{a/2}} e^{-\tau_\beta(z-d')} \leqslant \frac{C''}{n^a} e^{-\tau_\beta(d'-d)}.$$

Summing over $z \in \partial\Lambda_{2n^b}$, we get

$$\mathbf{P}^{\pm(d,d')}_{\Lambda_n(\Gamma),\beta}(\Gamma_{d,d'} \cap \Lambda_{2n^b} \neq \varnothing) \leqslant C \cdot n^{b-a/2},$$

which finishes the proof of (2.8). ■

2.4.2.3 Step 3: Every Ising measure is close to a convex combination of the two pure states

It remains now to combine the results of Steps 1 and 2, and the spatial relaxation in pure phases, proved in Proposition 1.28, to achieve the proof of Theorem 2.1.

From Step 1, we know that

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}_0) + \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}_1) = 1 + O_\beta(e^{-Cn^{2a-1}})$$

Moreover, conditionally on the event $\mathcal{J}_0$, all the open contours surround Λ_{n^a}. This implies the existence of a circuit surrounding Λ_{n^a} along which the spins take a constant value, $+1$ or -1. We denote by $\mathcal{J}_0^{\pm}$ these events. By Proposition 1.28, we have, uniformly in all $\mathcal{F}_{\Lambda_{n^{a/2}}}$-measurable functions f,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}(f \,|\, \mathcal{J}_0) = \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}_0^{+} \,|\, \mathcal{J}_0)\, \mathbf{P}^{+}_{\beta}(f) + \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}_0^{-} \,|\, \mathcal{J}_0)\, \mathbf{P}^{-}_{\beta}(f) + O\big(\|f\|_\infty e^{-C\,n^a}\big),$$

Similarly, conditionally on the event $\mathcal{J}_1$, all the open contours surround Λ_{n^b} with probability $1 - O(n^{b-a/2})$. We define $\mathcal{J}_1^{\pm}$ as above. Let $\mathcal{J}^{\pm} \doteqdot \mathcal{J}_0^{\pm} \cup \mathcal{J}_1^{\pm}$, we obtain finally, uniformly in $\mathcal{F}_{\Lambda_{n^b}}$-measurable functions f,

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}(f) = \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}^{+})\, \mathbf{P}^{+}_{\beta}(f) + \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{J}^{-})\, \mathbf{P}^{-}_{\beta}(f) + O\big(\|f\|_\infty n^{b-a/2}\big),$$

In particular, we recover the statement of the theorem. ■

The proof of the Aizenman-Higuchi theorem as a corollary of this finite volume estimate is elementary, and we refer to Chapter 3. As for showing that the rate $n^{-1/2}$ is the correct order, we actually think that the Dobrushin boundary condition (in the horizontal direction), which enforces the existence of a crossing contour, should lead the above rate, in view of the results [48]. We were not able to prove this fact with this precise boundary condition, but we present an example in Chapter 3 which is very close in spirit.

2.4.3 Sketch of the proof for the Potts model

The proof of Theorem 2.2 is based on a stronger lemma, which gives quantitative estimates on the closeness of the finite volume measure $\mathbb{P}^{\sigma}_{\Lambda_n,\beta}$ to the corresponding convex combination onto pure states, uniformly on the boundary condition σ around the finite box $\Lambda_n = \{-n, \dots, n\}^2$. It is the analog of Theorem 2.1 for the Ising model.

Lemma 2.1 *Let $q \geqslant 2$ and $\beta > \beta_c$ be fixed. Then for all $\varepsilon > 0$ there exists some constant $C_\varepsilon < \infty$ such that:*
for any boundary condition σ around $\partial\Lambda_n$ there exists a family of non-negative numbers $\alpha_1^n \ldots \alpha_q^n$ depending on (n, σ, β, q) only, such that

$$\left|\mathbb{P}^{\sigma}_{\Lambda_n,\beta}(f) - \sum_{i=1}^{q} \alpha_i^n \mathbb{P}^i_\beta(f)\right| \leqslant C_\varepsilon \|f\|_\infty n^{-\frac{1}{2}+\varepsilon} \tag{2.9}$$

for any function f with support inside Λ_{n^ε}.

The latter Lemma implies Theorem (2.2): indeed, let $\mathbb{P} \in \mathcal{G}_{\beta,q}$ and f be a local function, by the DLR equations (Proposition 2.2) and (2.9) we have:

$$\mathbb{P}(f) = \int d\mathbb{P}(\sigma)\mathbb{P}^{\sigma}_{\Lambda_n,\beta}(f) = \sum_{i=1}^{3}\left(\int d\mathbb{P}(\sigma)\alpha^i_n\right)\mathbb{P}^i_\beta(f) + O(n^{-1/2+\varepsilon}).$$

Applying this to $f = \sigma_0$, we conclude that $\int \alpha^i_n d\mathbb{P}(\sigma)$ converges to α_i as defined in (2.5) when $n \to \infty$. Hence $\mathbb{P}(f) = \sum_{i=1}^{3} \alpha_i \mathbb{P}^i_\beta(f) + O(n^{-1/2+\varepsilon})$. Theorem 2.2 follows by taking the limit $n \to \infty$.

The general philosophy which is behind the proof of Lemma (2.1) is the following. We can draw the separation lines between the different colored parts of the boundary of Λ_n (they will be called "interfaces" in the sequel). Observe that elementary macroscopic interfaces are now trees rather than lines.

We want to show that with high $\mathbb{P}^{\sigma}_{\Lambda_n,\beta}$ probability (the error term being given by the right hand side of (2.9)), whatever boundary condition we choose around the box Λ_n, any local part of $\mathbb{Z}^2$ stays at least at distance $O(n^{1/2})$ from all these interfaces (i.e deep inside a pure phase). Once this is proved, we can conclude with exponential relaxation into pure phases given by Proposition 1.28.

2.4.3.1 Step 1: Reformulation in terms of the random-cluster model

Unlike the Ising case, there is no useful tool in terms of the Potts spins, and no natural partial order distinguishing the colors. That is why we consider the conditioned random-cluster measure associated to the Potts model. Generalizing the Edwards-Sokal coupling for constant boundary condition in Proposition 1.10, we see that a general boundary condition σ for the Potts model gets rephrased as absence of connections (in the random-cluster configuration) between specified parts of the boundary of the domain. More precisely, if our initial Potts model lives on $(\mathbb{Z}^2)^\star$ the associated random-cluster measure has edge-weight $p^\star = 1 - e^{-\beta}$, cluster-weight q and wired boundary condition on $\partial\Lambda_n^\star$, conditioned on the following event:

> Writing $\mathfrak{Col}_i = \{x \in \partial\Lambda^\star_n : \sigma_x = i\}$, the sets $\mathfrak{Col}_i$ and $\mathfrak{Col}_j$ are not connected by open edges of $\Lambda^\star_n$, for every $i \neq j$ in $\{1, \ldots, q\}$. This event is called $\mathrm{Cond}_n[\sigma]$.

We denote this measure by $\mu^w_{\Lambda_n^\star}(\cdot \mid \mathrm{Cond}_n[\sigma])$. Note that the conditioning is a priori very complicated since we can impose up to $O(n)$ color changes along the boundary.

In the random-cluster representation, we have to prove that with high probability, no dual cluster connected to the boundary reaches a small box deep inside Λ_n (which, in particular, implies the same for the Potts interfaces). Since *dual* clusters are the relevant objects we want to consider, it is worth working with the dual random-cluster measure $\mu^f_{\Lambda_n}(\cdot \mid \mathrm{Cond}_n[\sigma])$ which is defined on $\mathbb{Z}^2$ and is subcritical ($p < p_c(q)$), see Proposition 1.16. For this measure, $\mathrm{Cond}_n[\sigma]$ is an increasing event which requires the existence of direct open paths disconnecting different dual $\mathfrak{Col}_i$-s. This hence reduces the problem to the study of the stochastic geometry of subcritical clusters, for which a lot of precise results are known.

If we reformulate the problem in terms of the subcritical random-cluster model, we have to prove that, for fixed $p < p_c(q)$ and some $\varepsilon \in (0,1)$, uniformly in the boundary condition σ,

$$\mu^f_{\Lambda_n}\big(C \cap \Lambda_{n^\varepsilon} \neq \varnothing \mid \mathrm{Cond}_n[\sigma]\big) = O(n^{-\frac{1}{2}+\varepsilon}) \tag{2.10}$$

where C is the set of sites connected to the boundary $\partial\Lambda_n$.

From now on, we give the sketch of the proof, without entering into technical details. Some of the statements will not be completely exact, but keep hopefully the main physical ideas of the proof.

The numerical constants will be written c or C. They are always non negative and can change from line to line. Let $\varepsilon > 0$ to be chosen later in the proof.

2.4.3.2 Step 2: Macroscopic flower domains

We first show that, no matter what the boundary condition σ is, with high probability only a bounded number of such interfaces is capable of reaching the box Λ_m, where m is a fraction of n. Furthermore, we argue that the number of sites on $\partial\Lambda_m$ which are connected to the original $\partial\Lambda_n$ is uniformly bounded (there could indeed be a finite number of crossings ending by little forks which could lead to a diverging number of such points). In terms of the original Potts model, this corresponds to the existence, with high probability, of a domain including the box Λ_m for which the boundary condition contains a uniformly bounded number of spin changes, see Figure (2.4).

Using exponential decay of the two points function and the FKG inequality, we first show that

$$\mu^f_{\Lambda_n}(\exists\, r \text{ disjoint crossings of } \Lambda_n \backslash \Lambda_{n/2}) \leqslant e^{-crn} \tag{2.11}$$

for some $c > 0$, where by r disjoint crossings we mean r clusters which are disjoints in $\Lambda_n \backslash \Lambda_{n/2}$. Indeed, by induction, if we explore the open clusters of $r-1$ crossings of $\Lambda_n \backslash \Lambda_{n/2}$, we have to upper bound the probability to have a crossing in the remaining part of the annulus, with free boundary conditions on the border of the previous exploration. By the FKG inequality this is upper bounded by the probability to have a crossing of $\Lambda_n \backslash \Lambda_{n/2}$ with wired boundary conditions on $\partial\Lambda_n \cup \partial\Lambda_{n/2}$, which is

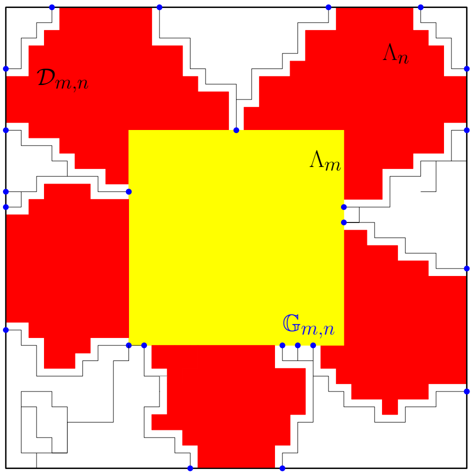


Figure 2.4 – We call a “flower domain” $\mathcal{D}_{m,n}$ the complement of the exploration of the open cluster C from $\partial\Lambda_n$ up to Λ_m. (On the picture $\mathcal{D}_{m,n}$ is the union of the heart Λ_m (in yellow) and the petals (in red); all the edges which are incident to the drawn open edges are closed). We write $\mathbb{G}_{m,n} = C \cap \partial\Lambda_m$.
Thanks to Paola for the esthetic advice, and see Figure 4.2 for a more realistic flower domain.

exponentially small in n by Corollary 1.5.
Together with exponential bounds on the size of subcritical clusters (point 2 of Corollary 1.5), (2.11) implies that with probability $1 - O(e^{-cn})$,

$$\exists\, m \in \left[\frac{n}{3}, \frac{n}{2}\right] \text{ and } M < \infty \text{ such that } |C \cap \partial\Lambda_m| \leqslant M \tag{2.12}$$

Indeed, if for all $m \in [n/3, n/2]$ we have $|\mathbb{G}_{m,n}| = |C \cap \partial\Lambda_m| > M$, either the annulus $\Lambda_{n/2} \backslash \Lambda_{n/3}$ contains more than r crossings, or one of the crossings has cardinality bigger than $Mn/6r$. The first possibility is exponentially decaying in nr by (2.11) and the second one in Mn/r by Corollary 1.5. We easily prove (2.12), for M and r sufficiently large, using the finite energy property: the probability of achieving $\mathrm{Cond}_n[\sigma]$ is bigger than p^n for some $p \in (0,1)$ (just open all the edges along $\partial\Lambda_n$).

Henceforth we write $\mathcal{D} = \mathcal{D}_{m,n}$, $\mathbb{G} = \mathbb{G}_{m,n}$ and $C_{\mathbb{G}} = C \cap \Lambda_m$. Note that $\mathrm{Cond}_n[\sigma]$ corresponds to the existence of certain connections between different sites of $\mathbb{G}$. We will write $\mathrm{Cond}(\underline{\mathbb{G}})$ for the event imposing connections between subsets in the partition $\underline{\mathbb{G}}$ of $\mathbb{G}$ (which is compatible with $\mathrm{Cond}_n[\sigma]$), see Figure (2.5).

Note that some of the points of $\mathbb{G}$ can be connected together from the outside (On Figure (2.4), it is the case of the two points on the upper part of $\partial\Lambda_m$). We simplify the problem with the following upper-bound:

$$\mu^f_{\Lambda_n}\big(C \cap \Lambda_{n^\varepsilon} \neq \varnothing \bigm| \mathrm{Cond}_n[\sigma]\big) \leqslant e^{-cn} + Cq^M \max \mu^f_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^\varepsilon} \neq \varnothing \bigm| \mathrm{Cond}(\underline{\mathbb{G}})\big), \tag{2.13}$$

where the maximum is over all flower domains $\mathcal{D}$ rooted at $m \in [n/3, n]$ with at most $|\mathbb{G}| \leqslant M$ color changes, and over all partitions $\underline{\mathbb{G}}$ of $\mathbb{G}$. The factor q^M bounds the Radon-Nikodym derivative between the measure resulting from the exploration of C up to Λ_m, which accounts for the external connections, and the measure $\mu^f_{\mathcal{D}}$.

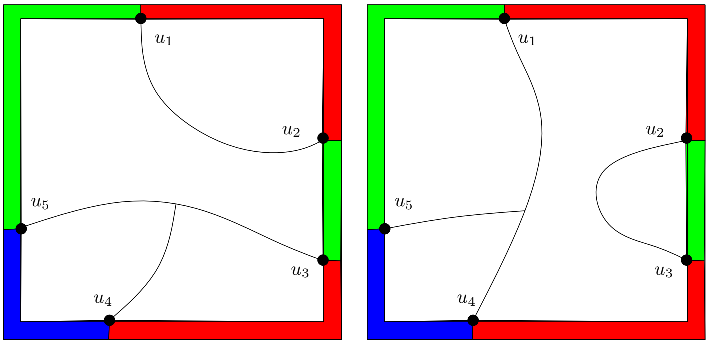


Figure 2.5 – Here is an example of two partitions $\underline{\mathbb{G}}_1 = \{\{u_1, u_2\}, \{u_3, u_4, u_5\}\}$ and $\underline{\mathbb{G}}_2 = \{\{u_2, u_3\}, \{u_1, u_4, u_5\}\}$ of $\mathbb{G} = \{u_1, u_2, u_3, u_4, u_5\}$, which are compatible with the coloring of the boundary condition.

2.4.3.3 Step 3: Macroscopic structure near the center of Λ_n

The next step of the proof consists in studying the macroscopic structure of the set of sites C connected to the boundary of Λ_n. We will see that if we coarse grain C at scale $K > 0$ sufficiently large, then we get (in a small fraction of the box Λ_n) a set that is either empty, or close to a segment, or close to a tripod (three segments coming out from a point).

After (2.13), we work with a given partition $\underline{\mathbb{G}}$ of the set $\mathbb{G}$ of points connected to the boundary of Λ_n, which is compatible with the initial Potts boundary condition σ. The natural objects we consider now are the graphs (seen as subsets of $\mathbb{R}^2$) which connects the points according to the partition and have minimal total τ length. We will prove that these "Steiner forests" are the structure on which the subcritical clusters (conditioned to $\mathrm{Cond}_n[\sigma]$) concentrate.

Recall Theorem 1.1. It turns out that very important features of the Steiner minimal graphs for the τ norm follow from the strict convexity of U_τ. General works about the Steiner problem such as [10] and [23] show that, for any given finite set $\mathbb{G}$ and any partition $\underline{\mathbb{G}}$:

1. There exists a finite number of such Steiner minimal graphs, which connect the sites of $\mathbb{G}$ according to $\underline{\mathbb{G}}$.

2. These minimal graphs are forests (i.e. collection of disjoint trees) such that each inner node has degree 3, and there exists an $\eta > 0$ such that the angle between two edges incident to an inner node is always larger than $\pi/2 + \eta$.

3. If a given graph has a total τ length δn close to the minimal one (fulfilling the partition), then this graph is $\delta' n$ close (for the τ Hausdorff distance) to one of the Steiner forests, for some $\delta, \delta' > 0$.

Let us now construct a coarse graining at scale K of the cluster $C_{\mathbb{G}}$ obtained under the measure $\mu^f_{\mathcal{D}}(\cdot | \mathrm{Cond}(\underline{\mathbb{G}}))$. Namely, cover iteratively the cluster $C_{\mathbb{G}}$ with τ balls of radius $K + c \log K$, written $\bar{\mathbf{B}}_K$, (where K is a constant that we will have to take sufficiently large depending on $|\mathbb{G}|$; the little "security corridor" of size $c \log K$ ensures

decoupling of the weights, and τ-balls of radius K are written $\mathbf{B}_K$):
center the first ball at some $u_i = x_0 \in \mathbb{G}$, and continue centering the $(n+1)$-th ball at the site where $C_{\mathbb{G}}$ exits $\cup_{i=1}^{n} \bar{\mathbf{B}}_K(x_i)$. This produces a graph $\mathcal{F}_K$ with vertices $\{x_i\}$ and such that $[x_i, x_j]$ (with $i > j$) is an edge if $C_{\mathbb{G}}$ exits $\cup_{k=1}^{i-1} \bar{\mathbf{B}}_K(x_k)$ through $\bar{\mathbf{B}}_K(x_j)$. Note that $\mathcal{F}_K$ is a family of *trees*, called "K-forest skeleton". Each tree $\mathcal{T}_K$ consists of a trunk $\mathfrak{t}_K$ and of branches $\mathfrak{B}_K$, see Figures 1.6 and 2.6).

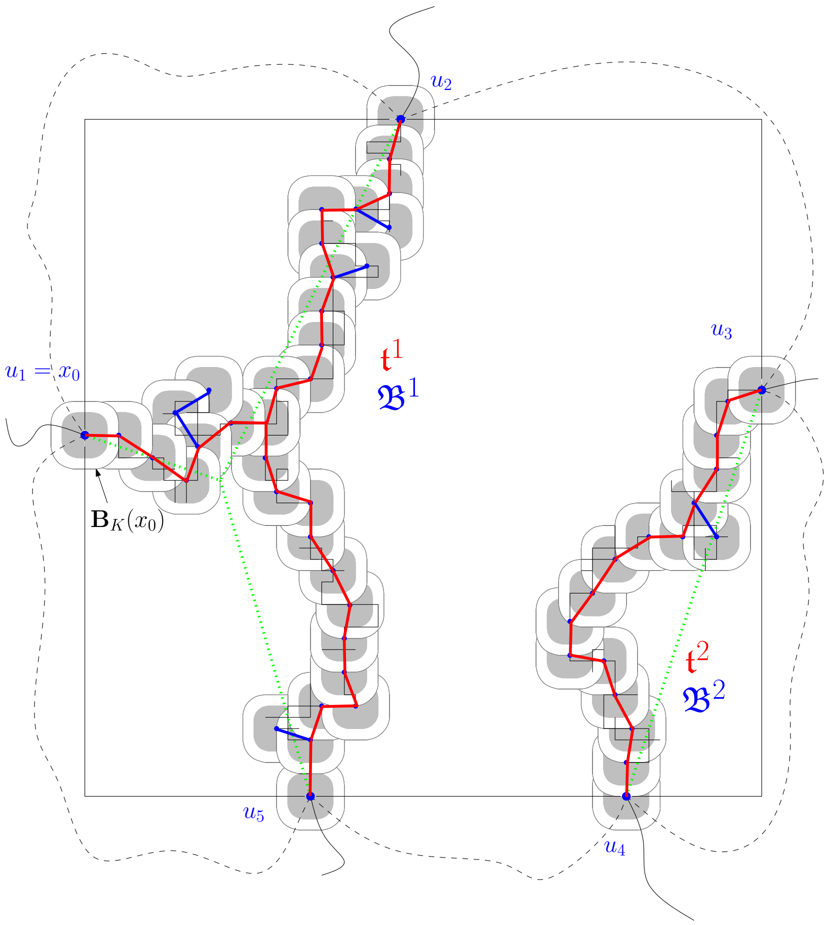


Figure 2.6 – Construction of the forest skeleton $\mathcal{F}_K = \{\mathcal{T}_K^1, \mathcal{T}_K^2\}$ of the cluster $C_{\mathbb{G}}$ (in black), consisting of the trees $\mathcal{T}_K^i = \{\mathfrak{t}^i, \mathfrak{B}^i\}$, $i = 1, 2$. The Steiner forest corresponding to the partition $\mathbb{G} = (\{u_1, u_2, u_5\}, \{u_3, u_4\})$ is drawn in dashed green.

Using an adaptation of [21, Section 2], we are able to prove that for all $\delta > 0$ there exists some $c = c(\delta) > 0$ such that:

$$\mu_{\mathcal{D}}^{f}\left(\min_{\mathcal{F} \in \{ \substack{\tau \text{ Steiner forests} \\ \text{corresp. to } \underline{\mathbb{G}}} \}} d_\tau(\mathcal{F}_K, \mathcal{F}) > \delta n \;\middle|\; \mathrm{Cond}(\underline{\mathbb{G}}) \right) \leqslant e^{-cn}, \tag{2.14}$$

where d_τ denotes the τ Hausdorff distance.

Indeed, by the FKG inequality, $\mu_{\mathcal{D}}^{f}(\mathrm{Cond}(\underline{\mathbb{G}})) \geqslant e^{-\tau_{\mathbb{G}}(1+o_n(1))}$, where $\tau_{\underline{\mathbb{G}}}$ is the τ length of the Steiner forest associated to the partition $\underline{\mathbb{G}}$. Moreover, for a given

K-forest skeleton $\mathcal{F}^*$ with trees t^*, writing $\bar{A}_i = \bar{\mathbf{B}}_K(x_i) \setminus \cup_{j=1}^{i-1} \mathbf{B}_K(x_j)$, and $A_i = \mathbf{B}_K(x_i) \setminus \cup_{j=1}^{i-1} \bar{\mathbf{B}}_K(x_j)$, we have the upper-bound:

$$\mu^f_{\mathcal{D}}(\mathcal{F}_K = \mathcal{F}^*) \leqslant \mu^f_{\mathcal{D}} \left(\bigcap_{t^* \in \mathcal{F}^*} \bigcap_{i \in t^*} x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i) \right)$$
$$\leqslant \prod_{t^* \in \mathcal{F}^*} \prod_{i \in t^*} \mu^w_{\bar{A}_i} \left(x_i \xleftrightarrow{A_i} \partial \mathbf{B}_K(x_i) \right) \leqslant e^{-\tau(\mathcal{F})(1 - o_K(1) - o_n(1))},$$

where in the second inequality we expand the probability of the intersection as a product of conditional expectations and then use the FKG inequality to compare these conditional expectations with the probability with wired boundary conditions. We also used that $\mu^w_{\mathbf{B}_K(x)}(x \leftrightarrow \partial \mathbf{B}_K(x)) = e^{-K(1-o_K(1))}$, see the proof of Corollary 1.5. If we now upper bound crudely the number of K-forest skeletons (which contain necessarily less than $C\tau(\mathcal{F})/K$ vertices) by $(cK)^{C\tau(\mathcal{F})/K}$, we prove (2.14) by comparison with the lower bound on the denominator.

The result (2.14), together with the known properties of τ Steiner forests, allows us to simplify drastically the problem: with overwhelming probability, a box Λ_m, where m is a sufficiently small fraction of n, intersects either nothing, or one segment, or one tripod of one of the Steiner forests $\mathcal{F}$ compatible with the partition $\underline{\mathbb{G}}$. We can show that moreover the set $\mathbb{G}_{m,n}$ consists in either 0 or 2 or 3 points on $\partial\Lambda_m$ (that must be connected together).

In the Potts counterpart (uniformly in the boundary condition σ), we have proved up to now that the macroscopic structure of the interfaces, in a box of size which is a fraction of n, is δn close to a Steiner forest connecting a finite number of points (with $\delta > 0$ as small as we want provided n is sufficiently large). This implies that in a smaller box of size which is still a fraction of n, only 3 cases occur: either the box sees a pure phase corresponding to a unique color, or it is crossed by an interface between two colors, or it stays at the triple point between three colors. All the other cases are exponentially suppressed with n, see Figure (2.7).

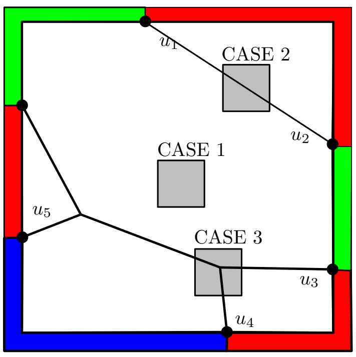


Figure 2.7 – The 3 cases to study.

2.4.3.4 Step 4: Fluctuation theory

It remains to study the fluctuations of $C_{\mathbb{G}}$ in the later three cases in order to ensure it intersects a box of size n^{ε} with vanishing probability.

- *Case 1: No imposed crossing*
  The first case boils down to exponential relaxation into pure phases (see Proposition 1.21). $\mathrm{Cond}(\underline{\mathbb{G}})$ is in this case the sure event (no conditioning inside Λ_m): we have
  $$\mu^f_{\mathcal{D}}(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}}) \leqslant O(e^{-cn}) \tag{2.15}$$

- *Case 2: One imposed crossing*
  Known results about the two point function in the subcritical random-cluster model allow us to handle the case where $\mathrm{Cond}(\underline{\mathbb{G}}) = \{u \leftrightarrow v\}$ for some points $u, v \in \partial\Lambda_m$, where m is a fraction of n. Indeed, [48] and Theorem 1.1 imply that the law of the cluster connecting u and v converges to the law of a Brownian bridge. Consequently, uniformly in u, v,
  $$\mu^f_{\mathcal{D}}(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \mid u \leftrightarrow v) \leqslant O(n^{-1/2+\varepsilon}) \tag{2.16}$$

- *Case 3: One tripod*
  This is the non trivial case where $\mathrm{Cond}(\underline{\mathbb{G}}) = \{C_{\{u_1,u_2,u_3\}} \neq \varnothing\}$ for some vertices $u_1, u_2, u_3 \in \partial\Lambda_m$. By the preceding step of the proof, we know that $C_{\{u_1,u_2,u_3\}}$ is $O(n)$ close (for the τ Hausdorff distance) to the unique Steiner tripod corresponding to this conditioning, denoted $\{u_1, u_2, u_3; x\}$ (the later is a short notation for the graph with vertices $\{u_1, u_2, u_3, x\}$ and edges $[u_1, x], [u_2, x], [u_3, x]$ such that its τ length is minimal). We will show that the actual tripod has Gaussian fluctuations and therefore intersects a small box with probability $O(n^{-1/2+\varepsilon})$.

  Using tools developed in [21] we can show that, with probability larger than $1 - O(e^{-cn^{\varepsilon}})$, the cluster $C_{\{u_1,u_2,u_3\}}$ coarse grained at scale n^{ε} is contained in a set which is a union of:

  - A central part $\Lambda^\star$ of diameter $O(n^{\varepsilon})$ centered at distance less than δn from x.
  - Three non degenerate cones based at $v_1, v_2, v_3 \in \partial\Lambda^\star$ which contain respectively u_1, u_2, u_3.

  We call this event S_{ε}, see Figure (2.8).

  If we indeed construct a coarse graining at scale n^{ε} of $C_{\{u_1,u_2,u_3\}}$, the forest skeleton consists of a unique tree $\mathcal{T}_{n^{\varepsilon}}$ whose multiple point is written x_{ε} (it is the cross point between the three "legs" of the tree: $\mathcal{T}_{n^{\varepsilon}} \backslash \{x_{\varepsilon}\} = \cup_{i=1}^{3} \mathcal{T}^i_{n^{\varepsilon}}$). We claim that there exists a box $\Lambda^\star$ of diameter $O(n^{\varepsilon})$, $\kappa > 0$ and $w_i \in \partial\Lambda^\star$ such that
  $$\mathcal{T}^i_{n^{\varepsilon}} \backslash \Lambda^\star \subseteq w_i + \mathcal{Y}_{i,\kappa} \quad \forall i = 1, 2, 3,$$
  where $\mathcal{Y}_{i,\kappa}$ is the cone $\{z : \measuredangle(z, u_i - x_{\varepsilon}) \leqslant \kappa\}$. Namely the first "cone point" w_i of each of the three legs $\mathcal{T}^i_{n^{\varepsilon}}$ is in a $O(n^{\varepsilon})$-neighborhood of the triple point x_{ε}.

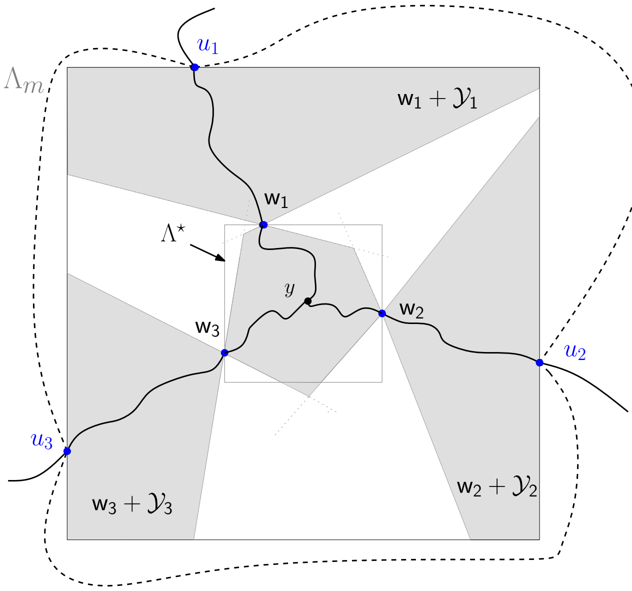


Figure 2.8 – The event S_ε. The diameter of $\Lambda^\star$ is of order $O(n^\varepsilon)$.

To see this, let us come back to coarse graining techniques we used for Step 3 of the proof at a rougher scale. We want to prove:

$$\mu^f_{\mathcal{D}}\Big(S_\varepsilon \;\Big|\; C_{\{u_1,u_2,u_3\}} \neq \varnothing\Big) \geqslant 1 - O(e^{-cn^\varepsilon}) \tag{2.17}$$

Again, by the FKG inequality and the Ornstein-Zernike two point function asymptotics (see Theorem 1.1), we have

$$\begin{aligned}\mu^f_{\mathcal{D}}(C_{\{u_1,u_2,u_3\}} \neq \varnothing) &\geqslant \prod_{i=1}^{3} \mu^f_{\mathcal{D}}(x \leftrightarrow u_i) \geqslant \prod_{i=1}^{3} \frac{C}{\sqrt{n}} e^{-\tau(u_i - x)} \\ &= \exp\left(-\sum_{i=1}^{3} \tau(u_i - x) - C\log n\right).\end{aligned}$$

Now, by the mass gap arguments of [21], which come essentially from strict convexity of the τ norm, the exponential cost for a leg $\mathcal{T}^i_{n^\varepsilon}$ not to have any cone point (between x_ε and u_i) is of order n. Let w_i denote the first cone point of the leg $\mathcal{T}^i_{n^\varepsilon}$. For each fixed realization $\mathcal{T}^{i,\star}$ of $\mathcal{T}^i_{n^\varepsilon}$ up to w_i such that $|w_i - x_\varepsilon| > Cn^\varepsilon$, we have (as in Step 3), using the OZ asymptotics:

$$\mu^f_{\mathcal{D}}\left(S_\varepsilon \cap \left\{ {w_i \text{ is the first cone point of } \mathcal{T}^i_{n^\varepsilon} \atop \mathcal{T}^i_{n^\varepsilon} = \mathcal{T}^{i,\varepsilon} \text{ up to } w_i} \right\}\right) \leqslant \exp\left(-\sum_{i=1}^{3} \tau(u_i - x_\varepsilon) - cn^\varepsilon\right) \tag{2.18}$$

where we used either the strict convexity of the τ norm in the case $w_i \in x_\varepsilon + \mathcal{Y}_{i,\kappa}$ which leads to $\tau(\mathcal{T}^{i,\star}) \geqslant \tau(w_i - x_\varepsilon) + C|w_i - x_\varepsilon|$, or the sharp triangle inequality in the case $w_i \notin x_\varepsilon + \mathcal{Y}_{i,\kappa}$ (see Proposition 1.3), which implies

$$\tau(w_i - x_\varepsilon) + \tau(u_i - w_i) - \tau(u_i - x_\varepsilon) \geqslant C|w_i - x_\varepsilon|.$$

As before we can upper bound the entropic factor related to the number of possible compatible realizations $\mathcal{T}^{i,\star}$ by $(n^\varepsilon)^{\sharp\text{balls}} \leqslant \exp\left(C|w_i - x_\varepsilon|\frac{\log n}{n^\varepsilon}\right)$, which is suppressed, proving (2.17).

In the Potts counterpart, we just proved that the interfaces associated to a boundary condition with 3 different colors (on macroscopic parts of Λ_n) have a tripod structure, with a central zone of diameter $O(n^\varepsilon)$ and center δn close to the Steiner triple point x.

Once we know that the event S_ε occurs with overwhelming probability, we can analyze the fluctuations of its middle point. Let $S_\varepsilon(y)$ denote the event S_ε with a prescribed middle point y. By performing a quadratic expansion of the function $\phi(y) = \sum_{i=1}^3 \tau(u_i - y)$ around x, and by showing that $\mu^f_{\mathcal{D}}(S_\varepsilon(y)) \simeq \mu^f_{\mathcal{D}}(S_\varepsilon(x))$ for all y in a $O(n^{1/2})$-neighborhood of x, we can prove that y fluctuates in a Gaussian way:

$$\mu^f_{\mathcal{D}}\left(S_\varepsilon(y) \,\middle|\, C_{\{u_1,u_2,u_3\}} \neq \varnothing\right) = O\left(n^{-1}\exp\left(-c\,\frac{|y-x|^2}{n}\right)\right) \tag{2.19}$$

Indeed, it is not difficult to argue, by homogeneity of τ, that $\phi(y) - \phi(x) \simeq C\frac{|y-x|^2}{n}$. This yields morally the following estimate:

$$\frac{\mu^f_{\mathcal{D}}(S_\varepsilon(y))}{\mu^f_{\mathcal{D}}(S_\varepsilon(x))} = \Theta\left(\exp\left(-c\,\frac{|y-x|^2}{n}\right)\right),$$

implying the upper bound:

$$\mu^f_{\mathcal{D}}\left(S_\varepsilon(y) \,\middle|\, C_{\{u_1,u_2,u_3\}} \neq \varnothing\right) \leqslant \exp\left(-c\,\frac{|y-x|^2}{n}\right)\frac{\mu^f_{\mathcal{D}}(S_\varepsilon(x))}{\mu^f_{\mathcal{D}}(C_{\{u_1,u_2,u_3\}} \neq \varnothing)}.$$

Next, by observing that $|y-x| \leqslant n^{1/2}$ implies that $\exp(-c|y-x|^2/n)$ is of order 1, by looking at the n sites which are at distance at most $n^{1/2}$ from x we deduce:

$$\frac{\mu^f_{\mathcal{D}}(S_\varepsilon(x))}{\mu^f_{\mathcal{D}}(C_{\{u_1,u_2,u_3\}} \neq \varnothing)} \leqslant O(n^{-1}),$$

which proves (2.19).

The later estimate allows us to compute explicitly the probability for the cluster $C_{\mathbb{G}}$ to reach the box Λ_{n^ε}, conditioned on CASE 3. Suppose for simplicity that the Steiner triple point x coincides with the origin, and for each $y \in (\Lambda_{Cn^\varepsilon})^c$

write $d = d_\tau(0, [y, u_i])$ where $[y, u_i]$ is the segment of the Steiner tripod which is the closest to the origin. Then,

$$
\begin{aligned}
\mu^f_{\mathcal{D}}(C_{\mathbb{G}} \cap \Lambda_{n^\varepsilon} \neq \varnothing \mid C_{\{u_1,u_2,u_3\}} \neq \varnothing)& \\
&\leqslant C \sum_{y \in \Lambda_n} n^{-1} \exp\left(-c\,\frac{|y|^2}{n}\right) \mu^f_{\mathcal{D}}\Big(C_{\mathbb{G}} \cap \Lambda_{n^\varepsilon} \neq \varnothing \,\Big|\, S_\varepsilon(y)\Big) \\
&\leqslant O(n^{-1+\varepsilon}) + Cn^{-1} \sum_{y \in \Lambda_{n^{1/2+\varepsilon}} \setminus \Lambda_{C'n^\varepsilon}} \frac{1}{\sqrt{|y|}} \exp\left(-c\frac{d^2}{|y|}\right) \\
&\leqslant O(n^{-1+\varepsilon}) + Cn^{-1} \sum_{\ell = C'n^\varepsilon}^{n^{1/2+\varepsilon}} \frac{O(\sqrt{\ell})}{\sqrt{\ell}} \\
&\leqslant O(n^{-\frac{1}{2}+\varepsilon}) \qquad (2.20)
\end{aligned}
$$

In the third line we used the Brownian scaling of the two point function (Corollary 1.4) when the center of the tripod is far enough from the box Λ_{n^ε} (i.e. outside $\Lambda_{C'n^\varepsilon}$), and the previous remark that sites y located outside $\Lambda_{n^{1/2+\varepsilon}}$ do not contribute to the sum because the term $\exp(-c|y|^2/n)$ is too small. In the fourth line we use that sites y at distance ℓ from the origin contributing substantially to the sum must satisfy $d = O(\sqrt{\ell})$.

Putting eveything together, (2.15), (2.16) and (2.20) imply the upperbound in (2.10). It is not difficult to show that this bound is (almost) optimal by taking a Potts boundary condition of the Dobrushin type (one color on $\Lambda_n \cap (\mathbb{Z} \times \mathbb{Z}^+)$ and another color on $\Lambda_n \cap (\mathbb{Z} \times \mathbb{Z}^-)$): the unique induced interface fluctuates like a Brownian bridge and gives the correct exponent $1/2$.

Chapter 3

Detailed proofs about the Ising model

This chapter is devoted to the rigorous proof of the result described in Section 2.2.2. It consists of an extended version of the article [LC4]. The introductive part is adapted to this manuscript in order to refer to results proved in details in Chapter 1. The article is a joint work with Yvan Velenik, and was published in the journal *Probability Theory and Related Fields* in June 2012.

3.1 The theorem

Let $\Omega \doteqdot \{-1, 1\}^{\mathbb{Z}^2}$ be the set of spin configurations, and Λ be a finite subset of $\mathbb{Z}^2$. For $A \subset \mathbb{Z}^2$, we denote by $\mathcal{F}_A$ the σ-algebra of all events depending only on the spins inside A. We recall that A probability measure $\mathbf{P}$ on Ω is an *infinite-volume Gibbs measure* for the 2 dimensional Ising model at inverse temperature β if and only if it satisfies the DLR equation

$$\mathbf{P}(\cdot\,|\mathcal{F}_{\Lambda^c})(\omega) = \mathbf{P}^{\omega}_{\Lambda,\beta}, \qquad \text{for } \mathbf{P}\text{-a.e. } \omega, \text{ and all } \Lambda \Subset \mathbb{Z}^2. \tag{3.1}$$

We denote by $\mathcal{G}_\beta$ the set of all such measures.

We set $\Lambda_r \doteqdot \{-\lfloor r \rfloor, \dots, \lfloor r \rfloor\}^2$. For $\Lambda \Subset \mathbb{Z}^2$, we denote by $\langle\cdot\rangle^{\omega}_{\Lambda,\beta}$ the expectation under the (finite-volume) measure $\mathbf{P}^{\omega}_{\Lambda,\beta}$ and by $\langle\cdot\rangle^{+}_{\beta}$, resp. $\langle\cdot\rangle^{-}_{\beta}$, the expectation under the (infinite-volume) measure $\mathbf{P}^{+}_{\beta}$, resp. $\mathbf{P}^{-}_{\beta}$.

We shall make use of the following notation: If R_1, R_2 and R_3 are three expressions, depending on various parameters (β, n, ω, etc.), and we write $R_1 = R_2 + O_\beta(R_3)$, this means that there exists a constant $C(\beta) < \infty$, depending on β only, such that $|R_1 - R_2| \leqslant C(\beta) R_3$.

During the whole chapter, the notation $\mathfrak{c}, \mathfrak{c}', \mathfrak{c}'', \dots$ will be used for positive constants that may change from line to line, while the notation $C_1, C_2, C_3, \dots$ will be used for positive constants defined once for all and recalled throughout the proofs.

Our main theorem is the following.

Theorem 3.1 *Let $\beta > \beta_c$, $b < 1/2$ and $\omega \in \Omega$. Then, for any $0 < \delta < 1/2 - b$, there exists $n_0 = n_0(\beta, b, \delta)$ such that, for all $n > n_0$, there exists a constant $\alpha^{n,\omega}(\beta) \in [0,1]$ such that, for all $\mathcal{F}_{\Lambda_{n^b}}$-measurable function f,*

$$\langle f\rangle^{\omega}_{\Lambda_n,\beta} = \alpha^{n,\omega}\langle f\rangle^{+}_{\beta} + (1-\alpha^{n,\omega})\langle f\rangle^{-}_{\beta} + O_\beta\big(\|f\|_\infty\, n^{-\delta}\big).$$

As a corollary we get the Aizenman-Higuchi theorem:

Corollary 3.1 *For any $\beta > \beta_c$, $\mathcal{G}_\beta = \{\alpha \mathbf{P}^+ + (1-\alpha)\mathbf{P}^- \,:\, 0 \leqslant \alpha \leqslant 1\}$.*

The estimate we have on the error term in Theorem 3.1 is essentially optimal:

Proposition 3.1 *Let $\beta > \beta_c$. There exist a local function f and a constant $c = c(\beta) > 0$ such that, for all n large enough, one can find $\omega \in \Omega$ with*

$$\inf_{\alpha\in[0,1]}\Big|\langle f\rangle^{\omega}_{\Lambda_n,\beta} - \alpha^{n,\omega}\langle f\rangle^{+}_{\beta} - (1-\alpha^{n,\omega})\langle f\rangle^{-}_{\beta}\Big| \geqslant c n^{-1/2}.$$

As explained in Section 2.4.2, the proof of Theorem 3.1 comprises two main steps: (i) Proving that, with high probability, at most one interface approaches the center of the box Λ_n, (ii) proving that this interface, when present, undergoes unbounded fluctuations (actually of order $\sqrt{n}$). It will then follow that any local observable, with support close to the center of the box, will lie, with high probability, deep inside the $+$ or $-$ phase.

3.2 Some tools and two lemmata

In this section we gather the results introduced in details in Chapter 1 which we use in the proof of Theorem 3.1.

3.2.1 Surface tension

We recall that the surface tension of the Ising model on $\mathbb{Z}^2$ satisfies the sharp triangle inequality, which follows from a combination of [77, Theorem 2.1] and [20, Theorem B]:
For any $\beta > \beta_c$, there exists a constant $\kappa_\beta > 0$ such that

$$\tau_\beta(x) + \tau_\beta(y) - \tau_\beta(x+y) \geqslant \kappa_\beta\big(\|x\|_2 + \|y\|_2 - \|x+y\|_2\big), \quad \forall x, y \in \mathbb{R}^2. \tag{3.2}$$

3.2.2 Random-line representation

We refer to Section 1.2.4.3 for a presentation of the random line representation. Let ω be a boundary condition around the box $\Lambda \Subset \mathbb{Z}^2$, and let $\mathbf{\Gamma}(\sigma) = (\Gamma_1(\sigma), \dots, \Gamma_M(\sigma))$ be the open contours of the configuration σ. The set of all endpoints of open contours, which is completely determined by the boundary condition ω, is denoted $\mathbf{b}(\omega) \equiv \{b_1, \dots, b_{2M}\}$.

