

# Phase Transition in Random-cluster and $O(n)$ -models

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À Raphaël, Maxime et Ruty

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# Résumé

Cette thèse traite des phénomènes critiques deux dimensionels. Plus précisément, nous étudions des modèles planaires de physique statistique qui exhibent une transition de phase, c'est-à-dire un changement brusque de leurs propriétés macroscopiques. L'étude se concentre sur deux familles de modèles: la *FK-percolation* et les modèles de boucles dénommés *modèles  $O(n)$* . Ces modèles englobent deux cas particuliers fondamentaux que sont le *modèle d'Ising* et les *marches auto-évitantes*. Cette thèse est donc à l'interface entre la physique statistique, les combinatoires et les probabilités. Elle s'articule en trois parties.

Dans un premier temps, nous identifions la phase critique de la FK-percolation. Ce résultat est le point de départ de notre étude, puisqu'il localise le point auquel la transition de phase de nos modèles a lieu. Nous étudions ensuite la transition de phase – en particulier son *ordre* – par le biais d'*observables parafermioniques*. Cette étude est l'opportunité d'introduire ces observables et de les étudier en détail. Elles sont au coeur des deux autres parties de la thèse.

La deuxième partie est dévolue au modèle d'Ising et son équivalent FK, le modèle FK-Ising (ces deux modèles constituent un modèle mathématique concret pour les phénomènes de magnétisme). L'observable parafermionique se révèle alors être holomorphe discrète. Ce fait important a été exploité par Smirnov puis Chelkak-Smirnov afin de montrer l'invariance conforme de ces deux modèles au point critique. Ce résultat primordial ouvre la voie à de nombreuses questions. Nous nous attachons à répondre à certaines d'entre elles. En particulier, nous étudions la géométrie de la phase critique, et les relations entre les phases critique et presque-critique.

La dernière partie traite des marches auto-évitantes. Ce modèle de polymères, introduit par Flory, est la source de difficiles problèmes, pour lesquels les outils mathématiques sont peu développés. Cependant, il est possible d'exhiber une observable parafermionique dans ce cas particulier également. Nous étudions cette observable afin d'estimer le nombre de marches auto-évitantes de longueur prescrite sur le réseau en nid d'abeille. Deux autres résultats concernant les marches auto-évitantes et leur limite d'échelle conjecturée complètent ce manuscrit.

Nous espérons que vous prendrez autant de plaisir à lire ces lignes que nous en avons eu à les écrire. *Bonne lecture!*

# Abstract

This thesis deals with two-dimensional planar phenomenon. More precisely, we study planar models of statistical physics that exhibit a phase transition, *i.e.* an abrupt change of their macroscopic properties. The study focuses on two families of models: *random-cluster models* and *loop  $O(n)$ -models*. These models encompass two fundamental cases: the *Ising model* and the *self-avoiding walk*. This thesis is at the interface between statistical physics, combinatorics and probabilities. It is organized in three parts.

In the first part, we identify the critical phase of the random-cluster model. This result is the starting point of our study, since it localizes the point at which the phase transition occurs. We then study the phase transition itself – in particular its *order* – by means of *parafermionic observables*. It also gives us the opportunity to introduce these observables and study them in detail. They are indeed at the hearts of the two next parts.

The second part is devoted to the Ising model and its random-cluster representation, the FK-Ising model (these two models constitute a concrete mathematical frame for the study of ferro-magnetism). The parafermionic observable appears to be discrete holomorphic in these cases. This important fact was harnessed by Smirnov and Chelkak-Smirnov in order to prove conformal invariance of these two models at criticality. This deep result paved the way to a complete study of the critical phase. In particular, we study the geometry of the critical phase, as well as the relation between the critical and near-critical phases.

The last part deals with the self-avoiding walk. This model of polymers, introduced by Flory, offers many difficult problems, for which mathematical tools are limited. Nevertheless, it is possible to exhibit a parafermionic observable once again in this particular case. We study this observable in order to estimate the number of self-avoiding walks with a prescribed length on the honeycomb lattice. Two other results dealing with self-avoiding walks and their conjecture scaling limits complete the study.

We hope you will enjoy reading these lines at least as much as we enjoyed writing them. *Bonne lecture!*

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# Organization of the thesis

Chapters 2, 3, 6 and 13 describe more-or-less standard theories and do not present new results. They contain the necessary background for understanding the other chapters. Chapter 7 contains Smirnov's proof of conformal invariance, which is used in several parts of the thesis. Other chapters describe joint works. We use published results as a basis for most of the chapters, even though the original articles have been modified in order to unify notation and concepts, and to avoid repetitions. New results are also added at several places in the thesis.

Let us now describe briefly the content of each chapter.

Chapter 1 is a general description of two-dimensional statistical physics intended for a large audience of mathematicians. This chapter is independent of the rest of the document.

Chapter 2 is a toolbox on discrete complex analysis. Theorems gathered in this chapter will be used extensively in the rest of the document. The first section surveys general definitions on graphs and should not be skipped.

**Part 1: Random-cluster models** Chapters 3, 4 and 5 form the first part of this thesis. They all deal with planar random-cluster models with  $q \geq 1$ .

Chapter 3 is a mathematical introduction to the random-cluster model. It studies its basic properties. We chose to restrict ourselves to the case of the random-cluster model on the square lattice. A particular emphasize is made on the existence of a phase transition and on planar duality.

Chapter 4 identifies rigorously the position of the phase transition. The proof harnesses two ingredients: first, the study of crossing probabilities in the torus at the so-called self-dual point (this is the equivalent of the celebrated Russo-Seymour-Welsh Theorem for percolation) and second, a sharp threshold argument to prove that crossing probabilities go to 0 or 1 away from the self-dual point. It is then possible to prove that the phase transition must occur at the self-dual point. A byproduct of the proof is the fundamental property of exponential decay of correlations in subcritical phase.

Chapter 5 dives into the study of the critical regime itself. We introduce the so-called parafermionic observable and use it to prove several properties on random-cluster models. We will see in the next chapters that special values of  $q$  are much better understood than

the general ones, yet it is possible to prove interesting results for every  $q \geq 1$ . In particular, we show that the phase transition is second order by proving that the correlation length  $\xi$  blows up when  $p$  approaches the critical point when  $1 \leq q \leq 4$ . We strengthen this property by showing that the susceptibility diverges when  $1 \leq q \leq 3$ . On the other hand, when  $q > 4$ , we give another identification of the critical point, and we provide evidences that the phase transition is of first order.

**Part 2: The FK-Ising and Ising models.** The second part of the thesis contains a more elaborated study of the FK-Ising model, *i.e.* the random-cluster model with  $q = 2$ . Indeed, this model, which can be coupled with the Ising model, satisfies special integrability properties that allow a much more precise understanding.

Chapter 6 presents the Ising and FK-Ising models in the planar case. Our goal is once again to focus on specific properties of these models and not to provide a general exposition. We will focus on the Edwards-Sokal coupling between the Ising and the FK-Ising model, and on the low and high-temperature expansions of the model, which leads the definition of the so-called spin fermionic observable.

Chapter 7 is an exposition of Smirnov's proof of conformal invariance for the FK-Ising model. The main ingredient is the discrete holomorphicity of the fermionic observable. In the scaling limit, the properly rescaled observable converges to a conformally covariant object, namely the solution to a certain Riemann-Hilbert boundary problem. We also include a sketch of Chelkak-Smirnov proof of conformal invariance for the Ising model. Let us insist on the fact that these proofs are not due to us.

Chapter 8 studies the observable away from the critical point. We show that it becomes massive harmonic. This massive harmonicity allows the computation of the correlation length of the model explicitly, and its comparison to large deviation estimates for the simple random walk, thus proving a link between Ising and random walks first noticed by Messikh.

Chapter 9 is devoted to the proof of Russo-Seymour-Welsh type bounds on crossing probabilities at criticality. The proof relates crossing probabilities on the boundary of a domain to the fermionic observable and to discrete harmonic measure. The novelty of this chapter with respect to crossing probabilities proved in Chapter 4 comes from the uniformity with respect to boundary conditions. This fact allows us to deduce several noteworthy results.

Chapter 10 investigates a generalization of the result of Chapter 9. Namely, we prove crossing probabilities in general discrete topological rectangles. While this result could appear technical, we believe it to be crucial in the proof of the so-called full scaling-limit of the FK-Ising and Ising models. As an application, we derive the universal arm exponents for FK-Ising, and in particular we show that the five-arm exponent is equal to 2. We deduce an alternative proof of convergence to SLE.

Chapter 11 presents a proof that interfaces of the FK-Ising and the Ising model converge to the Schramm-Loewner Evolution of parameters  $16/3$  and  $3$  respectively. The

main ingredients are contained in Chapter 7 and 9. We sketch two proofs, one invoking a result by Kemppainen and Smirnov (they proved it using this strategy in [KS10]), and the other invoking the estimation of the five-arm event.

Chapter 12 is a study of the near-critical regime. In particular, we identify the geometric correlation length of the FK-Ising. Contrarily to the percolation case, it is not possible to obtain the correlation length using the so-called four-arm exponents. In this case, one should consider an exponent related to the influence of an edge, an exponent which is different from the four-arm event. We discuss the mechanisms involved in this phenomenon.

**Part 3:  $O(n)$ -models and the Self-avoiding walk.** Chapter 13 recall general facts on the  $O(n)$ -model and the self-avoiding walk which will be used in the next section. In particular, we discuss the bridge decomposition of self-avoiding walks.

Chapter 14 presents a computation of the connective constant on the hexagonal lattice. We show Nienhuis's prediction that  $\mu = \sqrt{2 + \sqrt{2}}$ . We also study self-avoiding walks on the so-called  $3.12^2$  lattice and on a slightly modified lattice. These walks can easily be obtained from the self-avoiding walks on the hexagonal lattice via a so-called star-triangle transformation. Surprisingly, these three lattices are the only planar lattices for which a close formula is known (or even conjectured) for the connective constant. We also state a conjecture concerning the behavior of critical planar self-avoiding walks.

Chapter 15 studies supercritical self-avoiding walks. We show that these walks become space-filling in the scaling limit. The theorem is much more rigid than the previous one and applies on any lattice with sufficient symmetry and in any dimensions.

Chapter 16 studies the decomposition of SLE(8/3) (the conjectured scaling limit of self-avoiding walks) into bridges.

Last but not least, Chapter 17 gather open questions on the different subjects treated in this thesis. In particular, parafermionic observables are used to predict the critical behavior of random-cluster and  $O(n)$  models. In addition, we include a short discussion of the square lattice  $O(n)$ -model.



# Chapter 1

## Introduction

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**Abstract:** The first chapter is independent of the rest of the document. It contains a general presentation of planar statistical physics aimed for a wide audience of mathematicians. The main objective is neither completeness nor rigor, but rather to provide a soft introduction of the main concepts appearing in the thesis.

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### 1 Phase transitions

When heating a block of ice, it turns to water. This very familiar phenomenon hides a rather intricate one: the properties of  $H_2O$  molecules do not depend continuously on the temperature. More precisely, macroscopic properties of a large system of  $H_2O$  molecules evolve non-continuously when the temperature rises. For instance, when passing through 0 degree Celsius, the density increases from 0.91 to 1 (it is even more impressive when passing from water to vapor, where the density drops by a factor 1600).

This example of the every day life is an instance of *phase transition*. In a system composed of many particles interacting directly only with their neighbors, a phase transition occurs if a macroscopic property of the system changes abruptly as a relevant parameter (temperature, porosity, density) is varied continuously through a critical value. Understanding how local interactions can govern the behavior of the whole system is extremely hard in general, and involve all fields of physics.

In order to simplify the problem, one can introduce a *model*, *i.e.* an idealized system of particles following elementary rules, which should mimic the behavior of the real model. An example of model could be the following. In order to model the evolution of a large population, one can forget about mortality, fecundity or sex, and simply assume that every individual is hermaphrodite and dies after giving birth to exactly  $n$  children. It is then straightforward to see that such a population survives for ever if and only if  $n \geq 1$ . Of course, this model is pretty far from reality and can be improved in a number of ways. For instance, one can assume that every individual has a random number  $N \in \{0, 1, \dots\}$  of children. It is then possible to show that the population survives forever if and only

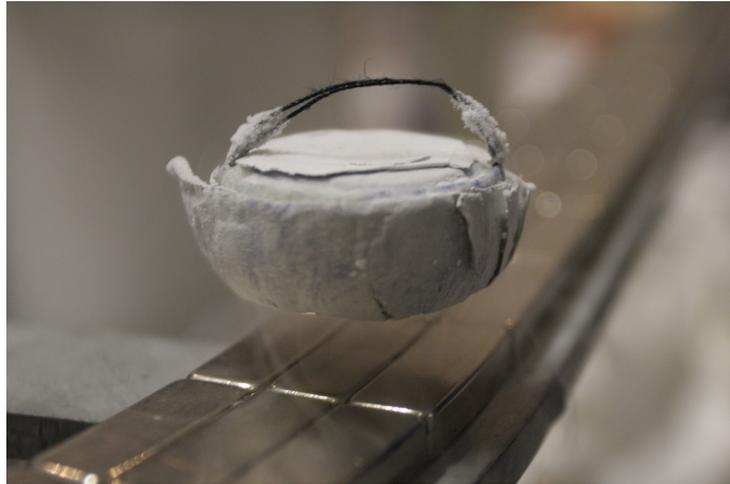


Figure 1.1: A high-temperature supraconductor levitating above a ferromagnet.

if  $\mathbb{E}[N] > 1$ , where  $\mathbb{E}[N]$  is the averaged number of children per individual. In human populations, it is usually admitted that  $\mathbb{E}[N]$  should be around 2.1 per couple to insure stability of the population. It exceeds the theoretical prediction and shows that other factors must be taken into consideration (which is not surprising). Nevertheless, the study of simplified models provides good guesses about the behavior of phenomenon in real life.

The area of science in charge of modeling large systems mathematically is called *statistical physics*. Before diving into mathematical models, let us mention other two classical phase transitions.

Another example of phase transition is given by superconductors. Superconductivity is the phenomenon of exact zero electrical resistance occurring in special materials at very low temperature. It was discovered by Heike Kamerlingh Onnes in 1911 when studying solid mercury at very low temperature (liquid helium was recently discovered, allowing to work with cryogenic temperatures). Below a certain critical temperature  $T_c = 4.2 K$ , the mercury loses its resistance abruptly (Kamerlingh also discovered, without noticing it, the superfluid transition of helium at  $T_c = 2.2 K$ ). Since then, superconductivity has been studied extensively, and the number of examples of superconductors has exploded. Practical applications are numerous, and everyone has the image of a superconductor levitating above a magnet in mind (Fig. 1.1).