We recall and gather the following properties of the weights $q_{\Lambda,\beta}$:

- Let $\beta < \beta_c$, and $\beta^\star > \beta_c$ the dual inverse-temperature. Then,

$$\frac{Z^{\omega}_{\Lambda,\beta}}{Z^{+}_{\Lambda,\beta}} = \sum_{\partial\mathbf{\Gamma}=\mathbf{b}(\omega)} q^{\omega}_{\Lambda,\beta}(\mathbf{\Gamma}) = \sum_{\mathbf{\Gamma}\sim(\omega,\Lambda)} q_{\Lambda^\star,\beta^\star}(\mathbf{\Gamma}) = \langle \sigma_{b_1}\cdots\sigma_{b_{2M}}\rangle_{\Lambda^\star,\beta^\star}, \tag{3.3}$$

  where $\langle\cdot\rangle_{\Lambda^\star,\beta^\star}$ denotes expectation with respect to the finite-volume Gibbs measure in $\Lambda^\star$ at inverse temperature $\beta^\star$ with free boundary condition,

- If $\Lambda_1 \subset \Lambda_2$ then

$$q_{\Lambda_1,\beta}(\mathbf{\Gamma}) \geqslant q_{\Lambda_2,\beta}(\mathbf{\Gamma}). \tag{3.4}$$

- We associate to an (ω,Λ)-compatible family of open contours the set $\mathfrak{F}(\Gamma_1,\dots,\Gamma_n)$ of all vertices of Λ whose spin value is completely determined by ω and these open contours, i.e., the maximal set such that, if σ' is another configuration compatible with ω such that $\Gamma_1,\dots,\Gamma_n \subset \mathbf{\Gamma}(\sigma')$, then $\sigma'_i = \sigma_i$, for all $i \in \mathfrak{F}(\Gamma_1,\dots,\Gamma_n)$. We set $\Lambda(\Gamma_1,\dots,\Gamma_n) \doteqdot \Lambda \setminus \mathfrak{F}(\Gamma_1,\dots,\Gamma_n)$, and say that $\Gamma_1,\dots,\Gamma_n$ *partition the box* Λ into the connected components of $\Lambda(\Gamma_1,\dots,\Gamma_n)$. We then have:

$$q_{\Lambda,\beta}(\Gamma_1,\dots,\Gamma_n,\Gamma_{n+1},\dots,\Gamma_m) = q_{\Lambda,\beta}(\Gamma_1,\dots,\Gamma_n)\, q_{\Lambda(\Gamma_1,\dots,\Gamma_n),\beta}(\Gamma_{n+1},\dots,\Gamma_m), \tag{3.5}$$

  for all (ω,Λ)-compatible family $\Gamma_1,\dots,\Gamma_m \subset \mathbf{\Gamma}(\sigma)$ of open contours.

- Let $\mathbf{b}$ be a subset of even cardinality of $\partial^\star\Lambda$, and b_1, b'_1, b_2, b'_2 four distinct vertices of $\mathbf{b}$. Let also $A_1 \subset \left\{\Gamma : b_1 \to b'_1\right\}$ and $A_2 \subset \left\{\Gamma : b_2 \to b'_2\right\}$. It follows easily from (3.3) and (3.5) that

$$\sum_{\substack{\Gamma_1\in A_1,\Gamma_2\in A_2,\mathbf{\Gamma}\\ \partial(\Gamma_1,\Gamma_2,\mathbf{\Gamma})=\mathbf{b}}} q_{\Lambda,\beta}(\Gamma_1,\Gamma_2,\mathbf{\Gamma}) \leqslant \sum_{\substack{\Gamma_1\in A_1,\Gamma_2\in A_2\\ \partial(\Gamma_1,\Gamma_2)=\{b_1,b'_1,b_2,b'_2\}}} q_{\Lambda,\beta}(\Gamma_1,\Gamma_2). \tag{3.6}$$

- Let $i, j \in \partial^\star\Lambda$. Then,

$$\sum_{\Gamma:i\to j} q_{\Lambda,\beta}(\Gamma) \leqslant e^{-\tau_\beta(j-i)}. \tag{3.7}$$

- Let $\mathbf{b}_1, \mathbf{b}_2$ be two disjoint subsets of even cardinality of $\partial^\star\Lambda$. Then we have the BK-type inequality:

$$\sum_{\substack{\partial\mathbf{\Gamma}_1=\mathbf{b}_1,\partial\mathbf{\Gamma}_2=\mathbf{b}_2\\ \partial(\mathbf{\Gamma}_1,\mathbf{\Gamma}_2)=\mathbf{b}_1\cup\mathbf{b}_2}} q_{\Lambda,\beta}(\mathbf{\Gamma}_1,\mathbf{\Gamma}_2) \leqslant \sum_{\partial\mathbf{\Gamma}_1=\mathbf{b}_1} q_{\Lambda,\beta}(\mathbf{\Gamma}_1) \sum_{\partial\mathbf{\Gamma}_2=\mathbf{b}_2} q_{\Lambda,\beta}(\mathbf{\Gamma}_2). \tag{3.8}$$

- Let $z \in \Lambda^\star$; we write $\Gamma : b \to z \to b'$ when $\Gamma : b \to b'$ and $\Gamma \ni z$. Then,

$$\sum_{\Gamma : b \to z \to b'} q_{\Lambda,\beta}(\Gamma) \leqslant \frac{C_1(\beta)}{\sqrt{\|z-b\|_2 \|z-b'\|_2}} e^{-\tau_\beta(z-b)-\tau_\beta(z-b')}. \tag{3.9}$$

3.2.3 Spatial relaxation in pure phases

We recall Proposition 1.28: for $\Lambda \subset \mathbb{Z}^2$ and any $\beta > \beta_c$, there exists $C_2(\beta) > 0$ such that, uniformly for any local function f with support $S(f)$ inside Λ,

$$\left|\langle f\rangle^+_{\Lambda,\beta} - \langle f\rangle^+_\beta\right| \leqslant \|f\|_\infty |S(f)|\, e^{-C_2 \cdot d(S(f),\Lambda^c)}. \tag{3.10}$$

3.2.4 Finite volume corrections to the surface tension

We recall the "ellipse's lemma". Let

$$\mathfrak{E}(x,y,\rho) \doteqdot \{t \in (\mathbb{Z}^2)^\star \text{ s.t. } \|x-t\| + \|y-t\| \leqslant \|x-y\| + \rho\}$$

be the ellipse in $\mathbb{R}^2$ with focuses x and y and big axis $2\rho + \|x-y\|$. Then,

$$\frac{\sum_{\substack{\lambda : x \to y \\ \lambda \not\subset \mathfrak{E}(x,y,\rho)}} q_{\beta^\star}(\lambda)}{\langle \sigma_x \sigma_y\rangle} = \frac{\sum_{\substack{\lambda : x \to y \\ \lambda \not\subset \mathfrak{E}(x,y,\rho)}} q_{\beta^\star}(\lambda)}{\sum_{\lambda : x \to y} q_{\beta^\star}(\lambda)} \leqslant \mathfrak{c} \cdot |\partial \mathfrak{E}(x,y,\rho)| \cdot \|x-y\|^{1/2} e^{-\kappa_{\beta^\star}\rho} \tag{3.11}$$

for a certain $\mathfrak{c} > 0$.

The next two lemmas provide informations on the finite-volume corrections to the surface tension τ_β and play a crucial role in our analysis. The first lemma provides a lower bound for the ratio of partition functions in a square box, when the endpoints are not both simultaneously close to one side of the box (in which case, the pre-factor would change). With slightly more work, this lower bound can be replaced by full Ornstein-Zernike asymptotics, using a variant of [20] similarly as in [48].
Let $\bar{\Lambda}_{n/2} \doteqdot [-n/2, n/2]^2 \subset \mathbb{R}^2$.

Lemma 3.1 *Let $\beta > \beta_c$. Then there exists a constant $C_3 > 0$ such that, as n tends to infinity, uniformly in vertices $i, j \in \partial^\star \Lambda_n$ such that $\overline{ij} \cap \bar{\Lambda}_{n/2} \neq \varnothing$,*

$$\frac{Z^{\pm(i,j)}_{\Lambda_n,\beta}}{Z^+_{\Lambda_n,\beta}} \geqslant C_3 n^{-1/2}\, e^{-\tau_\beta(j-i)},$$

Proof

The argument is an adaptation of the proof of the Ornstein-Zernike asymptotics given in [20]. Take a forward-cone (see the latter paper for definition) of sufficiently small opening to ensure that it is contained in the cone $\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geqslant 0, x_2 \geqslant 0\}$.

One can then proceed exactly as in [48]: By [20], typical contours appearing in the random-line representation admit a decomposition into a string of irreducible pieces. The distribution of the displacement random variables associated to the irreducible pieces has exponential tails. One then concatenates the $(\log n)^2$ left-most irreducible pieces into a single path, and does the same with the $(\log n)^2$ right-most pieces. Up to an event of negligible probability, the size of these two paths is bounded above by $(\log n)^4$. By construction, the two inner endpoints i' and j' of these two paths are at a distance at least $(\log n)^2$ from Λ^c; in particular, the ellipse $\mathfrak{E}(i', j', \log n)$ is entirely contained inside Λ. It follows that their $q_{\Lambda,\beta}$ weight can be replaced by their infinite-volume counterpart q_β with negligible cost. The analysis then proceeds as in [20, 48], the left-most and right-most paths constructed above affecting only the function $\Psi_{\Lambda,\beta}$ in the pre-factor.

■

When the endpoints i and j both lie too close to one of the sides of Λ_n, the above result does not apply (and is actually incorrect in general). It turns out that, for our purposes, the following rough lower bound is sufficient.

Lemma 3.2 *Let $\beta > \beta_c$. Then, for any $1/2 < \rho < 1$, there exists a constant $C_4 = C_4(\beta, \rho)$ such that, for all $i, j \in \partial^\star \Lambda_n$*

$$\frac{Z^{\pm(i,j)}_{\Lambda_n,\beta}}{Z^{+}_{\Lambda_n,\beta}} \geqslant e^{-C_4 n^\rho} e^{-\tau_\beta(j-i)}.$$

Proof First, by (3.3), we have

$$\frac{Z^{\pm(i,j)}_{\Lambda_n,\beta}}{Z^{+}_{\Lambda_n,\beta}} = \langle \sigma_i \sigma_j \rangle_{\Lambda_n^\star,\beta^\star}.$$

Let $i', j' \in \Lambda_{n-n^\rho}$ be the two vertices closest to i and j. Then, by the GKS inequality,

$$\langle \sigma_i \sigma_j \rangle_{\Lambda_n^\star,\beta^\star} \geqslant \langle \sigma_i \sigma_{i'} \rangle_{\Lambda_n^\star,\beta^\star} \langle \sigma_{i'} \sigma_{j'} \rangle_{\Lambda_n^\star,\beta^\star} \langle \sigma_{j'} \sigma_j \rangle_{\Lambda_n^\star,\beta^\star}.$$

On the one hand, by the GKS inequality and the finite energy property, we have

$$\langle \sigma_i \sigma_{i'} \rangle_{\Lambda_n^\star,\beta^\star} \geqslant \langle \sigma_i \sigma_{i_1} \rangle_{\Lambda_n^\star,\beta^\star} \langle \sigma_{i_1} \sigma_{i_2} \rangle_{\Lambda_n^\star,\beta^\star} \ldots \langle \sigma_{i_\ell} \sigma_{i'} \rangle_{\Lambda_n^\star,\beta^\star} \geqslant e^{-\mathfrak{c} \cdot \|i-i'\|},$$

where $\{i, i_1, i_2, \ldots i_\ell, i'\}$ is the approximation on $\mathbb{Z}^2$ of the segment $\overline{ii'}$. Doing the same for the third term, we get

$$\langle \sigma_i \sigma_{i'} \rangle_{\Lambda_n^\star,\beta^\star} \langle \sigma'_j \sigma_j \rangle_{\Lambda_n^\star,\beta^\star} \geqslant e^{-\mathfrak{c}\,(\|i-i'\|+\|j-j'\|)} = e^{-\mathfrak{c}\, n^\rho}.$$

By construction, $\mathfrak{E} = \mathfrak{E}(i', j', n^\rho)$ is included in $\bar{\Lambda}_n$. So, by the ellipse's lemma (3.11), we have,

$$\langle \sigma_{i'} \sigma_{j'} \rangle_{\Lambda_n^\star,\beta^\star} = (1 - O(e^{-n^\mathfrak{c}})) \langle \sigma_{i'} \sigma_{j'} \rangle_{\beta^\star}.$$

Then, since the infinite-volume 2-point function admits Ornstein-Zernike asymptotics, see Theorem 1.3, for n sufficiently large,

$$\langle \sigma_{i'}\sigma_{j'} \rangle_{\Lambda_n^\star,\beta^\star} \geqslant c \cdot \|j'-i'\|_2^{-1/2}\, e^{-\tau_\beta(j'-i')}.$$

It then follows from the continuity of τ_β as a function of the direction that

$$\langle \sigma_{i'}\sigma_{j'} \rangle_{\Lambda_n^\star,\beta^\star} \geqslant e^{-c\cdot n^\rho}\, e^{-\tau_\beta(j-i)}.$$

Finally, we get
$$\langle \sigma_i\sigma_j \rangle_{\Lambda_n^\star,\beta^\star} \geqslant e^{-C_4 n^\rho} e^{-\tau_\beta(j-i)}. \qquad \blacksquare$$

3.3 The proof

3.3.1 Step 1 : Typical configurations have at most one crossing interface

As explained before, we associate to the boundary condition ω the set of endpoints of the open contours induced by ω, written $\mathbf{b}(\omega) \equiv \{b_1, \ldots, b_{2M}\}$. We also denote by $\mathbf{\Gamma}(\sigma) \equiv \{\Gamma_1(\sigma), \ldots, \Gamma_M(\sigma)\}$ the set of the latter open contours in a configuration σ compatible with the boundary condition ω (their ordering is chosen according to some fixed, but arbitrary, rule). $\mathbf{\Gamma}$ induces a matching of the elements of $\mathbf{b}(\omega)$. Of course, not all possible matchings of $\mathbf{b}(\omega)$ can be realized in this way, and we denote by $\Pi(\omega)$ the set of all admissible matchings; a particular admissible matching, realized in a configuration σ, is denoted by $\pi(\sigma)$. The notation $(b, b') \in \pi(\sigma)$ means that b and b' are matched in $\pi(\sigma)$. The open contour with endpoints b and b' is denoted by $\Gamma_{b,b'}$.

Let $\bar{\Lambda}_{2n^a} \doteqdot [-2n^a, 2n^a]^2 \subset \mathbb{R}^2$. The next lemma shows that, for $\max\{2b, \frac{3}{4}\} < a < 1$, with high probability, a pair (b, b') in an admissible matching, whose associated open contour intersects the box $\Lambda^\star_{n^a}$, must be such that the segment $\overline{bb'}$ intersects $\bar{\Lambda}_{2n^a}$.

Lemma 3.3 *Let* $b < 1/2$ *and* $\max\{2b, \frac{3}{4}\} < a < 1$. *There exists* $C_5 = C_5(\beta) > 0$ *such that, for all* n *large enough,*

$$\mathbf{P}^\omega_{\Lambda_n,\beta}\big(\exists (b,b') \in \pi(\sigma) \,:\, \Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing, \ \textit{and} \ \ \overline{bb'} \cap \bar{\Lambda}_{2n^a} = \varnothing\big) \leqslant e^{-C_5 n^{2a-1}}.$$

Proof Let $(b, b') \in \mathbf{b}(\omega)$, such that $\overline{bb'} \cap \bar{\Lambda}_{2n^a} = \varnothing$. The line segment $\overline{bb'}$ splits Λ_n into two disjoint components Λ_n^1 and Λ_n^2 (with a fixed rule for attributing the vertices falling on the segment to one of these two sets), with $\bar{\Lambda}_{2n^a} \subset \Lambda_n^1$ (see Fig. 3.1). We denote by $\mathbf{b}^1(\omega)$ the subset of $\mathbf{b}(\omega) \setminus \{b, b'\}$ consisting of vertices lying on $\partial\Lambda_n^1$.

Let $\mathcal{C}_{b,b'}$ be the set of configurations of all open contours $\mathbf{\Gamma}^1(\sigma)$ with (both) endpoints in $\mathbf{b}^1(\omega)$ appearing in configurations σ for which $\Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing$.

Such a family $\mathbf{\Gamma}^1(\sigma)$ splits Λ_n into a number of connected components, only one of which contains b and b' along its boundary; we denote the latter component by

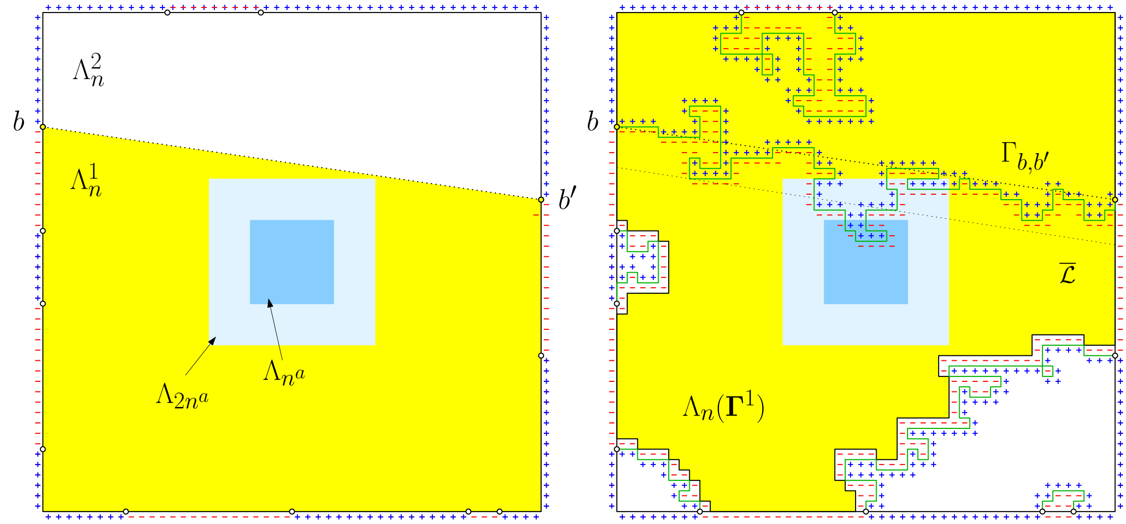


Figure 3.1 – The procedure in Lemma 3.3. The dots on the boundary represent $\mathbf{b}(\omega)$, which contain b, b'. Left: The yellow area is the sub-box Λ_n^1. Right: The yellow area is the box $\Lambda_n(\mathbf{\Gamma}^1)$; observe that when $\Gamma_{b,b'}$ intersects $\Lambda_{n^a}^\star$, there must be an s-path of $+$ spins starting from $\partial\Lambda_n^2 \cap \partial\Lambda_n$ and crossing $\bar{\mathcal{L}}$ (assuming that the boundary condition on $\partial\Lambda_n(\mathbf{\Gamma}^1) \setminus \partial\Lambda_n$ is $-$).

$\Lambda(\mathbf{\Gamma}^1(\sigma))$, and the corresponding boundary condition by $\omega(\mathbf{\Gamma}^1(\sigma))$ (see Fig. 3.1); we assume, without loss of generality, that the boundary condition along the random boundary $\partial\Lambda(\mathbf{\Gamma}^1(\sigma)) \setminus \partial\Lambda_n$ is given by $-$ spins. Using these notations and the DLR equation (3.1), we can write

$$\mathbf{P}^\omega_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing\big) = \sum_{\Gamma^1 \in \mathcal{C}_{b,b'}} \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \Gamma^1)\, \mathbf{P}^{\omega(\Gamma^1)}_{\Lambda(\Gamma^1),\beta}\big(\Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing\big).$$

Denote by $\bar{\mathcal{L}}$ the line parallel to $\overline{bb'}$ at distance n^a from the latter, and located on the same side as Λ_{2n^a}, and $\mathcal{L}$ a discrete approximation in $(\mathbb{Z}^2)^\star$ (say, the nearest neighbor path staying closest to $\bar{\mathcal{L}}$ in Hausdorff distance, with a fixed rule to break possible ties). On the event $\Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing$, there must be an s-path of $+$ spins connecting $(\partial\Lambda_n^2 \cap \partial\Lambda_n)$ to $\mathcal{L}$, $\partial\Lambda_n^2 \cap \partial\Lambda_n \xleftrightarrow{+} \mathcal{L}$. The latter event being increasing, it follows from stochastic domination of measures ($\mathbf{P}^-_\Lambda \preceq \mathbf{P}^\omega_\Lambda \preceq \mathbf{P}^+_\Lambda$) and the FKG inequality that

$$\begin{aligned}
\mathbf{P}^{\omega(\Gamma^1)}_{\Lambda(\Gamma^1),\beta}\big(\Gamma_{b,b'} \cap \Lambda^\star_{n^a} \neq \varnothing\big) &\leqslant \mathbf{P}^{\omega(\Gamma^1)}_{\Lambda(\Gamma^1),\beta}\big(\partial\Lambda_n^2 \cap \partial\Lambda_n \xleftrightarrow{+} \mathcal{L}\big) \\
&\leqslant \mathbf{P}^{\pm(b,b')}_{\Lambda(\Gamma^1),\beta}\big(\partial\Lambda_n^2 \cap \partial\Lambda_n \xleftrightarrow{+} \mathcal{L}\big) \\
&\leqslant \mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\partial\Lambda_n^2 \cap \partial\Lambda_n \xleftrightarrow{+} \mathcal{L}\big) \\
&\leqslant \mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \mathcal{L} \neq \varnothing\big), \qquad (3.12)
\end{aligned}$$

where the boundary condition $\pm(b,b')$ is given by -1 along $\partial\Lambda_n^1$ and $+1$ along $\partial\Lambda_n^2$.

To evaluate the latter probability, first observe that

$$\mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \mathcal{L} \neq \varnothing\big) \leqslant \frac{Z^{+}_{\Lambda_n,\beta}}{Z^{\pm(b,b')}_{\Lambda_n,\beta}} \sum_{z \in \mathcal{L}\cap\Lambda^\star_n} \sum_{\Gamma: b\to z\to b'} q_{\Lambda^\star_n,\beta^\star}(\Gamma). \tag{3.13}$$

On the one hand, applying Lemma 3.2 with $\rho \in (1/2, 2a-1)$, we obtain, for some constant $C_4 = C_4(\beta)$ that

$$\frac{Z^{\pm(b,b')}_{\Lambda_n,\beta}}{Z^{+}_{\Lambda_n,\beta}} \geqslant e^{-C_4 n^\rho} e^{-\tau_\beta(b-b')}.$$

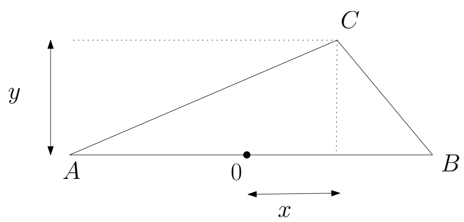
On the other hand, it follows from (3.9) that

$$\begin{aligned}\sum_{\Gamma: b\to z\to b'} q_{\Lambda^\star_n,\beta^\star}(\Gamma) &\leqslant \sum_{\Gamma_1: b\to z} q_{\Lambda^\star_n,\beta^\star}(\Gamma_1) \sum_{\Gamma_2: z\to b'} q_{\Lambda^\star_n,\beta^\star}(\Gamma_2) \\ &= \langle \sigma_b\sigma_z\rangle_{\Lambda^\star_n,\beta^\star}\langle \sigma_z\sigma_{b'}\rangle_{\Lambda^\star_n,\beta^\star} \\ &\overset{\text{GKS}}{\leqslant} \langle \sigma_b\sigma_z\rangle_{\beta^\star}\langle \sigma_z\sigma_{b'}\rangle_{\beta^\star} \\ &\leqslant e^{-\tau_\beta(z-b)-\tau_\beta(z-b')}.\end{aligned}$$

However, the sharp triangle inequality (3.2) implies that, uniformly in $z \in \mathcal{L}\cap\Lambda^\star_n$ and in b, b' such that $\overline{bb'} \cap \Lambda_{2n^a} = \varnothing$,

$$\tau_\beta(z-b)+\tau_\beta(z-b')-\tau_\beta(b'-b) \geqslant \kappa_\beta\big(\|z-b\|_2+\|z-b'\|_2-\|b'-b\|_2\big) \geqslant c\cdot n^{2a-1}.$$

Indeed, let $\mathcal{T} = ABC$ be an arbitrary triangle.



If we draw it in the plane, its base AB being horizontal and centered at the origin, and $C = (x, y)$ denoting the position of the third vertex, then it's easy to show that the function

$$\begin{aligned}T(x,y) &= \|C-A\|_2 + \|B-C\|_2 - \|B-A\|_2 \\ &= \sqrt{\big(\overline{AB}/2+x\big)^2+y^2} + \sqrt{\big(\overline{AB}/2-x\big)^2+y^2} - \overline{AB}\end{aligned}$$

in convex in x and y. This fact implies that the worst possible triangle bzb' (i.e. giving the smallest value of T) is isosceles and has a base bb' of length equal to $c\,n$ and height equal to $c'n^a$, so we get (by a Taylor expansion)

$$\begin{aligned}\|z-b\|_2+\|z-b'\|_2-\|b'-b\|_2 &= 2\sqrt{(cn/2)^2+(c'n^a)^2} - cn \\ &= cn\left[\sqrt{1+(2c'n^a/cn)^2}-1\right] \\ &\geqslant c''\cdot n^{2a-1}\end{aligned}$$

Since there are at most $4n$ vertices $z \in \mathcal{L}$, we conclude that, for n large enough,

$$\mathbf{P}^{\pm(b,b')}_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \mathcal{L} \neq \varnothing\big) \leqslant 4n \cdot e^{C_4 n^{\rho} - c\, n^{2a-1}} \leqslant e^{-c \cdot n^{2a-1}},$$

(Note that we use here the fact that $\rho < 2a - 1$). We thus obtain from (3.12) that, for all n large enough,

$$\begin{aligned}\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \Lambda^{\star}_{n^a} \neq \varnothing\big) &= \sum_{\mathbf{\Gamma}^1 \in \mathcal{C}_{b,b'}} \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \mathbf{\Gamma}^1)\, \mathbf{P}^{\omega(\mathbf{\Gamma}^1)}_{\Lambda(\mathbf{\Gamma}^1),\beta}\big(\Gamma_{b,b'} \cap \Lambda^{\star}_{n^a} \neq \varnothing\big)\\ &\leqslant e^{-c\, n^{2a-1}} \underbrace{\sum_{\mathbf{\Gamma}^1 \in \mathcal{C}_{b,b'}} \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \mathbf{\Gamma}^1)}_{\leqslant 1},\end{aligned}$$

and the conclusion follows, since there are at most $64n^2$ pairs b, b' :

$$\begin{aligned}\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\exists (b,b') \in \pi(\sigma) \,:\, \Gamma_{b,b'} \cap \Lambda^{\star}_{n^a} \neq \varnothing,\ \text{and } \overline{bb'} \cap \bar{\Lambda}_{2n^a} = \varnothing\big)&\\ \leqslant 64n^2 \cdot \mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\Gamma_{b,b'} \cap \Lambda^{\star}_{n^a} \neq \varnothing\big) \leqslant e^{-C_5 n^{2a-1}}&.\end{aligned}$$

■

Lemma 3.4 *Let us denote by* N_{cr} *the number of open contours intersecting* $\Lambda^{\star}_{n^a}$ *(which we call* crossing contours*). There exists* $C_6 = C_6(\beta) > 0$ *such that, for all* n *large enough,*

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{cr} \geqslant 2\big) \leqslant e^{-C_6 n^{2a-1}}.$$

Proof Thanks to Lemma 3.3, we can assume that all crossing contours have endpoints b, b' satisfying $\overline{bb'} \cap \bar{\Lambda}_{2n^a} \neq \varnothing$; let us denote by $\mathcal{D}$ this event.

Let $\Gamma_{b_1,b'_1}(\sigma), \dots, \Gamma_{b_m,b'_m}(\sigma)$ be the family of all crossing contours in a configuration $\sigma \in \mathcal{D}$, assuming that $m \geqslant 2$. Because we suppose that the event $\mathcal{D}$ is realized, these endpoints can be naturally split into two "diametrically opposed" families $b_1, \dots, b_m$ and $b'_1, \dots, b'_m$. The vertices $b_1, \dots, b_m$ are ordered clockwise (and thus the corresponding vertices $b'_1, \dots, b'_m$ counterclockwise). In particular, the crossing contours $\Gamma_{b_1,b'_1}(\sigma)$ and $\Gamma_{b_2,b'_2}(\sigma)$ are neighbors (i.e. there are no other crossing contours between them). Notice that, since $\mathcal{D}$ is supposed to hold, $\max\{\|b_1 - b_2\|_1, \|b'_1 - b'_2\|_1\} \leqslant c\, n^a$.

The segments $\overline{b_1 b'_1}$ and $\overline{b_2 b'_2}$ split the box Λ_n into 3 pieces. We denote by Λ^1_n and Λ^2_n the two non-neighboring ones (see Fig. 3.2). Let also $\mathbf{\Gamma}^1$, resp. $\mathbf{\Gamma}^2$, be the open contours with both endpoints on $\partial\Lambda^1_n$, resp. $\partial\Lambda^2_n$. These open contours split Λ_n into connected pieces, exactly one of which contains b_1, b'_1, b_2, b'_2 along its boundary (it also contain the random slice b_1, b'_1, b_2, b'_2); we denote this component by $\Lambda_n(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2)$, and the induced boundary condition on $\Lambda_n(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2)$ by $\omega(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2)$ (see Fig. 3.2). For definiteness and without loss of generality, we can assume that the boundary condition acting along $\partial\Lambda_n(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2) \setminus \partial\Lambda_n$ is given by $-$ spins.

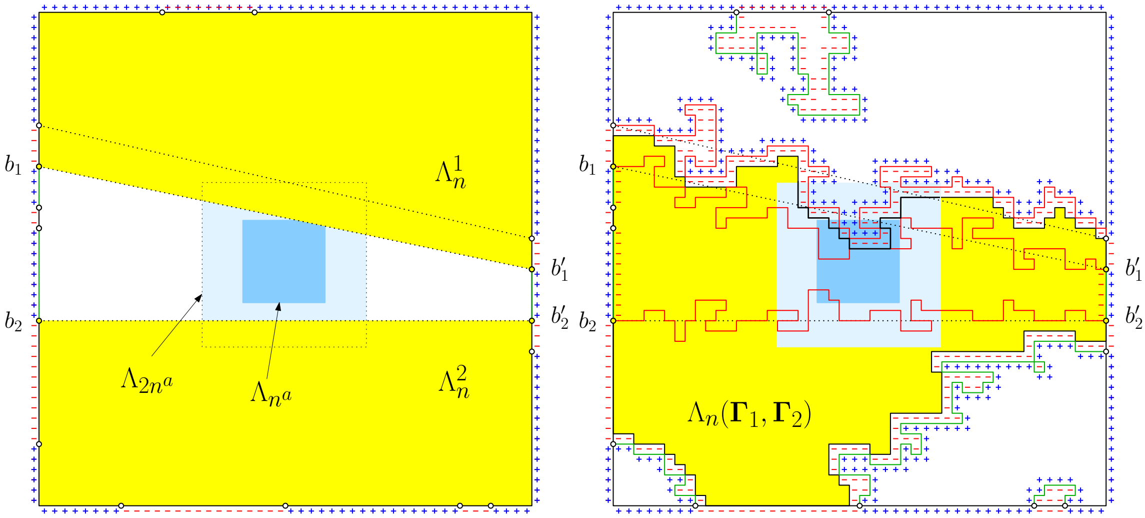


Figure 3.2 – Illustration of the procedure in the proof of Lemma 3.4.

Using the DLR equation (3.1), we have

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{\text{cr}} \geqslant 2, \mathcal{D}\big) \leqslant \sum_{b_1,b_1',b_2,b_2'} \sum_{\mathbf{\Gamma}^1,\mathbf{\Gamma}^2} \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \mathbf{\Gamma}^1, \mathbf{\Gamma}^2(\sigma) = \mathbf{\Gamma}^2)$$
$$\times\, \mathbf{P}^{\omega(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2)}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}\big(\Gamma_{b_1,b_1'} \text{ and } \Gamma_{b_2,b_2'} \text{ are crossing}\big).$$

Let $\{k_1, \ldots, k_\ell\} = \mathbf{b}(\omega(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2)) \setminus \{b_1, b_2, b_1', b_2'\}$ be the set of all endpoints of open contours induced by the boundary condition $\omega(\mathbf{\Gamma}^1, \mathbf{\Gamma}^2)$, apart from b_1, b_2, b_1', b_2' (they are situated on the small sides of the random slice $b_1 b_1' b_2 b_2'$). Using (3.6), we obtain

$$\mathbf{P}^{\omega(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2)}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}\big(\Gamma_{b_1,b_1'} \text{ and } \Gamma_{b_2,b_2'} \text{ are crossing}\big) \leqslant \frac{\mathbf{Z}^{+}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}}{\mathbf{Z}^{\omega(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2)}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}} \sum_{\substack{\Gamma_1: b_1 \to b_1' \\ \Gamma_2: b_2 \to b_2'}} q_{\Lambda_n^*(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta^*}(\Gamma_1, \Gamma_2).$$

On the one hand, we evidently have the lower bound

$$\frac{\mathbf{Z}^{\omega(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2)}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}}{\mathbf{Z}^{+}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}} \geqslant e^{-\mathfrak{c}' n^a},$$

for some constant $\mathfrak{c}' = \mathfrak{c}'(\beta) < \infty$, since $\max\{\|b_1 - b_2\|_1, \|b_1' - b_2'\|_1\} \leqslant \mathfrak{c}\, n^a$ and changing boundary conditions (from - to + on a total length less than $2\mathfrak{c}\, n^a$) add contours whose length are bounded below by $\mathfrak{c}' n^a$.
On the other hand, using (3.8) and (3.7), we deduce the following upper bound for the second term

$$\sum_{\substack{\Gamma_1: b_1 \to b_1' \\ \Gamma_2: b_2 \to b_2'}} q_{\Lambda_n^*(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta^*}(\Gamma_1, \Gamma_2) \leqslant \sum_{\Gamma_1: b_1 \to b_1'} q_{\Lambda_n^*(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta^*}(\Gamma_1) \sum_{\Gamma_2: b_2 \to b_2'} q_{\Lambda_n^*(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta^*}(\Gamma_2)$$
$$\leqslant e^{-\tau_\beta(b_1' - b_1) - \tau_\beta(b_2' - b_2)} \leqslant e^{-\mathfrak{c} \cdot n}.$$

for a certain $\mathfrak{c} = \mathfrak{c}(\beta)$ since $\min\{\|b_1' - b_1\|_1, \|b_2' - b_2\|_1\} \geqslant \mathfrak{c}' n$. Combining these estimates, we deduce that

$$\begin{aligned}
\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{\rm cr} \geqslant 2, \mathcal{D}\big) &\leqslant \sum_{b_1,b_1',b_2,b_2'} \sum_{\mathbf{\Gamma}^1,\mathbf{\Gamma}^2} \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \mathbf{\Gamma}^1, \mathbf{\Gamma}^2(\sigma) = \mathbf{\Gamma}^2) \\
&\qquad\qquad \times \mathbf{P}^{\omega(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2)}_{\Lambda_n(\mathbf{\Gamma}^1,\mathbf{\Gamma}^2),\beta}\big(\Gamma_{b_1,b_1'} \text{ and } \Gamma_{b_2,b_2'} \text{ are crossing}\big) \\
&\leqslant e^{\mathfrak{c}\, n^a} e^{-\mathfrak{c}' n} \sum_{b_1,b_1',b_2,b_2'} \underbrace{\sum_{\mathbf{\Gamma}^1,\mathbf{\Gamma}^2} \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathbf{\Gamma}^1(\sigma) = \mathbf{\Gamma}^1, \mathbf{\Gamma}^2(\sigma) = \mathbf{\Gamma}^2)}_{\leqslant 1} \\
&\leqslant \mathfrak{c}'' n^{2+2a} \cdot e^{\mathfrak{c}\, n^a} e^{-\mathfrak{c}' n} \leqslant e^{-\mathfrak{c}\, n}
\end{aligned}$$

for some constant $\mathfrak{c} = \mathfrak{c}(\beta) > 0$ and for all n large enough.
We then have

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(N_{\rm cr} \geqslant 2\big) = \underbrace{\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\{N_{\rm cr} \geqslant 2\} \cap \mathcal{D}\big)}_{\leqslant e^{-\mathfrak{c}\, n}} + \underbrace{\mathbf{P}^{\omega}_{\Lambda_n,\beta}\big(\{N_{\rm cr} \geqslant 2\} \cap \mathcal{D}^c\big)}_{\leqslant e^{-C_5 n^{2a-1}}} \leqslant e^{-C_6 n^{2a-1}}$$

■

3.3.2 Step 2 : When present, this interface has large fluctuations

We denote by $\mathcal{I}_1$ the event that there is a unique crossing contour. To deal with $\mathcal{I}_1$, we have to exploit the fact that the interface undergoes fluctuations of order $\sqrt{n}$ and will thus "miss", with high probability, a box of side-length n^b with $b < 1/2$. The next lemma implements this idea.

Lemma 3.5 *Denoting by Γ the unique crossing contour on the event $\mathcal{I}_1$, we have*

$$\mathbf{P}^{\omega}_{\Lambda_n,\beta}(\Gamma \cap \Lambda^{\star}_{2n^b} \neq \varnothing, \mathcal{I}_1) \leqslant C_7 n^{b-a/2},$$

for some constant $C_7 = C_7(\beta)$ and all n large enough.

Proof Let us denote by b and b' the endpoints of the unique crossing contour Γ. We denote by γ and γ' the parts of Γ connecting, respectively, b to $\partial^\star\Lambda_{n^a}$ and b' to $\partial^\star\Lambda_{n^a}$ (these are two open contours). Let also $\bar{\mathbf{\Gamma}}$ denote the set of all open contours of the configuration apart from Γ. The contours $\bar{\mathbf{\Gamma}}, \gamma, \gamma'$ split Λ_n in a number of connected components, only one of which contains Λ_{n^a}; we denote the latter by $\Lambda_n(\bar{\mathbf{\Gamma}}, \gamma, \gamma')$ (see Fig. 3.3). Let d, d' be the endpoints of γ and γ' on $\partial^\star\Lambda_{n^a}$. Observe that the boundary condition acting on $\Lambda_n(\bar{\mathbf{\Gamma}}, \gamma, \gamma')$ takes two different constant values along each of the two pieces between d and d'; we write $\pm(d, d')$ for this boundary condition (by symmetry, it does not matter which part is $+$ and which is $-$). We consider two cases.

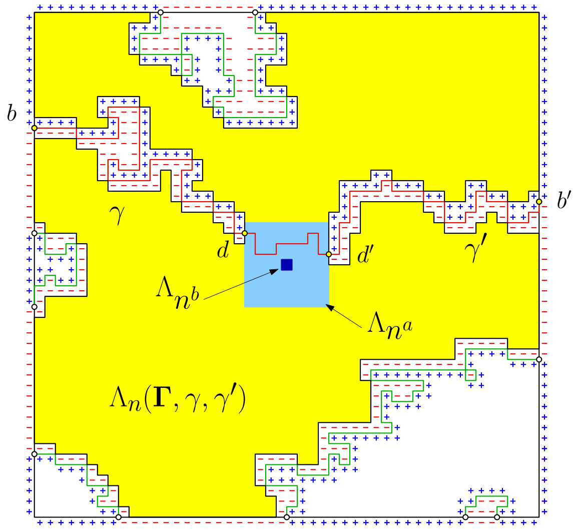


Figure 3.3 – The construction in Lemma 3.5.

Case 1: $\overline{dd'} \cap \bar{\Lambda}_{n^a/2} = \varnothing$. In that case, we argue exactly as in the proof of Lemma 3.3 to obtain that

$$\mathbf{P}^{\pm(d,d')}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}(\Gamma_{d,d'} \cap \Lambda^{\star}_{2n^b} \neq \varnothing) \leqslant e^{-c\,n^{a(2a-1)}} \leqslant e^{-c\,n^a}.$$

for some $c = c(\beta) > 0$. Indeed, one can check easily that the proof of Lemma 3.3 works for the extended box $\Lambda_n(\bar{\Gamma},\gamma,\gamma')$ containing Λ_{n^a}

Case 2: $\overline{dd'} \cap \bar{\Lambda}_{n^a/2} \neq \varnothing$. The argument is once more the same as the one used in the proof of Lemma 3.3 or Lemma 3.5 until getting an analog of expression (3.13) :

$$\mathbf{P}^{\pm(d,d')}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}(\Gamma_{d,d'} \cap \Lambda^{\star}_{2n^b} \neq \varnothing) \leqslant \frac{Z^{+}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}}{Z^{\pm(d,d')}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}} \sum_{z\in\partial^\star\Lambda_{2n^b}} \sum_{\Gamma:d\to z\to d'} q_{\Lambda^\star_n(\bar{\Gamma},\gamma,\gamma'),\beta^\star}(\Gamma).$$

But now, the sharp triangle inequality doesn't provide an exponentially small term uniformly over all $\Gamma_{d,d'}$ considered here, since the interface can be straight. We then have to keep track of the pre-factors. On the one hand, using duality, we get

$$\frac{Z^{\pm(d,d')}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}}{Z^{+}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}} = \langle\sigma_d\sigma_{d'}\rangle_{\Lambda^\star_n(\bar{\Gamma},\gamma,\gamma'),\beta^\star} \overset{GKS}{\geqslant} \langle\sigma_d\sigma_{d'}\rangle_{\Lambda^\star_{n^a},\beta^\star} = \frac{Z^{\pm(d,d')}_{\Lambda_{n^a},\beta}}{Z^{+}_{\Lambda_{n^a},\beta}}$$

As $\overline{dd'} \cap \bar{\Lambda}_{n^a} \neq \varnothing$, hypothesis of Lemma 3.1 is satisfied and we get

$$\frac{Z^{\pm(d,d')}_{\Lambda_{n^a},\beta}}{Z^{+}_{\Lambda_{n^a},\beta}} \geqslant \frac{C_3}{n^{a/2}} e^{-\tau_\beta(d'-d)}.$$

On the other hand, by (3.9),

$$\sum_{\lambda:d\to z\to d'} q_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}(\lambda) \leqslant \frac{c}{n^{a/2}}\frac{c'}{n^{a/2}}\, e^{-\tau_\beta(z-d)-\tau_\beta(z-d')} \leqslant \frac{c}{n^a}\, e^{-\tau_\beta(d'-d)}.$$

Summing over $z \in \partial^\star \Lambda_{2n^b}$, with $|\partial^\star \Lambda_{2n^b}| \leqslant \mathfrak{c}\, n^b$ shows that

$$\mathbf{P}^{\pm(d,d')}_{\Lambda_n(\bar{\Gamma},\gamma,\gamma'),\beta}(\Gamma_{d,d'} \cap \Lambda^\star_{2n^b} \neq \varnothing) \leqslant C_7 n^{b-a/2}.$$

∎

3.3.3 Step 3 : Every Ising measure is close to a convex combination of the two pure states

Proof [of Theorem 3.1] Let $\mathcal{I}_0$ be the event that there is no crossing interface, and, as before, $\mathcal{I}_1$ the event that there is a unique crossing contour. We know from Lemma 3.4 that,

$$\mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0) + \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_1) = 1 + O_\beta(e^{-C_6 n^{2a-1}})$$

which means that, uniformly in f,

$$\langle f\rangle^\omega_{\Lambda_n,\beta} = \langle f\,|\,\mathcal{I}_0\rangle^\omega_{\Lambda_n,\beta}\, \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0) + \langle f\,|\,\mathcal{I}_1\rangle^\omega_{\Lambda_n,\beta}\, \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_1) + O_\beta\big(\|f\|_\infty e^{-C_6 n^{2a-1}}\big). \quad (3.14)$$

Let us consider first the event $\mathcal{I}_0$. When the latter occurs, there must be a circuit surrounding Λ_{n^a} along which spins take a constant value. Let us denote by $\mathcal{I}_0^+(\gamma)$, $\mathcal{I}_0^-(\gamma)$ the events that the largest such circuit is given by γ, and the spins value along γ is 1, resp. -1. Let us also denote by $\Lambda(\gamma)$ the interior of the circuit γ. It then follows from (3.10) that, uniformly in all $\mathcal{F}_{\Lambda_{n^b}}$-measurable functions f,

$$\big|\langle f\rangle^+_{\Lambda(\gamma),\beta} - \langle f\rangle^+_\beta\big| \leqslant \|f\|_\infty\, |S(f)|\, e^{-\mathfrak{c}\cdot n^a - \mathfrak{c}'\cdot n^b} \leqslant \mathfrak{c}\|f\|_\infty e^{-\mathfrak{c}'\cdot n^a}.$$

Then,

$$\begin{aligned}\langle f\,|\,\mathcal{I}_0\rangle^\omega_{\Lambda_n,\beta} &= \sum_\gamma \big\{\mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0^+(\gamma)\,|\,\mathcal{I}_0)\,\langle f\rangle^+_{\Lambda(\gamma),\beta} + \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0^-(\gamma)\,|\,\mathcal{I}_0)\,\langle f\rangle^-_{\Lambda(\gamma),\beta}\big\}\\ &= \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0^+\,|\,\mathcal{I}_0)\,\langle f\rangle^+_\beta + \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_0^-\,|\,\mathcal{I}_0)\,\langle f\rangle^-_\beta + O_\beta\big(\|f\|_\infty e^{-\mathfrak{c}\, n^a}\big), \quad (3.15)\end{aligned}$$

where $\mathcal{I}_0^\pm = \bigcup_\gamma \mathcal{I}_0^\pm(\gamma)$.