Another experiment, which is perhaps even more important historically, was performed in 1895 by Pierre Curie. He showed that a ferromagnet loses its magnetization, when heated above a critical temperature, called the *Curie temperature*. The experiment is fairly simple theoretically: one attaches a rod of iron to an axis, near a large magnet. At room temperature, the rod is attracted by the magnet. When the rod gets hot enough, it abruptly comes back to vertical, witnessing a loss of magnetization, see Fig. 1.2. Practically, the difficulty of the experiment comes from the fact that this temperature equals

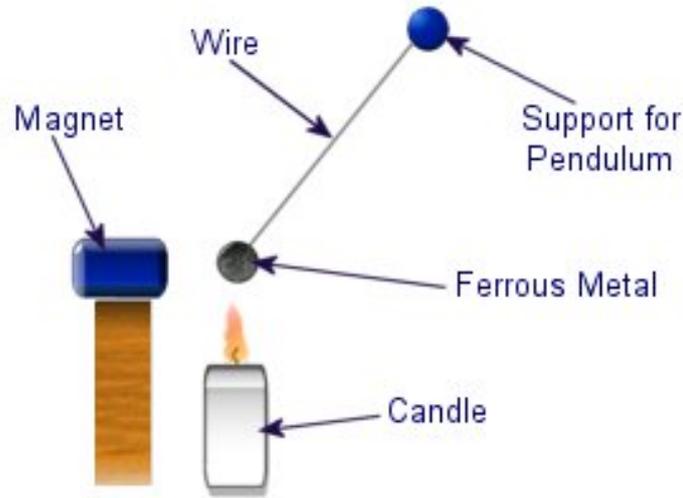


Figure 1.2: Experimental setup to find the Curie temperature of a ferromagnetic material.

770 degree celsius for iron. If the composition of the magnet is different, the critical temperature changes (it can be 30 degree celsius only), yet the phenomenon remains the same. The moral is: it is always possible to un-magnet a matter by heating it, which naturally leads to the following question: what is the microscopic phenomenon explanation this macroscopic behavior?

## 2 Three models of statistical physics

The previous examples illustrate the different kinds of phase transitions occurring in nature. We now aim for a theoretical study of phase transitions. The three following examples illustrate the different properties of statistical models we wish to study through the phase transition. Before starting, a warning: everything contained in this section is not necessarily proved mathematically! We simply plan to motivate the introduction of divers notions, such as *critical exponents*, *universality*, *correlation length*, *order* of a phase transition, *thermodynamical quantities* in a comprehensive way.

### 2.1 Percolation

**Definition and phase transition** Percolation is probably the simplest model of statistical physics. It was introduced by Broadbent and Hammersley in 1957 as a model for a fluid in a porous medium [BH57]. The medium contains a network of randomly arranged microscopic pores through which fluid can flow. One can interpret the  $d$ -dimensional medium as being a lattice (for instance the hypercubic lattice with  $\mathbb{Z}^d$  as vertex set and edges between nearest neighbors), each vertex being a possible hole in the medium. In

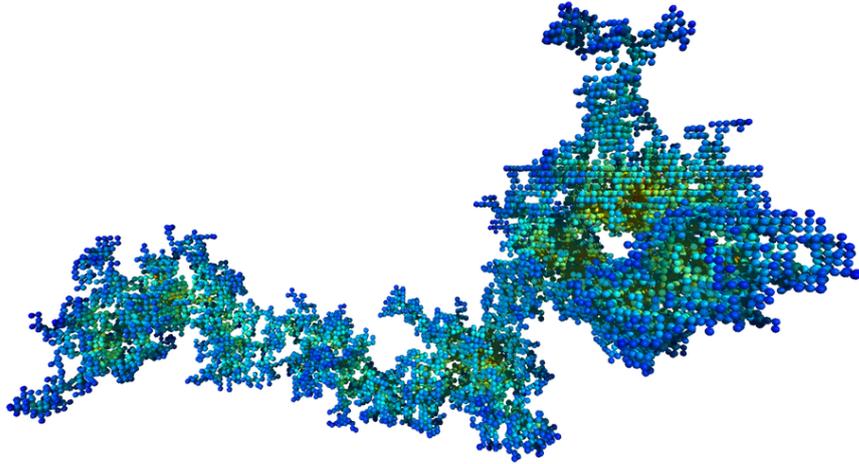


Figure 1.3: A three-dimensional percolation cluster on  $\mathbb{Z}^3$ .

our setting, a vertex is called *open* if it is a hole, and *closed* otherwise. One can then think of the open vertices together with the edges between them as a subgraph of  $\mathbb{Z}^d$ .

In order to model the randomness inside the medium, we simply state that each vertex is open with probability  $p$ , and closed with probability  $1 - p$ , and this independently of each others. The random graph obtained is called  $\omega_p$ , and the probability measure is denoted by  $\mathbb{P}_p$ .

For a fluid to flow through the medium there must exist a macroscopic set of connected open vertices. The phase transition in this model on  $\mathbb{Z}^d$  thus corresponds to the emergence of an infinite connected component (sometimes called *cluster*) of open vertices.

Intuitively, there are more and more open vertices in the graph when we increase  $p$ . It is thus not surprising that there exists a critical  $p_c = p_c(d) \in [0, 1]$  such that

- for  $p < p_c(d)$ , there is no infinite cluster,
- for  $p > p_c(d)$ , there is a infinite cluster. This cluster is unique on  $\mathbb{Z}^d$  (this result is due to [AKN87], alternative arguments were presented in [GGR88] and [BK89]).

Actually,  $p_c(1) = 1$ , since as soon as the vertex-density equals  $p < 1$ , there are always closed vertices to the right and left of every given vertex. Therefore, there is no phase transition in dimension 1. However, as soon as  $d > 1$ , the phase transition occurs in the sense that  $p_c(d)$  lies strictly between 0 and 1. The behavior changes drastically when the porosity parameter  $p$  evolves continuously through  $p_c(d)$ .

**Infinite-cluster density  $\theta(p)$  and universality** When  $p > p_c(d)$ , there exists a unique infinite cluster. Via invariance by translation, this cluster has a positive density  $\theta(p)$ , which can be defined as