Now let us consider the event $\mathcal{I}_1$. It follows from Lemma 3.5 that, conditionally on $\mathcal{I}_1$, there is, with high probability, a contour surrounding Λ_{2n^b} along which spins take a constant value. Denoting as before the largest such contour by γ, its interior by $\Lambda(\gamma)$, and introducing the events $\mathcal{I}_1^+(\gamma)$ and $\mathcal{I}_1^-(\gamma)$ similarly as above, we obtain in the same way (cf. (3.10)) that, for any $\mathcal{F}_{\Lambda_{n^b}}$-measurable function f,

$$\big|\langle f\rangle^+_{\Lambda(\gamma),\beta} - \langle f\rangle^+_\beta\big| \leqslant \|f\|_\infty\, |S(f)|\, e^{-\mathfrak{c}\cdot n^b}$$

and

$$\mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_1^+\,|\,\mathcal{I}_1) + \mathbf{P}^\omega_{\Lambda_n,\beta}(\mathcal{I}_1^-\,|\,\mathcal{I}_1) = 1 + O_\beta(n^{b-a/2}) \quad \text{where } \mathcal{I}_1^\pm = \bigcup_\gamma \mathcal{I}_1^\pm(\gamma).$$

Then, combining these two results similarly as in (3.15), we get

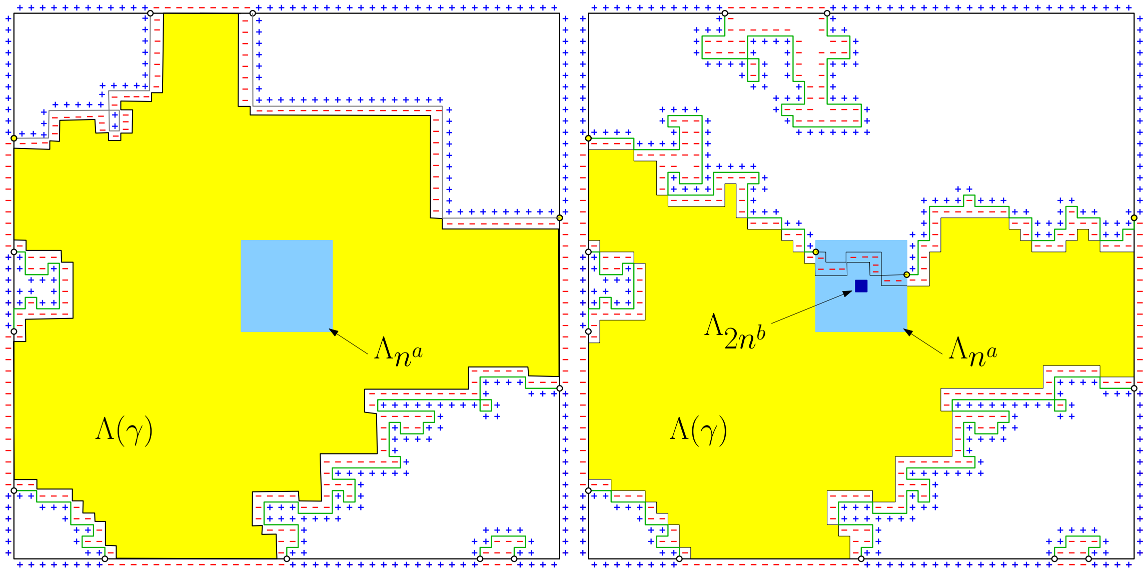


Figure 3.4 – Left: On the event $\mathcal{I}_0$, there is a region $\Lambda(\gamma)$ (yellow) containing Λ_{n^a} with constant spin value on its boundary. Right: On the event $\mathcal{I}_1$, there is a region $\Lambda(\gamma)$ (yellow) containing Λ_{2n^b} with constant spin value on its boundary.

$$\langle f \,|\, \mathcal{I}_1 \rangle^{\omega}_{\Lambda_n,\beta} = \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{I}_1^+ \,|\, \mathcal{I}_1)\, \langle f \rangle^+_\beta + \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{I}_1^- \,|\, \mathcal{I}_1)\, \langle f \rangle^-_\beta + O_\beta\big(\|f\|_\infty n^{b-a/2}\big), \tag{3.16}$$

Let $\mathcal{I}^\pm \doteqdot \mathcal{I}_0^\pm \cup \mathcal{I}_1^\pm$. Inserting (3.15) and (3.16) into (3.14), we obtain finally

$$\langle f \rangle^{\omega}_{\Lambda_n,\beta} = \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{I}^+)\, \langle f \rangle^+_\beta + \mathbf{P}^{\omega}_{\Lambda_n,\beta}(\mathcal{I}^-)\, \langle f \rangle^-_\beta + O_\beta\big(\|f\|_\infty n^{b-a/2}\big),$$

uniformly in $\mathcal{F}_{\Lambda_{n^b}}$-measurable functions f. In particular, we recover the statement of the theorem,

$$\langle f \rangle^{\omega}_{\Lambda_n} = \alpha^{n,\omega} \langle f \rangle^+ + (1-\alpha^{n,\omega}) \langle f \rangle^- + O_\beta(\|f\|_\infty\, n^{b-a/2}),$$

by choosing $a = 2(b+\delta)$. ■

Proof [of Corollary 3.1] For $\beta < \beta_c$, we know that there is a unique infinite-volume Gibbs measure and the statement is therefore trivial.
For $\beta > \beta_c$, let $\mathbf{P} \in \mathcal{G}_\beta$ be an infinite-volume Gibbs measure and f be a local function. Let n_0 be such that f is $\mathcal{F}_{\Lambda_{n^b}}$-measurable for all $n \geqslant n_0$.
Now from the DLR equation (3.1), we get that for all $n \geqslant 0$, and any function g,

$$\mathbf{P}(g) = \int \langle g \rangle^{\omega}_{\Lambda_n} \, d\mathbf{P}(\omega).$$

Theorem 3.1 implies that, for some $\delta > 0$ and uniformly in $\mathcal{F}_{\Lambda_{n^b}}$-measurable functions g,

$$\mathbf{P}(g) = A_n \langle g \rangle^+_\beta + (1-A_n) \langle g \rangle^-_\beta + O_\beta\big(n^{-\delta} \|g\|_\infty\big), \tag{3.17}$$

with $A_n = \int \alpha^{n,\omega}\, d\mathbf{P}(\omega)$. Applying this to the function $g = \sigma_0$, we deduce that

$$\mathbf{P}(\sigma_0) = (2A_n - 1) m^\star_\beta + O_\beta(n^{-\delta}),$$

where we have introduced the spontaneous magnetization $m^\star_\beta \doteqdot \langle\sigma_0\rangle^+_\beta$. This shows that

$$A_n = \frac{m^\star_\beta + \mathbf{P}(\sigma_0)}{2m^\star_\beta} + O_\beta(n^{-\delta}).$$

Let us set $\alpha \doteqdot (m^\star_\beta + \mathbf{P}(\sigma_0))/2m^\star_\beta$. Applying now (3.17) to the function $g = f$, we see that, for all $n > n_0$,

$$\mathbf{P}(f) = \alpha\langle f\rangle^+_\beta + (1-\alpha)\langle f\rangle^-_\beta + O_\beta(\|f\|_\infty n^{-\delta}).$$

Letting n tend to infinity, we conclude that $\mathbf{P}(f) = \alpha\langle f\rangle^+_\beta + (1-\alpha)\langle f\rangle^-_\beta$. Since this holds for any local function f, it follows that $\mathbf{P} = \alpha\mathbf{P}^+_\beta + (1-\alpha)\mathbf{P}^-_\beta$.

■

3.3.4 Optimality of the convergence rate

Proof [of Proposition 3.1] Let us consider the box $\Lambda_n = \{-n,\ldots,n\}^2$ and the boundary condition $\omega_i = +1$ if and only if $i = (i_1, i_2)$ with $i_2 > 0$ (Dobrushin boundary condition). We denote the corresponding expectation by $\langle\cdot\rangle^\pm_{\Lambda,\beta}$. The trick is to consider a local function f for which the expectation $\langle f\rangle^+_\beta = \langle f\rangle^-_\beta = 0$, since this trivializes the optimization over α. Let $f_i(\omega) = \omega_{(0,i)} - \omega_{(0,i-1)}$, and

$$F(\omega) = \sum_{i=-\lfloor Kn^{1/2}\rfloor+1}^{\lfloor Kn^{1/2}\rfloor} f_i(\omega) = \omega_{(0,\lfloor Kn^{1/2}\rfloor)} - \omega_{(0,-\lfloor Kn^{1/2}\rfloor)},$$

with K a large constant, to be chosen below. Thanks to translation invariance of $\mathbf{P}^+$ and $\mathbf{P}^-$, $\langle f_i\rangle^+_\beta = \langle f_i\rangle^-_\beta = 0$, for all i, and thus $\langle F\rangle^+_\beta = \langle F\rangle^-_\beta = 0$. Let us denote the only open contour by γ and its endpoints a and b. Let also

$$\mathcal{S} = \{(\tfrac{1}{2}, j) \in \Lambda^\star \,:\, |j| > \lfloor Kn^{1/2}\rfloor\},$$

We then have

$$\langle F\rangle^\pm_{\Lambda,\beta} \geqslant \langle F \,|\, \gamma\cap\mathcal{S} = \varnothing\rangle^\pm_{\Lambda,\beta}\, \mathbf{P}^\pm_{\Lambda,\beta}(\gamma\cap\mathcal{S} = \varnothing) - 2\mathbf{P}^\pm_{\Lambda,\beta}(\gamma\cap\mathcal{S} \neq \varnothing).$$

Now, FKG inequality implies that

$$\langle F \,|\, \gamma\cap\mathcal{S} = \varnothing\rangle^\pm_{\Lambda,\beta} \geqslant 2m^\star_\beta,$$

while, using (3.2), (3.9) and Lemma 3.1, we get

$$\begin{aligned}\mathbf{P}^\pm_{\Lambda,\beta}(\gamma\cap\mathcal{S} \neq \varnothing) &\leqslant \mathfrak{c}\sum_{z\in\mathcal{S}} \frac{\sqrt{|a-b|}}{\sqrt{|a-z|}\sqrt{|z-b|}} e^{-(\tau_\beta(a-z)+\tau_\beta(z-b)-\tau_\beta(a-b))}\\ &\leqslant \frac{\mathfrak{c}}{\sqrt{n}} \sum_{k\geqslant\lfloor K\sqrt{n}\rfloor} e^{-\kappa_\beta k^2/2n} \leqslant \mathfrak{c}\, e^{-\kappa_\beta K^2/2}.\end{aligned}$$

Since $|F| \leqslant 2$, we deduce from the above, choosing K large enough, that

$$\langle F\rangle^{\pm}_{\Lambda,\beta} \geqslant 2m^{\star}_{\beta}(1 - \mathfrak{c}\, e^{-\kappa_\beta K^2/2}) - 2\mathfrak{c}\, e^{-\kappa_\beta K^2/2} > \mathfrak{c}' > 0,$$

Now, F being a sum of $2\lfloor Kn^{1/2}\rfloor$ terms, there exists an index $j_0 = j_0(n)$ such that

$$\langle f_{j_0}\rangle^{\pm}_{\Lambda,\beta} > \frac{\mathfrak{c}}{2\lfloor Kn^{1/2}\rfloor} > \mathfrak{c}\, n^{-1/2},$$

for some constant $\mathfrak{c} > 0$. At this point, we have very little control on the location of the support of f_{j_0} inside Λ. To remedy this, let $\Delta_n^{j_0} = (0, j_0) + \{-\lfloor n/2\rfloor, \dots, \lfloor n/2\rfloor\}^2$. Using DLR equation, we can write (the averaging being over ω)

$$\langle f_{j_0}\rangle^{\pm}_{\Lambda,\beta} = \langle\, \langle f_{j_0}\rangle^{\omega}_{\Delta_n^{j_0},\beta}\, \rangle^{\pm}_{\Lambda,\beta} > \mathfrak{c}\, n^{-1/2},$$

so that there exists an $\tilde{\omega} = \tilde{\omega}(n)$ for which

$$\langle f_{j_0}\rangle^{\tilde{\omega}}_{\Delta_n^{j_0},\beta} > \mathfrak{c}\, n^{-1/2}.$$

This proves, albeit non-constructively, the existence of a constant $c = c(\beta) > 0$ and a sequence of boundary conditions $(\omega_m)_{m\geqslant 1}$ such that, for all m large enough,

$$\inf_{\alpha\in[0,1]} |\langle f\rangle^{\omega_m}_{\Delta_m,\beta} - \alpha\langle f\rangle^{+}_{\beta} - (1-\alpha)\langle f\rangle^{-}_{\beta}| > c\, m^{-1/2}, \tag{3.18}$$

where $f(\omega) = \omega_{(0,1)} - \omega_{(0,0)}$ and $\Delta_m = \{-m, \dots, m\}^2$. ■

Remark 3.1 *We actually expect that* (3.18) *is satisfied, for the same function* f*, with* ω *given by Dobrushin boundary condition.*

Chapter 4

Detailed proofs about the Potts model

This chapter is devoted to the rigorous proof of the result described in Section 2.3.3. It consists of a slightly extended version of the proof published in the article [LC1]. The introductive part is adapted to this manuscript in order to refer to results proved in details in Section 1. The article is a joint work with Hugo Duminil-Copin, Dima Ioffe and Yvan Velenik, it was accepted for publication in the journal *Probability Theory and Related Fields* in January 2013.

4.1 The theorem

Let $\Omega = \{1, \ldots, q\}^{\mathbb{Z}^2}$ be the space of configurations. Let Λ be a finite subset of $\mathbb{Z}^2$, and $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$ be its complement. For an arbitrary subset A of $\mathbb{Z}^2$, let $\mathcal{F}_A$ be the sigma-algebra generated by spins in A. We recall that a probability measure $\mathbb{P}$ on Ω is an *infinite-volume Gibbs measure* for the q-state Potts model at inverse temperature β if and only if it satisfies the following DLR condition:

$$\mathbb{P}(\cdot|\mathcal{F}_{\Lambda^c})(\sigma) = \mathbb{P}^{\sigma}_{\Lambda,\beta,q} \qquad \text{for } \mathbb{P}\text{-a.e. } \sigma \text{, and all finite subsets } \Lambda \text{ of } \mathbb{Z}^2.$$

with $\mathbb{P}^{\sigma}_{\Lambda,\beta,q}$ the finite volume Potts measure defined in (1.4). Let $\mathcal{G}_{q,\beta}$ be the space of infinite-volume q-state Potts measures.

In the present work, we determine all infinite-volume Gibbs measures for the q-state Potts models at inverse temperature $\beta > \beta_c(q)$ on $\mathbb{Z}^2$. More precisely, we show that every Gibbs state is a convex combination of infinite-volume measures with pure boundary condition:

Theorem 4.1 *For any $q \geqslant 2$ and $\beta > \beta_c(q)$,*

$$\mathcal{G}_{q,\beta} = \Big\{ \sum_{i=1}^{q} \alpha_i \mathbb{P}^i_\beta, \ \textit{where } \alpha_i \geqslant 0, \forall i \in \{1, \ldots, q\} \textit{ and } \sum_{i=1}^{q} \alpha_i = 1 \Big\}. \qquad (4.1)$$

where $\mathbb{P}^i_\beta$ is the pure phase with boundary condition "i" defined in (1.30). The parameter q is dropped from the notation since it is fixed throughout the chapter.

We recall the two important corollaries of this theorem (presented in Corollary 2.2 above): for $q \geqslant 2$ and $\beta > \beta_c(q)$, all elements of $\mathcal{G}_{q,\beta}$ are invariant under translations; moreover, the extremal elements of $\mathcal{G}_{q,\beta}$ are pure phases $\mathbb{P}^i_\beta, i \in \{1, \dots, q\}$.
As already explained in the heuristics in Section 2.4.3, our main result is stronger than Theorem 4.1. We give here the exact statement of the result [1]

Theorem 4.2 *Let $q \geqslant 2$ and $\beta > \beta_c(q)$, and set $\Lambda_n = \mathbb{Z}^2 \cap [-n, n]^2$. For any $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ such that, for any boundary condition σ on $\partial\Lambda_n$, we can find $\alpha^n_1, \dots, \alpha^n_q \geqslant 0$ depending on (n, σ, β, q) only, such that*

$$\Big| \mathbb{P}^\sigma_{\Lambda_n,\beta}[g] - \sum_{i=1}^{q} \alpha^n_i \, \mathbb{P}^i_\beta[g] \Big| \leqslant C_\varepsilon \|g\|_\infty n^{-\frac{1}{2}+14\varepsilon},$$

for any measurable function g of the spins in Λ_{n^ε}.

Above we write $\mu[f] = \int f d\mu$ for a measure μ and an integrable function f.
Note that the error term is essentially of the right order (which is $O(n^{-1/2})$); see [LC4] for a proof of this claim when $q = 2$.

The strategy of the proof is the following. We consider the conditioned random-cluster measure on Λ associated to the q-state Potts model with boundary condition σ. Boundary conditions for the Potts model get rephrased as absence of connections (in the random-cluster configuration) between specified parts of the boundary of Λ. In other words, boundary conditions for the Potts models correspond to conditioning on the existence of dual-clusters between some dual-sites on the boundary. Note that the conditioning can be very messy, since intricate boundary conditions correspond to microscopic conditioning on existence of dual-clusters. It will be seen that being a mixture of measures with pure boundary condition boils down to the fact that, with high probability, no dual-cluster connected to the boundary reaches a small box deep inside Λ (which, in particular, implies that the same is true for the Potts interfaces).

The techniques involved in the proof are two-fold. First, we use positivity of surface tension in the regime $\beta > \beta_c$, which was proved in [11], in order to get rid of the microscopic mess due to the conditioning and to show that, deep inside the box, the conditioning with respect to σ corresponds to the existence of macroscopic dual-clusters. The second part of the proof consists in proving that these clusters are very slim, and that they fluctuate in a diffusive way, so that the probability that they touch a small box centered at the origin is going to zero as the size of Λ goes to infinity. The crucial step here is the use of the Ornstein-Zernike theory of sub-critical FK clusters developed in [21].

[1] which differs from the heuristic Lemma 2.1 by 13ε in the exponent of the error term, where according to Dima Ioffe "13 is the number of steps of the proof".

4.1.1 Some notations

Each nearest-neighbor edge e of $\mathbb{Z}^2$ intersects a unique dual edge of $(\mathbb{Z}^2)^* = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, that we denote by e^*. Consider a subgraph $G = (V, E)$ of $\mathbb{Z}^2$, with vertex set V and edge set E. If E is a set of direct edges, then its dual is defined by $E^* = \{e^* : e \in E\}$. Furthermore, if G does not possess any isolated vertices, we can define the dual V^* as the endpoints of edges in E^*. Altogether, this defines a dual graph $G^* = (V^*, E^*)$.

Let Λ_n be the set of sites of $\mathbb{Z}^2 \cap [-n, n]^2$ and E_n be the set of all nearest-neighbor edges of Λ_n. The dual graph is denoted by (Λ_n^*, E_n^*). For $m < n$, the annulus $\Lambda_n \setminus \Lambda_m$ is denoted by $A_{m,n}$. The *vertex-boundary* ∂V of a graph (V, E) is defined by $\partial V = \{x \in V : \exists y \sim x \text{ such that } y \notin V\}$. The *exterior vertex-boundary* $\partial^{\text{ext}} V$ of a graph (V, E) is defined by $\partial^{\text{ext}} V = \cup_{x \in V} \{y \notin V : y \sim x\}$. The *edge-boundary* ∂E of a graph (V, E) is the set of edges between two adjacent points of ∂V.

It will occasionally be convenient to think about ∂E_m as a closed contour in $\mathbb{R}^2$ or, more generally, to think about subsets of E (clusters, paths, etc) in terms of their embedding into $\mathbb{R}^2$; we shall do it without further comments in the sequel.

All constants in the sequel depend on β and q only. We shall use the notation $f = O(g)$ if there exists $C = C(\beta, q) > 0$ such that $|f| \leqslant C|g|$. We shall write $f = \Theta(g)$ if both $f = O(g)$ and $g = O(f)$.

4.2 Step 1: From Potts model to random-cluster model

Il n'y a réellement ni beau style, ni beau dessin, ni belle couleur : il n'y a qu'une seule beauté, celle de la vérité qui se révèle.

Auguste Rodin

Let $G = (V(G), E(G))$ be a finite graph. An element $\omega \in \{0,1\}^{E(G)}$ is called a configuration. An edge e is said to be open in ω if $\omega(e) = 1$ and closed if $\omega(e) = 0$. We shall work with two types of boundary conditions: f-free and w-wired. Recall that the random-cluster measure with edge-weight p and cluster-weight q on G with $*$-boundary condition ($* = f, w$) is given by

$$\mu^*_{G,p,q}(\omega) = \mu^*_G(\omega) = \frac{p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}} q^{\#_* \text{ clusters}}}{Z^*_{G,p,q}},$$

where $Z^*_{G,p,q}$ is a normalizing constant and a cluster is a maximal connected component of the graph $(V(G), \{e \in E(G) : \omega(e) = 1\})$. The number $\#_f$ clusters counts all the disjoint clusters, whereas the number $\#_w$ clusters counts only those disjoint clusters which are not connected to the vertex boundary ∂V.

4.2.1 Coupling with a supercritical random-cluster model on $(\mathbb{Z}^2)^*$

We consider the q-state Potts model on the graph $(\mathbb{Z}^2)^*$ at inverse temperature $\beta > \beta_c(q)$. As the parameters β and q will always remain fixed, we drop them from the notation. Fix $\sigma \in \{1,\dots,q\}^{(\mathbb{Z}^2)^*}$. For each n, we define the Potts measure $\mathbb{P}^{\sigma}_{\Lambda_n^*}$ on Λ_n^* with boundary condition σ on the vertex boundary $\partial\Lambda_n^*$.

We recall that the Potts model can be coupled with a random-cluster configuration in the following way. From a configuration of spins $\eta \in \{1,\dots,q\}^{V(\Lambda_n^*)}$, construct a percolation configuration $\omega^* \in \{0,1\}^{E_n^*}$ by setting each edge in E_n^* to be

- closed if the two end-points have different spins,
- closed with probability $e^{-\beta}$ and open otherwise if the two end-points have the same spins.

The measure thus obtained is a random-cluster measure on $(\mathbb{Z}^2)^*$ with edge-weight $p^* = 1 - e^{-\beta}$, cluster-weight q and wired boundary condition on $\partial\Lambda_n^*$, conditioned on the following event, called $\mathrm{Cond}_n[\sigma]$: writing $S_i = \{x \in \partial\Lambda_n^* : \sigma(x) = i\}$, the sets S_i and S_j are not connected by open edges in E_n^*, for every $i \neq j$ in $\{1,\dots,q\}$. We denote this measure by $\mu^w_{\Lambda_n^*}(\cdot \mid \mathrm{Cond}_n[\sigma])$. When there is no conditioning, the random-cluster measure with wired (resp. free) boundary condition is denoted by $\mu^w_{\Lambda_n^*}$ (resp. $\mu^f_{\Lambda_n^*}$).

Reciprocally, the Potts measure can be obtained from $\mu^w_{\Lambda_n^*}(\cdot \mid \mathrm{Cond}_n[\sigma])$ by assigning to every cluster a spin in $\{1,\dots,q\}$ according to the following rule:

- For every $i \in \{1,\dots,q\}$, sites connected to S_i receive the spin i,
- The sites of a cluster which is not connected to S_i receive the same spin in $\{1,\dots,q\}$ chosen uniformly at random, independently of the spins of the other clusters.

Thanks to the connection between Potts measures and random-cluster measures, tools provided by the theory of random-cluster models can be used in this context. Note that the parameters of the corresponding random-cluster measure are supercritical ($p^* > p_c(q)$).

4.2.2 Coupling with the subcritical Random-Cluster model on $\mathbb{Z}^2$

Rather than working with the supercritical random-cluster measure on $(\mathbb{Z}^2)^*$, we will be working with its subcritical dual measure on $\mathbb{Z}^2$ (this is the reason for choosing to define the Potts model on $(\mathbb{Z}^2)^*$). We recall the natural one-to-one mapping between $\{0,1\}^{E_n^*}$ and $\{0,1\}^{E_n}$. Namely, set $\omega(e) = 1 - \omega(e^*)$. In this way, both direct and dual FK configurations are defined on the same probability space. In the sequel, the same notation will be used for percolation events in direct and dual configurations. For instance, $\omega \in \mathrm{Cond}_n[\sigma]$ means that $\omega^* \in \mathrm{Cond}_n[\sigma]$. The corresponding direct FK measure is $\mu^f_{\Lambda_n}(\cdot \mid \mathrm{Cond}_n[\sigma])$. We recall the relation between primal and dual parameters p and p^*: they satisfy $pp^*/[(1-p)(1-p^*)] = q$.

Since we are working with the low temperature Potts model, the random-cluster model on $(\mathbb{Z}^2)^*$ corresponds to $p^* > p_c(q)$ so that the random-cluster model on $\mathbb{Z}^2$ is subcritical ($p < p_c(q)$). For this measure, $\mathrm{Cond}_n[\sigma]$ is an increasing event which requires the existence of direct open paths disconnecting different dual S_i-s. This reduces the problem to the study of the stochastic geometry of subcritical clusters. In particular, this enables us to use known results on the subcritical model.

We recall Corollary 1.5 of the introduction, which we slightly adapt to the following proposition.

Proposition 4.1 *There exists $c > 0$ such that, for n large enough and $2k \leqslant n \leqslant m$,*

$$\mu^{w}_{A_{k,n}}(\textit{there exists a crossing of } A_{k,n}) \leqslant e^{-cn},$$
$$\mu^{w}_{\Lambda_n}(\textit{there exists a cluster of cardinality } m \textit{ in } \Lambda_{n/2}) \leqslant e^{-cm},$$

where a crossing *is a cluster of $A_{m,n}$ connecting the inner box to the outer box.*

A cluster surrounding the inner box of $A_{m,n}$ inside the outer box of $A_{m,n}$ is said to be a *circuit*. Note that the existence of a dual circuit is a complementary event to the existence of a crossing between the inner and outer boxes.

4.2.2.1 Surface tension

We recall that the surface tension in the supercritical dual model is the inverse correlation length in the primal sub-critical FK percolation. Let $p < p_c(q)$. The surface tension in direction x is defined by

$$\tau(x) = \tau_p(x) = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{\mathbb{Z}^2}(0 \leftrightarrow [kx]),$$

where $y \leftrightarrow z$ means that y and z belong to the same connected component. We will also refer to it as the τ-distance. By Proposition 4.1, τ is equivalent to the usual Euclidean distance on $\mathbb{R}^d$. Furthermore, we recall Corollary 1.3 of the introduction, it satisfies the "sharp triangle inequality", i.e. there exists $\rho = \rho(p) > 0$ such that

$$\tau(x) + \tau(y) - \tau(x+y) \geqslant \rho(|x| + |y| - |x+y|). \tag{4.2}$$

We define $d_\tau(A, B) = \sup_{a\in A} \inf_{b\in B} \tau(a-b)$ to be the τ-Hausdorff distance between the two sets $A, B \subset \mathbb{R}^2$.

4.2.3 Reformulation of the problem in terms of the subcritical random-cluster model

Theorem 4.3 *Fix $p < p_c(q)$ and let $\varepsilon \in (0,1)$. Then, uniformly in all boundary conditions σ,*

$$\mu^{f}_{\Lambda_n}\big(C \cap \Lambda_{n^\varepsilon} \neq \varnothing \mid \mathrm{Cond}_n[\sigma]\big) = O(n^{-\frac{1}{2}+14\varepsilon}) \tag{4.3}$$

where C is the set of sites connected to the boundary $\partial\Lambda_n$.

The proof of this theorem will be the core of the paper. Before delving into the proof, let us show how it implies Theorem 4.2. First of all, we recall (1.32), uniqueness of the infinite cluster on $(\mathbb{Z}^2)^*$ implies that for $\beta > \beta_c(q)$,

$$\mathbb{P}^f_{(\mathbb{Z}^2)^*} = \frac{1}{q}\sum_{i=1}^{q} \mathbb{P}^i_{(\mathbb{Z}^2)^*}. \tag{4.4}$$

Moreover, we recall Proposition 1.28: there exists $c > 0$ such that, for any $n > 0$ and any subdomain Ω^* of $(\mathbb{Z}^2)^*$ containing Λ^*_{2n},

$$\mathbb{P}^f_{\Omega^*}[g] = \mathbb{P}^f_{(\mathbb{Z}^2)^*}[g] + O(\|g\|_\infty e^{-cn}), \tag{4.5}$$

for any g depending only on spins in Λ^*_n. The same holds for pure boundary conditions $i \in \{1, \dots, q\}$.

In the sequel we will also need the so-called ratio strong mixing property for the dual random-cluster model (which follows from [7, Theorem 1.7(ii)]): If a percolation event A depends on edges from E_A and if B depends on edges from E_B, then,

$$\left| \frac{\mu^f_{(\mathbb{Z}^2)^*}(A \cap B)}{\mu^f_{(\mathbb{Z}^2)^*}(A)\mu^f_{(\mathbb{Z}^2)^*}(B)} - 1 \right| \leqslant \sum_{e_A \in E_A, e_B \in E_B} e^{-c\, d(e_A, e_B)}, \tag{4.6}$$

where $d(e_A, e_B)$ is a distance between edges e_A and e_B (for instance the distance between their mid-points).

Fix now some $n > 0$ and a boundary condition σ on $\partial\Lambda_n$. Fix $\varepsilon > 0$ small. We consider the coupling (η, ω) (the measure is denoted by ν) with marginals $\mathbb{P}^\sigma_{\Lambda_n}$ and $\mu^f_{\Lambda_n}(\cdot \mid \mathrm{Cond}_n[\sigma])$ described in the previous section. Let $\mathcal{E}$ be the event that ω contains an open crossing in $A_{2n^\varepsilon, n}$. Let $\mathcal{F}^f$ be the event that ω contains an open circuit in $A_{2n^\varepsilon, n}$. Let $\mathcal{F}^{(i)}$ be the event that ω contains neither an open crossing nor an open circuit in $A_{2n^\varepsilon, n}$, and that $(\Lambda_{2n^\varepsilon})^*$ is connected in the dual configuration to S_i. Note that

$$\nu(\mathcal{E}) = \mu^f_{\Lambda_n}(\mathcal{E} \mid \mathrm{Cond}_n[\sigma]) = O(n^{-\frac{1}{2}+14\varepsilon}),$$

by applying Theorem 4.3.

- (conditioning on $\mathcal{F}^f$). Let Γ^* be the connected component of $\partial\Lambda^*_n$ in ω^*. Denote the connected component of $\Lambda^*_{2n^\varepsilon}$ in $\Lambda^*_n \setminus \Gamma^*$ by Ω^*. We have $\Lambda^*_{2n^\varepsilon} \subset \Omega^*$. Conditioning on Γ^* we infer, using (4.5) and (4.4) that

$$\begin{aligned}\nu\big(g \mid \mathcal{F}^f\big) &= \nu\big(\mathbb{P}^f_{\Omega^*}[g] \mid \mathcal{F}^f\big) = \mathbb{P}^f_{(\mathbb{Z}^2)^*}[g] + O(\|g\|_\infty e^{-cn^\varepsilon}) \\ &= \frac{1}{q}\sum_{i=1}^{q} \mathbb{P}^{(i)}_{(\mathbb{Z}^2)^*}[g] + O(\|g\|_\infty e^{-cn^\varepsilon}).\end{aligned}$$

- (conditioning on $\mathcal{F}^{(i)}$). In this case, let us condition on the connected cluster Γ of $\partial\Lambda_n$. We view Γ as the set of bonds. Define Ω^* as the connected component of $\Lambda^*_{2n^\varepsilon}$ in $(E_n \setminus \Gamma)^*$. By construction, $\Lambda^*_{2n^\varepsilon} \subset \Omega^*$ and $\Omega^* \cap S_i \neq \varnothing$. Consequently, using (4.5) once again, we obtain

$$\nu\big(g \mid \mathcal{F}^{(i)}\big) = \nu\big(\mathbb{P}^{(i)}_{\Omega^*}[g] \mid \mathcal{F}^{(i)}\big) = \mathbb{P}^{(i)}_{(\mathbb{Z}^2)^*}[g] + O(\|g\|_\infty e^{-cn^\varepsilon}).$$

By summing all these terms,

$$\mathbb{P}^{\sigma}_{\Lambda_n}[g] = \nu[g] = \nu[g|\mathcal{E}]\,\nu[\mathcal{E}] + \nu[g|\mathcal{F}^f]\,\nu[\mathcal{F}^f] + \sum_{i=1}^{q} \nu[g|\mathcal{F}^{(i)}]\nu[\mathcal{F}^{(i)}]$$
$$= \sum_{i=1}^{q} \big(\tfrac{1}{q}\nu[\mathcal{F}^f] + \nu[\mathcal{F}^{(i)}]\big)\mathbb{P}^{(i)}_{(\mathbb{Z}^2)^*}[g] + O(\|g\|_\infty n^{-\frac{1}{2}+14\varepsilon}),$$

which implies Theorem 4.2 readily.

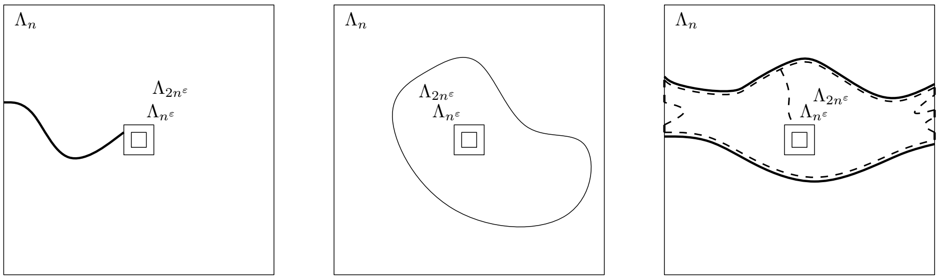


Figure 4.1 – On the left (resp. center, right), the event $\mathcal{E}$ (resp. $\mathcal{F}^f$, $\mathcal{F}^{(i)}$) is depicted.

4.3 Step 2: Macroscopic flower domains

Je n'ai qu'une fleur dans mon jardin
C'est une fleur que m'a fait le destin
[...]
Elle m'en fait voir de toutes les couleurs
Ma fleur, ma fleur

Claude Nougaro, "Ma fleur".

In the box Λ_n, the conditioning on $\mathrm{Cond}_n[\sigma]$ can be very messy. Indeed, as we mentioned before, it forces the existence of open paths separating the sets S_i. For instance, the number of such paths forced by an alternating boundary condition $1, 2, \ldots, q, 1, 2, \ldots$ is necessarily of order n.

We first show that, no matter what the boundary condition σ is, with high probability only a bounded number of such interfaces is capable of reaching an inner box Λ_m, where m is a fraction of n. Furthermore, we shall argue that the number of sites in $\partial\Lambda_m$ which are connected to the original $\partial\Lambda_n$ is uniformly bounded. In terms of the original Potts model, this corresponds to the existence, with high probability, of a domain including the box Λ_m for which the boundary condition contains a uniformly bounded number of spin changes. This will be called a flower domain below.

4.3.1 Definition of flower domains

Let $m < n$. For a configuration ω, let $C_{m,n} = C_{m,n}(\omega)$ be the set of sites connected to $\partial\Lambda_n$ in $\omega \cap (E_n \setminus E_m)$. Define the set of *marked vertices* by

$$\mathbb{G}_{m,n} = \mathbb{G}_{m,n}(\omega) = C_{m,n} \cap \partial\Lambda_m.$$

The set $\mathbb{G}_{m,n} \cup (\Lambda_n \setminus C_{m,n})$ may have several connected components, exactly one of them containing Λ_m. Let us call the latter the *flower domain* $\mathcal{D}_{m,n} = \mathcal{D}_{m,n}(\omega)$ *rooted at* m. Note that $\mathbb{G}_{m,n} = \partial\mathcal{D}_{m,n} \cap \partial\Lambda_m$, that is marked sites are unambiguously determined by the corresponding flower domains.

Fix a configuration ω. Let $\mathcal{C} = C_{m,n}(\omega)$ and let $\mathcal{D} = \mathcal{D}_{m,n}(\omega)$ be the corresponding flower domain. Let also $\mathbb{G} = \mathbb{G}_{m,n}(\omega)$. By construction, the restriction of the conditional measure $\mu^f_{\Lambda_n}(\,\cdot\,|C_{m,n} = \mathcal{C})$ to $\{0,1\}^{\mathcal{E}_\mathcal{D}}$, where $\mathcal{E}_\mathcal{D}$ is the set of edges of $\mathcal{D}$, is the FK measure with free boundary conditions on $\partial\mathcal{D} \setminus \mathbb{G}$ and wiring between sites of $\mathbb{G}$ inherited from connections in $\mathcal{C}$. We denote this restricted conditional measure as $\mu^{\text{flower}}_{\mathcal{D}}$. We also set $C_\mathbb{G}$ for the connected component of $\mathbb{G}$ in the restriction of ω to $\mathcal{E}_\mathcal{D}$.

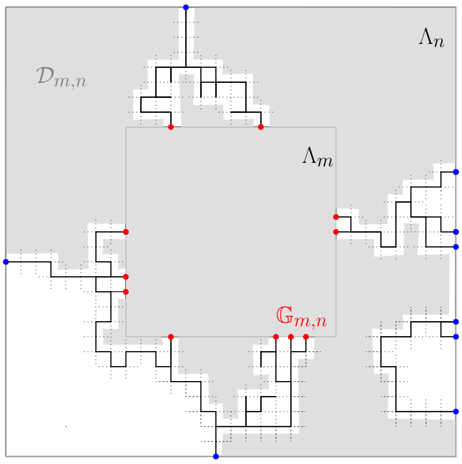


Figure 4.2 – Description of a flower domain $\mathcal{D}_{m,n}$ (light grey area). The blue points are locations of spin changes (i.e. separation between sets S_i), the red points constitute $\mathbb{G}_{m,n}$, the solid black lines in the annulus $\Lambda_n \backslash \Lambda_m$ constitute $C_{m,n}$.

4.3.2 Cardinality of $\mathbb{G}_{m,n}$

Flower domains have typically small sets $\mathbb{G}_{m,n}$, as the following proposition shows.

Proposition 4.2 *There exists* $M > 0$ *such that for any* $\delta > 0$

$$\mu^f_{\Lambda_n}\Big(\exists m \in \big[\tfrac{\delta n}{3}, \delta n\big] : |\mathbb{G}_{m,n}| \leqslant M \;\Big|\; \mathrm{Cond}_n[\sigma]\Big) \geqslant 1 - e^{-\delta n}, \qquad (4.7)$$

uniformly in σ *and* n *sufficiently large.*

The notation M will now be reserved for an integer $M > 0$ satisfying the previous proposition. We shall prove this Proposition for $\delta = 1$; the general case follows by a straightforward adaptation.

Definition 4.1 *Let $\mathcal{E}_r$ be the event that there exist r disjoint crossings of $A_{n/3,n/2}$.*

Lemma 4.1 *For all $r \geqslant 1$ and $n > 0$,*

$$\mu^f_{\Lambda_n}(\mathcal{E}_r) \leqslant e^{-crn},$$

where $c > 0$ is defined in Proposition 4.1.

Proof We prove that for all $r \geqslant 1$ and $n > 0$,

$$\mu^f_{\Lambda_n}(\mathcal{E}_r) \leqslant \left(\mu^w_{A_{n/3,n/2}}(\mathcal{E}_1)\right)^r. \tag{4.8}$$

The conclusion will then follow easily, since Proposition 4.1 implies that $\mu^w_{A_{n/3,n/2}}(\mathcal{E}_1)$ is smaller than $\exp(-cn)$.

In order to prove (4.8), we proceed by induction. First, note that $\mu^f_{\Lambda_n}$ restricted to $A_{n/3,n/2}$ is stochastically dominated by $\mu^w_{A_{n/3,n/2}}$.

Let $r \geqslant 1$ and consider $\mu^f_{\Lambda_n}(\mathcal{E}_{r+1}|\mathcal{E}_r)$. We number the vertices of the boundary of Λ_n as $\partial\Lambda_n = \{x_1, \dots, x_{4n+4}\}$ in clockwise order, starting at the bottom right corner. Let k be the smallest number such that there are r crossings among the clusters containing $x_1, \dots, x_k$. Denote by $\mathcal{S}$ the union of these clusters (which may contain isolated vertices). Observe that all edges in $A_{n/3,n/2} \setminus \mathcal{S}$ which are incident to vertices of $\mathcal{S}$ are closed. Therefore, the conditional measure $\mu^f_{\Lambda_n}(\cdot_{|A_{n/3,n/2}\setminus\mathcal{S}}|\mathcal{S})$ is stochastically dominated by $\mu^w_{A_{n/3,n/2}}(\cdot_{|A_{n/3,n/2}\setminus\mathcal{S}})$. In both instances above, the symbol $\nu(\cdot_{|B})$ means the restriction of ν to edges of the graph with the vertex set B. As a result, the probability, under $\mu^f_{\Lambda_n}(\cdot_{|A_{n/3,n/2}\setminus\mathcal{S}}|\mathcal{S})$, that there exists a crossing of $A_{n/3,n/2}$ is smaller than $\mu^w_{A_{n/3,n/2}}(\mathcal{E}_1)$. We obtain

$$\begin{aligned}\mu^f_{\Lambda_n}(\mathcal{E}_{r+1}) &= \mu^f_{\Lambda_n}(\mathcal{E}_{r+1}|\mathcal{E}_r)\mu^f_{\Lambda_n}(\mathcal{E}_r) = \mu^f_{\Lambda_n}[\mu^f_{\Lambda_n}(\mathcal{E}_{r+1}|\mathcal{S})]\mu^f_{\Lambda_n}(\mathcal{E}_r)\\ &\leqslant \mu^w_{A_{n/3,n/2}}(\mathcal{E}_1)\mu^f_{\Lambda_n}(\mathcal{E}_r) \leqslant \mu^w_{A_{n/3,n/2}}(\mathcal{E}_1)^{r+1}.\end{aligned}$$

■

Proof [of Proposition 4.2] Obviously,

$$\mu^f_{\Lambda_n}\left(\forall m \in \left[\tfrac{n}{3}, \tfrac{n}{2}\right] : |\mathbb{G}_{m,n}| > M \;\middle|\; \mathrm{Cond}_n[\sigma]\right) \leqslant \frac{\mu^f_{\Lambda_n}\left(\forall m \in [\frac{n}{3}, \frac{n}{2}] : |\mathbb{G}_{m,n}| > M\right)}{\mu^f_{\Lambda_n}(\mathrm{Cond}_n[\sigma])}. \tag{4.9}$$

Let us bound from below the denominator of (4.9). If all the edges of ∂E_n are open, then $\mathrm{Cond}_n[\sigma]$ occurs. Moreover, the measure $\mu^f_{\Lambda_n}$ stochastically dominates independent Bernoulli edge percolation on $\{0,1\}^{E_n}$ with $\tilde{p} = p/(p + (q-1)p)$, see [4, Theorem 4.1]. We deduce

$$\mu^{f}_{\Lambda_n}(\mathrm{Cond}_n[\sigma]) \geqslant \mu^{f}_{\Lambda_n}(\text{all the edges in } \partial E_n \text{ are open}) \geqslant \tilde{p}^{8n}. \tag{4.10}$$

Let us now bound from above the numerator of (4.9). First,

$$\mu^{f}_{\Lambda_n}\big(\forall m \in \big[\tfrac{n}{3}, \tfrac{n}{2}\big] : |\mathbb{G}_{m,n}| > M\big) \leqslant \mu^{f}_{\Lambda_n}\big(|C_{n/3,n} \cap A_{n/3,n/2}| \geqslant Mn/6\big).$$

Fix $R > 0$. If $|C_{n/3,n} \cap A_{n/3,n/2}| \geqslant Mn/6$, either $A_{n/3,n/2}$ contains more than R crossings or one of the crossing has cardinality larger than $Mn/(6R)$. Proposition 4.1 implies that the probability of having clusters with size larger than $Mn/(6R)$ in $\Lambda_{n/2}$ is smaller than $\exp[-cMn/(6R)]$ for n large enough. Lemma 4.1 together with (4.9) implies that, for n large enough,

$$\mu^{f}_{\Lambda_n}\big(\forall m \in \big[\tfrac{n}{3}, \tfrac{n}{2}\big] : |\mathbb{G}_{m,n}| > M \;\big|\; \mathrm{Cond}_n[\sigma]\big) \leqslant \tilde{p}^{-8n}[e^{-cRn} + e^{-cMn/(6R)}] \leqslant e^{-n},$$

provided that R and M be sufficiently large. ■

4.3.3 Reduction to FK measures on flower domains with free boundary condition

We define

$$\mathcal{M}_n = \max\{m \leqslant n : |\mathbb{G}_{m,n}| \leqslant M\}, \tag{4.11}$$

where the maximum is set to be equal to ∞ if there is no $m \leqslant n$ such that $|\mathbb{G}_{m,n}| \leqslant M$. With this notation, we actually proved that $\mathcal{M}_n \in [\frac{n}{3}, n]$ with probability bounded below by $1 - e^{-n}$.

Let $\mathcal{C}$ be a possible realization of $C_{m,n}$ and $\mathcal{D} = \mathcal{D}_{m,n}$ be the corresponding flower domain. The restriction of $\mu^{f}_{\Lambda_n}(\cdot \mid \mathcal{M}_n = m; C_{m,n} = \mathcal{C})$ to $\mathcal{D}$ is $\mu^{\text{flower}}_{\mathcal{D}}$. Furthermore,

$$\mathrm{Cond}_n[\sigma] \cap \{\mathcal{M}_n = m\} \cap \{C_{m,n} = \mathcal{C}\}$$

is a product event $\Omega_{\sigma,\mathcal{C}} \times \{\mathcal{M}_n = m; C_{m,n} = \mathcal{C}\}$, where $\Omega_{\sigma,\mathcal{C}} \subset \{0,1\}^{\mathcal{E}_{\mathcal{D}}}$. Then

$$\mu^{f}_{\Lambda_n}(C \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \mid \mathrm{Cond}_n[\sigma]; \mathcal{M}_n = m; C_{m,n} = \mathcal{C}) = \mu^{\text{flower}}_{\mathcal{D}}(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \mid \Omega_{\sigma,\mathcal{C}}). \tag{4.12}$$

The event $\Omega_{\sigma,\mathcal{C}}$ has an obvious structure. It corresponds to the existence of certain connections between different sites of $\mathbb{G} = \mathcal{C} \cap \partial\Lambda_m = \mathcal{D} \cap \partial\Lambda_m$. More precisely, let $\mathcal{P}_{\mathbb{G}}$ be the collection of different partitions of $\mathbb{G}$. Elements of $\mathcal{P}_{\mathbb{G}}$ are of the form $\underline{\mathbb{G}} = (\mathbb{G}_1, \dots, \mathbb{G}_\ell)$. Define

$$\Omega_{\underline{\mathbb{G}}} = \bigcap_{i} \bigcap_{u,v \in \mathbb{G}_i} \{u \leftrightarrow v\} \subset \{0,1\}^{\mathcal{E}_{\mathcal{D}}}.$$

Let us say that a partition $\underline{\mathbb{G}}$ is compatible with $\Omega_{\sigma,\mathcal{C}}$ if $\Omega_{\underline{\mathbb{G}}} \subseteq \Omega_{\sigma,\mathcal{C}}$. Note that we do not rule out that some elements $\mathbb{G}_i$ of a partition $\underline{\mathbb{G}}$ are singletons. If $\mathbb{G}_i$ is a singleton, then $\bigcap_{u,v \in \mathbb{G}_i}\{u \leftrightarrow v\}$ is, of course, a sure event, which could be dropped from the definition of $\Omega_{\underline{\mathbb{G}}}$. In other words, only non-singleton elements of $\underline{\mathbb{G}}$ are relevant for $\Omega_{\underline{\mathbb{G}}}$. Also note that the events $\Omega_{\underline{\mathbb{G}}}$ do not have to be disjoint. Still, for any σ,

$$\Omega_{\sigma,\mathcal{C}} = \bigcup_{\underline{\mathbb{G}} \in \mathcal{P}'_{\mathbb{G}}} \Omega_{\underline{\mathbb{G}}},$$

where the set $\mathcal{P}'_{\mathbb{G}}$ corresponds to partitions which are compatible with the occurrence of the event $\Omega_{\sigma,\mathcal{C}}$, and which are *maximal* in the sense that one cannot find a finer partition which would be still compatible with $\Omega_{\sigma,\mathcal{C}}$.

The previous section implies the following reduction, which we will now consider for the rest of this work.

Proposition 4.3 *Fix $\delta > 0$. Then, writing B_k for the k^{th} Bell number, which counts the number of partitions of a set of k elements,*

$$\mu^{f}_{\Lambda_n}\big(C \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \;\big|\; \mathrm{Cond}_n[\sigma]\big) \leqslant e^{-\delta n} + B_M q^M \max \mu^{f}_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \;\big|\; \Omega_{\underline{\mathbb{G}}}\big), \quad (4.13)$$

for all boundary conditions σ and n sufficiently large. The above maximum is over all flower domains $\mathcal{D}$ rooted at $m \in [\frac{n}{3}, n]$ with at most $|\mathbb{G}| \leqslant M$ marked points, and over all partitions $\underline{\mathbb{G}} \in \mathcal{P}'_{\mathbb{G}}$.

Above the term q^M comes from the fact that the elements of $\mathbb{G}$ are possibly wired together. It then bounds the Radon-Nikodym derivative between measures $\mu^{\mathrm{flower}}_{\mathcal{D}}$ and $\mu^{f}_{\mathcal{D}}$. The quantity B_M bounds from above the number of sub-partitions of $\mathbb{G}$ (the events $\Omega_{\underline{\mathbb{G}}}$ being not necessarily disjoint).

4.4 Step 3: Macroscopic structure near the center of the box

Je n'ai vu qu'un seul arbre, un seul, mais je l'ai vu,
Et je connais par cœur sa ramure touffue,
Et ce tout petit bout de branche me suffit :
Pour connaître une feuille, il faut toute une vie.

Georges Brassens, "Le fidèle absolu".

This section studies the macroscopic structure of the set C of sites connected to the boundary of Λ_n. Its main result, Proposition 4.4 below, implies that on a sufficiently small scale $\delta > 0$, the intersection $C \cap \Lambda_k$ for boxes with $k \in [\frac{\delta n}{3}, \delta n]$ is with an overwhelming probability either empty, or close to a segment, or close to a tripod (three segments coming out from a point).

Before starting, note that Proposition 4.3 enables us to restrict attention to a flower domain $\mathcal{D} = \mathcal{D}_{m,n}$ with $m \in [\frac{n}{3}, n]$ and $|\mathbb{G}_{m,n}| \leqslant M$. We set $\mathbb{G} = \mathbb{G}_{m,n}$. We now fix this flower domain and work under $\mu^{f}_{\mathcal{D}}\left(\cdot \;\middle|\; \Omega_{\underline{\mathbb{G}}}\right)$ for some $\underline{\mathbb{G}} \in \mathcal{P}'_{\mathbb{G}}$. All constants in this section are independent of $\mathcal{D}_{m,n}$ and $\underline{\mathbb{G}}$ as long as $|\mathbb{G}_{m,n}| \leqslant M$. We will often recall this independence by using the expression "uniformly in $(\mathcal{D}, \underline{\mathbb{G}})$ with $|\mathbb{G}| \leqslant M$".

Define $C_{k,\mathbb{G}}$ to be the set of edges connected to $\mathbb{G}$ in $\mathcal{D} \setminus \Lambda_k$ (which can consist of several connected components). Note that $\mathbb{G}_{k,n} = C_{k,n} \cap \partial\Lambda_k = C_{k,\mathbb{G}} \cap \partial\Lambda_k$. Given $v_1, v_2 \in \mathbb{R}^2$, we define $[v_1, v_2]$ to be the line segment with endpoints v_1 and v_2, and $\measuredangle(v_1, v_2)$ to be the angle between v_1 and v_2, seen as vectors in the plane. We refer to Fig. 4.3 for an illustration of the following definitions.

Definition 4.2 *For $k < m$, $\nu > 0$ and $\ell = 1, 2, 3$, let us say that $E^{\ell}_{\nu,k} \subset \{0,1\}^{\mathcal{E}_{\mathcal{D}}}$ occurs if $S\ell$ below happens:*

S1. $\mathbb{G}_{k,n} = \varnothing$.

S2. $\mathbb{G}_{k,n} = \mathbb{V}^1_{k,n} \cup \mathbb{V}^2_{k,n}$, *where* $\mathbb{V}^1_{k,n}, \mathbb{V}^2_{k,n}$ *are two disjoint sets of τ-diameter less than or equal to νk. Moreover,*

- *Each of the sets $\mathbb{V}^1_{k,n}$ and $\mathbb{V}^2_{k,n}$ is connected in $C_{k,\mathbb{G}}$.*
- *For any two vertices $v_i \in \mathbb{V}^i_{k,n}$; $i = 1, 2$, we have $[v_1, v_2] \cap \Lambda_{k/2} \neq \varnothing$.*

S3. $\mathbb{G}_{k,n} = \mathbb{V}^1_{k,n} \cup \mathbb{V}^2_{k,n} \cup \mathbb{V}^3_{k,n}$, *where* $\mathbb{V}^1_{k,n}$, $\mathbb{V}^2_{k,n}$ *and* $\mathbb{V}^3_{k,n}$ *are disjoint sets with τ-diameter less than or equal to νk. Moreover,*

- *Each of the sets $\mathbb{V}^1_{k,n}$, $\mathbb{V}^2_{k,n}$ and $\mathbb{V}^3_{k,n}$ is connected in $C_{k,\mathbb{G}}$,*
- *For any choice of $v_i \in \mathbb{V}^i_{k,n}$; $i = 1, 2, 3$. there exists $x \in \Lambda_{k/2}$ such that $\mathcal{T} = \{v_1, v_2, v_3; x\}$ is a Steiner tripod (see Definition 4.4 below). In particular, as it follows from P2 of Proposition 4.5 below, $\measuredangle(v_i - x, v_j - x) > \frac{\pi}{2} + \eta$ for every $i \neq j$.*

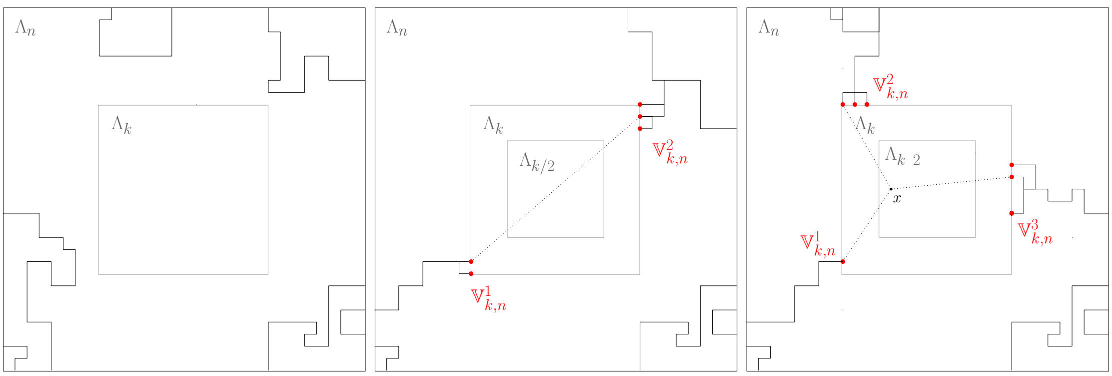


Figure 4.3 – Description of the events $E^{\ell}_{\nu,k}$, $\ell = 1, 2, 3$ from left to right. The set $\mathbb{G}_{k,n}$, partitioned into $\mathbb{V}^{\ell}_{k,n}$, $\ell = 1, 2, 3$, is indicated in red.

We are now in a position to state the main proposition.

Proposition 4.4 *For any $\nu > 0$, there exist $\delta = \delta(\nu, M) > 0$ and $\kappa = \kappa(\nu, M) > 0$ such that*

$$\mu^f_{\mathcal{D}}\Big(\bigcup_{k \geqslant \delta n} (E^1_{\nu,k} \cup E^2_{\nu,k} \cup E^3_{\nu,k}) \cap \big\{ |\mathbb{G}_{k,n}| \leqslant M \big\} \;\Big|\; \Omega_{\underline{\mathbb{G}}} \Big) \geqslant 1 - e^{-\kappa n}, \tag{4.14}$$

uniformly in $(\mathcal{D}, \underline{\mathbb{G}})$ with $|\mathbb{G}| \leqslant M$.

The proof of Proposition 4.4 comprises two steps: First, we show that the implied geometric structure is characteristic of deterministic objects called *Steiner forests*. Then, we show that, with high $\mu^f_{\mathcal{D}}(\,\cdot\,|\,\Omega_{\underline{\mathbb{G}}})$-probability, the cluster $C_{\mathbb{G}}$ sits in the vicinity of one such forest.

4.4.1 Steiner forests

Note that for every m the set $\mathcal{K}_m$ of all compact subsets of Λ_m is a Polish space with respect to the d_τ-distance.

We now recall the concept of Steiner forest. Consider $E \subseteq \partial\Lambda_m$ with $|E| \leqslant M$. Let $\underline{E} = (E_1, \ldots, E_i)$ be a partition of E and $\Omega_{\underline{E}}$ be the set of compact subsets of $\mathbb{R}^2$ such that E_j is included in one of their connected components for every $j \in \{1, \ldots, i\}$. For the trivial partition $\underline{E} = \{E\}$, we shall write Ω_E.

For a compact $\mathcal{S} \subset \mathbb{R}^2$, let $\tau(\mathcal{S})$ be the (one-dimensional) Hausdorff measure of $\mathcal{S}$ in the τ-norm. Explicitly,

$$\tau(\mathcal{S}) = \lim_{\varepsilon \to 0} \inf\Big\{ \sum \operatorname{diam}_\tau(A_i) \,:\, \mathcal{S} \subseteq \cup A_i,\, \operatorname{diam}_\tau(A_i) \leqslant \varepsilon \Big\}, \tag{4.15}$$

where $\operatorname{diam}_\tau(A) = \sup\{\tau(x-y) : x, y \in A\}$. Define the set of *Steiner forests* by

$$\Omega^{\min}_{\underline{E}} = \big\{ \mathcal{F} \in \Omega_{\underline{E}} \,:\, \tau(\mathcal{F}) = \min_{\mathcal{S} \in \Omega_{\underline{E}}} \tau(\mathcal{S}) \big\}.$$

We set

$$\tau_{\underline{E}} = \min_{\mathcal{S} \in \Omega_{\underline{E}}} \tau(\mathcal{S}) = \tau(\mathcal{F}),$$

for any Steiner forest $\mathcal{F} \in \Omega^{\min}_{\underline{E}}$.

In the sequel we shall work only with Steiner forests $\mathcal{F} \in \Omega^{\min}_{\underline{E}}$, when $\underline{E}$ is a partition of a set $E \subset \partial\Lambda_m$ of cardinality $|E| \leqslant M$. Let $\Omega^{\min}_{M,m}$ be the collection of all such forests.

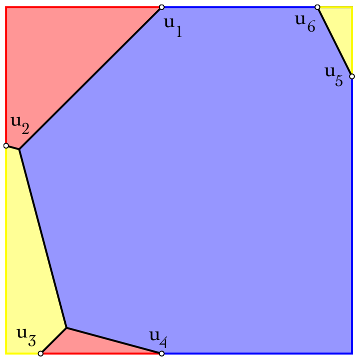


Figure 4.4 – A non trivial Steiner forest with a partition $\underline{E} = (E_1, E_2)$ with $E_1 = \{u_1, u_2, u_3, u_4\}$ and $E_2 = \{u_5, u_6\}$.

Proposition 4.5 *Fix $M > 0$. The following properties hold uniformly in m, in finite subsets $E \subseteq \partial\Lambda_m$ with $|E| \leqslant M$ and in partitions $\underline{E}$ of E:*

P1. ***(Number of Steiner forests and compactness of $\Omega^{\min}_{M,m}$)*** *There exists $k = k(M) < \infty$ such that $|\Omega^{\min}_{\underline{E}}| \leqslant k$. The set $\Omega^{\min}_{M,m}$ is a compact subset of $(\mathcal{K}_m, d_\tau)$.*

P2. ***(Structure of Steiner forests)*** *The sets $\mathcal{F} \in \Omega^{\min}_{\underline{E}}$ are forests (that is collections of disjoint trees). Each inner node (that is not belonging to E) of such $\mathcal{F}$ has degree 3. Furthermore, there exists an $\eta > 0$ such that the angle between two edges incident to an inner node of $\mathcal{F}$ is always larger than $\frac{\pi}{2} + \eta$.*

P3. ***(Well separateness of trees)*** *There exists $\delta_1 = \delta_1(M) > 0$ such that any $\mathcal{F} \in \Omega^{\min}_{\underline{E}}$ satisfies:*

(a) *for any Steiner tree $\mathcal{T} \in \mathcal{F}$, two different nodes of $\mathcal{T}$ in $\Lambda_{m/2}$ are at d_τ-distance at least $\delta_1 m$ of each other;*

(b) *if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two disjoint trees of $\mathcal{F}$, then $d_\tau\left(\mathcal{T}_1 \cap \Lambda_{m/2}, \mathcal{T}_2 \cap \Lambda_{m/2}\right) \geqslant \delta_1 m$.*

P4. ***(Stability)*** *For any $\delta_2 > 0$, there exists $\kappa_2 = \kappa_2(\delta_2, M) > 0$ such that, for any $|E| \leqslant M$, any partition $\underline{E}$ of E and any $\mathcal{S} \in \Omega_{\underline{E}}$,*

$$\tau(\mathcal{S}) \leqslant \tau_{\underline{E}} + \kappa_2 m \quad \textit{implies} \quad \min_{\mathcal{F} \in \Omega^{\min}_{M,m}} d_\tau(\mathcal{S}, \mathcal{F}) < \delta_2 m. \tag{4.16}$$

Proof We shall be rather sketchy since the arguments are presumably well understood. We shall consider the case $m = 1$ (the general case follows by homogeneity).

Let us start with P4. The functional τ in (4.15) is lower semi-continuous on $(\mathcal{K}_1, d_\tau)$ and has compact level sets (meaning sets of the form $\{\mathcal{S} : \tau(\mathcal{S}) \leqslant R\}$). See, for instance, [24, Proposition 3.1], where these facts are explained for the inverse correlation length of sub-critical Bernoulli bond percolation.

Assume that P4 is wrong. Then there exists $\delta > 0$ and two sequences; $\underline{E_k}$ and $\mathcal{S}_k \in \Omega_{\underline{E_k}}$, such that

$$\tau(\mathcal{S}_j) < \tau_{\underline{E_k}} + \frac{1}{k} \quad \text{but} \quad \min_{\mathcal{F} \in \Omega^{\min}_{M,m}} d_\tau(\mathcal{S}_j, \mathcal{F}) > \delta.$$

Since $|E_k| \leqslant M$, the sequence $\tau_{\underline{E_k}}$ is bounded. Hence $\{\mathcal{S}_j\}$ is pre-compact. Possibly passing to subsequence we may assume that $\underline{E_k}$ converges to $\underline{E}$ (points might collapse, but this is irrelevant since this preserves $|E| \leqslant M$), and that $\mathcal{S}_k$ converges to $\mathcal{S} \in \Omega_{\underline{E}}$. Both convergence are, of course, in the sense of Hausdorff distance. By minimality it is evident that $\tau_{\underline{E}} = \lim \tau_{\underline{E_k}}$. By lower-semicontinuity $\tau(\mathcal{S}) \leqslant \liminf \tau(\mathcal{S}_k)$. Which means that $\mathcal{S} \in \Omega^{\min}_{\underline{E}}$. A contradiction.

A proof of the first assertion of P1 can be found in [23, Theorem 1]. Compactness of $\Omega^{\min}_{M,1}$ follows from compactness of level sets of τ and the fact that if $\mathcal{F}_k \in \Omega^{\min}_{\underline{E_k}}$ converges to $\mathcal{F} \in \Omega_{\underline{E}}$, then, as was already mentioned above, $\tau_{\underline{E}} = \lim \tau_{\underline{E_k}}$, and hence, by the lower-semicontinuity of τ, $\mathcal{F} \in \Omega^{\min}_{\underline{E}}$.

A proof of P2 can be found in [10].

Let us turn to the proof of P3. For trivial partitions, Steiner forests are trees. Now, assume that there exists a sequence of Steiner trees $\mathcal{T}_k \in \Omega^{\min}_{E_k}$ such that $\mathcal{T}_k$ contains at least two inner nodes in $\Lambda_{1/2}$ at distance less or equal $\frac{1}{k}$. There is no loss of generality to assume that the sequence $\mathcal{T}_k$ converges to some $\mathcal{T}^*$. As it follows from P4, $\mathcal{T}^* \in \Omega^{\min}_{E^*}$, where E^* is the corresponding limit of E_k. Obviously, $|E^*|$ is still less or equal to M, since boundary points can only collapse under the limiting procedure.

The total number of nodes of each of $\mathcal{T}_k$ is uniformly bounded above. Hence by our assumption we can choose a number $\ell \geqslant 2$, a point $x \in \Lambda_{1/2}$, a radius $\varepsilon > 0$ and a sequence $\nu(k) \to 0$, so that
(a) each of $\mathcal{T}_k$ contains ℓ nodes in $\Lambda_{\nu(k)}(x) = x + \Lambda_{\nu(k)}$;
(b) none of $\mathcal{T}_k$ contains nodes in the annulus $A_{\nu(k),\varepsilon}(x)$.
Then the restriction of $\mathcal{T}_k$ to $\Lambda_\varepsilon(x)$ is a Steiner tree, whereas the cardinality of the intersection $|\partial\Lambda_\varepsilon(x) \cap \mathcal{T}_k| = \ell + 2$. By the minimality of $\mathcal{T}_k$ the points of $\partial\Lambda_\varepsilon(x) \cap \mathcal{T}_k$ are uniformly separated. Consequently, $|\partial\Lambda_\varepsilon(x) \cap \mathcal{T}^*| = \ell + 2 > 3$. We infer that the degree of x in the Steiner tree $\mathcal{T}^*$ is $\ell + 2 > 3$, which is impossible by P2. This proves P3(a).

Consider now two disjoint Steiner trees $\mathcal{T}_1 \in \Omega^{\min}_{E_1}$ and $\mathcal{T}_2 \in \Omega^{\min}_{E_2}$, such that the forest $\{\mathcal{T}_1, \mathcal{T}_2\}$ belongs to $\Omega^{\min}_{\{E_1,E_2\}}$. By the strict convexity of τ, the trees are confined to their convex envelopes: $\mathcal{T}_i \in \mathrm{co}\,(E_i)$ for $i = 1, 2$. Thus if both trees are disjoint and intersect $\Lambda_{1/2}$, it follows that $\mathrm{co}\,(E_1) \cap \mathrm{co}\,(E_2) = \varnothing$. Consequently, there exist $u_1, v_1 \in E_1$ and $u_2, v_2 \in E_2$, such that $\mathcal{T}_1$ lies below the interval $[u_1, v_1]$ and $\mathcal{T}_2$ lies above the interval $[u_2, v_2]$ (notions of above and below are with respect to the directions of normals). We are now facing two cases:

- $\mathcal{T}_1$ or $\mathcal{T}_2$ has an inner node in $\Lambda_{2/3}$. By P2, inner nodes are of degree three and angles between edges incident to inner nodes are at most $\pi - 2\eta$. This pushes inner nodes of $\mathcal{T}_i$ away from $[u_i, v_i]$ uniformly in $\mathcal{T}_1$ and $\mathcal{T}_2$. In such a case, P3 is satisfied.

- Both $\mathcal{T}_1$ and $\mathcal{T}_2$ do not contain nodes in $\Lambda_{2/3}$, but each contains an edge which crosses $\Lambda_{1/2}$. Having such edges close to each other (and hence running essentially in parallel across $\Lambda_{1/2}$) is easily seen to be incompatible with the minimality of $\mathcal{F}$.

This achieves the proof of P3(b). ■

4.4.2 Forest skeleton of the cluster $C_{\mathbb{G}}$

Let $\underline{\mathbb{G}}$ be a partition of $\mathbb{G}$. We now aim to show that, under $\mu^f_{\mathcal{D}}(\,\cdot \mid \Omega_{\underline{\mathbb{G}}})$, the cluster $C_{\mathbb{G}}$ stays typically close to one of the Steiner forests from $\Omega^{\min}_{M,m}$. In order to do that, we introduce the notion of forest skeleton of the cluster. This notion is a modification of the coarse-graining procedure developed in Section 2.2 of [21].

Let $\mathbf{U}_\tau$ be the unit ball in τ-norm. Fix a large number $c > 0$ and consider K such that $c \log K < K$. For any $y \in \mathbb{Z}^2$, set

$$\mathbf{B}_K(y) = (y + K \cdot \mathbf{U}_\tau) \cap \mathbb{Z}^2 \quad \text{and} \quad \hat{\mathbf{B}}_K(y) = \mathbf{B}_{K+c\log K}(y).$$

If $x \in A \subset \mathbb{Z}^2$ and $y \in A \cup \partial^{\text{ext}} A$, we shall use $\{x \xleftrightarrow{A} y\}$ to denote the event that x and y are connected by an open path from x to y whose vertices belong to A, with the possible exception of the terminal point y itself.

Let us construct the forest skeleton $\mathcal{F}_K$ of the cluster $C_{\mathbb{G}}$ (see Figure 4.5). Here and below, vertices in $\mathbb{Z}^2$ are ordered using the lexicographical ordering. In the following construction, we will often refer to the *minimal* vertex having some property.

Step 1. Set $r = 1, i = 1$. Set $x^1_0 = u_{i_1}$ be the minimal vertex of $\mathbb{G}$. Set $V = \{x^1_0\}$ and $\mathcal{C} = \hat{\mathbf{B}}_K(x^1_0)$. Go to Step 2.

Step 2. If there exists $x \in V$ and $u \in \mathbb{G}\setminus V$ such that $u \in \hat{\mathbf{B}}_{2K}(x)$, then choose $u^* \in \mathbb{G}\setminus V$ to be the minimal such vertex. Set $x^r_i = u^*, A^r_i = \mathbf{B}_K(x^r_i)$ and go to Step 3. Otherwise, go to Step 4.

Step 3. Update $V \to V \cup \{x^r_i\}$, $\mathcal{C} \to \mathcal{C} \cup \hat{\mathbf{B}}_K(x^r_i)$ and $i \to i+1$. Go to Step 2.

Step 4. If there is at least one vertex $y \in \partial^{\text{ext}}\mathcal{C}$ such that

$$y \xleftrightarrow{C_{\mathbb{G}}\setminus\mathcal{C}} \partial^{\text{ext}}\mathbf{B}_K(y) \setminus \mathcal{C},$$

then choose y^* to be the minimal such vertex, set $x^r_i = y^*, A^r_i = \mathbf{B}_K(x^r_i) \setminus \mathcal{C}$, and go to Step 3. Otherwise, go to Step 5.

Step 5. If $\mathbb{G} \subset V$, then terminate the construction. Otherwise, choose u^* to be the minimal vertex of $\mathbb{G} \setminus V$. Update $r \to r+1$ and set $x^r_0 = u^*$. Update $V \to V \cup \{x^r_0\}$ and $i = 1$. Go to Step 2.

Definition 4.3 *The above procedure produces* r *disjoint sets of vertices* $V^1 = \{x_0^1, x_1^1, \dots\}$, $V^2 = \{x_0^2, x_1^2, \dots\}, \dots, V^r = \{x_0^r, x_1^r, \dots\}$. *The vertices* x_i^j *constructed on* Step 4 *are equipped with sets* A_i^j, $j = 1 \dots r$. *Exit paths through such* A_i^j*-s contribute multiplicative factors* e^{-K} *each. Sets* A_i^j *for vertices* x_i^j *constructed on* Step 2 *play no role and are introduced for notational convenience only (see* (4.19) *below). By construction, there are at most* M *such vertices.*

The edges within each group $\ell = 1, \dots, r$ *are constructed as follows:* x_i^ℓ *is connected to the vertex of*

$$\left\{x_j^\ell \,:\, j < i \text{ and } x_i \in \hat{\mathbf{B}}_{2K}(x_j)\right\}$$

which has smallest index j.

This produces a graph which is a set of r *trees* $\mathcal{T}_K^1, \dots, \mathcal{T}_K^r$. *The union of the trees is called the* forest skeleton $\mathcal{F}_K = \cup_\ell \mathcal{T}_K^\ell$.

Note that we consider these graphs as compact subsets of $\mathbb{R}^2$. An example of forest squeleton is drawn on Figure 4.5.

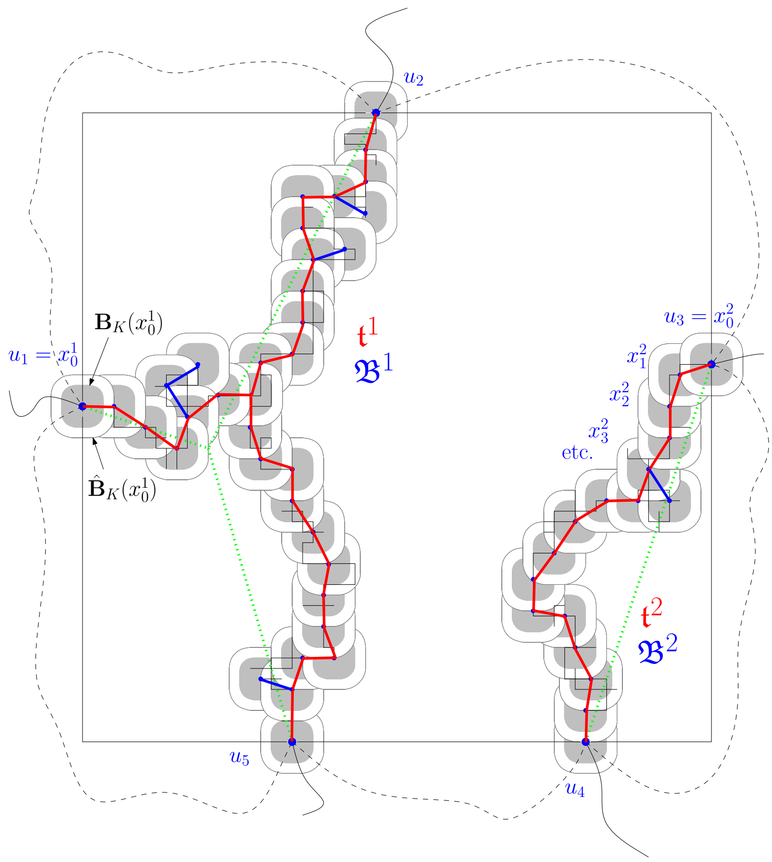


Figure 4.5 – Construction of the forest skeleton $\mathcal{F}_K = \{\mathcal{T}_K^1, \mathcal{T}_K^2\}$ of the cluster $C_{\mathbb{G}}$ (in black), consisting of the trees $\mathcal{T}_K^i = \{t^i, \mathfrak{B}^i\}$, $i = 1, 2$. The Steiner forest corresponding to the partition $\mathbb{G} = (\{u_1, u_2, u_5\}, \{u_3, u_4\})$ is drawn in dashed green.

The following result follows trivially from the construction of the forest skeleton.

Proposition 4.6 *Let $\mathcal{F}_K$ be the forest skeleton of $C_{\mathbb{G}}$, then*

1. *$\mathbb{G}$ is included in the vertices of $\mathcal{F}_K$.*
2. *Two vertices $u, v \in \mathbb{G}$ which were connected in $C_{\mathbb{G}}$ are also connected in $\mathcal{F}_K$.*
3. *$C_{\mathbb{G}} \subseteq \cup_{\ell,i} \hat{\mathbf{B}}_{2K}(x_i^\ell)$.*

4.4.3 Distance between $C_{\mathbb{G}}$ and Steiner forests

Proposition 4.7 *For every $\delta_3 > 0$, there exists $\kappa_3 = \kappa_3(M) > 0$ such that for n large enough,*

$$\mu_{\mathcal{D}}^f\Big(\min_{\mathcal{F}\in\Omega^{\min}_{M,m}} d_\tau\big(C_{\mathbb{G}}, \mathcal{F}\big) > \delta_3 n \;\Big|\; \Omega_{\underline{\mathbb{G}}}\Big) \leqslant e^{-\kappa_3 n},$$

uniformly in $(\mathcal{D}, \underline{\mathbb{G}})$ with $|\mathbb{G}| \leqslant M$.

Proof Let $\mathcal{F}_K$ be the forest skeleton of $C_{\mathbb{G}}$ at scale K (K will be chosen later). By the third item of Proposition 4.6,

$$d_\tau\left(C_{\mathbb{G}}, \mathcal{F}_K\right) \leqslant 2K + c\log 2K.$$

The proposition thus reduces to the following claim: for any $\delta_3 > 0$, there exist $K = K(M) > 0$ and $\kappa_3 = \kappa_3(M) > 0$ such that

$$\mu_{\mathcal{D}}^f\Big(\min_{\mathcal{F}\in\Omega^{\min}_{M,m}} d_\tau\big(\mathcal{F}_K, \mathcal{F}\big) > \delta_3 n \;\Big|\; \Omega_{\underline{\mathbb{G}}}\Big) \leqslant e^{-\kappa_3 n},$$

uniformly in $(\mathcal{D}, \underline{\mathbb{G}})$ with $|\mathbb{G}| \leqslant M$. We now prove this statement.

Writing $E \doteqdot \{\min_{\mathcal{F}\in\Omega^{\min}_{M,m}} d_\tau\big(\mathcal{F}_K, \mathcal{F}\big) > \delta_3 n\}$, we have

$$\mu_{\mathcal{D}}^f(E|\Omega_{\underline{\mathbb{G}}}) = \frac{\mu_{\mathcal{D}}^f(E\cap\Omega_{\underline{\mathbb{G}}})}{\mu_{\mathcal{D}}^f(\Omega_{\underline{\mathbb{G}}})} \leqslant \frac{\mu_{\mathcal{D}}^f(\tau(\mathcal{F}_K) \geqslant \tau_{\mathbb{G}} + \kappa_2 n)}{\mu_{\mathcal{D}}^f(\Omega_{\underline{\mathbb{G}}})} \tag{4.17}$$

where in the last inequality we used Property P4 of Proposition 4.5, applied with $\delta_2 = \delta_3$.

Let $\mathcal{F}$ be a Steiner forest in $\Omega^{\min}_{\underline{\mathbb{G}}}$ and $\mathcal{F}'$ be the forest obtained by replacing each inner node of $\mathcal{F}$ by the closest vertex of $\mathbb{Z}^2$. Now, by the FKG inequality, we can lower bound the denominator

$$\begin{aligned}\mu_{\mathcal{D}}^f(\Omega_{\underline{\mathbb{G}}}) &\geqslant \mu_{\mathcal{D}}^f\Big(\bigcap_{\{x,y\}\in\mathcal{E}(\mathcal{F}')} \{x \leftrightarrow y\}\Big) \geqslant \prod_{\{x,y\}\in\mathcal{E}(\mathcal{F}')} \mu_{\mathcal{D}}^f(x \leftrightarrow y) \\ &\geqslant \prod_{\{x,y\}\in\mathcal{E}(\mathcal{F}')} e^{-\tau(y-x)(1+o_{|y-x|}(1))} = e^{-\tau_{\mathbb{G}}(1+o_n(1))}.\end{aligned} \tag{4.18}$$

where $\lim_{k\to\infty} o_k(1) = 0$ by definition, and the product is taken over the set $\mathcal{E}(\mathcal{F}')$ of all the inner edges of the approximate Steiner forest $\mathcal{F}'$.

To obtain an upper bound on the numerator, we follow [21, Section 2]. Let $|V(\mathcal{F}_K)| = \sum_{\ell=1}^{r} |V^\ell|$ be the total number of vertices of the forest skeleton $\mathcal{F}$, then

$$\begin{aligned} e^{-2KM}\mu^f_{\mathcal{D}}(\mathcal{F}_K = \mathcal{F}) &\leqslant \mu^f_{\mathcal{D}}\Big(\bigcap_{\ell=1}^{r}\bigcap_{i=0}^{|V^\ell|} x_i^\ell \overset{A_i^\ell}{\leftrightarrow} \partial^{\text{ext}}\mathbf{B}_K(x_i^\ell)\Big) \leqslant \prod_{\ell=1}^{r}\prod_{i=1}^{|V^\ell|} \mu^w_{\hat{A}^i_k}\big(x_i^\ell \overset{A_i^\ell}{\leftrightarrow} \partial^{\text{ext}}\mathbf{B}_K(x_i^\ell)\big) \\ &\leqslant \big(e^{-K(1-o_K(1))}\big)^{\sum_{\ell=1}^r |V^\ell|} = e^{-K|V(\mathcal{F})|(1-o_K(1))} \\ &\leqslant e^{-\tau(\mathcal{F})(1-o_K(1)-o_n(1))}, \end{aligned} \tag{4.19}$$

where in the first inequality the term e^{-2MK} compensates (by the FKG inequality) the inclusion of events $x_i^\ell \overset{A_i^\ell}{\leftrightarrow} \partial^{\text{ext}}\mathbf{B}_K(x_i^\ell)$ for points $x_i^\ell \in \mathbb{G}$, whereas in the second inequality we expand the probability of the intersection as a product of conditional expectations and then use the FKG inequality to compare this conditional expectations with the probability with wired boundary conditions, and in the second line we use that $\mu^w_{\mathbf{B}_K(x)}(x \leftrightarrow \partial^{\text{ext}}\mathbf{B}_K(x)) = e^{-K(1-o_K(1))}$ (this follows from [21, Corollary 1.1], which is now known to be valid up to $p_c(q)$ thanks to Proposition 4.1). If we now upper bound crudely the number of forest K-skeletons rooted at $\mathbb{G}$ with $\tau(\mathcal{F}) = T$ (and so with less than C_1T/K vertices) by $(C_2K)^{C_3T/K}$, we get

$$\begin{aligned} \mu^f_{\mathcal{D}}(\tau(\mathcal{F}_K) \geqslant \tau_{\mathbb{G}} + \kappa_2 n) &= \sum_{\mathcal{F}:\tau(\mathcal{F})\geqslant\tau_{\mathbb{G}}+\kappa_2 n} \mu^f_{\mathcal{D}}(\mathcal{F}_K = \mathcal{F}) = \sum_{T\geqslant\tau_{\mathbb{G}}+\kappa_2 n}\ \sum_{\substack{\mathcal{F}:\\ \tau(\mathcal{F})=T}} \mu^f_{\mathcal{D}}(\mathcal{F}_K = \mathcal{F}) \\ &= \sum_{T\geqslant\tau_{\mathbb{G}}+\kappa_2 n} e^{(C_1T/K)\log(C_2K)-T(1-o_K(1)-o_n(1))} \\ &\leqslant C_4 \cdot e^{-(\tau_{\mathbb{G}}+\kappa_2 n)(1+o_K(1)+o_n(1))}, \end{aligned} \tag{4.20}$$

where we used (4.19) in the second line. The result follows by comparison with (4.18):

$$\mu^f_{\mathcal{D}}(E|\Omega_{\mathbb{G}}) \leqslant e^{-(\tau_{\mathbb{G}}+\kappa_2 n)(1-o_K(1)-o_n(1))+\tau_{\mathbb{G}}(1+o_n(1))} \leqslant e^{-n\kappa_3}.$$

Note that $\tau_{\mathbb{G}} o_n(1) = o(n)$ since $\tau_{\mathbb{G}} = O(n)$. The latter follows from the fact that $\tau_{\mathbb{G}}$ is bounded by the τ-length of the forest obtained by opening all the edges of ∂E_m (recall that τ is an equivalent norm on $\mathbb{R}^2$). ■

Proof [of Proposition 4.4]
Fix $\mathcal{D} = \mathcal{D}_{m,n}$ and $\mathbb{G} = \mathbb{G}_{m,n}$ with $m \geqslant \frac{n}{3}$ and $|\mathbb{G}| \leqslant M$. Let $\nu > 0$. Fix an arbitrary $0 < \delta \ll 1$ such that $\Lambda_{250\delta n} \subset \delta_1 n \mathbf{U}_\tau$, where δ_1 is given by P3. By definition of δ, we know that for any forest $\mathcal{F} \in \Omega^{\min}_{M,m}$, $\mathcal{F} \cap \Lambda_{250\delta n}$ is connected and contains at most one node. Therefore, we have three cases: either $\mathcal{F} \cap \Lambda_{2\delta n} = \varnothing$, or $\mathcal{F} \cap \Lambda_{2\delta n} \neq \varnothing$ but $\mathcal{F} \cap \Lambda_{20\delta n}$ contains only one edge, or $\mathcal{F} \cap \Lambda_{20\delta n}$ contains more than one edge. In the later case, the fact that edges incident to a node make an angle larger or equal to $\frac{\pi}{2} + \eta$ implies that $\mathcal{F} \cap \Lambda_{40\delta n}$ contains a node.

Also set $\delta_3 < \min\{\nu, \delta\}$. Proposition 4.7 implies that

$$\min_{\mathcal{F}\in\Omega^{\min}_{M,m}} d_\tau\big(C_{\mathbb{G}}, \mathcal{F}\big) \leqslant \delta n, \tag{4.21}$$

with probability larger than $1 - e^{-\kappa_3 n}$ for n large enough. We now assume that this inequality is indeed satisfied. Since, by P1 of Proposition 4.5 the set $\Omega^{\min}_{M,m}$ is compact, and since we are after an upper bound which vanishes with n, it will be enough to fix a Steiner forest $\mathcal{F} \in \Omega^{\min}_{M,m}$ and to assume that

$$d_\tau\big(C_{\mathbb{G}}, \mathcal{F}\big) \leqslant \delta n, \tag{4.22}$$

Let us treat the three previous cases separately.

C1. $\mathcal{F} \cap \Lambda_{2\delta n} = \varnothing$. In such case, (4.22) shows that $C_{\mathbb{G}} \cap \Lambda_{\delta n} = \varnothing$. Thus, $E^1_{\nu,\delta n}$ holds true and $\mathbb{G}_{\delta n, n} = \varnothing$.

C2. $\mathcal{F} \cap \Lambda_{20\delta n} = [u_1, u_2]$ with u_1 and u_2 on $\partial\Lambda_{20\delta n}$ and $[u_1, u_2] \cap \Lambda_{2\delta n} \neq \varnothing$. In such case, (4.22) shows that $C_{\mathbb{G}}$ intersects $\Lambda_{3\delta n}$ which in turns implies that $E^2_{\nu,k}$ holds for every $k \in [6\delta n, 18\delta n]$. Proposition 4.2 implies the existence of $k \in [6\delta n, 18\delta n]$ with $|\mathbb{G}_{k,n}| \leqslant M$ on an event of probability larger than $1 - e^{-18\delta n}$.

C3. There exists a node $x \in \Lambda_{40\delta n}$ and therefore $\mathcal{F} \cap \Lambda_{250\delta n} = [u_1, x] \cup [u_2, x] \cup [u_3, x]$ with u_1, u_2, u_3 on $\partial\Lambda_{250\delta n}$ such that $\measuredangle(u_i - x, u_j - x) > \frac{\pi}{2} + \eta$ for every $i \neq j$. In such case, (4.22) shows that $E^3_{\nu,k}$ holds for every $k \in [82\delta n, 246\delta n]$. Proposition 4.2 implies the existence of $k \in [82\delta n, 246\delta n]$ with $|\mathbb{G}_{k,n}| \leqslant M$ on an event of probability larger than $1 - e^{-246\delta n}$.

Altogether, we obtain the claim. ■

For later use, let us introduce the following definition:

Definition 4.4 *For u_1, u_2, u_3 in general position the function $\phi(y) \doteqdot \sum_{i=1}^{3} \tau(u_i - y)$ is strictly convex and quadratic around its minimum point; see [19, Lemma 3]. Let x be the unique minimizer of ϕ. In this way the notation $\mathcal{T}(u_1, u_2, u_3; x)$ is reserved for the minimal Steiner forest (in this case it is a tree) which contains u_1, u_2, u_3. It might happen, of course, that x coincides with one of the u_i-s. When, however, this is not the case, we shall refer to $\mathcal{T}(u_1, u_2, u_3; x)$ as a* Steiner tripod.