$$\theta(p) = \mathbb{P}_p(0 \text{ belongs to the infinite cluster}).$$

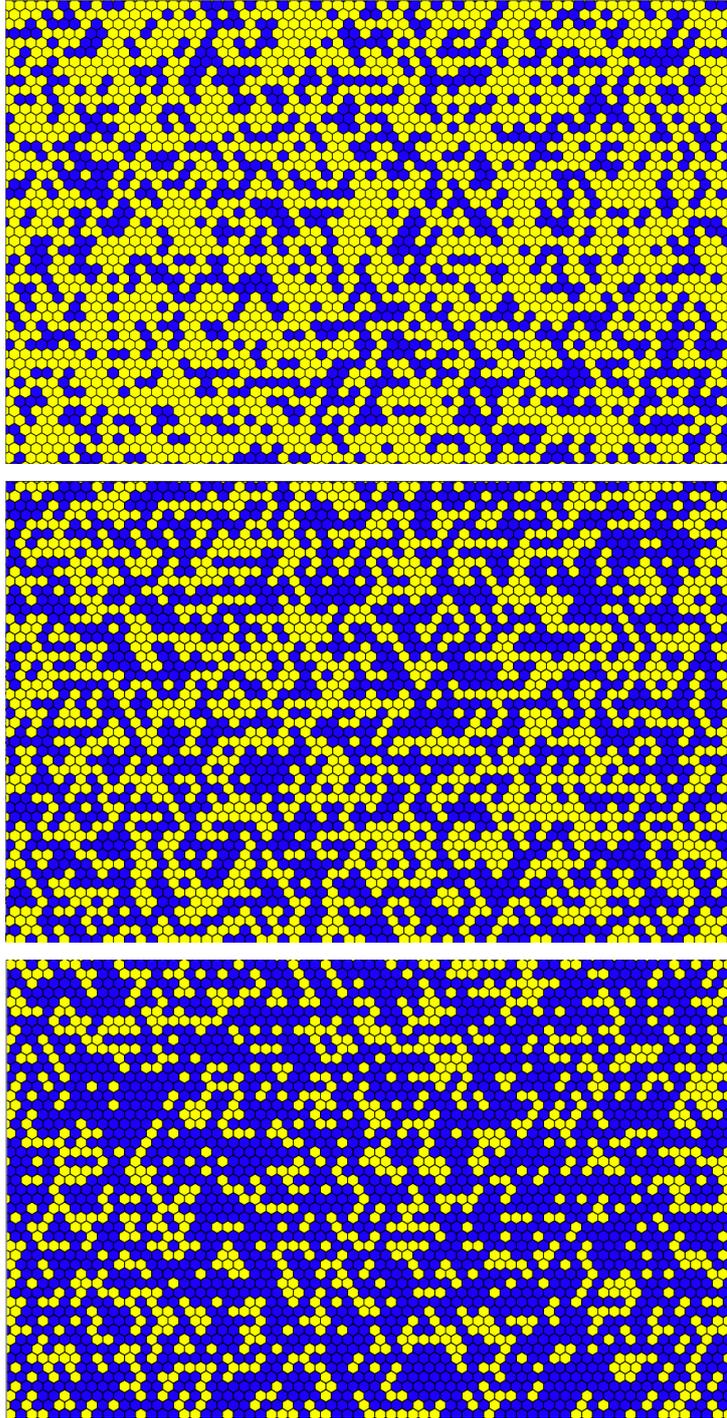


Figure 1.4: Percolation configurations on the triangular lattice for three different values of  $p$  (0.35, 0.5 and 0.65). For esthetic reasons, every vertex is replaced by an hexagonal face: a face is open (blue) with probability  $p$  and closed (yellow) with probability  $1 - p$ , independently of the others.

We are interested in the behavior of  $\theta(p)$  when  $p \searrow p_c(d)$ . This behavior is very similar in every dimensions, even though subtle differences do occur. More precisely,  $\theta(p)$  is predicted to always follow a power law decay in  $(p - p_c)$ . The power, usually named  $\beta$ , depends on the dimension in the following way:

$$\theta(p) \approx (p - p_c)^\beta \quad \text{where } \beta = \begin{cases} 5/36 & \text{if } d = 2 \\ \text{numerical value} & \text{if } d \in \{3, 4, 5\} \\ 1 & \text{if } d \geq 6 \end{cases}$$

The value  $\beta$  is called a *critical exponent*.

As mentioned earlier, one can consider percolation on the hypercubic lattice. Nevertheless, percolation can be defined on any graph or lattice. For instance, it could be defined on the triangular lattice or the triangular lattice in dimension two (see Fig. 1.4). A striking feature of percolation, and more generally of a relevant statistical model<sup>1</sup>, is that the behavior is *universal*: the microscopic properties of the model depend on the local geometry of the graph, while the macroscopic do not. It mimics real phase transitions: the critical temperature for superconductors ranges from a few degrees Kelvin to thirty or even more degrees Kelvin, yet the phase transition is similar. In the case of percolation, connectivity properties between two neighbors in the square or the hexagonal lattices are not the same, yet the thermodynamical properties, such as the infinite-cluster density, behave similarly. Thus, the exponent  $\beta$  is expected to be the same for any lattice of a fixed dimension. For instance,  $\beta$  equals  $5/36$  for the hexagonal, triangular and square lattices.

**Correlation length  $\xi(p)$  and order of a phase transition** As a matter of fact, a phase transition always occurs in infinite volume. To illustrate this, let us make a brief detour and discuss the physical notion of correlation length. It is also an opportunity to introduce an additional critical exponent.

Consider the percolation of parameter  $p$  on a box of size  $N \in (0, \infty]$ . Can we decide with high probability if  $p$  is supercritical or not? When  $N = \infty$  (in other words, we look at the percolation on  $\mathbb{Z}^d$  itself), it is sufficient to check the existence of an infinite cluster. Now, if  $N$  is finite, the situation is more intricate. Indeed, when  $N$  is very small, it is even difficult to give good bounds on  $p$ , see the left-side picture in Fig. 1.5, while when  $N$  is very large, the configuration looks pretty much like the one on  $\mathbb{Z}^d$ . Roughly speaking, the *correlation length* is the smallest  $N = N(p)$  for which we can recognize with good probability if  $p$  is supercritical or not. Similarly, the correlation length in subcritical phase (when  $p < p_c(d)$ ) is the smallest  $N = N(p)$  for which we can decide if  $p$  is subcritical or not.

Mathematically, the correlation length is defined in a *a priori* completely different fashion. We will discuss the relation between the previous non-rigorous definition and the mathematical one in Chapter 12. Formally, when  $p < p_c(d)$ , largest connected components

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<sup>1</sup>we will encounter other examples further on

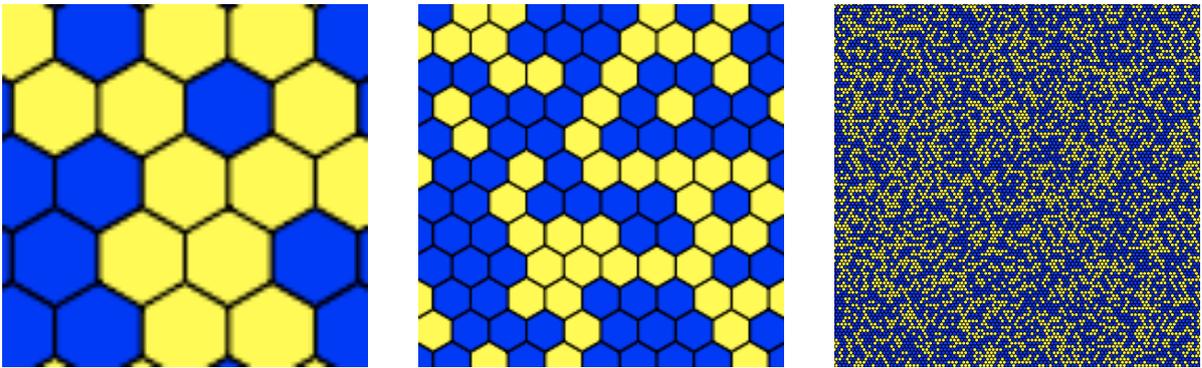


Figure 1.5: Pictures of three percolation configurations with  $p = 0.58$ . In the first one, the size of the box is so small that the number of blue hexagons is not even a good indicator of the value of  $p$ . In the second one, blue hexagons are in majority, yet connectivity properties of the configuration are still 'in favor' of the yellow hexagons (for instance, there is an open yellow path crossing the rectangle from top to bottom). In the last one, the size is big enough that there exists a net of open paths crossing the square from one side to the other. These connected paths are reminiscent of the infinite cluster, and it becomes natural to expect  $p$  to be supercritical. These three pictures are respectively much below, around and much above the correlation length. If  $p$  was closer to  $p_c$ , the third picture would have to be taken much bigger to be sure to find such a net of open paths.