4.5 Step 4: Fluctuation theory

We are now in a position to prove Theorem 4.3. Let $\nu > 0$ small enough to be fixed later. By (4.13) and (4.14), we can assume that there exist $\delta = \delta(\nu) > 0$ and $k \geqslant \delta n$ such that $|\mathbb{G}_{k,n}| \leqslant M$ and $E^\ell_{\nu,k}$ holds true for some $\ell \in \{1, 2, 3\}$. Let

$$\mathcal{R}_n = \max\left\{k \geqslant \delta n : |\mathbb{G}_{k,n}| \leqslant M \text{ and } E^1_{\nu,k} \cup E^2_{\nu,k} \cup E^3_{\nu,k}\right\} \in [\delta n, n]\,.$$

Let $\mathcal{C}$ be a possible realization of $C_{k,n}$ and $\mathcal{D} = \mathcal{D}_{k,n}$ be the corresponding flower domain. We also set $\mathbb{G} = \mathbb{G}_{k,n}$. The restriction of $\mu^f_{\Lambda_n}(\cdot \mid \mathcal{R}_n = k; C_{k,n} = \mathcal{C})$ to $\mathcal{D}$ is $\mu^{\text{flower}}_{\mathcal{D}}$. Exactly as in Section 4.3.3,

$$\mathrm{Cond}_n[\sigma] \cap \{\mathcal{R}_n = k\} \cap \{C_{k,n} = \mathcal{C}\} = \Omega_{\sigma,\mathcal{C}} \times \{\mathcal{R}_n = k; C_{k,n} = \mathcal{C}\},$$

where $\Omega_{\sigma,\mathcal{C}} = \cup_{\underline{\mathbb{G}}\in\mathcal{P}'_{\mathbb{G}}}\Omega_{\underline{\mathbb{G}}}$ is defined as in Section 4.3.3. This reduction shows that it is sufficient to prove that

$$\mu^{f}_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \bigm| \Omega_{\sigma,\mathcal{C}}\big) = O(n^{\varepsilon-1/2}),$$

uniformly in the possible realizations of $\mathcal{D}$, $\mathcal{C}$ and $\mathbb{G}$.

From now on, we fix $k \geqslant \delta n$ such that $|\mathbb{G}_{k,n}| \leqslant M$ and $E^{\ell}_{\nu,k}$ holds true for some $\ell \in \{1,2,3\}$. We set $\mathcal{D} = \mathcal{D}_{k,n}$, $\mathcal{C} = \mathcal{C}_{k,n}$ and $\mathbb{G} = \mathbb{G}_{k,n}$.

Since each set $\mathbb{V}^{i}$ is already assumed to be connected outside of $\mathcal{D}$ (since $E^{1}_{\nu,k} \cup E^{2}_{\nu,k} \cup E^{3}_{\nu,k}$ occurs), partitions $\underline{\mathbb{G}} \in \mathcal{P}'_{\mathbb{G}}$ can be of four different types (recall that they are maximal in the sense defined in the previous section): singletons only, singletons together with one pair of elements in two different $\mathbb{V}^{i}$ (this cannot occur in $E^{1}_{\nu,k}$), singletons together with one triplet of elements in three different $\mathbb{V}^{i}$ (this can occur only in $E^{3}_{\nu,k}$), singletons together with two pairs (u,v) and (u',w), where u and u' belong to the same $\mathbb{V}^{i}$, and v and w belong to the other $\mathbb{V}^{j}$ (this can occur only in $E^{3}_{\nu,k}$). Let $\mathcal{P}^{*}_{\mathbb{G}}$ be the set of partitions in $\mathcal{P}'_{\mathbb{G}}$ of one of the first three types. If the configuration is in $\Omega_{\sigma,\mathcal{C}} \setminus \cup_{\underline{\mathbb{G}}\in\mathcal{P}^{*}_{\mathbb{G}}}\Omega_{\underline{\mathbb{G}}}$, there are two different clusters connecting two pairs of vertices (u,v) and (u',w) satisfying the conditions described above. By choosing $\nu > 0$ small enough, the assumption that $E^{3}_{\nu,k}$ holds implies that $\tau(u-v) + \tau(u'-w) \geqslant (1+\varepsilon)\tau_{\mathbb{G}}$ (where $\varepsilon = \varepsilon(\delta_3,\nu) > 0$) uniformly in the possible pairs (u,v) and (u',w). As in the proof of Proposition 4.7, one obtains after a small computation that

$$\mu^{f}_{\mathcal{D}}\big(\Omega_{\sigma,\mathcal{C}} \setminus \cup_{\underline{\mathbb{G}}\in\mathcal{P}^{*}_{\mathbb{G}}}\Omega_{\underline{\mathbb{G}}} \bigm| \Omega_{\sigma,\mathcal{C}}\big) = O(e^{-ck}),$$

for some constant $c > 0$. Hence, a reduction in the spirit of Proposition 4.3 shows that Theorem 4.3 would follow from the bound

$$\mu^{f}_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \bigm| \Omega_{\underline{\mathbb{G}}}\big) = O(n^{\varepsilon-1/2}), \tag{4.23}$$

where the right-hand side is uniform in the possible realizations of $\mathcal{D}$ and in the $\underline{\mathbb{G}} \in \mathcal{P}^{*}_{\mathbb{G}}$. We decompose the proof of (4.23) into three cases, depending on the type of $\underline{\mathbb{G}}$.

4.5.1 Scenario S1: No imposed crossing

This occurs in the following two cases (cf. Definition (4.2)): (i) $E^{1}_{\nu,k}$ occurs; (ii) $E^{2}_{\nu,k} \cup E^{3}_{\nu,k}$ occurs and the partition $\underline{\mathbb{G}}$ is composed of singletons only. In this case, the measure $\mu^{f}_{\mathcal{D}_{k,n}}(\,\cdot\,|\Omega_{\underline{\mathbb{G}}})$ is unconditioned (i.e. $\Omega_{\underline{\mathbb{G}}} = \Omega$). Proposition 4.1 then implies that $\mu^{f}_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^{\varepsilon}} \neq \varnothing \bigm| \Omega_{\underline{\mathbb{G}}}\big)$ decays exponentially with n.

4.5.2 Scenario S2: One imposed crossing

This occurs when $E^{2}_{\nu,k} \cup E^{3}_{\nu,k}$ occurs and $\underline{\mathbb{G}}$ is composed of singletons together with a unique pair (u,v), where $u \in \mathbb{V}^{i}, v \in \mathbb{V}^{j}$ with $i \neq j$. In other words, $\Omega_{\underline{\mathbb{G}}} = \{u \leftrightarrow v\}$. In this case, the cluster $\mathcal{C}_{\mathbb{G}} \subseteq \mathcal{D}$ may contain several connected components, but, up to exponentially small (in k) $\mu^{f}_{\mathcal{D}}(\cdot|u \leftrightarrow v)$-conditional probabilities, only one of them, namely the connected cluster $C(u,v)$ of $\{u,v\}$ is capable of reaching Λ_{n^ε}. However, the law of the cluster connecting u and v converges to the law of a Brownian bridge. In fact, one obtains the following stronger result:

$$\mu^f_{\mathcal{D}}(x \in C(u,v)|u \leftrightarrow v) \leqslant \frac{C}{\sqrt{|u-v|}} \exp\Big(-\kappa \frac{d_\tau(x,[u,v])^2}{|u-v|}\Big), \tag{4.24}$$

where κ and C are constants depending on p only, and $[u,v]$ denotes the segment between u and v. In the case of Ising interfaces, such bound was obtained in [48, (3.31)]. The proof relies on the positive curvature of the surface tension and on the effective random walk with exponentially decaying step distribution representation of the interface. The theory developed in [21] enables a literal adaptation to the case of sub-critical FK-clusters, see Theorems C and E and Subsections 4.4 and 4.5 in [21]. Consequently,

$$\mu^f_{\mathcal{D}}\big(C_{\mathbb{G}} \cap \Lambda_{n^\varepsilon} \neq \varnothing \bigm| \Omega_{\underline{\mathbb{G}}}\big) = O(n^{2\varepsilon-1/2}). \tag{4.25}$$

4.5.3 Scenario S3: One tripod

This can only happen when $E^3_{\nu,k}$ occurs and $\underline{\mathbb{G}}$ is composed of singletons together with one triplet (u_1,u_2,u_3) with $u_1 \in \mathbb{V}^1, u_2 \in \mathbb{V}^2, u_3 \in \mathbb{V}^3$. Thus, in this case $\Omega_{\underline{\mathbb{G}}} = \{C(u_1,u_2,u_3) \neq \varnothing\}$, where $C(u_1,u_2,u_3)$ is the joint connected cluster of $\{u_1,u_2,u_3\}$. Again, $C = C_{\mathbb{G}} \subseteq \mathcal{D}$ may contain several connected components, but, up to exponentially small (in k) $\mu^f_{\mathcal{D}}(\cdot|C(u_1,u_2,u_3) \neq \varnothing)$-conditional probabilities, only one of them, namely $C(u_1,u_2,u_3)$ itself, is capable of reaching Λ_{n^ε}. By definition, there exists a unique $x = x(u_1,u_2,u_3) \in \Lambda_{k/2}$ (see Definition 4.4) such that $\mathcal{T}_x = \{u_1,u_2,u_3;x\}$ is a Steiner tripod. To lighten the notation, we set

$$E(u_1,u_2,u_3,x) = \{u_1,u_2,u_3 \text{ are connected and } d_\tau(C_{\mathbb{G}},\mathcal{T}_x) \leqslant \nu k\}$$

and redefine $C = C(u_1,u_2,u_3)$. Thanks to Propositions 4.4 and 4.7, we now aim at proving the bound

$$\mu^f_{\mathcal{D}}\big(C \cap \Lambda_{n^\varepsilon} \neq \varnothing \bigm| E(u_1,u_2,u_3,x)\big) = O(n^{\varepsilon-1/2}). \tag{4.26}$$

This bound will imply Theorem 4.3.

Proving (4.26) is more complicated than proving (4.25). Nevertheless, the idea remains the same: The tripod has Gaussian fluctuations, therefore it intersects a small box with probability going to 0. In the case of percolation, fluctuations of tripods on the level of local limit results were studied in [19]. We are not after a full local limit picture here, and merely explain how techniques from [21] allow to derive (4.26). Let us write $\Lambda_r(x)$ for $x + \Lambda_r$.

Definition 4.5 *(Cones $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$)*
Since, by Property P2 of the Steiner forests, for every $i \neq j$,

$$\measuredangle(u_i - x, u_j - x) \geqslant \frac{\pi}{2} + \eta,$$

there exist disjoint cones $\mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{Y}_3$ such that each $\mathcal{Y}_i$ contains exactly one lattice direction in its interior (i.e., *one of the four vectors* $(1,0)$, $(0,1)$, $(-1,0)$ *and* $(0,-1)$, *denoted by* f_i), *and there exists* $\varepsilon_1 > 0$ *such that* $u_i \in \mathrm{int}\,(y + \mathcal{Y}_i)$ *for every* $y \in \Lambda_{\varepsilon_1 k}(x)$ *and every* $i \in \{1,2,3\}$, *and* $u_i \in \mathrm{int}\,(u_j - \mathcal{Y}_i)$ *for every* $i \neq j$.

Definition 4.6 *(Event* $S(t,y)$*)*
Given $y \in \Lambda_{\varepsilon_1 k}(x)$ *and* $t \in \mathbb{N}$, *let* $\mathcal{S}(t,y)$ *be the event that the following three conditions occur:*

R1. u_1, u_2 *and* u_3 *are pairwise disconnected in* $C \setminus \Lambda_t(y)$,

R2. C *intersects* $\partial\Lambda_t(y)$ *in exactly three vertices.*

For $i = 1,2,3$, *let* $C_i(t,y)$ *be the connected component of* $C \setminus \Lambda_t(y)$ *containing* u_i, *and* $v_i(t,y) = C_i(t,y) \cap \partial\Lambda_t(y)$. *Define* $C_0(t,y) = C \setminus \big(C_1(t,y) \cup C_2(t,y) \cup C_3(t,y)\big)$. *We will drop the reference to* t *and* y *when no confusion is possible.*

R3. C_0 *is contained in* $\bigcap_{i=1}^{3} (v_i - \mathcal{Y}_i)$ *and* $C_i \subset (v_i + \mathcal{Y}_i)$ *for* $i \in \{1,2,3\}$.

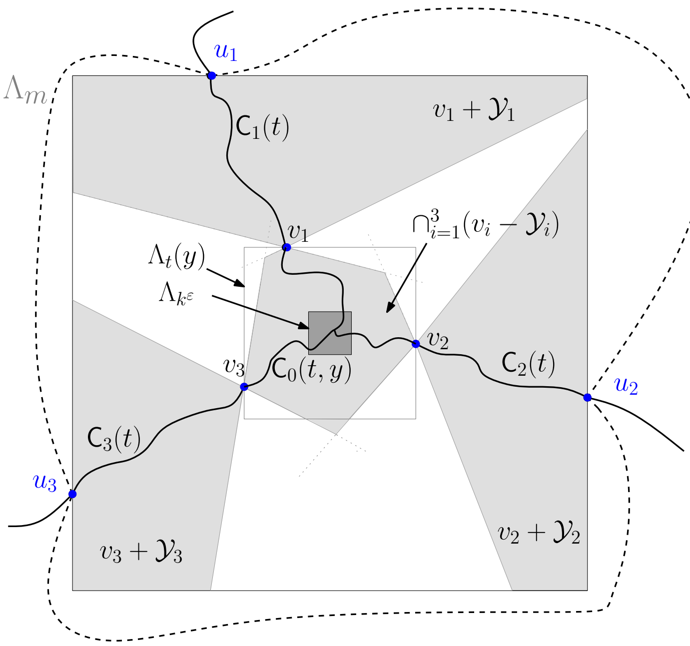


Figure 4.6 – Description of the event $S(t,y)$, namely the cones and the decomposition of the cluster C into $C_i(t)$ and $v_i(t)$, $i = 1,2,3$.

Lemma 4.2 *Fix* $\varepsilon_1 > 0$ *and let* $\varepsilon > 0$ *be sufficiently small. There exists* $C > 0$ *such that*

$$\mu^f_{\mathcal{D}}\Big(\bigcup_{t \leqslant Ck^\varepsilon} \bigcup_{y \in \Lambda_{\varepsilon_1 k}(x)} \mathcal{S}(t,y) \,\Big|\, E(u_1,u_2,u_3,x) \Big) \geqslant 1 - O(e^{-k^\varepsilon}). \tag{4.27}$$

Proof [of Lemma 4.2] First of all, we notice that coarse-graining on the k^ε-scale enables a reduction to particularly simple geometric structures. Consider a forest skeleton of the cluster C at scale k^ε. Note that, conditionally on $E(u_1, u_2, u_3, x)$, this forest is in fact a tree $\mathcal{T}_{k^\varepsilon}$.

We define the *trunk* t_ε of $\mathcal{T}_{k^\varepsilon}$ as the minimal subtree of $\mathcal{T}_{k^\varepsilon}$ which spans $\{u_1, u_2, u_3\}$.

We define the *branches* of $\mathcal{T}_{k^\varepsilon}$ as $\mathfrak{B}_\varepsilon = \mathcal{T}_{k^\varepsilon} \backslash t_\varepsilon$. In this case, we obtain the following reduced geometry of typical $\mathcal{T}_{k^\varepsilon}$, which holds uniformly in all situations in question, up to probabilities which are exponentially small in k^ε:

T1. $\mathcal{T}_{k^\varepsilon}$ does not have branches. This means that the tree $\mathcal{T}_{k^\varepsilon}$ consists only of a trunk which is a tripod, i.e. with one vertex of degree 3 and all other vertices of degree at most 2. We will write x_ε for the only triple point of $\mathcal{T}_{k^\varepsilon}$, and $\mathcal{T}^i_{k^\varepsilon} = \{u^{n_i}_{i,\varepsilon}, \ldots, u^1_{i,\varepsilon} = x_\varepsilon\}$, $i = 1, 2, 3$, for the three legs of $\mathcal{T}_{k^\varepsilon}$. Note that $u_i \in \hat{\mathbf{B}}_{2k^\varepsilon}(u^{n_i}_{i,\varepsilon})$.

T2. Fix $\kappa > 0$ small. For every $\varepsilon > 0$ and each $\varepsilon' \in (0, \varepsilon/2)$, the skeletons $\mathcal{T}^i_{k^{\varepsilon'}} \setminus \Lambda_{k^\varepsilon}(x_{\varepsilon'}) \subseteq x_{\varepsilon'} + \mathcal{Y}_{i,2\kappa}$ as soon as k becomes sufficiently large, where cones $\mathcal{Y}_{i,2\kappa}$ are defined via
$$\mathcal{Y}_{i,r} = \left\{ z : \measuredangle(z, u_i - x_{\varepsilon'}) \leqslant r \right\}. \tag{4.28}$$
That is, the vertices of each of the three branches of $\mathcal{T}_{k^{\varepsilon'}}$ outside the box Λ_{k^ε} are confined to the respective cones $x_{\varepsilon'} + \mathcal{Y}_{i,2\kappa}$.

Before proving Properties T1 and T2, let us describe how they can be used to prove the lemma. First of all, note that, by Proposition 4.7, we may assume that $|x_{\varepsilon'} - x| \leqslant \delta_1 k$ with $\delta_1 > 0$ fixed as small as we wish. In particular, we may assume that $u_i \in \mathrm{int}(x_{\varepsilon'} + \mathcal{Y}_i)$ (see Definition 4.5) and, consequently, that $\mathcal{Y}_{i,2\kappa} \subset \mathcal{Y}_i$.
By Proposition 4.6, the connected cluster C is included in $\mathcal{T}_{k^{\varepsilon'}} + 2k^{\varepsilon'}\mathbf{U}_\tau$. Therefore, Properties T1 and T2 imply that $C \setminus \Lambda_{k^\varepsilon}(x_{\varepsilon'}) = \tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3$, where $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are the clusters (in $C \setminus \Lambda_{k^\varepsilon}(x_{\varepsilon'})$) of u_1, u_2 and u_3 respectively. Note that, by T2, clusters C_i are confined to the sets (actually truncated cones) $(x_{\varepsilon'} + \mathcal{Y}_{i,2\kappa} + 2k^{\varepsilon'}\mathbf{U}_\tau) \setminus \Lambda_{k^\varepsilon}(x_{\varepsilon'})$, which are well separated on the k^ε-scale. Consequently coarse-graining estimates developed in [21, Section 2] apply to each of C_i separately. As a result, the claim of Lemma 4.2 follows by a straightforward adaptation of the mass-gap arguments of [21, Section 2] applied separately to each of the three disjoint clusters $\tilde{C}_1, \tilde{C}_2$ and $\tilde{C}_3$. For instance, one can show the following: Fix r large enough so that $\Lambda_{k^{\varepsilon'}}(x_{\varepsilon'}) \subset v - \mathcal{Y}_i$ for any $v \in (x_{\varepsilon'} + \mathcal{Y}_{i,2\kappa}) \cap (\Lambda_{2rk^\varepsilon}(x_{\varepsilon'}) \setminus \Lambda_{rk^\varepsilon}(x_{\varepsilon'}))$ and $i = 1, 2, 3$. Then, up to probabilities which are exponentially small in k^ε, there exists $t \in [rk^\varepsilon, 2rk^\varepsilon]$ such that each of the clusters $\tilde{C}_i$ contains a $\mathcal{Y}_i$-break point on $\partial\Lambda_t(x_{\varepsilon'})$. That is,

- for $i = 1, 2, 3$, the intersection $v_i = \tilde{C}_i \cap \partial\Lambda_t(x_\varepsilon)$ is a singleton;
- for $i = 1, 2, 3$ the cluster $\tilde{C}_i \subset (v_i + \mathcal{Y}_i) \cup (v_i - \mathcal{Y}_i)$.

This ensures $\mathcal{S}(t, y)$ for some $y \in \Lambda_{\varepsilon_1 k}$ and (4.27) follows. ■

Proof [of Property T1.] We follow the idea of [21, Lemma 2.1 and 2.2], which consists of two steps. First we show that, up to probability of order e^{-ck}, the trunk of $\mathcal{T}_{k^\varepsilon}$ is ballistic, namely $N(t) \leqslant (1+\delta)k^{1-\varepsilon}$ for some $\delta > 0$, where $N(t)$ denotes the number of vertices of t. Second we show that, up to probability of order $e^{-ck^{\varepsilon}}$, a typical trunk t has no branches. For the first step,

$$\begin{aligned}
\mu^f_{\mathcal{D}}\big(N(t) > (1+\delta)k^{1-\varepsilon} \bigm| E(u_1,u_2,u_3,x)\big) &= \sum_{m>(1+\delta)k^{1-\varepsilon}} \sum_{t\,:\,N(t)=m} \frac{\mu^f_{\mathcal{D}}(t \cap E(u_1,u_2,u_3,x))}{\mu^f_{\mathcal{D}}(E(u_1,u_2,u_3,x))} \\
&\leqslant \sum_{m>(1+\delta)k^{1-\varepsilon}} (C_1 k^{\varepsilon})^m \exp\big[(-k^{\varepsilon} m + \tau_{\mathbb{G}})(1+o_k(1))\big] \\
&\leqslant C_2 \exp(-k(C_3(1+\delta) - C_4)) < e^{-C_5 k} \quad \text{for some } \delta > 0 \text{ sufficiently large.}
\end{aligned}$$

We used the rough upper bound $(C_1 k^{\varepsilon})^m$ on the number of t for which $N(t) = m$, and the fact that $\tau_{\mathbb{G}} = O(k)$. For the second step, let us denote by $N(\mathfrak{B})$ the number of edges of $\mathfrak{B}$. Neglecting branches with more than one edge (which have an additional cost of order $e^{-ck^{\varepsilon}}$) each branch has $N(t)$ possibilities to be attached to the trunk. This implies

$$\begin{aligned}
&\mu^f_{\mathcal{D}}\big(N(\mathfrak{B}) > 1 \bigm| E(u_1,u_2,u_3,x)\big) \\
&\quad \leqslant \mu^f_{\mathcal{D}}\big(N(\mathfrak{B}) > 1 \bigm| N(t) \leqslant (1+\delta)k^{1-\varepsilon}\,,\ E(u_1,u_2,u_3,x)\big) + O(e^{-k}) \\
&\qquad \leqslant \big(1 + O(e^{-k^{\varepsilon}})\big) \sum_{m=1}^{(1+\delta)k^{1-\varepsilon}} \binom{(1+\delta)k^{1-\varepsilon}}{m} e^{-C_1 k^{\varepsilon} m} \leqslant e^{-C_6 k^{\varepsilon}}.
\end{aligned}$$

■

Proof [of Property T2.] Let us start with a lower bound on $\mu^f_{\mathcal{D}}(E(u_1,u_2,u_3,x))$ which will be used later as a test threshold quantity for ruling out improbable events. Let y be a lattice approximation of x. By the FKG inequality,

$$\mu^f_{\mathcal{D}}(E(u_1,u_2,u_3,x)) \geqslant \mu^f_{\mathcal{D}}\Big(\bigcap_{i=1}^{3} \{y \overset{\mathcal{D}}{\leftrightarrow} u_i\}\Big) \geqslant \prod_{i=1}^{3} \mu^f_{\mathcal{D}}(y \overset{\mathcal{D}}{\leftrightarrow} u_i).$$

Theorem A in [21] gives sharp asymptotics of quantities $\mu^f(y \leftrightarrow u_i)$. These sharp asymptotics are built upon an effective random walk representation of events $\{y \leftrightarrow u\}$ as described in Subsection 4.1 of the paper. Steps of this random walk have effective drift from u_i towards y, and, since $\Lambda_k \subset \mathcal{D}$, it is easy to adjust the arguments therein in order to show that

$$\mu^f_{\mathcal{D}}(y \overset{\mathcal{D}}{\leftrightarrow} u) \geqslant \frac{C_0}{\sqrt{k}} e^{-\tau(u-y)},$$

uniformly in $y \in \Lambda_{\frac{k}{2}}$ and $u \in \partial\Lambda_k$, where C_0 (and, similarly, C_1, C_2, ... below) is a universal constant, in the sense that (4.29) applies uniformly in all the situations in question as soon as k is sufficiently large. Consequently,

$$\mu^f_{\mathcal{D}}(E(u_1,u_2,u_3,x)) \geqslant \exp\Big(-\sum_{i=1}^{3} \tau(u_i - x) - C_1 \log k\Big), \tag{4.29}$$

also uniformly in all the situations in question as soon as k is sufficiently large.

Next, let us say that $w \in \mathcal{T}^i_{k^{\varepsilon'}}$ is a 2κ-cone point of $\mathcal{T}^i_{k^{\varepsilon'}}$ if $\mathcal{T}^i_{k^{\varepsilon'}} \subset (w - \mathcal{Y}_{i,2\kappa}) \cup (w + \mathcal{Y}_{i,2\kappa})$. In our notation,

$$\tau(\mathcal{T}_{k^{\varepsilon'}}) = \sum_{i=1}^{3} \tau(\mathcal{T}^i_{k^{\varepsilon'}})$$

Since τ is a strictly convex norm ([21, Subsection 1.3.2]) ,

$$\tau(\mathcal{T}^i_{k^{\varepsilon'}}) \geqslant \tau(u_i - x_{\varepsilon'}) + C_2 k, \tag{4.30}$$

whenever $\mathcal{T}^i_{k^{\varepsilon'}}$ does not contain 2κ-cone points at all. This is essentially Lemma 2.4 of [21]. In view of (4.19), and in view of the lower bound (4.29), we are entitled to ignore the situation when any of the $\mathcal{T}^i_{k^{\varepsilon'}}$ does not have 2κ-cone points at all.

In the sequel, we use w_i^* to denote the first 2κ-cone point of $\mathcal{T}^i_{k^{\varepsilon'}}$ (starting at $x_{\varepsilon'}$) and N_i to denote its serial number; that is, $w_i^* = u^{N_i}_{i,\varepsilon'}$. Define $\mathcal{T}^{i,*}_{k^{\varepsilon'}} = \{u^1_{i,\varepsilon'}, \dots, u^{N_i}_{i,\varepsilon'} = w_i^*\}$ as the portion of $\mathcal{T}^i_{k^{\varepsilon'}}$ up to w_i^*. Given y and $\underline{w} = (w_1, w_2, w_3)$, define the percolation event $E_{\varepsilon'}(y, \underline{w}) \subset E(u_1, u_2, u_3, x)$ as

$$E_{\varepsilon'}(y, \underline{w}) = \big\{x_{\varepsilon'} = y\,;\ w_i^* = w_i \text{ for } i = 1, 2, 3\big\}.$$

In view of (4.29), Property T2 will follow as soon as we shall have checked that

$$\mu^f_{\mathcal{D}}\left(E_{\varepsilon'}(y, \underline{w})\right) \leqslant e^{-\sum_i \tau(u_i - y) - C_3 k^{\varepsilon}}, \tag{4.31}$$

uniformly in k, tripods $\mathcal{T}_x$, y and $\underline{w} \not\subset \Lambda_{k^\varepsilon}(y)$. For fixed realizations $\mathcal{T}^{i,*}$ of $\mathcal{T}^{i,*}_{k^{\varepsilon'}}$ we have

$$\begin{aligned}\mu^f_{\mathcal{D}}&\big(E_{\varepsilon'}(y, \underline{w}); \mathcal{T}^{i,*}_{k^{\varepsilon'}} = \mathcal{T}^{i,*} \text{ for } i = 1, 2, 3\big)\\ &\leqslant \exp\Big\{-\sum_{i=1}^{3}\big\{\tau(u_i - w_i) + \tau(\mathcal{T}^{i,*})(1 - o_{k^{\varepsilon'}}(1))\big\} + C_4 k^{2\varepsilon'}\Big\}.\end{aligned} \tag{4.32}$$

This follows from (4.19) and from the finite energy property (applied for configurations on $\Lambda_{C_5 k^{\varepsilon'}}(w_i)$) of the FK measures. Indeed, the finite energy property and the exponential ratio mixing property (4.6) enable to decouple between the event $\bigcap_i \big\{\mathcal{T}^{i,*}_{k^{\varepsilon'}} = \mathcal{T}^{i,*}\big\}$ and the events $\bigcap_i \big\{w_i \overset{w_i + \mathcal{Y}_{i,2\kappa}}{\longleftrightarrow} u_i\big\}$.

Assume, for instance, that $w_1 \notin \Lambda_{k^\varepsilon}(y)$. There are two cases to consider:

Case 1: $w_1 \in y + \mathcal{Y}_{1,\kappa}$. Then, as by definition $\mathcal{T}^{1,*}$ does not contain any 2κ cone points, exactly as in (4.30) we get $\tau(\mathcal{T}^{1,*}) \geqslant \tau(w_1 - y) + C_6\,|w_1 - y|$. As in (4.20) we can upper bound the entropic factor related to the number of possible compatible realizations $\mathcal{T}^{1,*}$ by

$$(k^{\varepsilon'})^{N_1} \leqslant \exp\left(C_7 |w_1 - y| \frac{\log k}{k^{\varepsilon'}}\right). \tag{4.33}$$

The entropic factor (4.33) as well as the factor $C_4 k^{2\varepsilon'}$ of (4.32) are suppressed if we choose $\varepsilon > 2\varepsilon'$, and (4.31) follows.

Case 2: $w_1 \in (y+\mathcal{Y}_{1,2\kappa})\setminus(y+\mathcal{Y}_{1,\kappa})$. By construction, $\tau(\mathcal{T}^{1,*}) \geqslant \tau(w_1-y)$. However, by the sharp triangle inequality (4.2),

$$\tau(w_1-y)+\tau(u_1-w_1)-\tau(u_1-y) \geqslant C_8\,|w_1-y|\,,$$

uniformly in w_1 under consideration. Again, the entropic factor (4.33) as well as the factor $C_4 k^{2\varepsilon'}$ of (4.32) are suppressed if we choose $\varepsilon > 2\varepsilon'$, and (4.31) follows. ■

Lemma 4.3 *Let* $S_\varepsilon(y) = \bigcup_{t\leqslant Ck^\varepsilon} S(t,y)$. *There exist two universal constants* $\kappa > 0$ *and* $C < \infty$ *such that*

$$\mu^f_{\mathcal{D}}\big(S_\varepsilon(y)\;\big|\; E(u_1,u_2,u_3,x)\big) = O\Big(k^{12\varepsilon-1}\exp\big(-\kappa\frac{|y-x|^2}{k}\big)\Big), \tag{4.34}$$

uniformly in $y \in \Lambda_{\varepsilon_1 k}(x)$.

Proof Decompose

$$\mathcal{S}(t,y) = \bigcup_{W} \mathcal{S}^W(t,y)$$

according to the triple $W = \{v_1-y, v_2-y, v_3-y\} \subset \partial\Lambda_t$ which shows up in the definition. From now on, we set $w_1 = v_1 - y$, $w_2 = v_2 - y$ and $w_3 = v_3 - y$.

Since, under the event $S^W(t,y)$, we have that $C_0 \subseteq \bigcap_{i=1}^3 (v_i - \mathcal{Y}_i)$ and that the points u_i lie deep in the interior of the corresponding cones $v_i + \mathcal{Y}_i$, with $v_i \in \partial\Lambda_t(y)$ and $t \leqslant Ck^\varepsilon$, the Ornstein-Zernike asymptotics of [21, Theorem A] imply that

$$\mu^f_{\mathcal{D}}\Big(\bigcap_{i=1}^3\{C_i \subset v_i+\mathcal{Y}_i\}\;\Big|\; C_0(t,y)\Big) = \Theta\big(k^{-3/2}\,e^{-\sum_{i=1}^3 \tau(u_i-v_i)}\big), \tag{4.35}$$

uniformly in any possible realization C_0 of $C_0(t,y)$ compatible with $S^W(t,y)$. Note that if $C_0(t,y)$ is compatible with $S^W(t,y)$, then shifts $C_0^u \stackrel{\Delta}{=} C_0 + u$ are compatible with shifted events $S^W(t, y+u)$.

Recall Definition 4.4 of $\phi(y)$. Given a triple $W = \{w_1, w_2, w_3\} \subset \Lambda_{Ck^\varepsilon}$, let us define

$$\phi_W(y) = \sum_{i=1}^3 \tau(u_i - w_i - y).$$

Together with (4.35), we obtain

$$\begin{aligned}\frac{\mu^f_{\mathcal{D}}\big(\mathcal{S}^W(t,y)\big)}{\mu^f_{\mathcal{D}}\big(\mathcal{S}^W(t,z)\big)} &= \frac{\sum_{C_0}\mu^f_{\mathcal{D}}\Big(\bigcap_{i=1}^3\{C_i\subset y+w_i+\mathcal{Y}_i\}\;\Big|\;C_0\Big)\mu^f_{\mathcal{D}}(C_0(t,y)=C_0)}{\sum_{C_0}\mu^f_{\mathcal{D}}\Big(\bigcap_{i=1}^3\{C_i\subset z+w_i+\mathcal{Y}_i\}\;\Big|\;C_0\Big)\mu^f_{\mathcal{D}}(C_0(t,z)=C_0^{z-y})} \\ &= \Theta\big(e^{\phi_W(y)-\phi_W(z)}\big),\end{aligned} \tag{4.36}$$

uniformly in $t \leqslant Ck^{\varepsilon}$, $W = \{w_1, w_2, w_3\} \subset \partial\Lambda_t$ and $y, z \in \Lambda_{\varepsilon_1 k}$, where the sum is over C_0 compatible with $S^W(t, y)$ and where in the last line we used classical ratio-mixing properties of subcritical random-cluster measures [6, Theorem 3.4] and (4.6) to compare $\mu^f_{\mathcal{D}}(C_0(t,y) = C_0)$ and $\mu^f_{\mathcal{D}}(C_0(t,z) = C_0^{z-y})$.

The function ϕ_W has a non-degenerate quadratic minimum at some $x_{\min}(W)$ (see [19, Lemma 3]). In view of the homogeneity of τ, a quadratic expansion around $x_{\min}$ yields

$$\phi_W(y) - \phi_W(x_{\min}) = \Theta\Big(\frac{|y - x_{\min}|^2}{k}\Big), \tag{4.37}$$

uniformly in all situations in question. Since $|\phi(y) - \phi_W(y)| = O(k^{\varepsilon})$, its minimizers $x_{\min}(W)$ solve

$$F(x, W) \triangleq \nabla_x \phi_W(x) = 0.$$

Since $\mathrm{Hess}(\phi)$ is non-degenerate at x, the implicit function theorem applies. As a result, $|x_{\min}(W) - x| = O(\sum_i |w_i|) = O(k^{\varepsilon})$ uniformly in all W in question. This fact, together with (4.37), yields that

$$\phi_W(y) - \phi_W(x) = \Theta\Big(\frac{|y - x|^2}{k} + k^{2\varepsilon - 1}\Big). \tag{4.38}$$

Since there are at most $O(k^{3\varepsilon})$ possible choices for W and $O(k^{\varepsilon})$ possible choices for t, we deduce from (4.36) and (4.38) that

$$\frac{1}{O(k^{4\varepsilon})} \exp\big(-C_1 \frac{|y - x|^2}{k}\big) \leqslant \frac{\mu^f_{\mathcal{D}}(\mathcal{S}_{\varepsilon}(y))}{\mu^f_{\mathcal{D}}(\mathcal{S}_{\varepsilon}(x))} \leqslant O(k^{4\varepsilon}) \exp\big(-C_2 \frac{|y - x|^2}{k}\big). \tag{4.39}$$

Above $\mathcal{S}_{\varepsilon}(x)$ means in fact $\mathcal{S}_{\varepsilon}(\lfloor x \rfloor)$. We can now compute

$$\begin{aligned} \mu^f_{\mathcal{D}}(S_{\varepsilon}(y) \,|\, E(u_1, u_2, u_3, x)) &= \frac{\mu^f_{\mathcal{D}}(S_{\varepsilon}(y))}{\mu^f_{\mathcal{D}}(S_{\varepsilon}(x))} \cdot \frac{\mu^f_{\mathcal{D}}(S_{\varepsilon}(x))}{\mu^f_{\mathcal{D}}(E(u_1, u_2, u_3, x))} \\ &\leqslant O(k^{4\varepsilon}) \exp\Big(-C_2 \frac{|y - x|^2}{k}\Big) \frac{\mu^f_{\mathcal{D}}(S_{\varepsilon}(x))}{\mu^f_{\mathcal{D}}(E(u_1, u_2, u_3, x))} \end{aligned} \tag{4.40}$$

where we used the second inequality in (4.39). In order to see that the rightmost term in (4.40) is of the right order, observe that $|y - x| \leqslant k^{1/2-\varepsilon}$ implies that $e^{-C_2|y-x|^2/k}$ is of order 1, and therefore the ratio in (4.39) is smaller than $O(k^{4\varepsilon})$. Therefore, by looking at the $k^{1-2\varepsilon}$ sites which are at distance at most $k^{1/2-\varepsilon}$ from x, we deduce, using the first inequality in (4.39), that

$$\mu^f_{\mathcal{D}}(S_{\varepsilon}(x)) \leqslant O(k^{-1+6\varepsilon}) \sum_{y \in \Lambda_{k^{1/2-\varepsilon}}(x)} \mu^f_{\mathcal{D}}(S_{\varepsilon}(y)) \leqslant O(k^{-1+8\varepsilon}) \mu^f_{\mathcal{D}}(E(u_1, u_2, u_3, x)),$$

where in the second inequality, we used the fact that in a given configuration there are at most $O(k^{2\varepsilon})$ sites y such that the corresponding events $\mathcal{S}_{\varepsilon}(y)$ occur. This implies that

$$\frac{\mu^f_{\mathcal{D}}\big(\mathcal{S}_{\varepsilon}(x)\big)}{\mu^f_{\mathcal{D}}\big(E(u_1, u_2, u_3, x)\big)} \leqslant O(k^{8\varepsilon - 1}).$$

Together with (4.40), we obtain (4.34). ■

Lemmas 4.2 and 4.3 imply that

$$\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, E(u_1,u_2,u_3,x)\big)\leqslant \\ O(k^{12\varepsilon-1})\sum_{y\in\Lambda_{\varepsilon_1 k}(x)} e^{-\kappa|y-x|^2/k}\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, \mathcal{S}_\varepsilon(y)\big)+O\big(e^{-k^\varepsilon}\big). \quad (4.41)$$

It remains to provide an upper bound on $\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, \mathcal{S}_\varepsilon(y)\big)$. There are two cases to consider:

Case 1: $y\in\Lambda_{2Ck^\varepsilon}$. In this case, we simply use

$$\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, \mathcal{S}_\varepsilon(y)\big)\leqslant 1. \quad (4.42)$$

The total contribution to the right-hand side of (4.41) is then bounded by $O(k^{14\varepsilon-1})$, which is negligible with respect to our target estimate (4.26).

Case 2: $y\notin\Lambda_{2Ck^\varepsilon}$. In this case, Λ_{k^ε} can intersect at most one of the $\Lambda_{Ck^\varepsilon}(y)+\mathcal{Y}_i$, and therefore can be hit by only one cluster C_i. Without loss of generality, let us assume that $C_i=C_1$. Conditioning on the smallest t such that $\mathcal{S}(t,y)$ occurs as well as on C_0, C_2 and C_3, the cluster C_1 obeys, as was explained after (4.24), a diffusive scaling. In particular,

$$\mu^f_{\mathcal{D}}\big(z\in C_1 \,\big|\, S(t,y),C_0,C_2,C_3\big)=O\Big(\frac{1}{\sqrt{|v_1-z|}}\exp\Big[-\kappa'\frac{d_\tau(z,[v_1,u_1])^2}{|v_1-z|}\Big]\Big).$$

In the previous inequality, $v_1=v_1(t,y)$. We find:

$$\begin{aligned}\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, S(t,y),C_0,C_2,C_3\big)&\leqslant\sum_{z\in\partial\Lambda_{k^\varepsilon}}\mu^f_{\mathcal{D}}\big(z\in C_1 \,\big|\, S(t,y),C_0,C_2,C_3\big)\\&\leqslant\sum_{z\in\partial\Lambda_{k^\varepsilon}}O\Big(\frac{1}{\sqrt{|v_1-z|}}\exp\Big[-\kappa'\frac{d_\tau(z,[v_1,u_1])^2}{|v_1-z|}\Big]\Big)\\&=O\Big(\frac{k^\varepsilon}{\sqrt{|y|}}\exp\Big\{-\kappa''\frac{d_\tau(0,[y,u_1])^2}{|y|}\Big\}\Big). \quad (4.43)\end{aligned}$$

In the last line, we used the fact that $y,v_1\notin\Lambda_{Ck^\varepsilon}$ and $|v_1-y|\leqslant Ck^\varepsilon$. Let us substitute (4.43) into the sum on the right-hand side of (4.41) to obtain

$$\mu^f_{\mathcal{D}}\big(C\cap\Lambda_{k^\varepsilon}\neq\varnothing \,\big|\, E(u_1,u_2,u_3,x)\big)\leqslant O(e^{-k^\varepsilon})+O(k^{14\varepsilon-1}) \\ +O(k^{13\varepsilon-1})\sum_{y\in\Lambda_{\varepsilon_1 k}(x)\setminus\Lambda_{2Ck^\varepsilon}}\frac{1}{\sqrt{|y|}}\exp\Big[-\kappa\frac{|y-x|^2}{k}-\kappa''\frac{d_\tau(0,[y,u_1])^2}{|y|}\Big]. \quad (4.44)$$

After a simple estimate, we see easily that the sum on the right is bounded above as

$$2 \sum_{y \in \Lambda_{k^{1/2+\varepsilon}}(x) \setminus \Lambda_{2Ck^{\varepsilon}}} \frac{1}{\sqrt{|y|}} \exp\Big[-\kappa'' \frac{d_\tau(0,[y,u_1])^2}{|y|}\Big] \leqslant \sum_{\ell=\max\{2Ck^{\varepsilon}, |x|-k^{1/2+\varepsilon}\}}^{|x|+k^{1/2+\varepsilon}} \frac{O(\sqrt{\ell})}{\sqrt{\ell}} = O(k^{\frac{1}{2}+\varepsilon}),$$

uniformly in x. In order to obtain the first inequality, we used the fact that the term $\exp(-\kappa|y-x|^2/k)$ is very small for sites outside of $\Lambda_{k^{1/2+\varepsilon}}(x)$. For the second, observe that sites y at distance ℓ contributing substantially to this sum must satisfy the condition that 0 is at distance $O(\sqrt{\ell})$ of $[y,u_1]$. There are $O(\sqrt{\ell})$ of them. This concludes the proof. ■

Part 2

A few notes on the discrete Gaussian Free Field with disordered pinning on $\mathbb{Z}^d$, $d \geqslant 2$

Derrière les ennuis et les vastes chagrins
Qui chargent de leur poids l'existence brumeuse,
Heureux celui qui peut d'une aile vigoureuse
S'élancer vers les champs lumineux et sereins.

Charles Baudelaire, "Elévation".

Chapter 5

A short review of homogenous and disordered systems related to our model

In this chapter we present a short overview of the state of the art concerning models of pinned interfaces. First of all, we describe the discrete Gaussian free field (GFF), which we choose as the free (unpinned) interface model. Then, we present known results about the homogenous pinning of the GFF. Finally, we review the present knowledge about disordered pinning of interface models: the 1-dimensional case covers the so-called "polymers" in random environment, which have been studied a lot in the last decade, while in dimension 2 and more, much less is known. We end by introducing our new results which constitute some progress in the latter case, and by stating some open problems.

5.1 The discrete Gaussian free field

The discrete Gaussian free field is an interface model in $\mathbb{Z}^d \times \mathbb{R}$, which can also be seen as a spin model where the spins take value in a non-compact set, $\mathbb{R}$. A configuration of the GFF is an application $\varphi : \mathbb{Z}^d \to \mathbb{R}$. We write the configuration space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. Let $\Lambda \Subset \mathbb{Z}^d$, the Hamiltonian in Λ associated to the GFF is

$$\mathcal{H}_\Lambda(\varphi) = \frac{1}{4d} \sum_{\substack{\{x,y\} \cap \Lambda \neq \emptyset \\ x \sim y}} (\varphi_x - \varphi_y)^2, \tag{5.1}$$

The Gibbs measure of the GFF in Λ with boundary condition $\bar{\varphi} \in \Omega$ is the probability measure on Ω given by

$$\mu_\Lambda^{\bar{\varphi}}(d\varphi) = \frac{1}{Z_\Lambda^{\bar{\varphi}}} \exp\left(-\beta \mathcal{H}_\Lambda(\varphi)\right) \prod_{x \in \Lambda} d\varphi_x \prod_{y \in \Lambda^c} \delta_{\bar{\varphi}}(d\varphi_y), \tag{5.2}$$

where δ_a is the Dirac measure at a, and $Z_\Lambda^{\bar{\varphi}}$ is the normalizing constant. The parameter β is actually irrelevant since we can make the change of variables $\varphi \to \varphi/\sqrt{\beta}$ and obtain the same model with $\beta = 1$. For this reason, we will henceforth assume $\beta = 1$ and omit it from the notations.