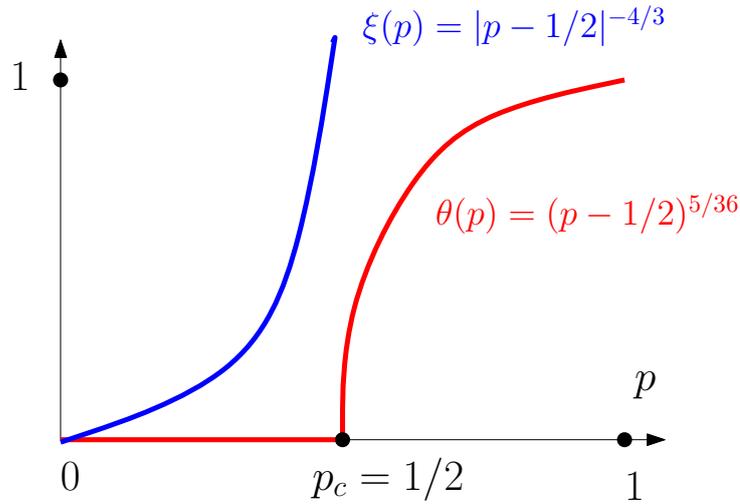


Figure 1.6: Behavior of thermodynamical quantities through the critical value.

in boxes of size  $N$  are typically of size  $\log N$ . Equivalently, the probability for points  $(0, 0, \dots, 0)$  and  $(N, 0, \dots, 0)$  to be connected by a path of adjacent open vertices decays exponentially fast and more precisely like

$$\mathbb{P}_p((0, 0, \dots, 0) \leftrightarrow (N, 0, \dots, 0)) = e^{-N/(\xi(p) + o_N(1))}$$

where  $\xi(p) \in (0, \infty)$  is called the *correlation length*<sup>2</sup>. In supercritical, a corresponding definition can be introduced.

In the case of percolation, the correlation length is finite when  $p \neq p_c$  and goes to infinity when  $p \nearrow p_c$ . This is not the case for every model and it is a proof of a *second order phase transition*. Once again, the behavior of  $\xi(p)$  is expected to follow a power law governed by a critical exponent:

$$\xi(p) \approx |p - p_c|^{-\nu} \quad \text{where } \nu = \begin{cases} 4/3 & \text{if } d = 2 \\ \text{numerical value} & \text{if } d \in \{3, 4, 5\} \\ 1/2 & \text{if } d \geq 6 \end{cases}$$

## 2.2 Ising model

The celebrated Lenz-Ising model is one of the simplest models in statistical physics exhibiting an order-disorder transition. It was introduced by Lenz in [Len20] and studied

<sup>2</sup>The mathematical justification of this definition is the following: for every  $x, y$ ,  $\mathbb{P}_p(0 \leftrightarrow x)\mathbb{P}_p(x \leftrightarrow x+y) \leq \mathbb{P}_p(0 \leftrightarrow x \text{ and } x \leftrightarrow x+y) \leq \mathbb{P}_p(0 \leftrightarrow x+y)$  (the right inequality is direct, while the left one follows from the fact that conditionally on the existence of one of the two paths, the second is more likely to exist, see the FKG inequality in Chapter 3 for more details). Now, translation invariance implies that  $\mathbb{P}_p(x \leftrightarrow x+y) = \mathbb{P}_p(0 \leftrightarrow y)$ , thus giving that the sequence  $u_N = \mathbb{P}_p(0 \leftrightarrow Ne_1)$  is submultiplicative. A classical use of Fekete's subadditive lemma allows us to define  $\xi(p)$ .

by his student Ising in his thesis [Isi25]. It is a model for ferromagnetism as an attempt to explain Curie's temperature. See [Nis05, Nis09] for a historical review of the classical theory.

**Definition** The definition is slightly more intricate than for percolation. In the Ising model, iron is modeled as a collection of atoms with fixed positions on a crystalline lattice. In order to simplify, each atom has a magnetic 'spin', pointing in one of two possible directions. We will set the spin to be equal to 1 or  $-1$ . Each configuration of spins has an intrinsic energy, which takes into account the fact that neighboring sites prefer to be aligned (meaning that they have the same spin), exactly like magnets tend to attract or repel themselves.

Formally, fix a box  $\Lambda$  of size  $n$  in dimension  $d$ . let  $\sigma \in \{-1, 1\}^\Lambda$  be a configuration of spins 1 or  $-1$ , the energy of the configuration  $\sigma$  is given by the Hamiltonian

$$E_\Lambda(\sigma) := - \sum_{x \sim y} \sigma_x \sigma_y$$

where  $x \sim y$  means that  $x$  and  $y$  are neighbors in  $\Lambda$ . Note that up to an additive constant equal to  $-|\Lambda|$ ,  $E_\Lambda$  is twice the number of disagreeing neighbors<sup>3</sup>.

Following a fundamental principle of physics, we wish to construct a model of random spin configurations that favor configurations with small energy. A natural choice is to sample a random configuration proportionally to its Boltzman weight: at a temperature  $T$ , the probability  $\mu_{T,\Lambda}$  of a configuration  $\sigma$  satisfies

$$\mu_{T,\Lambda}(\sigma) := \frac{e^{-\frac{1}{T}E_\Lambda(\sigma)}}{Z_{T,\Lambda}}$$

where

$$Z_{T,\Lambda} := \sum_{\tilde{\sigma} \in \{-1, 1\}^\Lambda} e^{-\frac{1}{T}E_\Lambda(\tilde{\sigma})}$$

is the so-called *partition function* defined in such a way that the sum of the weights over all possible configurations equals 1.

Note that the configurations minimizing the energy, and therefore the most likely, are the extremal ones: either all  $+1$  or all  $-1$ . Nevertheless, there are only two of them, thus the probability to see them in the nature is tiny. In other words, there is a competition between *energy* and *entropy*. The number of configurations for some level of energy can balance the decrease of energy. Finally, properties of a typical configurations are not trivial to study, and depends on the temperature. For instance, if  $T$  converges to  $\infty$ , the configurations become equally likely and the model is almost equivalent to a percolation model (on sites this time) where sites are independent. This phase is called *disordered*. On the contrary, when  $T$  goes to 0, the energy outdoes the entropy and configurations with a large majority or  $+1$  (or  $-1$ ) become typical. This phase is called *ordered*. The existence of two different phases suggests a phase transition.

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<sup>3</sup> $-\sigma_x \sigma_y + 1$  equals 2 if the two sites agree, and 0 otherwise.

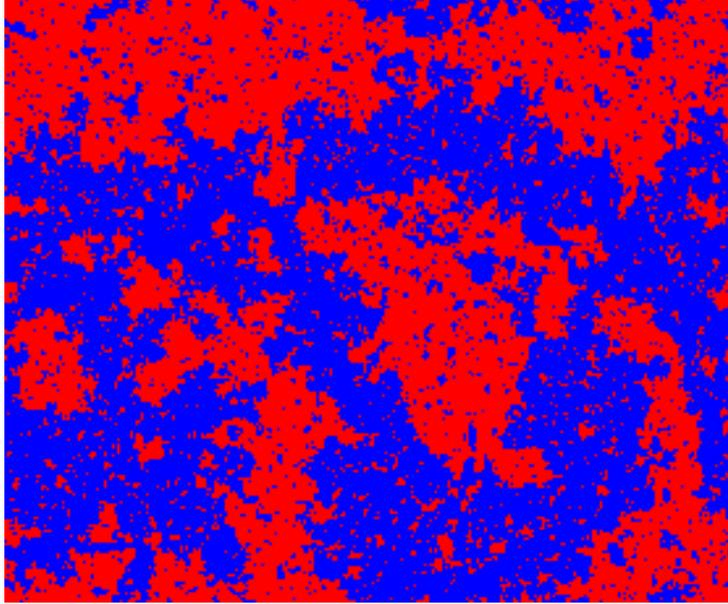


Figure 1.7: A configuration of the Ising model on the square lattice ((© S. Smirnov).

**Phase transition of the Ising model** Assume that spins on the boundary of the box  $\Lambda$  are forced to be +1 (we denote the measure thus obtained by  $\mu_{T,\Lambda}^+$ ) and define the *magnetization* at the origin in the box  $\Lambda$  by

$$M_\Lambda(T) := \mu_{T,\Lambda}^+(\sigma_0).$$

Since the boundary favors pluses, this magnetization is positive. Now, when letting the size of the box go to infinity, the magnetization decreases and converges to a limiting quantity, called the (*spontaneous*) *magnetization*  $M(T) := \lim_{\Lambda \nearrow \mathbb{Z}^d} M_\Lambda(T)$ .

The phase transition in dimension  $d \geq 2$  is the following: there exists a *critical temperature*  $T_c = T_c(d) \in (0, \infty)$  such that

- when  $T > T_c$ ,  $M_T = 0$ ,
- when  $T < T_c$ ,  $M_T > 0$ .

In other words, when the temperature is large, the spin at zero forgets about the boundary conditions: there is no long-range memory. When the temperature is low, the spin keeps track of the boundary conditions at infinity and is still plus with probability larger than  $1/2$ .

We are now in a position to explain Curie's experiment. A magnet imposes an exterior field on an iron rod, forcing exterior sites to be align within it. At low temperature, sites deep inside 'remember' that boundary sites are aligned, while at high temperature, they do not. Therefore, sites become globally aligned at low temperature, hence explaining the magnetization and the attraction.

In his thesis, Ising proved that there is no phase transition when  $d = 1$ . In other words, at any positive temperature, the spontaneous magnetization equals 0. He predicted the absence of phase transition to be the norm in every dimension. This belief was widely shared, and motivated Heisenberg to introduce a famous alternative model where spins take value in the three-dimensional sphere  $\mathbb{S}_3$  (in fact, this is the classical counterpart, first studied in [HK34] of the quantum Heisenberg model).

However, some years later Peierls [Pei36] used estimates on the length of interfaces between spin clusters to disprove the conjecture, showing a phase transition in the two dimensional case. In fact, a phase transition occurs in every dimension  $d \geq 2$ <sup>4</sup>, making the prediction of Ising among the wrongest generalizations in mathematics. The funny thing is that the name 'Ising model' was coined by Peierls in his publication. Ising retired from academics and discovered only 25 years later that his model became famous. Today, the Ising model is widely believed to be the most celebrated model in statistical physics.

**Physical phase transition** Fixing boundary conditions to be +1 or -1 is not completely satisfying physically. In order to mimic the real life experiment, let us add a magnetic field  $h$  in the following way: redefine the energy to be

$$E_{\Lambda,h}(\sigma) := - \sum_{x \sim y} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x.$$

Obviously,  $h$  favors pluses when it is positive (the energy decreases for each spin +1), and minuses when it is negative. Exactly as before, the measure  $\mu_{\Lambda,T,h}$  is defined by assigning to each configuration a weight proportional to  $e^{-\frac{1}{T}E_{\Lambda,h}(\sigma)}$ <sup>5</sup>. As expected,  $M(T,h)$  is strictly positive when  $h > 0$  and strictly negative when  $h < 0$ , but what about  $h$  going to 0? This operation corresponds to removing the magnetic field in the model. A phase transition occurs, at the same critical temperature  $T_c$  as above, in the following way:

- When  $T > T_c$ ,  $M(T,h)$  goes to 0 when  $h$  goes to 0.
- When  $T < T_c$ ,  $M(T,h)$  goes to  $M(T) > 0$  when  $h$  goes to 0 from above, and to  $-M(T)$  when  $h$  goes to 0 from below.

Therefore, at low temperature, the magnet keeps a *spontaneous magnetization*.

**Can we find the equivalent of the percolation critical exponent  $\beta$ ?** Let us study the phase transition, and in particular try to recover critical exponents. Exactly as in the percolation case, the behavior of the magnetization  $M(T,0)$  when  $T$  approaches  $T_c$  from below follows a power law:

$$M(T,0) \approx (T_c - T)^\beta \quad \text{where } \beta = \begin{cases} 1/8 & \text{if } d = 2 \\ \text{real number} & \text{if } d = 3 \\ 1/2 & \text{if } d \geq 4 \end{cases}.$$

<sup>4</sup>It occurs whenever the lattice is not quasi-isometric to  $\mathbb{Z}$ . Here we are cheating a little since this result is not yet known on Cayley graphs with intermediate growth.

<sup>5</sup>Boundary spins are not compelled to be +1 anymore.

The critical exponent  $\beta$  can be compared to the infinite-cluster density of percolation. We will see in Chapter 6 that they are related via the class of random-cluster models.

## 2.3 Self-avoiding walks

In 1953, Nobel prize winner Paul Flory introduced *self-avoiding walks* as a model for ideal polymers<sup>6</sup> [Flo53]. The model is very simple. Consider a lattice (for instance the hypercubic lattice): a *self-avoiding walk* is a self-avoiding sequence of neighboring vertices.

**Enumeration of self-avoiding walks** Of course, the first question that comes to mind deals with the number of self-avoiding walks of length  $n$ . More precisely, define  $\Omega_n$  to be the set of self-avoiding walks of length  $n$  on  $\mathbb{Z}^d$ , and  $c_n$  to be its cardinality.

Counting self-avoiding walks has a long history, see [MS93]. Let us consider the case of the hypercubic lattice  $\mathbb{Z}^3$ . Orr [Orr47] counted them up to  $n = 6$  by hand. For instance,

$$c_6 = 16\,926$$

Computers opened a new scope by offering computational power, yet they reached their full capacity very quickly. The difficulty comes from the fact that there is an exponential number of self-avoiding walks of length  $n$  (we leave to the reader the pleasure to prove the following bounds  $d^n \leq c_n \leq 2d(2d-1)^{n-1}$ ). In 1959, Fisher and Sykes [FS59] enumerated 3D self-avoiding walks up to  $n = 9$ . In 1987, Guttman [Gut87] pushed the computation up to  $n = 20$ . Recently, [SBB11] used a new algorithm together with 50000 hours of computing time to count self-avoiding walks up to  $n = 36$ :

$$c_{36} = 2\,941\,370\,856\,334\,701\,726\,560\,670.$$

Even though it seems hopeless to compute  $c_n$  explicitly for every  $n$ , it is possible to study its asymptotic behavior. Since a  $(n+m)$ -step self-avoiding walk can be uniquely cut into a  $n$ -step self-avoiding walk and a parallel translation of a  $m$ -step self-avoiding walk, we infer that

$$c_{n+m} \leq c_n c_m,$$

from which it follows that there exists  $\mu_c \in (0, +\infty)$  such that

$$\mu_c := \lim_{n \rightarrow \infty} c_n^{\frac{1}{n}}.$$

The positive real number  $\mu_c$  is called the *connective constant* of the lattice. We thus obtain that  $c_n = \mu_c^{n+o(n)}$  and the computation of the connective constant becomes a tempting question... Unfortunately, explicit formulæ for  $\mu_c$  are not expected to be frequent, and mathematicians and physicists only possess numerical predictions for the most common lattices with the exception of the hexagonal lattice, for which  $\mu_c = \sqrt{2 + \sqrt{2}}$ .

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<sup>6</sup>*i.e.* long chains of identical monomers like DNA.