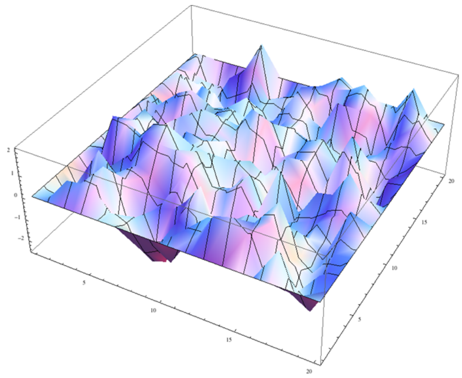


Figure 5.1 – Interpretation of the 2d Gaussian free field as an interface in $\mathbb{R}^3$.

The GFF measure (5.2) is Gaussian, and thus characterized by its mean and covariance matrix. There exists a useful representation of the latter in terms of the Green function of the simple symmetric random walk on $\mathbb{Z}^d$.

Proposition 5.1 *Let $((X_n)_n, P_x)$ be a simple symmetric random walk on $\mathbb{Z}^d$ starting at x, and τ_Λ be its exit time from the box Λ. Then,*

$$\mu_\Lambda^{\bar{\varphi}}(\varphi_x) = E_x\left[\bar{\varphi}_{X_{\tau_\Lambda}}\right]$$
$$\mu_\Lambda^{\bar{\varphi}}(\varphi_x \varphi_y) = E_x\left[\sum_{n=0}^{\tau_\Lambda - 1} \mathbb{1}_{[X_n = y]}\right].$$

We refer to [25] for the proof. As the simple random walk is recurrent in $d = 2$ and transient in $d \geqslant 3$, the variances diverge in $d = 2$, and stay bounded for $d \geqslant 3$ as $\Lambda \uparrow \mathbb{Z}^d$. More precisely, we have the following estimates

Proposition 5.2 *There exists constants* $c(d) > 0$ *such that the variance of the GFF in the box* $\Lambda = \{-n, \dots, n\}^d$ *satisfies, as* $n \to \infty$,

$$\mu_\Lambda^{\bar\varphi}(\varphi_x^2) - (\mu_\Lambda^{\bar\varphi}(\varphi_x))^2 = \begin{cases} (c(1) + o(1)) \cdot n & \text{for } d = 1 \\ (c(2) + o(1)) \cdot \log n & \text{for } d = 2 \\ c(d) + o(1) & \text{for } d \geqslant 3 \end{cases}$$

independently of the boundary condition $\bar\varphi$.

We do not focus on the set of Gibbs measures of the models in this part of the manuscript, but note that the later proposition implies in particular that there are no infinite volume Gibbs measures for the GFF in dimension 1 and 2. In higher dimensions, $d \geqslant 3$, there exist uncountably many infinite-volume Gibbs measures, even when considering only bounded boundary conditions. See [25].

An important property of the GFF is the FKG inequality. There exists indeed a partial order on the configurations in Ω: we say that

$$\varphi \leqslant \tilde\varphi \quad \text{iff} \quad \varphi_x \leqslant \tilde\varphi_x \quad \forall x \in \mathbb{Z}^d$$

A function $f : \Omega \to \mathbb{R}$ is said to be increasing if $f(\varphi) \leqslant f(\tilde\varphi)$ for $\varphi \leqslant \tilde\varphi$.

Proposition 5.3 (FKG inequality) *Let* f *and* g *be two increasing functions from* Ω *to* $\mathbb{R}$, *and* $\Lambda \Subset \mathbb{Z}^d$. *Then, for any boundary condition* $\bar\varphi$,

$$\mu_\Lambda^{\bar\varphi}(f \cdot g) \geqslant \mu_\Lambda^{\bar\varphi}(f) \cdot \mu_\Lambda^{\bar\varphi}(g)$$

This remains true for measures with density with respect to $\mu_\Lambda^{\bar\varphi}$ *of the form* $\exp(\sum_x h(\varphi_x))$, *for* $h \in \mathcal{C}^2$.

We refer to [25] for a proof and for more informations about the GFF. See also [84] for random walks estimates.

5.2 Homogenous pinning of the discrete Gaussian free field

The following model breaks the continuous symmetry of the GFF Hamiltonian, by pinning the field at height 0. The configuration space is still $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ and the so-called "delta-pinning" measure with intensity ε and boundary condition 0 (dropped from the notation) is:

$$\mu_\Lambda^\varepsilon(d\varphi) = \frac{1}{Z_\Lambda^\varepsilon} \exp\left(-\mathcal{H}_\Lambda(\varphi)\right) \prod_{x \in \Lambda} \left(d\varphi_x + \varepsilon\delta_0(d\varphi_x)\right) \prod_{y \in \Lambda^c} \delta_0(d\varphi_y), \tag{5.3}$$

with $\varepsilon \geqslant 0$ and $\mathcal{H}_\Lambda$ given by (5.1).

To analyze the properties of the model with pinning ($\varepsilon > 0$), it is convenient to map it onto a model of a random walk in an annealed random environment of killing obstacles. To achieve that, we can expand the product $\prod_{x\in\Lambda} d\varphi_x + \varepsilon\delta_0(d\varphi_x)$, as in a high temperature expansion. Let $\mathcal{A} = \{x \in \Lambda : \varphi_x = 0\}$ be the random variable describing the set of pinned sites. By simple computations, for any integrable function $f : \mathbb{R}^\Lambda \mapsto \mathbb{R}$, we have:

$$\mu_\Lambda^\varepsilon(f) = \sum_{A\subset\Lambda} \mu_\Lambda^\varepsilon(f \mid \mathcal{A} = A) \cdot \mu_\Lambda^\varepsilon(\mathcal{A} = A) = \sum_{A\subset\Lambda} \mu_{\Lambda\setminus A}^0(f) \cdot \mu_\Lambda^\varepsilon(\mathcal{A} = A). \tag{5.4}$$

For simplicity we denote $\nu_\Lambda^\varepsilon(A) \doteqdot \mu_\Lambda^\varepsilon(\mathcal{A} = A)$. Conditionally on $\mathcal{A}$, the measure μ_Λ^ε is Gaussian with covariances described by the random walk representation above. We get for example an expression for the covariance of the field φ under the measure μ_Λ^ε in terms of the Green function of the random walk in an annealed random environment of killing traps of distribution ν_Λ^ε.

Important results concerning the pinned sites distribution were obtained in [16]. First of all, the measure ν_Λ^ε is strong FKG in the sense of [35], with a partial order on the set of pinned sites which is the inclusion. Moreover, it can be compared with i.i.d. Bernoulli distributions. Let $\mathbf{B}_\Lambda^\alpha$ be the Bernoulli product measure (or site percolation) of parameter $\alpha \in [0, 1]$ in the box Λ, namely the measure on subsets of Λ given by $\mathbf{B}_\Lambda^\alpha(\mathcal{A} = A) = \alpha^{|A|}(1-\alpha)^{|\Lambda|-|A|}$.

Proposition 5.4 *There exist constants* $0 < c_+(d) < c_-(d) < \infty$ *such that for any* Λ*, any* $B \subset \Lambda$*, and* ε *sufficiently small, we have*

$$\mathbf{B}_\Lambda^{c_-(d)g(\varepsilon)}(\mathcal{A} \cap B = \varnothing) \leqslant \nu_\Lambda^\varepsilon(\mathcal{A} \cap B = \varnothing) \leqslant \mathbf{B}_\Lambda^{c_+(d)g(\varepsilon)}(\mathcal{A} \cap B = \varnothing), \tag{5.5}$$

where

$$g(\varepsilon) = \begin{cases} \varepsilon|\log\varepsilon|^{-1/2} & d = 2, \\ \varepsilon & d \geqslant 3. \end{cases}$$

For $d \geqslant 3$, an even stronger statement is true; the distribution of the pinned sites is strongly stochastically dominated by $\mathbf{B}_\Lambda^{c_+(d)\varepsilon/(1+c_+(d)\varepsilon)}$ and strongly stochastically dominates $\mathbf{B}_\Lambda^{c_-(d)\varepsilon/(1+c_-(d)\varepsilon)}$, namely,

Proposition 5.5 *Let* $d \geqslant 3$*. For any* $C \subset \Lambda$*,*

$$\frac{c_-(d)\varepsilon}{1 + c_-(d)\varepsilon} \leqslant \nu_\Lambda^\varepsilon(x \in \mathcal{A} \mid \mathcal{A}\setminus\{x\} = C) \leqslant \frac{c_+(d)\varepsilon}{1 + c_+(d)\varepsilon}. \tag{5.6}$$

The proof can be found in [16, Theorem 2.4].

Concerning the behavior of the interface, it is known that an arbitrarily weak pinning ε is sufficient to localize the interface. Indeed, in [28], Deuschel and Velenik proved that the infinite volume Gibbs measure μ^ε exists in all $d \geqslant 1$ and that for any ε small enough and all K large enough [1],

$$-\log \mu^\varepsilon(\varphi_0 > K) \asymp_d \begin{cases} K & d = 1, \\ K^2/\log K & d = 2, \\ K^2 & d \geqslant 3. \end{cases}$$

The so-called mass, or rate of exponential decay of the two-point function, associated to the infinite volume Gibbs measure $\mu^{\varepsilon,0}$ is defined, for any $x \in \mathbb{S}^{d-1}$, by

$$m^\varepsilon(x) \doteqdot -\lim_{k\to\infty} \frac{1}{k} \log \mathbb{C}\mathrm{ov}_\varepsilon(\varphi_0 \varphi_{[kx]}).$$

where $[x]$ is the vector of integer parts of x's coordinates. In [57] Ioffe and Velenik showed that for any $d \geqslant 1$,

$$\inf_{x\in\mathbb{S}^{d-1}} m^\varepsilon(x) > 0.$$

The localization of the interface becomes weaker as $\varepsilon \to 0$, we can quantify this by studying the behavior of the variance and the mass of the field in this limit. The most precise results were proved by Bolthausen and Velenik in [16], and can be stated as follows. For $d = 2$ and ε small enough,

$$\mu^\varepsilon(\varphi_0^2) = \frac{1}{\pi}|\log \varepsilon| + O(\log|\log \varepsilon|).$$

For ε small enough,

$$m^\varepsilon \asymp_d \begin{cases} \sqrt{\varepsilon}|\log \varepsilon|^{-3/4} & d = 2, \\ \sqrt{\varepsilon} & d \geqslant 3. \end{cases}$$

5.3 Similar disordered systems

The goal of this section is to present known results about some variant of the interface models presented above, but which have another random feature, usually called random environment.

For most of them, it is the pinning parameter which is randomly chosen from one site to the other. When a realization of this random environment is chosen at random and fixed, the quantities associated to the model are called "quenched". When the average over the environment is performed before taking expectation values (w.r.t. Gibbs measures), the associated quantities are called "annealed".

The rate of exponential volume-growth of the partition function of a model (up to a constant) is called the free energy. It will be defined rigorously in Section 5.4. As we will see, the quenched and annealed free energies are important quantities in the study of statistical mechanics systems in random environment. The natural questions to be asked in this field will appear throughout the text.

[1] $a \asymp_d b$ means that there exist two constants $0 < c_1 \leqslant c_2 < \infty$, depending only on d, such that $c_1 b \leqslant a \leqslant c_2 b$.

5.3.1 Dimension 1: Random polymers in random media

In [9], Alexander and Sidoravicius studied the following 1-dimensional model of a random walk called "polymer" which is randomly attracted or repulsed at a certain height. More generally, they consider a polymer, with monomer locations modeled by the trajectory of an integer valued Markov chain $(X_i)_{i\in\mathbb{Z}}$, in the presence of a potential (usually called a "defect line") that interacts with the polymer when it visits 0. Formally, let V_i be an i.i.d. sequence of 0-mean random variables, the model is given by weighting the realization of the chain with the Boltzmann term

$$\exp\left(\beta\sum_{i=1}^{n}(u+V_i)\mathbb{1}_{[X_i=0]}\right).$$

Their purpose was to study the localization transition in this model. If a positive fraction of monomers is at 0, we say that the polymer is pinned. In the plane β vs. u, critical lines are defined: for β fixed, let $u_c^q(\beta)$ (resp. $u_c^a(\beta)$) the quenched (resp. annealed) critical value of u above which the polymer is pinned with probability 1 (for the quenched (resp. annealed) measure). They show that the quenched free energy and critical point are nonrandom, computed the critical point for a deterministic interaction (i.e. $V_i \equiv 0$) and proved that the critical point in the quenched case is strictly smaller.
Note that when the underlying chain is a symmetric simple random walk on $\mathbb{Z}$, the deterministic critical point is known to be 0, so having the quenched critical point $(u_c(\beta))$ strictly negative means that, even when the disorder is repulsive on average, the chain is pinned. This result was obtained by Galluccio and Graber in [39] for a periodic potential, which is frequently used in the physics literature as a "toy model" for random environment.
In [47], Giacomin and Toninelli investigated the order of the localization transition in general models of directed polymers pinned on a defect line.
They prove that for quite a general class of models, as soon as disorder is present, the transition is at least of the second order, i.e. the free energy is differentiable at the critical line and the order parameter (which is the density of pinned sites) vanishes continuously at the transition.
This is particularly interesting as there are examples of non-disordered systems with first order transition (cf. for example [44], Proposition 1.6, for $(1+d)$-dimensional directed polymers and $d \geqslant 5$). The result thus implies that the introduction of a disorder may have a smoothening effect on the transition. For 1-dimensional models, the renewal structure of the return times to 0 plays important role, in particular it simplifies a lot of computations.

In [8], Alexander emphasized this fact by assuming that the tails of the excursion length between consecutive returns of X to 0 are as $n^{-c}\phi(n)$ (for some $1 < c < 2$ and slowly varying ϕ). He analyzed the quenched and annealed critical curves in the plane (u, β) for different values of c, showing that for $c > 3/2$ at high temperature the quenched and annealed curves differ significantly only in a very small neighborhood of the critical point, whereas for $c < 3/2$ the quenched and annealed critical points are equal. This was a prediction made by theoretical physicists on the basis of the so-called Harris criterion (see [44], Section 5.5, for more informations). The relevant case in the framework of this paper is the case of the Markov chain given by a simple symmetric random walk on $\mathbb{Z}$, which corresponds to $c = 3/2$ and $\phi(n) \sim K$ for $K > 0$, which is borderline.

Progress on this question has been made recently by Giacomin, Lacoin and Toninelli in [45] and [46]. They prove that in the borderline case $c = 3/2$, the disorder is relevant in the sense that the quenched critical point in shifted with respect to the annealed one. They consider i.i.d. Gaussian disorder in the first paper and extend the result to more general i.i.d. laws, as well as refine the lower bound on the shift, in the second paper. Note that this case includes pinning of a directed polymer in dimension (1+1) as already mentioned[2], but also the classical models of two-dimensional wetting of a rough substrate, pinning of directed polymers on a defect line in dimension (3 + 1) and pinning of an heteropolymer by a point potential in three-dimensional space.

5.3.2 Dimension 2 (and more): Random surfaces in random media

5.3.2.1 Diluted pinning, known results

The only result about random pinning in more than one dimension we are aware of is [59]. In this paper, Janvresse, De La Rue and Velenik considered a version of the delta-pinning model (5.3) in dimension 1 and 2 where the parameter ε can be either 0 or η on each site of the lattice. This is a model for an interface interacting with an attractive diluted potential. They show that the interface is localized in a sufficiently large but finite box (in the sense that there is a density of pinned sites) if and only if the sites at which the pinning potential is non-zero have positive density. Note that in this paper they characterize the set of realizations of the environment for which pinning holds (the disorder is fixed, not sampled from some given distribution), which is stronger than an almost sure result.

5.3.2.2 Random magnetic fields, known results

We also mention a series of papers by Kuelske *et al.* ([64], [86], [65] which study a model with disordered magnetic field (instead of disordered pinning potential). For example, Kuelske and Orlandi studied the following model in dimension 2

$$\mu_\Lambda^{\varepsilon,\eta}(d\varphi) = \frac{1}{Z_\Lambda^{\varepsilon,\eta}} \exp\left(-\frac{1}{4d}\sum_{x\sim y} V(\varphi_x - \varphi_y) + \right.$$
$$\left. + \sum_{x\in\Lambda} \eta_x \varphi_x\right) \prod_{x\in\Lambda} (d\varphi_x + \varepsilon\delta_0(d\varphi_x)) \prod_{y\in\Lambda^c} \delta_0(d\varphi_y),$$

where $\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$ is an arbitrary fixed configuration of external fields and V is not growing too slowly at infinity. Without disorder ($\boldsymbol{\eta} \equiv 0$), the interface is localized

[2] We speak about "polymer in dimension $(d+1)$" when the state space of the Markov chain X is of dimension d.

for any $\varepsilon > 0$ [16]. One could expect that in presence of disorder and at least for very large ε the interface is pinned. However, the authors show that this is not the case: the interface diverges regardless of the pinning strength. This implies that an infinite-volume Gibbs measure for this model does not exist. One could hope for the existence of the so-called gradient Gibbs measure (Gibbs distributions of the increments of the interface). In [86] Van Enter and Kuelske proved that such (infinite volume) measures do not exist in the random field model in dimension 2. Note that gradient Gibbs measures may exist, even when the corresponding Gibbs measure does not. This happens when the interface is locally smooth, although at large scales its fluctuations diverge. This is the case for the two-dimensional Gaussian free field.

5.4 New results

In collaboration with Piotr Miłoś we studied two models belonging to the so-called effective interface class. As above, for $\Lambda \Subset \mathbb{Z}^d$, let $\varphi = (\varphi_x)_{x\in\Lambda}$ represent the heights of the interface above or below sites in Λ. We always refer to φ as "the interface" or "the field". The models are defined in terms of a Gaussian pair potential and a random potential term at interface height zero (or close to zero). Both models hence contain two levels of randomness. The first one is $\boldsymbol{e}$ which we refer to as "the environment", and the second one is the actual interface model, depending on $\boldsymbol{e}$.

The dimensions 1 and 2 are physically relevant as interface models. In our work we focus on $d \geqslant 2$ which is a step towards the understanding of random (hyper-)surfaces in random environment. We recall that one-dimensional models have been well-studied in the last decade, but only a few works are about higher dimensions (see Section 5.3.1).

5.4.1 The $\delta-$pinning model (attractive potential)

In the first model, which is an easy case we can consider as a warm up for the questions we treat, we consider only attractive potential and random δ-pinning intensities which follow a Bernoulli law:

$$\mu_\Lambda^{\boldsymbol{e}}(d\varphi) = \frac{1}{Z_\Lambda^{\boldsymbol{e}}} \exp\left(-\mathcal{H}_\Lambda(\varphi)\right) \prod_{x\in\Lambda} \left(d\varphi_x + e_x \delta_0(d\varphi_x)\right) \prod_{y\in\Lambda^c} \delta_0(d\varphi_y), \tag{5.7}$$

where $\mathcal{H}_\Lambda(\varphi)$ is given by (5.1) and as usual $Z_\Lambda^{\boldsymbol{e}}$ is the partition function, i.e. it normalizes $\mu_\Lambda^{\boldsymbol{e}}$ so it is a probability measure. The environment $\boldsymbol{e} \doteqdot (e_x)_{x\in\Lambda}$ is given by positive i.i.d. Bernoulli random variables on $\mathbb{Z}^d$ standing for the quenched disorder, which thus represents a random attraction strength at height 0:

$$\mathbb{P}(e_x = \underline{e}) = p = 1 - \mathbb{P}(e_x = \overline{e}) \quad \text{for some } 0 < \underline{e} < \overline{e} \text{ and } p \in (0,1)$$

We write

$$e^\star \doteqdot p\underline{e} + (1-p)\overline{e} \quad \text{and} \quad \sigma^2 \doteqdot p\underline{e}^2 + (1-p)\overline{e}^2 - (e^\star)^2$$

for the average and variance of e_x, respectively.

The annealed model is obtained by averaging the weights over the disorder. It thus corresponds to a model with homogenous delta-pinning of parameter $e^\star$. In particular,

$$\mathbb{E}Z_\Lambda^{\boldsymbol{e}} = Z_\Lambda^{e^\star}$$

5.4.2 The attractive / repulsive model

In the second model, we consider an attractive/repulsive square potential, and a general environment law:

$$\mu_\Lambda^{\boldsymbol{e}}(d\varphi) = \frac{1}{Z_\Lambda^{\boldsymbol{e}}} \exp\left(-\mathcal{H}_\Lambda(\varphi) + \sum_{x\in\Lambda}(b\cdot e_x + h)\mathbb{1}_{[\varphi_x\in[-a,a]]}\right)\prod_{x\in\Lambda} d\varphi_x \prod_{y\in\Lambda^c}\delta_0(d\varphi_y) \tag{5.8}$$

where $a, b > 0$, $h \in \mathbb{R}$ and $\mathcal{H}_\Lambda(\varphi)$ is given by (5.1). We consider the environment $\boldsymbol{e} \doteqdot (e_x)_{x\in\Lambda}$ to be given by any i.i.d family of random variables such that $\mathbb{E}(e_x) = 0$, $\mathbb{V}\mathrm{ar}(e_x) = 1$, and

$$\mathbb{E}(e^{b\cdot e_x + h}) < \infty.$$

The parameter b is usually called the "intensity of the disorder", while h is its average. The disordered potential attracts or repells the field at heights belonging to $[-a, a]$. Observe that the annealed model corresponds to a model with homogenous potential height equal to $\boldsymbol{\ell} \doteqdot \log\mathbb{E}(e^{be_x+h})$. In particular,

$$\mathbb{E}Z_\Lambda^{\boldsymbol{e}} = Z_\Lambda^{\boldsymbol{\ell}}$$

Note that $\boldsymbol{\ell}$ depends only on the law of $\boldsymbol{e}$. The assumption $\mathbb{E}(e^{be_x+h}) < \infty$ is the minimal one in order for the annealed model to be well defined.

5.4.3 Existence of the free energy

The quenched and annealed free energies per site in $\Lambda \Subset \mathbb{Z}^d$ are defined by

$$f_\Lambda^{q}(\boldsymbol{e}) \doteqdot \frac{1}{|\Lambda|}\log\left(\frac{Z_\Lambda^{\boldsymbol{e}}}{Z_\Lambda^0}\right), \quad f_\Lambda^{a} \doteqdot \frac{1}{|\Lambda|}\log\left(\frac{\mathbb{E}Z_\Lambda^{\boldsymbol{e}}}{Z_\Lambda^0}\right),$$

where Z_Λ^0 denotes the partition function of the Gaussian free field. This normalization is chosen such that the free energy of the model with no pinning (which is the GFF) is zero.

The questions we are addressing in our work are the usual ones concerning statistical mechanics models in random environment:
Is the quenched free energy non-random? Does it differ from the annealed one? Can we give a physical meaning to the strict positivity (resp. vanishing) of the free energy? What can be said concerning the quenched and annealed critical lines (surfaces) in the space of the relevant parameters of the system?

For both models we proved that $(f_\Lambda^q(\boldsymbol{e}))_\Lambda$ converges to a non-random quantity as $\Lambda \uparrow \mathbb{Z}^d$ provided the Van Hove criterion is satisfied, namely $|\partial\Lambda|/|\Lambda| \to 0$. It is called the infinite volume quenched free energy, and is denoted by f^q. The same proof can be used to show that $f_\Lambda^a \to f^a$ as $\Lambda \uparrow \mathbb{Z}^d$ under the same condition.

Theorem 5.1 *For models* (5.7) *and* (5.8), *the following limit exists*

$$f^q_\Lambda(\boldsymbol{e}) \to f^q(\boldsymbol{e})$$

almost surely and in L^2 *as* $\Lambda \uparrow \mathbb{Z}^d$ *provided* $|\partial\Lambda|/|\Lambda| \to 0$.
Moreover, the quenched free energy is deterministic, namely

$$f^q(\boldsymbol{e}) = \mathbb{E} f^q(\boldsymbol{e}) \quad \text{a.s.}$$

The proof of this result in dimension 1 can be found in [47].

Remark 5.1 *In the attractive/repulsive model, the quenched and annealed free energies are convex functions of* h, *for fixed* b. *This can be immediately checked by noting that in finite volume* $\partial^2_h f^q_\Lambda(\boldsymbol{e}) = \mathbb{V}ar(\sum_{x\in\Lambda} \mathbb{1}_{[|\varphi_x|\leqslant a]}) \geqslant 0$. *Hence, by elementary properties of convex functions,* f^q *and* f^a *are increasing functions of* h, *since* $\partial^+_h f^q = \mu^e(\sum_x \mathbb{1}_{[|\varphi_x|\leqslant a]}) \geqslant 0$.

5.4.4 Strict inequality between quenched and annealed free energies

Comparing the free energy for the quenched and annealed models is useful for studying the effect of the environment on the system. The case when they are different indicates that the disorder of the environment has some macroscopic effect.

By the Jensen inequality and the L^1 convergence of $f^q_\Lambda(\boldsymbol{e})$, it is easy to see that

$$f^q \leqslant f^a \quad \text{a.s.}$$

Theorem 5.2 *For models* (5.7) *and* (5.8),

$$f^q < f^a \quad \text{a.s.} \quad \textit{whenever} \quad f^a > 0.$$

We provide also some estimates on the difference $f^a - f^q$ in terms of $\sigma^2 = \mathbb{V}\mathrm{ar}(\boldsymbol{e})$ for model (5.7), and in terms of $\boldsymbol{\ell}$ for model (5.8).

5.4.5 Attraction by a repulsive in average environment

Intuitively a model with positive free energy has a positive density of pinned points while a model with zero free energy is delocalised. We prove this statement rigorously for both models. In particular, we prove that the quenched free energy is always strictly positive in case of model (5.7), while it is strictly positive for $\boldsymbol{\ell} > 0$ and vanishes for $\boldsymbol{\ell} \leqslant 0$ in case of model (5.8).

This fact, together with Remark 5.1, motivates the definition of critical lines in the case of model (5.8), which are delimiting the region where $f^q = 0$ (resp. $f^a = 0$) from the region $f^q > 0$ (resp. $f^a > 0$).

$$h^q_c(b) \doteqdot \sup\{h \in \mathbb{R} : f^q(\boldsymbol{e}) = 0\} \quad \text{and} \quad h^a_c(b) \doteqdot \sup\{h \in \mathbb{R} : f^a = 0\}$$

We are interested in describing the behavior of these quantities in the phase diagram described by the plane (b, h). Knowing the behavior of the homogenous model for positive pinning [16], we easily deduce that the annealed critical line is given by the equation $\ell = 0$. Note that $f^q \leqslant f^a$ implies that $h_c^q(b) \geqslant h_c^a(b)$.

Theorem 5.3 *For model* (5.8)*, in the Bernoulli case* $\mathbb{P}(e_x = -1) = \mathbb{P}(e_x = +1) = 1/2$*, the quenched critical line* $h_c^q(b)$ *lies strictly below the axis* $h = 0$ *in the neighborhood of* $b = 0$ *for all dimensions* $d \geqslant 2$.

We also provide some estimates on $h_c^q(b)$ near $b = 0$. The extension of this result to more general environment laws is expected. Our result shows in particular that there exists a non trivial region where $h < 0$, $b > 0$ and $f^q > 0$, i.e. where the field is localized though it is repulsed on average by the environment, see Figure 5.2.

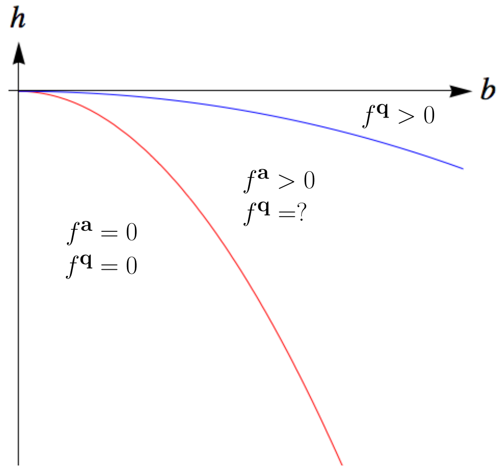


Figure 5.2 – Phase diagram of the model. The red curve is the annealed critical line; the blue one is our bound on the quenched critical line.

Note that we do not have any estimate on the behavior of $h_c^q(b) - h_c^a(b)$. Hence our results do not allow us to treat the questions concerning critical exponents of the system, nor the order of the phase transition, in presence (absence) of disorder. This has been done for a certain class of 1-dimensional systems (see Section 5.3.1), and is a much more difficult problem in dimension 2 and above.

5.5 Open problems

The models we studied lend to number of extensions. We list here a selection, with brief comments.

- **Path-wise description of the interface.** In the positive free energy region, for model (5.8), one expects localization, i.e. the finite variance of φ_x and exponential decay of correlations. A much more difficult question concerns the behavior

of the interface near the critical line. Does it behave the same as in the homogenous case (i.e. second order transition with the density of pinned sites decreasing linearly for $d \geqslant 3$, and with a logarithmic correction for $d = 2$) ? Or does the presence of disorder have a smoothening effect on the transition (as it was proven for certain 1-dimensional models) ?
In the zero free energy region, we expect the behavior similar to the entropic repulsion for the GFF: in a box of size n the interface should be repelled at height $\pm \log n$ in $d = 2$ and $\pm\sqrt{\log n}$ in $d \geqslant 3$ (see [27] and references therein for details). The $\pm$ stems from the fact that the model is symmetric with respect to reflection at zero height, hence with probability $1/2$ it either goes upwards or downwards.

- **Description for non Gaussian pair-potential.** A natural conjecture is that the behavior of the models (5.7) and (5.8) is the same if we change the Gaussian term $(\varphi_x - \varphi_y)^2$ to any other uniformly convex potential $V(\varphi_x - \varphi_y)$. Some results can be extended to this case in the homogenous setting [16].

- **Non-nearest neighbors interactions.** We restricted our work to the case of the nearest neighbors interactions. We suspect that the results hold true for fast decaying interactions, at least with condition like in [16, (2.1)] (which ensures a control of the random walk's behavior in the random walk representations). Note that As the behavior of the homogenous pinning model beyond this regime is not known, we are unable to pose any further conjectures.

- **Non i.i.d. environment laws.** Going beyond the i.i.d. case is a very interesting direction. Two natural cases would be the stationary Bernoulli field or the quenched chessboard like configuration. These questions may be closely connected to convexity/concavity properties of the annealed free energy function in terms of the homogenous pinning parameter. The understanding of this case is still limited. It would be interesting to know if finite range environment laws change the picture, as it is sometimes the case in models with bulk disorder but it seems difficult to answer this question rigorously.

- **Geometry of pinned sites.** The geometry of the pinned sites is still not fully understood in the homogenous case. For the $d \geqslant 3$ the law of pinned sites resembles a Bernoulli point process. It is conjectured that once the pinning tends to zero, under suitable re-scaling, this field converges to Poisson point process. For $d = 2$ the situation is not clear at all, since it is expected that the dependency between the points will be preserved in the limit (implying the limit being non-Poissonian). Not only these questions propagate to the non-homogenous case but also new ones arise. E.g. for model (5.8) it would be interesting to study the joint geometry of attractive and repulsive sites.

- **Models with wetting transition.** The effects of introducing a disorder in other models with pinning might be interesting, for example in models exhibiting a wetting phenomenon. In the case of the massless Gaussian model in $d = 2$, it is known [22] that the wetting transition takes place at a non-trivial point. A natural question to ask is, if adding disorder shifts this point.

Chapter 6

Random Bernoulli δ-pinning (attractive environment)

This chapter treats the case of a "toy model" we studied in order to get intuition about the addition of a random environment to a pinning model. The proof of existence of the free energy is much shorter in this case and contains the heuristics of the proof in the general case (given in the next chapter). That is why we present it here.

6.1 Pinned sites representation

We recall the notations introduced in the previous chapter. We denote by $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ the set of configurations of the model. For $\Lambda \Subset \mathbb{Z}^d$, the measure of the discrete Gaussian Free Field with random δ-pinning $\boldsymbol{e}$, with 0 boundary condition, is

$$\mu_\Lambda^{\boldsymbol{e}}(d\varphi) = \frac{1}{Z_\Lambda^{\boldsymbol{e}}} e^{-\beta \mathcal{H}_\Lambda(\varphi)} \prod_{x\in\Lambda} \left(d\varphi_x + \sqrt{\beta} e_x \delta_0(d\varphi_x) \right) \prod_{y\in\Lambda^c} \delta_0(d\varphi_y), \tag{6.1}$$

where the Hamiltonian $\mathcal{H}_\Lambda$ is given by (5.1). The non-homogenous pinning environment is denoted by $\boldsymbol{e} \doteqdot (e_x)_{x\in\Lambda}$. We consider the law of $\boldsymbol{e}$ given by a Bernoulli product measure. Namely, independently on each site $x \in \Lambda$:

$$e_x = \begin{cases} \underline{e} & \text{with probability } p \\ \overline{e} & \text{with probability } (1-p), \end{cases} \tag{6.2}$$

for some fixed triplet $(\underline{e}, \overline{e}, p)$ such that $0 < \underline{e} < \overline{e}$ and $p \in (0,1)$. We consider only positive values of the environment, meaning purely attractive environment. Let

$$e^\star \doteqdot p\underline{e} + (1-p)\overline{e}, \quad \text{and} \quad \sigma^2 \doteqdot p\underline{e}^2 + (1-p)\overline{e}^2 - (e^\star)^2. \tag{6.3}$$

be the average and variance of e_x, respectively.

Remark 6.1 *1. The measure (6.1) is the weak limit of the measures $\mu^{e^n}_{\Lambda,a^n}$, given by* (5.8), *once we choose $(a^n)_{n\geqslant 1}, (e^n)_{n\geqslant 1}$ such that $a^n \to 0$ and $2a^n(e^{e^n_x}-1) = e_x$.*

2. *The temperature parameter enters only in a trivial way. If we replace the field $(\varphi_x)_{x\in\Lambda}$ by $(\sqrt{\beta}\phi_x)_{x\in\Lambda}$, and $(e_x)_{x\in\Lambda}$ by $(\sqrt{\beta}e_x)_{x\in\Lambda}$ we have transformed the model to temperature parameter $\beta = 1$. Thus in the sequel, we will assume $\beta = 1$.*

In this section we extend the decomposition (5.4) to our inhomogenous model:

$$\begin{aligned}
\mu^{e}_{\Lambda}(d\varphi) &= \frac{1}{Z^{e}_{\Lambda}} e^{-\mathcal{H}_\Lambda(\varphi)} \prod_{x\in\Lambda} (d\varphi_x + e_x\delta_0(d\varphi_x)) \prod_{y\in\Lambda^c} \delta_0(d\varphi_y) \\
&= \frac{1}{Z^{e}_{\Lambda}} e^{-\mathcal{H}_\Lambda(\varphi)} \sum_{A\subset\Lambda} \left(\prod_{x\in A} e_x\delta_0(d\varphi_x) \prod_{y\in\Lambda\setminus A} d\varphi_y \right) \prod_{z\in\Lambda^c} \delta_0(d\varphi_z) \\
&= \sum_{A\subset\Lambda} \underbrace{\left(\prod_{x\in A} e_x\right) \frac{Z^0_{\Lambda\setminus A}}{Z^{e}_{\Lambda}}}_{\doteqdot \nu^{e}_{\Lambda}(A)} \mu^0_{\Lambda\setminus A}(d\varphi).
\end{aligned} \tag{6.4}$$

In other words ν^{e}_{Λ} is the marginal of the measure $\mu^{e}_{\Lambda}(d\varphi)$ giving the distribution of pinned points. We can obtain a formula for the ratio of partition functions appearing in the free energy:

$$\frac{Z^{e}_{\Lambda}}{Z^0_{\Lambda}} = \sum_{A\subset\Lambda} \left(\prod_{x\in A} e_x\right) \frac{Z^0_{\Lambda\setminus A}}{Z^0_{\Lambda}}. \tag{6.5}$$

6.2 Existence of quenched free energy

Theorem 6.1 *For $d \geqslant 2$, let $f^q_\Lambda(\boldsymbol{e}) \doteqdot |\Lambda|^{-1}\log(Z^{e}_{\Lambda}/Z^0_\Lambda)$ be the quenched free energy per site in $\Lambda \Subset \mathbb{Z}^d$. Then, the limit*

$$f^q(\boldsymbol{e}) \doteqdot \lim_{\Lambda\uparrow\mathbb{Z}^d} f^q_\Lambda(\boldsymbol{e}),$$

exists almost surely and in L^2, and does not depend on the sequence $\Lambda \uparrow \mathbb{Z}^d$ provided it satisfies $|\partial\Lambda|/|\Lambda| \to 0$. Moreover, $f^q(\boldsymbol{e})$ is non-random, i.e.

$$f^q(\boldsymbol{e}) = \mathbb{E}(f^q(\boldsymbol{e})) \quad a.s.$$

Proof We prove the existence of the limit along the sequence of boxes $B_n \doteqdot \Lambda_{2^n-1}$. The generalization to all sequences $\Lambda \uparrow \mathbb{Z}^d$ such that $|\partial\Lambda|/|\Lambda| \to 0$ is rather standard (cf. for example [88]).

Let us write $\alpha_{\boldsymbol{e}}(d\varphi) \doteqdot \prod_{x\in B_n}(d\varphi_x + e_x\delta_0(d\varphi_x))\prod_{y\notin B_n}\delta_0(d\varphi_y)$. We recall also notation (5.1) for the Hamiltonian. We will cut B_n in 2^d sub-boxes denoted by $B_{n-1}^{(i)}$. Let $X \doteqdot (\bigcup_{i=1}^{2^d}\partial B_{n-1}^{(i)})\backslash\partial B_n$ be the interface between the sub-boxes. In order to prove the existence of the limit along $(B_n)_n$, we first derive a "decoupling property". Namely there exist $c_n \geqslant 0$ such that $\sum_n c_n < \infty$ and $|Z_{B_n}^{\boldsymbol{e}} - \prod_{i=1}^{2^d} Z_{B_{n-1}}^{\boldsymbol{e}^{(i)}}| \leqslant c_n$ for any realization of $\boldsymbol{e}$, where $\boldsymbol{e}^{(i)}$ is the restriction of $\boldsymbol{e}$ to the box $B_{n-1}^{(i)}$. This allows us to prove that expectation of $f_{B_n}^{q}(\boldsymbol{e})$ converges, and its variance tends to zero, i.e. $f_{B_n}^{q}(\boldsymbol{e}) \to c \in \mathbb{R}$ almost surely, in L^1 and in L^2.

6.2.1 Lower bound on $Z_{B_n}^{\boldsymbol{e}}$

We have,

$$Z_{B_n}^{\boldsymbol{e}} = \int_{\varphi\in\mathbb{R}^{B_n}} e^{-\mathcal{H}(\varphi)}\alpha_{\boldsymbol{e}}(d\varphi) \geqslant \boldsymbol{e}^X \int_{\varphi|_X\equiv 0} e^{-\mathcal{H}(\varphi)}\alpha_{\boldsymbol{e}}(d\varphi)$$

$$\geqslant \underline{e}^{|X|}\prod_{i=1}^{2^d} Z_{B_{n-1}}^{\boldsymbol{e}^{(i)}} = \underline{e}^{d2^{n(d-1)}}\prod_{i=1}^{2^d} Z_{B_{n-1}}^{\boldsymbol{e}^{(i)}},$$

where $\boldsymbol{e}^{(i)}$ denotes $\boldsymbol{e}$ restricted to $B_{n-1}^{(i)}$. Hence,

$$f_{B_n}^{q}(\boldsymbol{e}) = 2^{-nd}\log\left(\frac{Z_{B_n}^{\boldsymbol{e}}}{Z_{B_n}^{0}}\right) \geqslant \frac{1}{2^d}\sum_{i=1}^{2^d} f_{B_{n-1}}^{\boldsymbol{e}^{(i)}} + C_{\min}2^{-n}.$$

for some constant $C_{\min} > 0$.

6.2.2 Upper bound on $Z_{B_n}^{\boldsymbol{e}}$

We will first prove that with high probability $|\varphi_i| \leqslant 2^{n\delta}$ for all $i \in X$ and some small $\delta > 0$. This will allow us to "force" φ to be zero on X for small energetic cost.

Lemma 6.1 *There exists $C_1, C_2 > 0$ such that for all $i \in \Lambda_n$ and for all n sufficiently large,*

$$\mu_n^{\boldsymbol{e}}(|\varphi_i| > T) \leqslant C_1 e^{-C_2 T^2/\log n}.$$

Proof Using (6.4),

$$\mu_n^{\boldsymbol{e}}(|\varphi_i| > T) \;=\; \sum_{A\subset\Lambda_n}\nu_n^{\boldsymbol{e}}(A)\cdot\mu_{\Lambda_n\backslash A}^{0}(|\varphi_i| > T)$$

Now, $\mu_{\Lambda_n\backslash A}^{0}$ is Gaussian, therefore, $\varphi_i \sim \mathcal{N}(0,\sigma_A)$ with

$$\sigma_A \doteqdot \mathrm{Var}_{\Lambda_n\backslash A}^{0}(\varphi_i) \leqslant \mathrm{Var}_{\Lambda_n}^{0}(\varphi_0) \leqslant \tilde{C}\log n$$

for some $\tilde{C} > 0$ and n large (cf. [25]). We can use the Gaussian tail estimate:

$$\mu^0_{\Lambda_n \setminus A}(|\varphi_i| > T) \leqslant C_1 e^{-C_2 T^2/\log n}.$$

■

In the sequel, the notation C, C', C'' will be used for positive constants that may change from line to line. Let us define $\tilde{X} \doteqdot X \cup (\partial X \cap B_n)$, which is a thickening of X consisting of 3 "layers". Then, Lemma 6.1 allow us to control the height of the field on $\tilde{X} \subset B_n$:

$$\mu^{\boldsymbol{e}}_{B_n}(\exists i \in \tilde{X} : |\varphi_i| > 2^{\delta n}) \leqslant \sum_{i \in \tilde{X}} \mu^{\boldsymbol{e}}_{B_n}(|\varphi_i| > 2^{\delta n}) \leqslant C 2^{n(d-1)} e^{-C_2 2^{\delta n}} \leqslant e^{-C 2^{\delta n}} \tag{6.6}$$

Hence,

$$\begin{aligned}
Z^{\boldsymbol{e}}_{B_n} &= (1 + C' e^{-C 2^{\delta n}}) \int_{\varphi|_{\tilde{X}} \in [-2^{\delta n}, 2^{\delta n}]} e^{-\mathcal{H}(\varphi)} \alpha_{\boldsymbol{e}}(d\varphi) \\
&\leqslant (1 + C' e^{-C 2^{\delta n}}) C'' e^{2^{2\delta n} 2^{n(d-1)}} \int_{\varphi|_X \equiv 0} \int_{\varphi|_{\partial X \cap B_n} \in [-2^{\delta n}, 2^{\delta n}]} e^{-\mathcal{H}(\varphi)} \alpha_{\boldsymbol{e}}(d\varphi) \\
&\leqslant (1 + C' e^{-C 2^{\delta n}}) C'' e^{2^{2\delta n} 2^{n(d-1)}} 2^{\delta n 2^{n(d-1)}} \int_{\varphi|_{B_n \setminus X}} e^{-\mathcal{H}(\varphi|_{B_n \setminus X})} \alpha_{\boldsymbol{e}}(d\varphi|_{B_n \setminus X}) \\
&= (1 + C' e^{-C 2^{\delta n}}) C'' e^{2^{n(2\delta + d - 1)}} 2^{\delta n 2^{n(d-1)}} \prod_{i=1}^{2^d} Z^{\boldsymbol{e}^{(i)}}_{B_{n-1}}.
\end{aligned}$$

This leads to

$$f^{\mathbf{q}}_{B_n}(\boldsymbol{e}) \leqslant \frac{1}{2^d} \sum_{i=1}^{2^d} f^{\boldsymbol{e}^{(i)}}_{B_{n-1}} + C 2^{n(\delta - d)} + C' 2^{n(2\delta - 1)} + C'' 2^{-n}.$$

Combining our two bounds, we obtain:

$$C_{\min} 2^{-n} \leqslant f^{\mathbf{q}}_{B_n}(\boldsymbol{e}) - \frac{1}{2^d} \sum_{i=1}^{2^d} f^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e}^{(i)}) \leqslant C_{\max} 2^{n(2\delta - 1)} \tag{6.7}$$

for some constants $C_{\min}, C_{\max} > 0$.