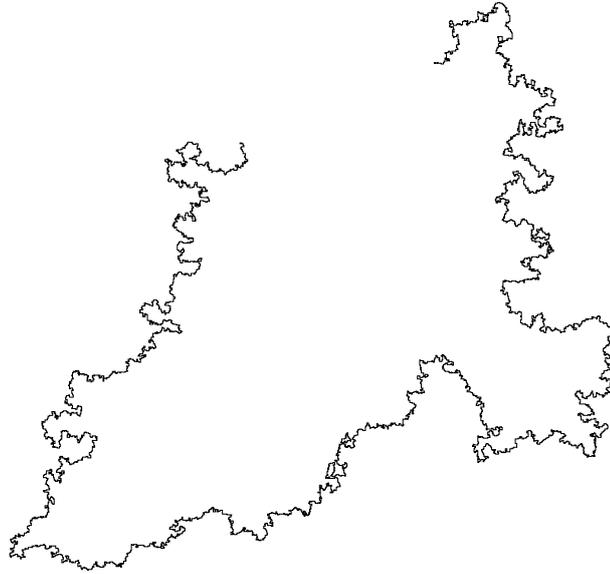


Figure 1.8: A 1000-step self-avoiding walk (© Vincent Beffara).

Overcoming the deception due to the absence of explicit formula for  $\mu_c$ , one can use  $\mu_c$  to get sharper predictions on the behavior of  $c_n$ . Physicists (always one step ahead) conjecture that

$$c_n \approx n^{\gamma-1} \mu_c^n \quad \text{where } \gamma = \begin{cases} 43/32 & \text{if } d = 2 \\ 1.162\dots & \text{if } d = 3 \\ 1 & \text{if } d \geq 4 \text{ (with logarithmic correction for } d = 4) \end{cases} .$$

Once again,  $\gamma$  is a universal exponent depending only on the dimension of the lattice. In this context, universality seems even more surprising: it implies that even though the number of self-avoiding walks is growing exponentially at different speeds for say the hexagonal and the square lattice, the correction to the exponential growth is the same for both lattices.

**Mean-square displacement** Flory was not interested in the combinatorial aspect of self-avoiding walks but rather in its geometry. He predicted that the averaged squared euclidean distance between the ending point and the origin for self-avoiding walks of length  $n$

$$\langle |\gamma(n)|^2 \rangle := \frac{1}{c_n} \sum_{\gamma \in \Omega_n} |\gamma(n)|^2$$

behaves like  $n^{3/2}$  in dimension 2, where  $\gamma(n)$  is the last step of a  $n$ -steps self-avoiding walk. Later, physicists provided strong evidences that

$$\langle \gamma(n) \rangle \approx n^{2\nu} \quad \text{where } \nu = \begin{cases} 3/4 & \text{if } d = 2 \\ 0.59.. & \text{if } d = 3 \\ 1/2 & \text{if } d \geq 4 \end{cases}$$

It is now a good place to compare self-avoiding walks to the *simple random walks model*. A *walk* is a trajectory in  $\mathbb{Z}^d$ , possibly self-crossing. The number of walks of length  $n$  is obviously  $(2d)^n$  and the uniform measure on the family of walks of length  $n$  has a nice interpretation. It corresponds to the random walk constructed as follows: every step, the walker chooses a neighbor uniformly at random. This model is much better understood than the self-avoiding walk. For instance,  $\frac{1}{(2d)^n} \sum_{\gamma \in \Omega_n} |\gamma_n|^2$  behaves asymptotically like  $n$ .

Self-avoiding walks are more spread (they go further) than simple random walks in dimensions 2 and 3. This fact is natural, since a self-avoiding trajectory repulses itself. Interestingly, it is no longer true when the dimension becomes larger. It is actually possible to guess that this would occur, since the simple random walk itself becomes macroscopically self-avoiding at large scales when  $d \geq 4$ .

**Phase transition for self-avoiding walks** So far, the self-avoiding walk did not fit in the framework of statistical physics since it does not depend on any parameter and does not exhibit a phase transition. Thus, let us restate the model in a slightly different way.

Imagine we are now modeling a polymer in a solvent tied between two points  $a, b$  on the boundary of a domain  $\Omega$ . We can model these polymers by self-avoiding walks on a fine lattice  $\Omega_\delta := \delta\mathbb{Z}^d \cap \Omega$  of meshsize  $\delta \ll 1$ . In order to take into consideration the properties of the solvent, let  $x$  be a real positive number. Our polymer will be a curved picked at random among every possible self-avoiding paths in  $\Omega_\delta$  from  $a_\delta$  to  $b_\delta$  ( $a_\delta$  and  $b_\delta$  are the closest points to  $a$  and  $b$  on  $\Omega_\delta$ ), with probability proportional to  $x^{|\gamma|}$ , where  $|\gamma|$  is the length of the self-avoiding walk  $\gamma$ <sup>7</sup>. More precisely, let  $\Gamma_\delta(\Omega, a, b)$  be the set of self-avoiding trajectories from  $a_\delta$  to  $b_\delta$  in  $\Omega_\delta$ . The random polymer will have the law

$$\mathbb{P}_{\mu, \delta}(\gamma_\delta) := \frac{x^{|\gamma_\delta|}}{\sum_{\gamma \in \Gamma_\delta(\Omega, a, b)} x^{|\gamma|}}.$$

This model of random interface exhibits a phase transition when  $x$  varies<sup>8</sup>. On the one hand, when  $x$  is very small, the walk is penalized very much by its length, and it tends to be as straight as possible. On the other hand, if  $x$  is very large, the walk is favored by its length and tends to be as long as possible. Therefore, there exists  $x_c$  such that:

- When  $x < x_c$ ,  $\gamma_\delta$  (which is a random curve) becomes ballistic when  $\delta$  goes to 0: it converges to the (deterministic) geodesic between  $a$  and  $b$  in  $\Omega$ .

<sup>7</sup>This is similar to the Ising model, the energy is equal to the number of vertices on the walk, and the 'temperature' parameter  $T = -1/\log x$

<sup>8</sup>Here,  $\delta \rightarrow 0$  replaces the passage to the infinite-volume  $n \rightarrow \infty$  for percolation and Ising.

- When  $x > x_c$ ,  $\gamma_\delta$  converges to a random continuous curve filling the whole domain  $\Omega$  when  $\delta$  goes to 0.

It is possible to prove that  $x_c = 1/\mu_c$ . In other words, in order to obtain a critical model, one should penalize a walk of length  $n$  by  $\mu_c^{-n}$  (which is intuitive, since there are roughly  $\mu_c^n$  of them). When  $x = x_c$ , the sequence  $(\gamma_\delta)$  should converge in the space of random continuous curve when  $\delta$  goes to 0. In particular, the possible limiting curves should be invariant under scaling. Typical objects having the scale-invariance property are fractals, and it is conjectured that the scaling limit of self-avoiding walk at  $x = x_c$  is a random fractal.

**Flory's exponents and mean-field approximation** For the anecdote, let us present Flory's original determination of  $\nu$  (a little bit of sweetness in the hostile world of critical exponents). We aim to identify the typical distance  $N$  of the last site  $\gamma_n$  of a  $n$ -step self-avoiding walk. In order to do so, we compute the probability of  $|\gamma_n| = N$  in two different ways

First, let us make the assumption that sites are roughly spread on the box of size  $N$ <sup>9</sup>, and that all sites play symmetric roles with respect to each other. We thus know that at each step  $k + 1 \leq n$ , a random walker must avoid the  $k$  previous sites if it wants to remain self-avoiding, so that it must choose one of the  $N^d - k$  available sites. Thus, the probability that  $\gamma$  is still self-avoiding after  $n$  steps is of order

$$\prod_{k=0}^{n-1} \left( \frac{N^d - k}{N^d} \right) \approx \exp \left( - \sum_{k=0}^{n-1} k/N^d \right) \approx \exp \left( - \frac{n^2}{2N^d} \right)$$

as long as  $n \ll N^d$ . The assumption consisting to forget geometry (we do not require that the  $k$ -th point is a neighbor of the  $k - 1$ -th one) is called the *mean-field approximation*.