6.2.3 Expectation of $f^{\mathbf{q}}_{B_n}(\boldsymbol{e})$ converges and its variance tends to zero

By (6.7), it is easy to see that:

$$|\mathbb{E}(f^{\mathbf{q}}_{B_n}(\boldsymbol{e})) - \mathbb{E}(f^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e}))| \leqslant |\mathbb{E}(f^{\mathbf{q}}_{B_n}(\boldsymbol{e})) - \mathbb{E}(\frac{1}{2^d} \sum_{i=1}^{2^d} f^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e}^{(i)}))| \leqslant C 2^{n(2\delta - 1)} \tag{6.8}$$

The right hand side is summable for $\delta < 1/2$ hence $\mathbb{E}(f^{q}_{B_n}(e))$ converges as $n \to \infty$. Using independence of environment among the boxes $B^{(i)}_{n-1}$, we can deal with the variance. Let us write

$$f^{q}_{B_n}(e) = 2^{-d}\sum_{i=1}^{2^d} f^{q}_{B_{n-1}}(e^{(i)}) + \mathcal{E}_n$$

where $\mathcal{E}_n$ is the above error term, $|\mathcal{E}_n| \leqslant C' 2^{n(2\delta-1)}$. Then,

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(f^{q}_{B_n}(e)) &= \mathbb{V}\mathrm{ar}(2^{-d}\sum_{i=1}^{2^d} f^{q}_{B_{n-1}}(e^{(i)})) + \mathbb{V}\mathrm{ar}(\mathcal{E}_n) + 2\mathbb{C}\mathrm{ov}(2^{-d}\sum_{i=1}^{2^d} f^{q}_{B_{n-1}}(e^{(i)}), \mathcal{E}_n) \\
&= 2^{-d}\mathbb{V}\mathrm{ar}(f^{q}_{B_{n-1}}(e)) + \mathbb{V}\mathrm{ar}(\mathcal{E}_n) + 2\mathbb{C}\mathrm{ov}(f^{q}_{B_{n-1}}(e), \mathcal{E}_n) \\
&\leqslant 2^{-d}\mathbb{V}\mathrm{ar}(f^{q}_{B_{n-1}}(e)) + \mathbb{V}\mathrm{ar}(\mathcal{E}_n) + C' 2^{n(2\delta-1)}\mathbb{E}(f^{q}_{B_{n-1}}(e))
\end{aligned}$$

Now, since $(\mathbb{E}(f^{q}_{B_{n-1}}(e)))_n$ converges, we get the following upper bound on the variance:

$$\mathbb{V}\mathrm{ar}(f^{q}_{B_n}(e)) \leqslant 2^{-d}\mathbb{V}\mathrm{ar}(f^{q}_{B_{n-1}}(e)) + C 2^{2n(2\delta-1)} + C' 2^{n(2\delta-1)} \tag{6.9}$$

We deduce $\mathbb{V}\mathrm{ar}(f^{q}_{B_n}(e)) \to 0$, and conclude $f^{q}_{B_n}(e) \to c \in \mathbb{R}$ in L^2. The sequence converges also almost surely, since all error terms are summable. ■

Exactly the same proof until (6.7) yields

Theorem 6.2 *For* $d \geqslant 2$, *let* $f^{a}_{\Lambda} \doteqdot |\Lambda|^{-1}\log(\mathbb{E}Z^{e}_{\Lambda}/Z^{0}_{\Lambda})$ *be the annealed free energy per site in* $\Lambda \Subset \mathbb{Z}^d$. *Then, the limit*

$$f^{a} \doteqdot \lim_{\Lambda\uparrow\mathbb{Z}^d} f^{a}_{\Lambda}$$

exists and does not depend on the sequence $\Lambda \uparrow \mathbb{Z}^d$ *provided it satisfies* $|\partial\Lambda|/|\Lambda| \to 0$.

6.3 Bounds on the quenched free energy

6.3.1 Strict positivity of the quenched free energy

In this section we prove that the free energy of the model is strictly positive for all environments $(\underline{e}, \overline{e}, p)$ such that $0 < \underline{e} < \overline{e}$ and $p \in (0,1)$. That implies the localization of the field, namely the strict positivity of the limiting density of pinned sites. Our results are valid for all $d \geqslant 2$.

Fact 6.1 *For all* $(\underline{e}, \overline{e}, p)$ *such that* $0 < \underline{e} < \overline{e}$ *and* $p \in (0,1)$, *we have* $f^{q} > 0$.

Proof Using (6.5), and the fact that $0 < \underline{e} < \overline{e}$, we can write

$$f^{q}(\boldsymbol{e}) = \lim_{n\to\infty} n^{-d} \log\left(\sum_{A\subset\Lambda_n}\prod_{x\in A} e_x \frac{Z^0_{\Lambda_n\setminus A}}{Z^0_{\Lambda_n}}\right) \geqslant \lim_{n\to\infty} n^{-d} \log\left(\sum_{A\subset\Lambda_n} \underline{e}^{|A|}\frac{Z^0_{\Lambda_n\setminus A}}{Z^0_{\Lambda_n}}\right)$$

which is the free energy in the deterministic case $e_x = \underline{e}$ for all x. Moreover, it follows from [16] Theorem 2.4, that the asymptotic density of pinned points under $\mu^{\underline{e}}_{\Lambda_n}$ is strictly positive uniformly in Λ_n for n large enough. This immediately implies by definition of the free energy that the righmost term is strictly positive. ■

Fact 6.2 *Let* $0 < \underline{e} < \overline{e}$ *and* $p \in (0,1)$ *then there exists* $c > 0$ *such that for almost every realization* $\boldsymbol{e}$ *we have*

$$\lim_{n\to\infty} \mu^{\boldsymbol{e}}_{\Lambda_n}\left(n^{-d}\sum_{x\in\Lambda_n} \mathbb{1}_{[\varphi_x=0]} > c\right) = 1.$$

Remark 6.2 *More is known for* $d = 1$. *By [9] there exists* $\rho = \rho(e^\star, \sigma)$ *such that*

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \mu^{\boldsymbol{e}}_{\Lambda_n}\left(n^{-d}\sum_{x\in\Lambda_n} \mathbb{1}_{[\varphi_x=0]} \in (\rho-\epsilon, \rho+\epsilon)\right) = 1.$$

Proof Let $\delta > 0$, and $A_\delta \doteqdot \{\sum_{x\in\Lambda_n} \mathbb{1}_{[\varphi_x=0]} \leqslant \delta n^d\}$. Let us fix some $\epsilon > 0$. By definition we have

$$\mu^{\boldsymbol{e}}_{\Lambda_n}(A_\delta) = \frac{Z^\epsilon_{\Lambda_n}}{Z^0_{\Lambda_n}}\frac{Z^0_{\Lambda_n}}{Z^{\boldsymbol{e}}_{\Lambda_n}}(Z^\epsilon_{\Lambda_n})^{-1}\int \mathbb{1}_{[A_\delta]} e^{-\mathcal{H}(\varphi)}\prod_{x\in\Lambda}(d\varphi + e_x\delta_0(d\varphi_x)).$$

Using the pinned sites decomposition (6.4) we get:

$$\frac{Z^{\boldsymbol{e}}_{\Lambda_n}}{Z^\epsilon_{\Lambda_n}} = \sum_{A\subset\Lambda_n} \boldsymbol{e}^A \frac{Z^0_{\Lambda_n\setminus A}}{Z^\epsilon_{\Lambda_n}} = \sum_{A\subset\Lambda_n}\frac{\boldsymbol{e}^A}{\epsilon^{|A|}}\epsilon^{|A|}\frac{Z^0_{\Lambda_n\setminus A}}{Z^\epsilon_{\Lambda_n}} = \nu^\epsilon_n\left(\boldsymbol{\gamma}^{\mathcal{A}}\right), \tag{6.10}$$

where $\boldsymbol{e}^A \doteqdot \prod_{x\in A} e_x$, $\boldsymbol{\gamma} = (\gamma_x)_{x\in\Lambda_n}$ and $\gamma_x \doteqdot e_x/\epsilon$. Similarly,

$$(Z^\epsilon_{\Lambda_n})^{-1}\int \mathbb{1}_{[A_\delta]} e^{-\mathcal{H}(\varphi)}\prod_{x\in\Lambda}(d\varphi + e_x\delta_0(d\varphi_x)) = \nu^\epsilon_n\left(\boldsymbol{\gamma}^{\mathcal{A}}; |\mathcal{A}| \leqslant n^d\delta\right) \leqslant (\overline{e}/\epsilon)^{n^d\delta}.$$

We thus have

$$n^{-d}\log\mu^{\boldsymbol{e}}_{\Lambda_n}(A_\delta) \leqslant \delta\log(\overline{e}/\epsilon) + n^{-d}\log\frac{Z^\epsilon_{\Lambda_n}}{Z^0_{\Lambda_n}} - n^{-d}\log\frac{Z^{\boldsymbol{e}}_{\Lambda_n}}{Z^0_{\Lambda_n}}.$$

By Fact 6.1 the last term $n^{-d}\log(Z^{\mathbf{e}}_{\Lambda_n}/Z^{0}_{\Lambda_n})$ converges to a strictly positive quantity. Moreover, it is not difficult to show that $Z^{\epsilon}_{\Lambda_n} \leqslant Z^{0}_{\Lambda_n}(1+C\epsilon)^{n^d}$ for some $C>0$ uniformly in n, hence the second term tends to 0 uniformly in n as $\epsilon \to 0$. Choosing $\epsilon > 0$ and $\delta > 0$ small enough we get

$$\limsup_{n\to+\infty} n^{-d}\log \mu^{\mathbf{e}}_{\Lambda_n}(A_\delta) < 0.$$

and the result follows. ■

6.3.2 Strict inequality between quenched and annealed free energies

Theorem 6.3 *Take* $(p, \underline{e}, \overline{e})$ *such that* $p \in (0,1)$ *and* $0 < \underline{e} < \overline{e}$. *Let* $\mathbf{e}$ *be an i.i.d. Bernoulli environment of parameters* $(p, \underline{e}, \overline{e})$ *as described in (6.2). We recall the notations* $e^\star = p\underline{e} + (1-p)\overline{e}$ *and* $\sigma^2 = \underline{e}^2 p + \overline{e}^2(1-p) - e^{\star 2}$.

(a) Let $d \geqslant 3$. *Then,*

$$f^q < f^a.$$

Moreover, there exists $c_3(d,p) > 0$ *such that,*

$$f^a - f^q \geqslant c_3(d,p)\sigma + O(\sigma^2) \quad \text{as} \quad \sigma \to 0.$$

(b) Let $d = 2$. *There exists* $\epsilon > 0$ *uniform in* $p, \underline{e}, \overline{e}$ *such that if* $e^\star < \epsilon$, *then*

$$f^q < f^a.$$

Moreover, under the same hypothesis on $e^\star$, *there exist* $c_2(p, e^\star) > 0$ *such that,*

$$f^a - f^q \geqslant c_2(p, e^\star)\sigma + O(\sigma^2) \quad \text{as} \quad \sigma \to 0.$$

The proof will be presented in more generality in the next chapter.

Chapter 7

Generalisation: square potential with attractive/repulsive environment of arbitrary law

This chapter contains the main part of the results obtained in collaboration with Piotr Miłoś. The corresponding article [LC2] has been accepted for publication in the journal *Stochastic Processes and their Applications* in April 2013.

7.1 The model

Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ be the set of configurations. The finite volume Gibbs measure in Λ for the discrete Gaussian Free Field with disordered square-well potential, and 0 boundary conditions, is the probability measure on Ω defined by:

$$\mu_\Lambda^{e}(d\varphi) = \frac{1}{Z_\Lambda^{e}} \exp\left(-\beta\mathcal{H}_\Lambda(\varphi) + \beta\sum_{x\in\Lambda}(b\cdot e_x + h)\mathbb{1}_{[\varphi_x\in[-a,a]]}\right)\prod_{x\in\Lambda} d\varphi_x \prod_{y\in\Lambda^c}\delta_0(d\varphi_y). \tag{7.1}$$

where $a, \beta, b > 0$, $h \in \mathbb{R}$ and $\mathcal{H}_\Lambda(\varphi)$ is given by (5.1). We consider the environment $\boldsymbol{e} \doteqdot (e_x)_{x\in\Lambda}$ to be an i.i.d. family of random variables such that $\mathbb{E}(e_x) = 0$, $\mathbb{V}\mathrm{ar}(e_x) = 1$, and

$$\mathbb{E}(e^{b\cdot e_x + h}) < \infty. \tag{7.2}$$

We recall that $\mathbb{E}Z_\Lambda^{e} = Z_\Lambda^{\ell}$ with $\ell \doteqdot \log(\mathbb{E}(e^{b\cdot e_x+h}))$.

The inverse temperature parameter β enters only in a trivial way. Indeed, if we replace the field $(\varphi_x)_{x\in\Lambda}$ by $(\sqrt{\beta}\phi_x)_{x\in\Lambda}$, a by $\sqrt{\beta}a$, and $(b\cdot e_x + h)_{x\in\Lambda}$ by $(\beta(b\cdot e_x + h))_{x\in\Lambda}$ we have transformed the model to temperature parameter $\beta = 1$. In the sequel, we will therefore work with $\beta = 1$.

Below we prove that the quenched free energy exists, is non-random, and strictly smaller than the annealed free energy whenever the annealed free energy is positive.

7.2 New results

7.2.1 Existence of the free energy

Theorem 7.1 *Let* $d \geqslant 2$ *and* $\boldsymbol{e}$ *be an environment such that* (7.2) *holds. Then the limit*

$$f^{q}(\boldsymbol{e}) \doteqdot \lim_{\Lambda \uparrow \mathbb{Z}^d} f^{q}_{\Lambda}(\boldsymbol{e}) \in \mathbb{R},$$

exists almost surely and in L^2*, and does not depend on the sequence* $\Lambda \uparrow \mathbb{Z}^d$ *provided that* $|\partial\Lambda|/|\Lambda| \to 0$.
Moreover, $f^{q}(\boldsymbol{e})$ *is deterministic, i.e.*

$$f^{q}(\boldsymbol{e}) = \mathbb{E}(f^{q}(\boldsymbol{e})) \quad a.s.$$

The following is a straightforward corollary.

Corollary 7.1 *Let* $d \geqslant 2$ *and* $\boldsymbol{e}$ *be an environment such that* (7.2) *holds. Then the limit*

$$f^{a} \doteqdot \lim_{\Lambda \uparrow \mathbb{Z}^d} f^{a}_{\Lambda},$$

exists in $\mathbb{R}$ *and does not depend on the sequence* $\Lambda \uparrow \mathbb{Z}^d$ *provided that* $|\partial\Lambda|/|\Lambda| \to 0$.

Below we present the proof of Theorem 7.1. The heuristics is close to the one developed in the last chapter for delta-pinning. However, there are a few technical issues which need to be clarified. They stem for the square potential and the fact that we put only minimal conditions of the environment.

In the main part of the proof it will be easier to deal with environments with bounded support. Thus we start with a truncation argument. Let $H > 1$, we define a new environment $\boldsymbol{e}^H$ by $e^H_x \doteqdot \tilde{e}_x \mathbb{1}_{[\tilde{e}_x \in [-H,H]]}$ with $\tilde{e}_x \doteqdot b e_x + h$. We claim that:

$$\limsup_{H \uparrow \infty} \limsup_{\Lambda \uparrow \mathbb{Z}^d} |f^{q}_{\Lambda}(\boldsymbol{e}) - f^{q}_{\Lambda}(\boldsymbol{e}^H)| = 0. \tag{7.3}$$

Proof By definition of the free energy and a direct calculation we obtain

$$\begin{aligned} f^{q}_{\Lambda}(\boldsymbol{e}) - f^{q}_{\Lambda}(\boldsymbol{e}^H) &= |\Lambda|^{-1} \log \mu^{\boldsymbol{e}^H}_{\Lambda}\left(\exp\left(\sum_{x\in\Lambda} \tilde{e}_x \mathbb{1}_{[\tilde{e}_x \in [-H,H]^c]} \mathbb{1}_{[\varphi_x \in [-a,a]]}\right)\right) \\ &\leqslant |\Lambda|^{-1} \log \mu^{\boldsymbol{e}^H}_{\Lambda}\left(\exp\left(\sum_{x\in\Lambda} \tilde{e}_x \mathbb{1}_{[\tilde{e}_x \in [H,\infty)]} \mathbb{1}_{[\varphi_x \in [-a,a]]}\right)\right) \leqslant |\Lambda|^{-1} \sum_{x\in\Lambda} \tilde{e}_x \mathbb{1}_{[\tilde{e}_x \in [H,\infty)]}. \end{aligned}$$

As $\Lambda \uparrow \mathbb{Z}^d$ the last expression converges to $\mathbb{E}(\tilde{e}_0 \mathbb{1}_{[\tilde{e}_0 \in [H,\infty)]})$, which in turn converges to 0 as $H \to \infty$. Further we observe that

$$f^{q}_{\Lambda}(\boldsymbol{e}) - f^{q}_{\Lambda}(\boldsymbol{e}^H) \geqslant |\Lambda|^{-1} \log \mu^{\boldsymbol{e}^H}_{\Lambda} \left(\exp\left(\sum_{x\in\Lambda} \tilde{e}_x \mathbb{1}_{[\tilde{e}_x \in [H,\infty)]} \mathbb{1}_{[\varphi_x \in [-a,a]]} \right) \mathbb{1}_{[\varphi_x \notin [-a,a]\ \forall x \in A]} \right),$$

where $A = \{x \in \Lambda : \tilde{e}_x \leqslant -H\}$. Let $a_1, a_2, \ldots, a_N$ be some enumeration of point in A and $A_i = \{a_1, a_2, \ldots, a_i\}$, where $i \in \{1, 2, \ldots, N\}$. We write

$$\begin{aligned} f^{q}_{\Lambda}(\boldsymbol{e}) - f^{q}_{\Lambda}(\boldsymbol{e}^H) &\geqslant |\Lambda|^{-1} \log \mu^{\boldsymbol{e}^H}_{\Lambda}(\varphi_x \notin [-a,a]\ \forall x \in A) \\ &= |\Lambda|^{-1} \sum_{i=1}^{N} \log \mu^{\boldsymbol{e}^H}_{\Lambda}(\varphi_{x_i} \notin [-a,a] | \varphi_x \notin [-a,a]\ \forall x \in A_{i-1}). \end{aligned} \tag{7.4}$$

For any $x_i \in A$, by the spatial Markov property, we have

$$\mu^{\boldsymbol{e}^H}_{\Lambda}(\varphi_{x_i} \notin [-a,a] | \varphi_x, x \sim x_i) = Z^{-1} \mathbf{E}\left(\mathbb{1}_{[G \notin [-a,a]]} e^{\tilde{e}_{x_i} \mathbb{1}_{[G \in [-a,a]]}} \right), \tag{7.5}$$

where G denotes a Gaussian random variable $\mathcal{N}((\sum_{x\sim x_i} \varphi_x)/2d, 1)$ (which law is denoted $\mathbf{P}$ and the corresponding expectation $\mathbf{E}$) and Z is the normalizing constant $Z \doteqdot \mathbf{E}\left(e^{\tilde{e}_{x_i} \mathbb{1}_{[G\in[-a,a]]}}\right)$. We recall that at site x_i we have necessarily $\tilde{e}_{x_i} < 0$ hence $Z \leqslant 1$. Finally, the expression (7.5) is lower bounded by

$$\mathbf{E}\left(\mathbb{1}_{[G \notin [-a,a]]} e^{\tilde{e}_{x_i} \mathbb{1}_{[G \in [-a,a]]}} \right) = \mathbf{P}(G \notin [-a,a]) \geqslant C,$$

for some $C > 0$ independent of $(\sum_{x\sim x_i} \varphi_x)/2d$. We use this to estimate (7.4) as follows

$$f^{q}_{\Lambda}(\boldsymbol{e}) - f^{q}_{\Lambda}(\boldsymbol{e}^H) \geqslant |\Lambda|^{-1} N \log C \to \mathbb{P}(\tilde{e}_x \leqslant -H) \cdot \log C \quad \text{as} \quad \Lambda \uparrow \mathbb{Z}^d.$$

Like previously this converges to 0 as $H \to \infty$, which concludes the proof of (7.3). ■

Further we will assume that the law of the environment has a bounded support. Moreover, by the well-known fact that the free energy of the Gaussian free field exists, i.e. $\lim_{\Lambda\uparrow\mathbb{Z}^d} |\Lambda|^{-1} \log Z^0_{\Lambda}$ exist and is finite, it is enough to prove the existence of the limit of $\tilde{f}^{q}_{\Lambda}(\boldsymbol{e})$ given by

$$\tilde{f}^{q}_{\Lambda}(\boldsymbol{e}) \doteqdot |\Lambda|^{-1} \log Z^{\boldsymbol{e}}_{\Lambda}.$$

We prove convergence only along a sequence of boxes $B_n = \Lambda_{2^n-1}$. The extension to the case of general sequences is standard. We will cut B_n in 2^d sub-boxes denoted by B^i_{n-1}. Let $X = (\bigcup_{i=1}^{2^d} \partial B^i_{n-1}) \setminus \partial B_n$ be "the border" between the sub-boxes. In order to prove the existence of the limit along $(B_n)_n$, we first derive a "decoupling property". Namely, there exists $c_n \geqslant 0$ such that $\sum_n c_n < \infty$ and

$$|Z^{\boldsymbol{e}}_{B_n} - \prod_{i=1}^{2^d} Z^{\boldsymbol{e}^i}_{B^i_{n-1}}| \leqslant c_n,$$

for any realization of $\boldsymbol{e}$, where $\boldsymbol{e}^i$ is the restriction of $\boldsymbol{e}$ to the box B^i_{n-1}.

The next lemma provides us with control over the concentration of the field.

Lemma 7.1 *There exist* $C_1, C_2, C_3 > 0$ *such that for* n *sufficiently large, for all* $x \in \Lambda_n$ *and* $T > C_3 \log n$ *we have*

$$\mu^{e}_{\Lambda_n}(|\varphi_x| > T) \leqslant C_1 e^{-C_2 T^2/\log n}.$$

Proof Let $\mathcal{A} \doteqdot \{x : \varphi_x \in [-a, a]\}$. We first use the following decomposition, which is a reordering of the pinning contributions over subsets of Λ:

$$\mu^{e}_{\Lambda_n}(|\varphi_x| > T) \;=\; \sum_{A \subset \Lambda} \left(\left(\prod_{x \in A} e^{b e_x + h} \right) \frac{Z^0_{\Lambda_n}(\mathcal{A} = A)}{Z^{e}_{\Lambda_n}} \right) \mu^0_{\Lambda_n}(|\varphi_x| > T \mid \mathcal{A} = A).$$

It is sufficient to upper-bound the rightmost term uniformly in A. Using the FKG inequality (see e.g. [43, Section B.1]) it is standard to check that

$$\mu^0_{\Lambda_n}(d\varphi_x \mid \mathcal{A} = A) \prec \mu^0_{\Lambda_n}(d\varphi_x \,|\, \forall y \in \Lambda_n,\ \varphi_y \geqslant a),$$

where $\prec$ denotes the stochastic domination. Further, we intend to use the Brascamp-Lieb inequality. To this end we first estimate

$$A_n \doteqdot \mu^0_{\Lambda_n}(\varphi_x \,|\, \forall y \in \Lambda_n,\ \varphi_y \geqslant a).$$

For $d \geqslant 3$ it follows easily by [43, Theorem 3.1] that $A_n \leqslant C\sqrt{\log n}$ for some $C > 0$. For $d = 2$ let us assume first that $A_n \geqslant C \log n$ for some large C. By [43, (B.14)] we have $\mu^0_{\Lambda_n}(\varphi_i^2 \,|\, \forall y \in \Lambda_n,\ \varphi_y \geqslant a) \leqslant \log n + A_n^2$. Using the Paley-Zygmund inequality we get

$$\mu^0_{\Lambda_n}(\varphi_i \geqslant A_n/2 \,|\, \forall y \in \Lambda_n,\ \varphi_y \geqslant a) \geqslant \frac{1}{4} \cdot \frac{A_n^2}{\log n + A_n^2} \geqslant 1/8,$$

for n large enough. Increasing C further, if necessary, we get a contradiction with [14, Theorem 4]. Thus, $A_n \leqslant C \log n$. Finally by the Brascamp-Lieb inequality [43, Section B.2] we obtain

$$\mu^0_{\Lambda_n}(\varphi_x \geqslant T \,|\, \forall y \in \Lambda_n,\ \varphi_y \geqslant a) \;\leqslant\; \exp(-\tilde{C}_2 (T - A_n)^2/\log n) \;\leqslant\; C_1 \exp(-C_2 T^2/\log n)$$

as announced. ∎

Let us define $\tilde{X} = X \cup (\partial X \cap B_n)$, which is a thickening of X consisting of 3 “layers”. Then, Lemma 7.1 allows us to control the height of the field on $\tilde{X} \subset B_n$. Let us fix some $\delta > 0$, we have

$$\mu^{e}_{B_n}(\exists i \in \tilde{X} : |\varphi_i| > 2^{\delta n}) \leqslant \sum_{i \in \tilde{X}} \mu^{e}_{B_n}(|\varphi_i| > 2^{\delta n}) \leqslant e^{-C 2^{\delta n}}, \tag{7.6}$$

for some $C > 0$. We denote the full Hamiltonian of our system (i.e. not only the Gaussian part) by $\mathcal{K}(\varphi)$. Using (7.6) we deduce

$$Z^{e}_{B_n} \leqslant (1 + 2e^{-C 2^{\delta n}}) \int_{\mathbb{R}^{B_n}} \mathbb{1}_{[|\varphi_x| \leqslant 2^{\delta n}, \forall x \in \tilde{X}]} e^{-\mathcal{K}(\varphi)} d\varphi.$$

Given φ we define $\tilde{\varphi}$ by setting $\varphi_x = 0$ for $x \in X$ and $\tilde{\varphi}_x = \varphi_x$ otherwise. It is easy to check that for φ fulfilling $|\varphi_x| \leqslant 2^{\delta n}$ for any $x \in \tilde{X}$ we have $|\mathcal{K}(\varphi) - \mathcal{K}(\tilde{\varphi})| \leqslant C|X|2^{2\delta n}$, for some $C > 0$ (here we used the fact that the support of the law of the environment is bounded). Therefore

$$Z^{\boldsymbol{e}}_{B_n} \leqslant (1 + C' e^{-C2^{\delta n}}) e^{C|X|2^{2\delta n}} \int_{[-2^{\delta n}, 2^{\delta n}]^X} \prod_i \int_{\mathbb{R}^{B^i_n}} \mathbb{1}_{[|\varphi_x| \leqslant 2^{\delta n}, \forall x \in \tilde{X}]} e^{-\mathcal{K}(\tilde{\varphi})} d\varphi.$$

We notice that $\tilde{\varphi}$ "enforces" 0 boundary conditions on X. Consequently, each of the inner integrals is bounded from above by $Z^{\boldsymbol{e}^i}_{B^i_n}$. This leads to

$$\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}) \leqslant \frac{1}{2^d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i) + C_{max} 2^{n(2\delta - 1)},$$

for some $C_{max} \geqslant 0$. Now let us prove a bound from below. We have

$$Z^{\boldsymbol{e}}_{B_n} = \int_{\mathbb{R}^{B_n}} e^{-\mathcal{K}(\varphi)} d\varphi \geqslant \int_{\mathbb{R}^{B_n}} \mathbb{1}_{[\varphi \in A]} e^{-\mathcal{K}(\varphi)} d\varphi,$$

where $A = \{\varphi : \varphi_x \in (-a, a), \forall x \in X \text{ and } |\varphi_x| \leqslant n^3, \forall x \in \tilde{X}\}$. Let $\varphi \in A$ we define $\tilde{\varphi}$ by setting $\varphi_x = 0$ for $x \in X$ and $\tilde{\varphi}_x = \varphi_x$ otherwise. It is easy to check that $|\mathcal{K}(\varphi) - \mathcal{K}(\tilde{\varphi})| \leqslant C|X|n^3$, for some $C > 0$ (again we use the fact that the environment is bounded). Therefore

$$Z^{\boldsymbol{e}}_{B_n} \geqslant e^{-C|X|n^3} \int_{[-a, a]^X} \prod_i \int_{\mathbb{R}^{B^i_n}} \mathbb{1}_{[\varphi \in A]} e^{-\mathcal{K}(\tilde{\varphi})} d\varphi.$$

Let us consider an integral in the product. Using Lemma 7.1 similarly as before we check that it is bounded from below by $\left(1 - |X| e^{-Cn}\right) Z^{\boldsymbol{e}^i}_{B^i_n}$. Hence

$$\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}) \geqslant \frac{1}{2^d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i) - C_{\min} 2^{-n/2}.$$

for some constant $C_{\min} \geqslant 0$. Combining our two bounds above we obtain

$$C_{\min} 2^{-n/2} \leqslant \tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}) - \frac{1}{2^d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i) \leqslant C_{\max} 2^{n(2\delta - 1)}. \tag{7.7}$$

By (7.7), it is easy to see that:

$$|\mathbb{E}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e})) - \mathbb{E}(\tilde{f}^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e}))| \leqslant \left| \mathbb{E}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e})) - \mathbb{E}\Big(\frac{1}{2^d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i) \Big) \right| \leqslant 2^{-cn}, \tag{7.8}$$

for some $c > 0$ as soon as $\delta < 1/2$ and n is large enough. The right hand side is thus summable hence $\mathbb{E}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}))$ converges as $n \to \infty$ to some limit $L \in \mathbb{R}$. Using independence of environment among the boxes B^i_{n-1}, we can estimate the variance. Let us write

$$\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}) = 2^{-d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i) + \mathcal{E}_n,$$

where $\mathcal{E}_n$ is the (random) error term above. We have $|\mathcal{E}_n| \leqslant 2^{-cn}$. Thus

$$\begin{aligned} \mathbb{V}\mathrm{ar}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e})) &= \mathbb{V}\mathrm{ar}\Big(2^{-d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i)\Big) + \mathbb{V}\mathrm{ar}(\mathcal{E}_n) + \mathbb{C}\mathrm{ov}\Big(2^{-d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i), \mathcal{E}_n\Big) \\ &\leqslant 2^{-d}\mathbb{V}\mathrm{ar}(\tilde{f}^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e})) + \mathbb{V}\mathrm{ar}(\mathcal{E}_n) + 2^{-cn}\mathbb{E}\Big(2^{-d} \sum_{i=1}^{2^d} \tilde{f}^{\mathbf{q}}_{B^i_{n-1}}(\boldsymbol{e}^i)\Big). \end{aligned}$$

Now, since $(\mathbb{E}(\tilde{f}^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e})))_n$ converges, we get the following inductive upper bound on the variance:

$$\mathbb{V}\mathrm{ar}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e})) \leqslant 2^{-d}\mathbb{V}\mathrm{ar}(\tilde{f}^{\mathbf{q}}_{B_{n-1}}(\boldsymbol{e})) + 2^{-2cn} + 2L2^{-cn},$$

as long as n is large enough. We deduce that for some $b \in (0,1)$ and n large enough we have $\mathbb{V}\mathrm{ar}(\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e})) \leqslant b^n$. This yields $\tilde{f}^{\mathbf{q}}_{B_n}(\boldsymbol{e}) \to L$ both in L^2 and almost surely.

7.2.2 Positivity of the free energy

Fact 7.1 *Let $d \geqslant 2$ and $\mathbf{e}$ be an environment such that* (7.2) *holds. Then,*

$$f^{\mathbf{q}} \geqslant 0.$$

Proof By definition of the free energy we have

$$\begin{aligned} f^{\mathbf{q}} &= \lim_{n\to\infty} n^{-d} \log \mu^0_{\Lambda_n}\left(\exp\left(\sum_{x\in\Lambda_n} (b\cdot e_x + h)\mathbb{1}_{[|\varphi_x|\leqslant a]}\right)\right) \\ &\geqslant \lim_{n\to\infty} n^{-d} \log \mu^0_{\Lambda_n}\left(\varphi_x \geqslant a\,,\, \forall x \in \Lambda_n\right). \end{aligned}$$

One easily checks that

$$\begin{aligned} \mu^0_{\Lambda_n}(\varphi_x \geqslant a\,,\, \forall x\in\Lambda_n) \quad &\geqslant \quad \mu^0_{\Lambda_n}(\varphi_x \geqslant a\,,\, \forall x \in \Lambda_{n-1} | \varphi_x \in [a, a{+}1]\,,\, \forall x \in \partial\Lambda_{n-1}) \\ &\qquad \times \mu^0_{\Lambda_n}(\varphi_x \in [a, a+1]\,,\, \forall x \in \partial\Lambda_{n-1}). \end{aligned}$$

The second factor is of order $e^{-Cn^{d-1}}$ and thus is negligible. The first one is bounded from below by $I_n = \mu^0_{\Lambda_{n-1}}(\varphi_x \geqslant 0\,,\, \forall x \in \Lambda_{n-1})$. Using Gaussian tail estimates and the FKG inequality, we have for n large enough:

$$\begin{aligned} I_n &\geqslant \mu^0_{\Lambda_{n-1}}(\varphi_x \geqslant 0,\ \forall x \in \Lambda_{n-1} \,|\, \varphi_x > \log n,\ \forall x \in \partial\Lambda_{n-2}) \cdot e^{-Cn^{d-1}(\log n)^2} \\ &\geqslant \left(1 - \sum_{x \in \Lambda_{n-2}} \mu^0_{\Lambda_{n-2}}(|\varphi_x| > \log n)\right) \cdot e^{-Cn^{d-1}(\log n)^2} \\ &\geqslant (1 - e^{-c(\log n)^2}) \cdot e^{-Cn^{d-1}(\log n)^2} \\ &\geqslant 1/2 \cdot e^{-Cn^{d-1}(\log n)^2} \end{aligned}$$

We obtain $\liminf_{n\to\infty} n^{-d} \log I_n \geqslant 0$, which concludes the proof. ■

7.2.3 Strict inequality between quenched and annealed free energies

In this section we state the main results of our work. Before that we recall that we assume $b > 0$ hence the disorder is always non-trivial.

Theorem 7.2 *Let $d \geqslant 2$ and $\mathbf{e}$ be an environment such that (7.2) holds. Then*

$$f^q < f^a \quad \textit{whenever} \quad f^a > 0.$$

Moreover, the following quantitative bounds hold. Let $\gamma \doteqdot \exp(b \cdot e_0 + h - \ell)$, then

(a) For $d \geqslant 3$ we have

$$f^q - f^a \leqslant \mathbb{E} \log(\lambda\gamma + 1 - \lambda),$$

where $\lambda \doteqdot \frac{C_1 \ell}{1 + C_1 \ell}$ for some $C_1 = C_1(a) > 0$.

(b) For $d = 2$ we have

$$f^q - f^a \leqslant \mathbb{E} \log\left(\frac{\lambda}{|\log \lambda|}\gamma + 1 - \frac{\lambda}{|\log \lambda|}\right),$$

for some $\lambda = \lambda(\ell) > 0$ which equal to $\frac{C_2 \ell}{\sqrt{|\log \ell|}}$ with $C_2 = C_2(a) > 0$ for ℓ small enough.

For the annealed model, using a variant of the argument in the proof of [16, Theorem 2.4], it is possible to show that

Fact 7.2 *The annealed free energy is a non-decreasing function of* ℓ *such that* $f^a = 0$ *whenever* $\ell < 0$. *Moreover, for* $d \geqslant 3$ *there exist a constant* $C_d > 0$ *such that*

$$f^a = C_d \ell(1 + o(1)) \quad \text{as } \ell \to 0,$$

For $d = 2$ *there exists a constant* C_2 *such that*

$$f^a = C_2 \frac{\ell}{\sqrt{|\log(\ell)|}}(1 + o(1)) \quad \text{as } \ell \to 0$$

The constant C_d *can be computed explicitely for all* $d \geqslant 2$.

Remark 7.1 1. *The explicit expression for* $\lambda(\ell)$ *for large* ℓ *in dimension 2 could be a priori derived by a method similar to the one developed in [16, 57]. One should keep track, though, of the dependency in* ℓ *of the size of all the boxes. This information is of little relevance here.*

2. *It follows from Fact 7.2 that in all dimensions* $d \geqslant 2$ *we have* $\lambda = \tilde{C}_d f^a$ *for* ℓ *small enough and some constants* $\tilde{C}_d = \tilde{C}_d(a) > 0$.

As an example, we can compute the bounds for concrete environment laws. Let the environment be given by the Bernoulli random variables $\mathbb{P}(e_x = -1) = \mathbb{P}(e_x = 1) = 1/2$. Then there exists a constant $C > 0$ such that for b, h small enough and $\ell > 0$ we have

$$f^q - f^a \leqslant \begin{cases} -Cb^2(b^2/2 + h) + o(b^2(b^2/2 + h)) & \text{in } d \geqslant 3, \\ -Cb^2 \frac{b^2/2+h}{|\log(b^2/2+h)|^{3/2}} + o\left(b^2 \frac{b^2/2+h}{|\log(b^2/2+h)|^{3/2}}\right) & \text{in } d = 2 \end{cases}$$

We recall that for $d \geqslant 3$, we have $f^a \approx \ell \approx b^2/2 + h$. We note that the condition $\ell > 0$ yields $b^2/2 + h > 0$ hence the expression above is well-defined.

The same estimates hold for the Gaussian environment i.e. $e_x \sim \mathcal{N}(0, 1)$.

Proof [of Theorem 7.2] We perform calculations which resemble the so-called high-temperature expansion

$$\begin{aligned}
\mu_\Lambda^{e}(d\varphi) &= \frac{1}{Z_\Lambda^{e}} \exp\left(-\mathcal{H}_\Lambda(\varphi) + \sum_{x\in\Lambda} (b \cdot e_x + h)\mathbb{1}_{[\varphi_x \in [-a,a]]}\right) \prod_{x\in\Lambda} d\varphi_x \prod_{y\in\Lambda^c} \delta_0(d\varphi_y) \\
&= \frac{1}{Z_\Lambda^{e}} \exp\left(-\mathcal{H}_\Lambda(\varphi)\right) \prod_{x\in\Lambda} \left((e^{b\cdot e_x + h} - 1)\mathbb{1}_{[\varphi_x\in[-a,a]]} + 1\right) \prod_{x\in\Lambda} d\varphi_x \prod_{y\in\Lambda^c} \delta_0(d\varphi_y) \\
&= \sum_{A\subset\Lambda} \underbrace{\left(\prod_{x\in A} (e^{b\cdot e_x + h} - 1) \frac{Z_\Lambda^0(\varphi_x \in [-a,a], \forall x \in A)}{Z_\Lambda^{e}}\right)}_{=\nu_\Lambda^{e}(A)} \times \qquad (7.9) \\
&\qquad \times \quad \mu_\Lambda^0(d\varphi \mid \varphi_x \in [-a,a], \forall x \in A).
\end{aligned}$$

We observe that when $b \cdot e_x + h \geqslant 0$ for all x in the domain Λ, then ν^e_Λ is a probability measure (otherwise some weights $e^{b \cdot e_x + h} - 1$ are negative).
For any homogenous environment $e_x = \varepsilon$ for all $x \in \Lambda$, it is known [16] that ν^ε_Λ is strong FKG in the sense of [35], and that it can be stochastically majored and minored by two Bernoulli product measures (the precise statement needed in our work will appear later on).

By Theorem 7.1 and Corollary 7.1 we have

$$
\begin{aligned}
f^q - f^a &= \lim_{n\to\infty} n^{-d} \mathbb{E} \log \left(\frac{Z^e_{\Lambda_n}}{Z^\ell_{\Lambda_n}} \right) \\
&= \lim_{n\to\infty} n^{-d} \mathbb{E} \log \mu^\ell_{\Lambda_n} \left(\exp \left(\sum_{x \in \Lambda_n} (b \cdot e_x + h - \ell) \mathbb{1}_{[\varphi_x \in [-a,a]]} \right) \right). \quad (7.10)
\end{aligned}
$$

For the rest of the proof $\mathcal{A}$ will denote the set of "pinned points" of a given configuration $\varphi \in \Omega$, namely $\mathcal{A} = \{x \in \Lambda_n : \varphi_x \in [-a, a]\}$. By (7.10) we conclude that our goal is to prove

$$
\limsup_{n\to\infty} n^{-d} \mathbb{E} \log \left(\mu^\ell_n \left(\boldsymbol{\gamma}^{\mathcal{A}} \right) \right) < 0,
$$

where μ^ℓ_n is a simplified notation for $\mu^\ell_{\Lambda_n}$ and we denote

$$
\boldsymbol{\gamma}^A \doteqdot \prod_{x \in A} \gamma_x \quad \text{with} \quad \gamma_x \doteqdot \exp(b \cdot e_x + h - \ell).
$$

Let us now comment on the proof strategy. Let us observe that the calculations would be simple if $\mathcal{A}$ was distributed according to an i.i.d. Bernoulli(λ) product measure. Indeed, in such a case, the above limit does not depend on n:

$$
n^{-d} \mathbb{E} \log \mu^\ell_n (\boldsymbol{\gamma}^{\mathcal{A}}) = n^{-d} \mathbb{E} \log \left(\prod_{x \in \Lambda_n} (\lambda \gamma_x + 1 - \lambda) \right) = \mathbb{E} \log(\lambda \gamma_0 + 1 - \lambda) < 0. \quad (7.11)
$$

The last inequality follows by the strict concavity of the logarithm and the Jensen inequality. However, the interaction between the geometry of $\mathcal{A}$ and the one of the environment in $\gamma^{\mathcal{A}}$ might be potentially complicated and hard to analyze. Exploiting the fact that the environment is i.i.d. we will introduce an additional randomization. This will simplify the problem so that only the information about the cardinality of $\mathcal{A}$ will matter. The last trick we use is to compare the distribution of $\mathcal{A}$ under μ^ℓ_n with the measure ν^ℓ_n defined in (7.9). This will enable to use the stochastic domination results announced above and to get an expression similar to (7.11). Further, calculations are standard though little tiresome since we work with general laws of environments.

Let us now introduce the randomization. Let π be a permutation of the vertices of Λ_n chosen uniformly at random. We will denote the corresponding expectation by $\tilde{\mathbb{E}}$. It is easy to check that for any i.i.d. pinning law $\mathbb{E}\tilde{\mathbb{E}}(\cdot) = \mathbb{E}(\cdot)$. By the Jensen inequality we have

$$
\begin{aligned}
n^{-d}\mathbb{E}\log\left(\mu_n^{\ell}\left(\gamma^{\mathcal{A}}\right)\right) &= n^{-d}\mathbb{E}\tilde{\mathbb{E}}\log\mu_n^{\ell}\left(\prod_{i\in\mathcal{A}}\gamma_{\pi(i)}\right) \qquad (7.12)\\
&\leqslant n^{-d}\mathbb{E}\log\tilde{\mathbb{E}}\mu_n^{\ell}\left(\prod_{i\in\mathcal{A}}\gamma_{\pi(i)}\right) = n^{-d}\mathbb{E}\log\tilde{\mathbb{E}}\mu_n^{\ell}\left(\gamma^{\pi(\mathcal{A})}\right),
\end{aligned}
$$

where $\pi(A) = \{\pi(i) : i \in A\}$. Intuitively, $\tilde{\mathbb{E}}\mu_n^{\ell}$ is the expectation of the distribution of pinned sites "scattered" by a random permutation. Thanks to this we can work with a uniformly distributed set of pinned points, provided we know its cardinality. More precisely,

$$
\tilde{\mathbb{E}}\mu_n^{\ell}\left(\gamma^{\pi(\mathcal{A})}\right) = \sum_{k=0}^{n^d}\binom{n^d}{k}^{-1}\left(\sum_{A\subset\Lambda_n:|A|=k}\gamma^{A}\right)\mu_n^{\ell}\left(|\mathcal{A}|=k\right). \qquad (7.13)
$$

We recall measure ν_n^{ℓ} defined in (7.9). Paper [16] provides us with stochastic domination results which will be useful in our estimations. To this end we make the following elementary calculations:

$$
\begin{aligned}
\mu_n^{\ell}(|\mathcal{A}|\leqslant k) &= \sum_{A\subset\Lambda_n}\nu_n^{\ell}(A)\cdot\mu_n^{0}(|\mathcal{A}|\leqslant k\,|\,\forall x\in A, |\varphi_x|\leqslant a)\\
&= \sum_{A\subset\Lambda_n,|A|\leqslant k}\nu_n^{\ell}(A)\cdot\mu_n^{0}(|\mathcal{A}|\leqslant k\,|\,\forall x\in A, |\varphi_x|\leqslant a)\\
&\leqslant \sum_{A\subset\Lambda_n,|A|\leqslant k}\nu_n^{\ell}(A) = \nu_n^{\ell}(|\mathcal{A}|\leqslant k). \qquad (7.14)
\end{aligned}
$$

The next difficulty is that $\{|\mathcal{A}| = k\}$ appearing in (7.13) is not an increasing event. This will be handled differently for $d \geqslant 3$ and $d = 2$. Let us start with the former.

7.2.3.1 Case $d \geqslant 3$

By [16, Theorem 2.4, (2.15)], there exists some $C_1 > 0$ such that ν_n^{ℓ} stochastically dominates a Bernoulli product measure, denoted $\mathbf{B}_n^{\lambda}$, with a specific intensity λ depending on a and $\boldsymbol{\ell}$. More precisely,

$$
\nu_n^{\ell} \succ \mathbf{B}_n^{\lambda} \quad \text{with} \quad \lambda \doteqdot C_1\boldsymbol{\ell}/(1+C_1\boldsymbol{\ell}). \qquad (7.15)
$$

Below we will write $\mathbf{B}_n^{\lambda}(|\mathcal{A}| = k) = b_{n,\lambda}(k) = \binom{n^d}{k}\lambda^k(1-\lambda)^{n^d-k}$.

Observe that by [16] we know that $f^{a}(\boldsymbol{e}) > 0$ as soon as $\boldsymbol{\ell} > 0$ and consequently $\lambda > 0$. As the event $\{|\mathcal{A}| \leqslant k\}$ is decreasing, we have the following upper-bound, using (7.14) and (7.15):

$$\begin{aligned} \mu_n^{\ell}\left(|\mathcal{A}|=k\right) &\leqslant \mu_n^{\ell}\left(|\mathcal{A}|\leqslant k\right)\leqslant \nu_n^{\ell}\left(|\mathcal{A}|\leqslant k\right)\leqslant \mathbf{B}_n^{\lambda}\left(|\mathcal{A}|\leqslant k\right) \\ &= \sum_{j=0}^{k} b_{n,\lambda}(j) = b_{n,\lambda}(k)\left(1+\sum_{j=0}^{k-1}\frac{b_{n,\lambda}(j)}{b_{n,\lambda}(k)}\right) \\ &= b_{n,\lambda}(k)\left(1+\sum_{j=0}^{k-1}\prod_{i=j}^{k-1}\frac{b_{n,\lambda}(i)}{b_{n,\lambda}(i+1)}\right), \end{aligned} \tag{7.16}$$

Now, for $i \leqslant \lfloor \lambda n^d \rfloor$,

$$\frac{b_{n,\lambda}(i)}{b_{n,\lambda}(i+1)} = \frac{i+1}{n^d-i}\frac{1-\lambda}{\lambda} \leqslant 1.$$

This gives an upper bound for $k \leqslant \lfloor \lambda n^d \rfloor$, namely

$$\mu_n^{\ell}\left(|\mathcal{A}|=k\right) \leqslant n^d \cdot \mathbf{B}_n^{\lambda}\left(|\mathcal{A}|=k\right). \tag{7.17}$$

By Stirling's formula for $k \in [\lceil \lambda n^d \rceil, n^d]$ we have $b_{n,\lambda_k}(k) \geqslant c_2 n^{-d/2} > 0$, where $\lambda_k \doteqdot k n^{-d}$. Trivially

$$\mu_n^{\ell}\left(|\mathcal{A}|=k\right) \leqslant n^d \cdot \mathbf{B}_n^{\lambda_k}\left(|\mathcal{A}|=k\right),$$

for n large enough. Using the above estimates we treat (7.13) as follows

$$\begin{aligned} \tilde{\mathbb{E}}\mu_n^{\ell}\left(\boldsymbol{\gamma}^{\pi(\mathcal{A})}\right) &\leqslant n^d \sum_{k=0}^{n^d}\binom{n^d}{k}^{-1}\left(\sum_{A\subset\Lambda_n:|A|=k}\boldsymbol{\gamma}^{A}\right)\left[\mathbf{B}_n^{\lambda}\left(|\mathcal{A}|=k\right)+\sum_{j=\lceil\lambda n^d\rceil}^{n^d}\mathbf{B}_n^{\lambda_j}\left(|\mathcal{A}|=k\right)\right] \\ &\leqslant n^{2d}\max_{\alpha\in[\lambda,1]}\sum_{k=0}^{n^d}\binom{n^d}{k}^{-1}\left(\sum_{A\subset\Lambda_n:|A|=k}\boldsymbol{\gamma}^{A}\right)\cdot \mathbf{B}_n^{\alpha}(|\mathcal{A}|=k) \\ &\leqslant n^{2d}\max_{\alpha\in[\lambda,1]}\mathbf{B}_n^{\alpha}\left(\boldsymbol{\gamma}^{\mathcal{A}}\right) = n^{2d}\max_{\alpha\in[\lambda,1]}\prod_{x\in\Lambda_n}\mathbf{B}_n^{\alpha}\left(\gamma_x^{\mathbb{1}_{[x\in\mathcal{A}]}}\right). \end{aligned}$$

Hence,

$$n^{-d}\log\tilde{\mathbb{E}}\mu_n^{\ell}\left(\boldsymbol{\gamma}^{\pi(\mathcal{A})}\right) \leqslant \left(n^{-d}\max_{\alpha\in[\lambda,1]}\sum_{x\in\Lambda_n}\log(\alpha\gamma_x+(1-\alpha))\right)+o(1), \quad \text{as } n\to\infty.$$

We recall (7.10) and (7.12). Taking the expectation with respect to the environment we get

$$f^q - f^a \leqslant \lim_{n\to\infty} n^{-d}\mathbb{E}\left[\max_{\alpha\in[\lambda,1]}\sum_{x\in\Lambda_n}\log\left(\alpha(\gamma_x-1)+1\right)\right]. \tag{7.18}$$

At this stage the knowledge of the law of the environment can simplify the calculations. But we want to keep the generality and treat all the possible laws which satisfy assumption (7.2).

Let $r, R \in \mathbb{R}$ such that $0 < r < 1 < R < \infty$. The right hand side of (7.18) is bounded from above by $I_1(r) + I_2(r, R) + I_3(R)$ where

$$I_1(r) = \lim_{n\to\infty} n^{-d}\mathbb{E}\left[\max_{\alpha\in[\lambda,1]} \sum_{x\in\Lambda_n} \log\left(\alpha(\gamma_x - 1) + 1\right) \mathbb{1}_{[0\leqslant\gamma_x\leqslant r]}\right],$$

and I_2, I_3 are defined analogously by exchanging $\mathbb{1}_{[0<\gamma_x\leqslant r]}$ with $\mathbb{1}_{[r<\gamma_x<R]}$ and $\mathbb{1}_{[R\leqslant\gamma_x]}$ respectively. As $r < 1$, we have $I_1(r) \leqslant 0$. Further as $R > 1$ the term $I_3(R)$ is maximized at $\alpha = 1$, consequently

$$I_3(R) = \mathbb{E}\log(\gamma_0)\mathbb{1}_{[R\leqslant\gamma_0]}.$$

We observe that $\lim_{R\to\infty} I_3(R) = 0$ and further we proceed to I_2. Firstly we denote

$$f_n(\alpha, r, R) \doteqdot n^{-d} \sum_{x\in\Lambda_n} \log(\alpha\gamma_x + 1 - \alpha)\mathbb{1}_{[r<\gamma_x<R]},$$

and further let $X_n(\alpha, r, R) = f_n(\alpha, r, R) - \mathbb{E}f_n(\alpha, r, R)$. That is

$$X_n(\alpha, r, R) = n^{-d} \sum_{x\in\Lambda_n} \left[\log(\alpha\gamma_x + 1 - \alpha)\mathbb{1}_{[r<\gamma_x<R]} - \mathbb{E}\left(\log(\alpha\gamma_x + 1 - \alpha)\mathbb{1}_{[r<\gamma_x<R]}\right)\right].$$

The summands are centered independent and bounded. By Hoeffding's inequality we get

$$\mathbb{P}(|X_n(\alpha, r, R)| > t) \leqslant 2e^{-2t^2n^d/C_2(\alpha,r,R)^2}, \tag{7.19}$$

for any $t > 0$, where $C_2(\alpha, r, R) = \log(\alpha R + 1 - \alpha) - \log(\alpha r + 1 - \alpha)$. Further we observe that there exists $C_1(\lambda, r, R) > 0$ (deterministic) such that

$$\max_{\alpha\in[\lambda,1]} \partial_\alpha f_n(\alpha, r, R) \leqslant n^{-d} \sum_{x\in\Lambda_n} \max_{\alpha\in[\lambda,1]} \left(\frac{\gamma_x - 1}{\alpha\gamma_x + 1 - \alpha}\mathbb{1}_{[r<\gamma_x<R]}\right) \leqslant C_1(\lambda, r, R).$$

This let us work with a finite number of values of α. Let $N \in \mathbb{N}$ and $\alpha_i \doteqdot (1-\frac{i}{N})\lambda+\frac{i}{N}$. We have

$$\max_{\alpha\in[\lambda,1]} f_n(\alpha, r, R) \leqslant \max_{i=0,\dots,N} f_n(\alpha_i, r, R) + C_1(\lambda, r, R)/N.$$

Let $\bar{C}_2(r, R) = \max_{\alpha\in[\lambda,1]} C_2(\alpha, r, R)^2 < \infty$. We use (7.19) and the union bound to get

$$\mathbb{P}(\max_{i=0,\dots,N} |X_n(\alpha_i, r, R)| > t) \leqslant 2(N + 1)e^{-2t^2n^d/\bar{C}_2(r,R)}. \tag{7.20}$$

Now we may write

$$\begin{aligned} I_2(r, R) &= \lim_{n\to\infty} \mathbb{E} \max_{\alpha\in[\lambda,1]} f_n(\alpha, r, R) \leqslant \\ &\leqslant \lim_{n\to\infty} \mathbb{E}(\max_{i=1,\dots,N} X_n(\alpha_i, r, R)) + \lim_{n\to\infty} \max_{\alpha\in[\lambda,1]} \mathbb{E}f_n(\alpha, r, R) + \frac{C_1(\lambda, r, R)}{N}. \end{aligned}$$

The first term vanishes by (7.20). Taking the limit $N \uparrow \infty$ we have

$$I_2(r, R) \leqslant \lim_{n\uparrow\infty} \max_{\alpha\in[\lambda,1]} \mathbb{E} f_n(\alpha, r, R).$$

Taking now limits $r \downarrow 0$, and $R \uparrow \infty$ and coming back to (7.18), we obtain

$$f^q - f^a \leqslant \max_{\alpha\in[\lambda,1]} \mathbb{E}\log(\alpha\gamma_0 + 1 - \alpha) = \mathbb{E}\log(\lambda\gamma_0 + 1 - \lambda), \tag{7.21}$$

where the second equality will become apparent shortly. Indeed, we denote $h(\alpha) \doteqdot \mathbb{E}\log(\alpha\gamma_0 + 1 - \alpha)$. Obviously, $h'(\alpha) = \mathbb{E}\frac{\gamma_0 - 1}{\alpha(\gamma_0-1)+1} \leqslant \frac{\mathbb{E}(\gamma_0-1)}{\alpha\mathbb{E}(\gamma_0-1)+1} = 0$, which follows by the Jensen inequality applied to $x/(\alpha x + 1)$ and the facts that $\gamma_0 \geqslant 0$ and $\mathbb{E}\gamma_0 = 1$. We notice that (7.21) is precisely the bound announced in Theorem 7.2 part a). The strict inequality between the quenched and annealed free energies ($d \geqslant 3$) follows easily by the strict Jensen inequality and the fact that $\mathbb{E}\gamma_0 = 1$. ■

7.2.3.2 Case $d = 2$

The case $d = 2$ requires some slight modification as the stochastic domination of ν_n^ℓ by Bernoulli product measures holds in a weaker sense. Consequently, we cannot use the same argument as in (7.16). Indeed, by [16, Theorem 2.4, (2.13)], for any $\ell > 0$ and any set $B \subset \Lambda_n$ we have

$$\nu_n^\ell(\mathcal{A} \cap B = \varnothing) \leqslant (1-\lambda)^{|B|}, \tag{7.22}$$

with some $\lambda > 0$ depending on a and ℓ, which takes the form

$$\lambda \doteqdot C_1 \ell |\log \ell|^{-1/2} \tag{7.23}$$

for some $C_1 > 0$ when ℓ is small enough. Now, using (7.14) and (7.22) we have

$$\begin{aligned} \mu_n^\ell(|\mathcal{A}| = k) &\leqslant \mu_n^\ell(|\mathcal{A}| \leqslant k) \leqslant \nu_n^\ell(|\mathcal{A}| \leqslant k) \\ &= \nu_n^\ell(\exists B \subset \Lambda_n : |B| = n^d - k \text{ and } \mathcal{A} \cap B = \varnothing) \\ &\leqslant \sum_{B\subset\Lambda_n : |B| = n^d - k} \nu_n^\ell(\mathcal{A} \cap B = \varnothing) \leqslant \binom{n^d}{k}(1-\lambda)^{n^d - k}. \end{aligned}$$

By a direct calculations one checks that for $\tilde{\lambda} \doteqdot \frac{\lambda}{|\log\lambda|}$ we have

$$\binom{n^d}{k}(1-\lambda)^{n^d-k} \leqslant \mathbf{B}_n^{\tilde{\lambda}}(|\mathcal{A}| = k), \qquad \text{for } k \leqslant \lfloor \tilde{\lambda} n^d \rfloor.$$

Further one may process as for $d \geqslant 3$ by using $\tilde{\lambda}$ instead of λ. Consequently we deduce

$$f^q - f^a \leqslant \mathbb{E}\log\left(\frac{\lambda}{|\log\lambda|}\gamma_0 + 1 - \frac{\lambda}{|\log\lambda|}\right).$$

This finishes the proof for $d = 2$. ■

Chapter 8

Attraction by a repulsive in average environment (Bernoulli case)

This chapter contains the second part of the results obtained in collaboration with Piotr Miłoś. The corresponding article [LC3] can be found on the *arXiv*.

8.1 The model

We study the discrete Gaussian Free Field with a disordered square-well potential defined in (7.1):

$$\mu_{\Lambda}^{e,0}(d\varphi) = \frac{1}{Z_{\Lambda}^{e,0}} \exp\left(-\mathcal{H}_{\Lambda}(\varphi) + \sum_{x\in\Lambda}(b\cdot e_x + h)\mathbb{1}_{[\varphi_x\in[-a,a]]}\right)\prod_{x\in\Lambda} d\varphi_x \prod_{y\in\Lambda^c}\delta_0(d\varphi_y). \tag{8.1}$$

The superscript 0 reminds the boundary condition, it is added to the notation compared to (7.1) because it will be useful below. We consider here e given by i.i.d. random variables

$$\mathbb{P}(e_x = -1) = \mathbb{P}(e_x = +1) = 1/2$$

In the previous chapter, we proved under minimal assumptions on the law of the environment, that the quenched free energy associated to this model exists in $\mathbb{R}^+$, is deterministic, and strictly smaller than the annealed free energy whenever the latter is strictly positive.

Here we investigate the phase diagram of the model: in the plane (b, h), we prove that the quenched critical line (separating the phases of positive and zero free energy) lies strictly below the line $h = 0$. Thus there exists a non trivial region where the field is localized though repelled on average by the environment.

As we mentioned in Section 5.3.1, the same type of result has been proven for (1-dimensional) polymer models in great generality in [9].

A much shorter proof with explicit bounds can be found in [44], in less generality, but [26] contains a revisited proof with explicit estimates and weakening the assumptions on the underlying model.

Note that for polymers, or discrete height interfaces, one need a coarse graining procedure to achieve the proof. In our case, as we will see in the next section, we can shift the continuous interface where the environment is unfavorable, and this has a small cost in dimension $d \geqslant 3$. The procedure is a bit more complicated in dimension 2 and we have to localize the field before by introducing a small mass.

8.2 New result

We recall the definitions of the critical lines.

$$h_c^q(b) \doteqdot \sup\{h \in \mathbb{R} : f^q = 0\} \quad \text{and} \quad h_c^a(b) \doteqdot \sup\{h \in \mathbb{R} : f^a = 0\}$$

Knowing the behavior of the homogenous model for positive pinning [16], we easily deduce that the annealed critical line is given by the equation $\ell = 0$.

Recall that $f^q \leqslant f^a$ implies that $h_c^q(b) \geqslant h_c^a(b)$.

Theorem 8.1 *Let $e \sim \otimes_{x \in \mathbb{Z}^d} Bernoulli_{1/2}(-1,+1)$. Then,*

For $d \geqslant 2$, the quenched critical line $h_c^q(b)$ is located in the quadrant

$$\{(b,h) : b \geqslant 0, h < 0\}.$$

More precisely, there exists some $C, C' > 0$ depending on d, a only and $\epsilon \in (0,1)$ such that for any environment e which fulfills $b + h > 0$, $-\epsilon < -b + h < 0$ and

$$\begin{cases} h > \frac{C'(-b+h)^2}{\log(b-h)} & \text{for } d = 2 \\ h > -C \cdot (-b+h)^2 & \text{for } d \geqslant 3, \end{cases}$$

we have $f^q > 0$.

Remark 8.1 1. *A sketch of these bounds in the plane (b,h) can be seen on Figure 8.1. Moreover, the bound for $d \geqslant 3$ can be rewritten as $h > -C''(d,a) \cdot b^2$.*

2. *Jensen's inequality gives us an upper bound on C, C'. Indeed, as $f^a \geqslant f^q$, if $f^a = 0$ then $f^q = 0$. In particular, we must have $-C \leqslant \frac{\partial^2}{\partial b^2} h_c|_{b=0} < 0$. Our result gives thus an upper-bound on the behavior of the quenched critical line near $b = 0$.*

The dimensions $d \geqslant 3$ and $d = 2$ are treated differently so we split the proof accordingly.

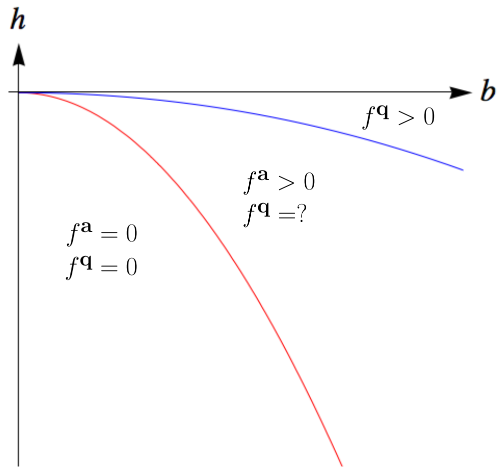


Figure 8.1 – Phase diagram of the model. The red curve is the annealed critical line; the blue one is our bound on the quenched critical line.

8.2.1 The case $d \geqslant 3$

We assume $-b + h < 0 < b + h$ i.e. the environment is repulsive if $e_x = -1$ while it is attractive if $e_x = +1$.

The idea is to tilt the measure such that the field φ is shifted up of an amount s on the sites x for which $b \cdot e_x + h < 0$. In this way the shift of the field follows the environment. For some technical reasons, we need to work with the measure with boundary condition a, so perform two changes of measure (first one changing boundary condition and the second one to follow the environment). Let $s > 0$ (to be fixed later).

$$f^{q}_{\Lambda_n}(\boldsymbol{e}) = n^{-d} \log \mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}\left(\frac{d\mu^{0,a}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}}\frac{d\mu^{0,0}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n}} \exp\left(\sum_{x\in\Lambda_n}(b\cdot e_x + h)\mathbb{1}_{[\varphi_x\in[-a,a]]}\right)\right),$$

where $(\varphi_x)_{x\in\Lambda_n}$ under $\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}$ is distributed as $\left(\varphi_x + s\mathbb{1}_{[(b\cdot e_x+h)<0]}\right)_{x\in\Lambda_n}$ under $\mu^{0,a}_{\Lambda_n}$. More formally, introducing $T_{\boldsymbol{e},s} : ((\varphi_x)_{x\in\Lambda_n}) \mapsto \left(\varphi_x + s\mathbb{1}_{[(b\cdot e_x+h)<0]}\right)_{x\in\Lambda_n}$, we define $\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}$ as $\mu^{0,a}_{\Lambda_n} \circ T^{-1}_{\boldsymbol{e},s}$.
Using Jensen's inequality, we get

$$\begin{aligned} f^{q}_{\Lambda_n}(\boldsymbol{e}) &= \frac{1}{n^d}\log\left[\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}\exp\left(\sum_{x\in\Lambda_n}(be_x+h)\mathbb{1}_{[\varphi_x\in[-a,a]]} + \log\frac{d\mu^{0,a}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}} + \log\frac{d\mu^{0,0}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n}}\right)\right] \\ &\geqslant n^{-d}\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}\left(\sum_{x\in\Lambda_n}(b\cdot e_x+h)\mathbb{1}_{[\varphi_x\in[-a,a]]} + \underbrace{\log\frac{d\mu^{0,a}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}}}_{(1)} + \underbrace{\log\frac{d\mu^{0,0}_{\Lambda_n}}{d\mu^{0,a}_{\Lambda_n}}}_{(2)}\right) \end{aligned}$$

As $Z^{0,0}_{\Lambda_n,\boldsymbol{e},s} = Z^{0,0}_{\Lambda_n}$ (which follows by change of variables in the Gaussian integral), the first term can be written as

$$(1) = -\frac{1}{4d} \sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing \\ x\sim y}} (\varphi_x - \varphi_y)^2 - (\hat{\varphi}_x - \hat{\varphi}_y)^2$$

where $\hat{\varphi}_x \doteqdot \varphi_x + s\mathbb{1}_{[b\cdot e_x + h<0]}$. Hence, using the definition of $\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}$,

$$\begin{aligned} n^{-d}\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}((1)) &= -\frac{n^{-d}}{4d}\mu^{0,a}_{\Lambda_n}\left(\sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing \\ x\sim y}} (\hat{\varphi}_x - \hat{\varphi}_y)^2 - (\varphi_x - \varphi_y)^2\right) \\ &= -\frac{s^2 n^{-d}}{4d} \sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing \\ x\sim y}} (\mathbb{1}_{[b\cdot e_x+h<0]} - \mathbb{1}_{[b\cdot e_y+h<0]})^2 \end{aligned}$$

The second term contains only boundary contribution of order n^{d-1}. Indeed,

$$(2) = 2a \sum_{x\in\partial\Lambda_n} \varphi_x - a^2|\partial\Lambda_n|$$

Hence,

$$n^{-d}\mu^{0,a}_{\Lambda_n,\boldsymbol{e},s}((2)) \geqslant 2a\cdot n^{-d} \sum_{x\in\partial\Lambda_n} \mu^{0,a}_{\Lambda_n}(\hat{\varphi}_x) - Cn^{-1} \geqslant s \sum_{x\in\partial\Lambda_n} \mathbb{1}_{[b\cdot e_x+h<0]} - Cn^{-1} \geqslant -Cn^{-1}$$

We get

$$\begin{aligned} f^{\mathfrak{q}}_{\Lambda_n}(\boldsymbol{e}) \geqslant\ & n^{-d} \sum_{x\in\Lambda_n} (b\cdot e_x + h)\mu^{0,a}_{\Lambda_n}(\hat{\varphi}_x \in [-a,a]) \\ & -\frac{s^2 n^{-d}}{4d} \sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing \\ x\sim y}} (\mathbb{1}_{[b\cdot e_x+h<0]} - \mathbb{1}_{[b\cdot e_y+h<0]})^2 - Cn^{-1} \end{aligned}$$

Now we use the fact that the marginal laws of all φ_x, $x\in\Lambda_n$ under $\mu^{0,a}_{\Lambda_n}$ are Gaussian variables centered at a, i.e. $\varphi_x \sim \mathcal{N}(a,\sigma^x_n)$ where $\sigma^x_n = \mathrm{Var}^{0,a}_{\Lambda_n}(\varphi_x) \leqslant \mathrm{Var}^{0,a}_{\infty}(\varphi_x) \leqslant c(d) < \infty$ for $d\geqslant 3$. Therefore,

$$\begin{aligned} \mu^{0,a}_{\Lambda_n}(\varphi_x \in [-a,a]) - \mu^{0,a}_{\Lambda_n}(\varphi_x + s \in [-a,a]) & \\ = \mu^{0,0}_{\Lambda_n}(\varphi_x \in [-2a,0]) - \mu^{0,0}_{\Lambda_n}(\varphi_x \in [-2a-s,-s]) & \\ = C\left(\int_{-s}^{0} - \int_{-2a-s}^{-2a}\right) e^{-y^2/2\sigma^{x\,2}_n} dy & \\ \asymp s \quad \text{as } n\to\infty, & \qquad (8.2) \end{aligned}$$

for $c(d) \gg s$. In particular we will use that:

$$\mu^{0,a}_{\Lambda_n}(\varphi_x \in [-a,a]) - \mu^{0,a}_{\Lambda_n}(\varphi_x + s \in [-a,a]) \geqslant C_1(d,a)\cdot s,$$

for some $C_1(d,a) > 0$.

$$f_n^{\mathfrak{q}}(\boldsymbol{e}) \geqslant n^{-d} \sum_{x \in \Lambda_n} (b \cdot e_x + h) \left(\mu_{\Lambda_n}^{0,a}(\varphi_x \in [-a,a]) - C_1(d,a) s \mathbb{1}_{[b \cdot e_x + h < 0]} \right) - \frac{s^2 n^{-d}}{4d} \sum_{\substack{\{x,y\} \cap \Lambda_n \neq \varnothing \\ x \sim y}} \left(\mathbb{1}_{[b \cdot e_x + h < 0]} - \mathbb{1}_{[b \cdot e_y + h < 0]} \right)^2 - C n^{-1}$$

Observe that $\mu_{\Lambda_n}^{0,a}(\varphi_x \in [-a,a]) = \mu_{\Lambda_n}^{0,0}(\varphi_x \in [-2a,0]) \geqslant \mu_{\infty}^{0,0}(\varphi_x \in [-2a,0]) \geqslant C_2(d,a)$ for some $C_2(d,a) > 0$.

By taking the expectation with respect to the environment, using the bounded convergence theorem and the fact that $f^{\mathfrak{q}}(\boldsymbol{e}) = \mathbb{E}(f^{\mathfrak{q}}(\boldsymbol{e}))$ (cf. previous chapter) we get:

$$f^{\mathfrak{q}} = \lim_{n \to \infty} \mathbb{E} f_{\Lambda_n}^{\mathfrak{q}}(\boldsymbol{e}) \quad \geqslant \quad h C_2(d,a) - \frac{s C_1(d,a)}{2}(-b+h) - \frac{s^2}{16} \tag{8.3}$$

We may optimize over s as the left hand side does not depend on it. Doing this one checks that $f^{\mathfrak{q}}(\boldsymbol{e}) > 0$ as soon as

$$h > -\frac{C_1(d,a)}{C_2(d,a)} \cdot (-b+h)^2 \doteqdot -K(d,a) \cdot (-b+h)^2$$

This gives the implicit equation in terms of the variance b^2 of $b \cdot e_x + h$:

$$h > b - \frac{1}{2K} + \frac{1}{2}\sqrt{\frac{1}{K^2} - \frac{8b}{K}} = -Kb^2 + O(b^3)$$

The annealed critical curve as well as this bound are drawn on Figure 8.1. We recall that (8.2) is valid under assumption that s is small. The maximum of (8.3) is realized at $s_{\max} = -4C_1 \cdot (-b+h)$, thus it is enough to assume that $(-b+h)$ is small. ■

8.2.2 The case $d = 2$

In the case $d = 2$, the variance of the Gaussian free field diverges with the size of the box, so we cannot use the previous estimates. To circumvent this problem we introduce the so-called massive free field. Let $m > 0$,

$$\mu_{\Lambda_n,m}^{0,\zeta}(d\varphi) = \frac{1}{Z_{\Lambda_n,m}^{0,\zeta}} \exp\left(-\mathcal{H}_{\Lambda_n}(\varphi) - m^2 \sum_{x \in \Lambda_n} (\varphi_x - \zeta)^2 \right) \prod_{x \in \Lambda_n} d\varphi_x \prod_{x \in \partial\Lambda_n} \delta_0(d\varphi_x), \tag{8.4}$$

where $\mathcal{H}_\Lambda(\varphi)$ is defined in (5.1). Known facts about this model can be found in [25, Section 3.3]. In particular, the random walk representation for the massless GFF (Proposition 5.1) is still true, but for a random walk Y_t that is killed with rate $\xi(m) = \frac{m^2}{1+m^2}$, namely at each time ℓ, if the walk has not already been killed, it is killed with probability $\xi(m)$, where the killing is independent of the walk. We write its law P_x when it starts at x.

Lemma 8.1 *Let* $d = 2$*. Then,*

1. *There exists some* $C_1 > 0$ *such that for* n *large enough,* $m > 0$ *small enough and all* $x \in \Lambda_n$*,*
$$\mu^{0,0}_{\Lambda_n,m}(\varphi_x^2) \leqslant C_1 |\log(m)|.$$

2. *There exists some* $C_2 > 0$ *such that for* n *large enough and* $m > 0$ *small enough, we have*
$$n^{-2} \log \frac{Z^{0,0}_{\Lambda_n,m}}{Z^{0,0}_{\Lambda_n}} \geqslant -C_2 m^2 |\log(m)|.$$

Proof These bounds are rather standard. We give here the main steps of the proofs with some references. For the first claim, we use the random walk representation [25] to write

$$\mu^{0,0}_{\Lambda_n,m}(\varphi_x^2) = \sum_{\ell=0}^{\infty} P_x(Y_\ell = x\,,\, \tau_{\Lambda_n} \wedge \aleph > \ell) = \sum_{\ell=0}^{\infty} (1 - \xi(m))^\ell P_x(Y_\ell = x\,,\, \tau_{\Lambda_n} > \ell)$$

where τ_{Λ_n} is the first exit time of Λ_n and $\aleph$ is the killing time of the random walk Y_t. Hence,

$$\mu^{0,0}_{\Lambda_n,m}(\varphi_x^2) \leqslant \mu^{0,0}_{\Lambda_n,m}(\varphi_0^2) \leqslant \sum_{\ell=1}^{\infty} (1 - \xi(m))^\ell P_0(X_\ell = 0) \tag{8.5}$$

where X_ℓ is a simple random walk (without killing). The projections of X_ℓ onto the canonical basis rotated by $\pi/4$ are two independent 1−dimensional random walks X^1_ℓ and X^2_ℓ, then by Stirling formula,

$$P_0(X_{2\ell} = 0) = (P_0(X^1_{2\ell} = 0))^2 = \left(\binom{2\ell}{\ell} 2^{-2\ell}\right)^2 = \frac{1}{\pi\ell}(1 + o(1)) \quad \text{as } \ell \to \infty$$

The asymptotics of (8.5) for small m gives the desired upper-bound.
To prove the second claim we use the representation of the partition function described in [15, p.542] (it applies to the massive GFF with an obvious modification). We denote by $\tilde{P}$ the coupling of a random walk X_n and a killed random walk Y_n such

that $Y_n = X_n$ up to its killing time $\aleph$,

$$\begin{aligned}
\frac{1}{|\Lambda_n|}\log\frac{Z^{0,0}_{\Lambda_n}}{Z^{0,0}_{\Lambda_n,m}} &= \frac{1}{2|\Lambda_n|}\sum_{x\in\Lambda_n}\sum_{\ell=1}^{\infty}\frac{1}{2\ell}\Big(\tilde{P}_x(X_{2\ell}=x,\tau_{\Lambda_n}>2\ell)\\
&\qquad\qquad -\tilde{P}_x(Y_{2\ell}=x,\tau_{\Lambda_n}\wedge\aleph>2\ell)\Big)\\
&\leqslant \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{2\ell}\Big(\tilde{P}_0(X_{2\ell}=0,\tau_{\Lambda_n}>2\ell)-\tilde{P}_0(Y_{2\ell}=0,\tau_{\Lambda_n}\wedge\aleph>2\ell)\Big)\\
&= \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{2\ell}\tilde{P}_0(X_{2\ell}=0,\tau_{\Lambda_n}>2\ell,\aleph\leqslant 2\ell)\\
&\leqslant \frac{1}{2}\sum_{\ell=1}^{\infty}\frac{1}{2\ell}\tilde{P}_0(X_{2\ell}=0)\left(1-(1-\xi(m))^{2\ell}\right) \qquad (8.6)
\end{aligned}$$

Using the same estimate as in (8.5), the asymptotics of (8.6) for small m gives the desired upper-bound. ∎

The idea is to tilt the measure, as in the proof for $d\geqslant 3$, first to work with the massive measure, and second to follow the environment such that the field φ is shifted up of an amount s on the sites x for which $b\cdot e_x+h<0$. For some technical reason, we need to work with the measure with boundary condition a, so we perform three changes of measure (first one for changing boundary condition, a second one for adding mass, and a third one for following the environment).
Let $s>0$ and $m>0$ to be fixed later.

$$\begin{aligned}
f^{q}_{\Lambda_n}(\boldsymbol{e}) &= n^{-2}\log\mu^{0,0}_{\Lambda_n}\left(\exp\sum_{x\in\Lambda_n}(b\cdot e_x+h)\mathbb{1}_{[\phi_x\in[-a,a]]}\right)\\
&= n^{-2}\log\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}\left(\exp\left(\sum_{x\in\Lambda_n}(b\cdot e_x+h)\mathbb{1}_{[\phi_x\in[-a,a]]}\right.\right.\\
&\qquad\left.\left.+\log\frac{d\mu^{0,0}_{\Lambda_n}}{d\mu^{0,\zeta}_{\Lambda_n}}+\log\frac{d\mu^{0,\zeta}_{\Lambda_n}}{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}}+\log\frac{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}}{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}}\right)\right)
\end{aligned}$$

where $(\varphi_x)_{x\in\Lambda_n}$ under $\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}$ is distributed as $\left(\varphi_x+s\mathbb{1}_{[(b\cdot e_x+h)<0]}\right)_{x\in\Lambda_n}$ under $\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}$; more formally, introducing $T_{\boldsymbol{e},s}:((\varphi_x)_{x\in\Lambda_n})\mapsto\left(\varphi_x+s\mathbb{1}_{[(b\cdot e_x+h)<0]}\right)_{x\in\Lambda_n}$, we define $\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}$ as $\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}\circ T^{-1}_{\boldsymbol{e},s}$. Using Jensen's inequality, $f^{q}_{\Lambda_n}(\boldsymbol{e})$ is bounded below by

$$n^{-2}\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}\left(\sum_{x\in\Lambda_n}(b\cdot e_x+h)\mathbb{1}_{[\varphi_x\in[-a,a]]}+\underbrace{\log\frac{d\mu^{0,0}_{\Lambda_n}}{d\mu^{0,\zeta}_{\Lambda_n}}}_{(1)}+\underbrace{\log\frac{d\mu^{0,\zeta}_{\Lambda_n}}{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}}}_{(2)}+\underbrace{\log\frac{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m}}{d\boldsymbol{\mu}^{0,\zeta}_{\Lambda_n,m,\boldsymbol{e},s}}}_{(3)}\right)$$

As in the proof for $d \geqslant 3$, we have

$$n^{-2}\mu^{0,\zeta}_{\Lambda_n,m,e,s}((1)) \geqslant -Cn^{-1}.$$

By Lemma 8.1, we have $\frac{Z^{0,\zeta}_{\Lambda_n,m}}{Z^{0,\zeta}_{\Lambda_n}} = \frac{Z^{0,\zeta}_{\Lambda_n,m}}{Z^{0,0}_{\Lambda_n,m}}\frac{Z^{0,0}_{\Lambda_n,m}}{Z^{0,0}_{\Lambda_n}}\frac{Z^{0,0}_{\Lambda_n}}{Z^{0,\zeta}_{\Lambda_n}} \geqslant -Cn - C_2n^2m^2|\log m|$, and then

$$(2) = \log\left(\frac{Z^{0,\zeta}_{\Lambda_n,m}}{Z^{0,\zeta}_{\Lambda_n}}\right) + m^2\sum_{x\in\Lambda_n}\varphi_x^2 \geqslant -Cn - C_2n^2m^2|\log m| + m^2\sum_{x\in\Lambda_n}\varphi_x^2$$

hence,

$$n^{-2}\mu^{0,\zeta}_{\Lambda_n,m,e,s}((2)) \geqslant -Cn^{-1} - C_2m^2|\log m|.$$

Finally, noticing that $Z^{0,\zeta}_{\Lambda_n,m,e,s} = Z^{0,\zeta}_{\Lambda_n,m}$ (just perform a change of variables in the Gaussian integral), we can compute the third term.

$$(3) = -\frac{1}{8}\sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing\\ x\sim y}}(\varphi_x-\varphi_y)^2 - (\hat\varphi_x-\hat\varphi_y)^2 - m^2\sum_{x\in\Lambda_n}(\varphi_x-s)^2-(\hat\varphi_x-s)^2$$

where $\hat\varphi_x \doteqdot \varphi_x + s\mathbb{1}_{[b\cdot e_x+h<0]}$. Now we will use the fact that the marginal laws of all φ_x, $x\in\Lambda_n$ under $\mu^{0,\zeta}_{\Lambda_n,m}$ are Gaussian variables, i.e. $\varphi_x \sim \mathcal{N}(\mu_n^x, {\sigma_n^x}^2)$ where $\mu_n^x \approx \zeta$ except for x close to the boundary of the box. Indeed, by the random walk representation of the mean, there is $C>0$ such that $|\mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x) - \zeta| \leqslant C(1+m^2)^{-\|x-\partial\Lambda_n\|}$. Moreover, $(\sigma_n^x)^2 = \mathbf{V}\mathrm{ar}^{0,\zeta}_{\Lambda_n,m}(\varphi_x) \leqslant C_1|\log m|$. Using the definition of $\mu^{0,\zeta}_{\Lambda_n,m,e,s}$, and computing the terms as in the proof for $d\geqslant 3$,

$$\begin{aligned} n^{-2}\mu^{0,\zeta}_{\Lambda_n,m,e,s}((3)) &\geqslant -\frac{s^2}{8n^2}\sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing\\ x\sim y}}\left(\mathbb{1}_{[b\cdot e_x+h<0]} - \mathbb{1}_{[b\cdot e_y+h<0]}\right)^2 \\ &\quad -\frac{m^2s^2}{n^2}\sum_{x\in\Lambda_n}\mathbb{1}_{[b\cdot e_x+h<0]} + \frac{C}{n}. \end{aligned}$$

We get, for n large enough and m small enough

$$\begin{aligned} f^{\mathfrak{q}}_{\Lambda_n}(e) &\geqslant n^{-2}\sum_{x\in\Lambda_n}(b\cdot e_x+h)\mu^{0,\zeta}_{\Lambda_n,m}(\hat\varphi_x\in[-a,a]) \\ &\quad -\frac{s^2}{8n^2}\sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing\\ x\sim y}}\left(\mathbb{1}_{[b\cdot e_x+h<0]} - \mathbb{1}_{[b\cdot e_y+h<0]}\right)^2 \\ &\quad -\frac{m^2s^2}{n^2}\sum_{x\in\Lambda_n}\mathbb{1}_{[b\cdot e_x+h<0]} - C'm^2|\log m| - Cn^{-1}, \end{aligned}$$

for some C and $C'>0$. Note that for $O(n^2)$ sites x, we have

$$\mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x\in[-a,a]) - \mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x+s\in[-a,a]) \asymp \Phi'_{\zeta,b}(a)\cdot s \quad \text{as } n\to\infty, \tag{8.7}$$

for $s \ll a \leqslant \zeta = C_1|\log m|$, and $b = C_1|\log m|$. Above $\Phi_{\zeta,b}$ stands for the p.d.f. of the above Gaussian distribution with mean ζ and variance b^2. In particular, for a positive fraction of x (close to 1) and m sufficiently small, we have the upper bound:

$$\mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x \in [-a,a]) - \mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x + s \in [-a,a]) \geqslant \frac{C_1(a)}{|\log m|} \cdot s,$$

for some $C_1(a) > 0$. Now we can compute:

$$\begin{aligned} f^{q}_{\Lambda_n}(\boldsymbol{e}) \geqslant\ & n^{-2}\sum_{x\in\Lambda_n}(b\cdot e_x + h)\Big(\mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x \in [-a,a]) - \frac{C_1(a)}{|\log m|} s \mathbb{1}_{[b\cdot e_x + h<0]}\Big) \\ & -\frac{s^2}{8n^2}\sum_{\substack{\{x,y\}\cap\Lambda_n\neq\varnothing \\ x\sim y}}(\mathbb{1}_{[b\cdot e_x+h<0]} - \mathbb{1}_{[b\cdot e_y+h<0]})^2 - \frac{m^2s^2}{n^2}\sum_{x\in\Lambda_n}\mathbb{1}_{[b\cdot e_x+h<0]} \\ & -C'm^2|\log m| - Cn^{-1} \end{aligned}$$

Observe that $\mu^{0,\zeta}_{\Lambda_n,m}(\varphi_x \in [-a,a]) \geqslant 2a\cdot\Phi'_{\zeta,b^2}(-a) = \frac{\tilde{C}_1(a)}{|\log m|}$ uniformly in $x \in \Lambda_n$. Let us take the expectation with respect to the environment, we get:

$$\begin{aligned} f^{q} = \lim_{n\to\infty}\mathbb{E}f^{q}_{\Lambda_n}(\boldsymbol{e}) \geqslant\ & h\frac{\tilde{C}_1(a)}{|\log m|} - s\cdot\frac{C_1(a)(-b+h)}{2|\log m|} - \frac{s^2m^2}{2} \\ & -\frac{s^2}{16} - C'm^2|\log m|. \end{aligned} \tag{8.8}$$

Our aim now is to show that the right hand side can be positive even when h is negative. In the above expression s, m are free parameters which we may vary. However, we have to remember that both s and m need to be small enough, which makes standard optimization analysis cumbersome. We are going to show that there exists $C > 0$ and $\epsilon > 0$ such that for any b, h such that $(-b+h) \in (-\epsilon, 0)$ and

$$h \doteqdot C\frac{(-b+h)^2}{\log(-(-b+h))},$$

there exist small s and m such that the r.h.s. of (8.8) is positive. Notice that the result will imply that for any $h \geqslant C\frac{(-b+h)^2}{\log(-(-b+h))}$ the free energy is positive. Let us choose the value of s which maximizes (8.8) for fixed m, i.e.

$$s = -\frac{C_1(a)(-b+h)}{(m^2+1/4)|\log(m)|}. \tag{8.9}$$

and for m let us take

$$m^2 = -k/(\log k)^3, \text{ where } k \doteqdot -h\tilde{C}_1(a)/C'. \tag{8.10}$$

One can verify that with the above choice of parameters both s and m are as small as we want. Let us first put (8.9) into the r.h.s. of (8.8) and obtain

$$f^{q} \geqslant h\frac{\tilde{C}_1(a)}{|\log m|} + \frac{C_1(a)^2(-b+h)^2}{(m^2+1/4)(\log m)^2} - C'm^2|\log m|.$$

For k and consequently m small enough we have

$$f^q \geqslant h\frac{\tilde{C}_1(a)}{|\log m|} + \frac{C_1(a)^2(-b+h)^2}{2(\log m)^2} - C'm^2|\log m|.$$

Further let us multiply both sides by $(\log m)^2$ and insert (8.10).

$$\begin{aligned}
f^q(\log m)^2 &\geqslant -h\tilde{C}_1(a)\log m + \frac{C_1(a)^2(-b+h)^2}{2} + C'm^2(\log m)^3 \\
&= -\frac{h\tilde{C}_1(a)}{2}(\log k - 3\log(|\log k|)) + \frac{C_1(a)^2(-b+h)^2}{2} \\
&\quad -C'\frac{k(\log k - 3\log(|\log k|))^3}{8(\log k)^3} \\
&= -h\frac{\tilde{C}_1(a)}{2}\log(-h\tilde{C}_1(a)/C') + \frac{C_1(a)^2(-b+h)^2}{2} + h\frac{\tilde{C}_1(a)}{8} + o(h)
\end{aligned}$$

as $|h| \to 0$. From the last claim it is straightforward to conclude existence of C (sufficiently small) and ϵ with the properties described above. ■

Articles presented for the PhD

[LC1] L. Coquille, H. Duminil-Copin, D. Ioffe, and Y. Velenik, *On the Gibbs states of the non-critical Potts model on* $\mathbb{Z}^2$, Probability Theory and Related Fields, http://dx.doi.org/10.1007/s00440-013-0486-z (2013).

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