Second, make the natural assumption that the end-point of the walk is distributed as a Gaussian, the probability for a walk to be at distance  $N$  from the origin is then equal to

$$N^{d-1} \cdot \frac{1}{N^{d/2}} \exp(-N^2/n).$$

Equating the two quantities, we find that  $n^3 \approx N^{d+2}$  *i.e.*  $N \approx n^{3/(d+2)}$ . It gives the following predictions for  $d = 1, 2, 3, 4$ :

$$\nu_{Flory} = \begin{cases} 1 & \text{if } d = 1 \\ 3/4 & \text{if } d = 2 \\ 3/5 & \text{if } d = 3 \\ 1/2 & \text{if } d = 4 \end{cases}.$$

Flory's argument is slightly more evolved and checks in particular that the reasoning cannot be valid when  $d > 4$ . Surprisingly, the prediction is true for  $d = 1, 2$  and 4. It is slightly off for  $d = 3$ . In fact, the prediction is obvious when  $d = 1$ . For  $d = 4$ , the

<sup>9</sup>actually one could take  $cst \cdot N$  with a very large constant instead of  $N$ , but this would not matter.

mean-field approximation is valid, even though its rigorous justification is a very hard problem. Funnily, the prediction in dimension 2 is saved by the surprising cancellation of two large mistakes. The probability to be self-avoiding is much smaller than the one described above. In the same time the Gaussian behavior of the walk is also completely wrong.

Flory's argumentation (especially in dimension 4) emphasizes an important fact of statistical physics: the mean-field approximation (*i.e.* assuming that the system lives on the complete graph) provides tractable ways to predict values for critical exponents and in large enough dimensions, these predictions are right. The reason for this connection is actually much deeper than Flory's argument. Roughly speaking, high-dimensional lattices behave with respect to statistical models like sparse graphs. Making the assumption that we are on the complete graph is then a small mistake. In the case of the self-avoiding walk, the comparison with the simple random walk illustrates this phenomenon. The dimension at which lattice exponents start to equal mean-field exponents is called the *upper critical dimension*  $d_c$ . It is equal to 4 for the self-avoiding walk and the Ising model, while it is 6 for percolation.

In low dimensions, the behavior does not correspond to the mean-field one. Interestingly, the critical exponents in two dimensions are all rational and fairly simple, which suggests a specific feature of two-dimensions that we shall discuss now.

### 3 Why two dimensions?

In the previous section, we self-avoiding walk by studying three very different models of statistical physics, that they shared properties concerning their phase transitions. On the one hand, critical exponents become independent of the dimension when exceeding the upper critical dimension of the model. On the other hand, exponents have rational values in two dimensions, which suggests the existence a deep underlying mechanism coming from physical laws. Our goal is to understand the phase transition in the latter case and **we now fix  $d = 2$  for the rest of the manuscript**. Mathematicians also make the assumption that **models are critical**.

This latter assumption is not very dramatic. In order to study the phase transition, in particular the critical exponents related to thermodynamical quantities, it is sufficient to study the critical phase. Indeed, critical exponents are not independent: they are connected via the so-called *scaling relations*, which do not depend on the model. One example of scaling relation is given by  $\beta = \nu\eta$ , where  $\beta$  and  $\nu$  were defined in the context of percolation, but also exist for other statistical models, and  $\eta$  is the one-arm exponent <sup>10</sup>.

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<sup>10</sup>The exponent  $\eta$  can be introduced in most statistical physics models. In the case of percolation or Ising, it is defined as follows:

- for percolation at criticality, there is no infinite cluster and the probability for two points to be connected converges to 0 when their distance goes to  $\infty$ . In fact, the behavior should be

$$\mathbb{P}_{p_c}(0 \leftrightarrow x) \approx \frac{1}{|x|^{d-2+\eta}},$$

Therefore, critical exponents depending only on the critical phase are often sufficient to understand the other critical exponents (such as  $\beta$  or  $\nu$ ).

### 3.1 Exactly solvable models and Conformal Field Theory:

The planar Ising model has been the subject of experimentation for both mathematical and physical theories for almost a century. Through a short historic of this model, we shall explain two physical perspectives on statistical physics.

**Exactly solvable models** It all started with Peierls's proof of the existence of a phase transition. This argument (the first of the kind) paved the way to the study of the critical regime<sup>11</sup>. The next step was achieved by Onsager in 1944. In a series of seminal papers [Ons44, KO50], he<sup>12</sup> computed the partition function of the model and proved the equality  $\beta = 1/8$ . This result represented a shock for the community: it was the first mathematical proof that the mean-field behavior was inaccurate in low dimensions! Moreover, in the physical approach to statistical models, the computation of the partition function is the first step towards a deep understanding of the model, enabling for instance the computation of the free energy. The formula provided by Onsager led to an explosion in the number of results on the planar Ising model (papers published on the Ising model can now be counted by thousands). Among the most noteworthy results, Yang derived rigorously the spontaneous magnetization [Yan52] (the result was derived non rigorously by Onsager himself), McCoy and Wu [MW73] computed many important quantities of the Ising model, including several critical exponents, culminating with the derivation of two-point correlations  $\mu_T(\sigma_0\sigma_x)$  between sites 0 and  $x = (n, n)$  in the whole plane. See the more recent book of Palmer for an exposition of these and other results [Pal07].

The computation of the partition function was accomplished later by several other methods and the model became the most prominent example of an exactly solvable model. The most classical techniques include the transfer-matrices technique developed by Lieb and Baxter [Lie67, Bax89], the Pfaffian method, initiated by Fisher and Kasteleyn, using a connection with dimers models [Fis66, Kas61], and the combinatorial approach to the Ising model, initiated by Kac and Ward [KW52] and then developed by Sherman [She60] and Vdovichenko [Vdo65], see also the more recent [DZM<sup>+</sup>99, Cim10].

Despite the number of results that can be obtained using the partition function, the impossibility to compute it explicitly enough in finite volume makes the geometric study of the model very hard to perform while using the classical methods. The lack of understanding of the geometric nature of the model remained unsatisfying for years.

- 
- for the Ising model, the magnetization equals 0 and we have

$$\mu_{T_c}(\sigma_0\sigma_x) \approx \frac{1}{|x|^{d-2+\eta}}.$$

<sup>11</sup>Peierls argument was later extended to many other statistical models.

<sup>12</sup>later helped by Kaufman.

**Renormalization group theory and Conformal Field Theory** The arrival of the renormalization group formalism (see [Fis98] for a historical exposition) led to a better physical and geometrical understanding, albeit mostly non-rigorous. It suggests that block-spin renormalization transformation (coarse-graining, *e.g.* replacing a block of neighboring sites by one site having a spin equal to the dominant spin in the block) corresponds to appropriately changing the scale and the temperature of the model. The critical point arises then as the fixed point of the renormalization transformations. In particular, under simple rescaling the Ising model at the critical temperature should converge to a scaling limit, a 'continuous' version of the originally discrete Ising model, corresponding to a quantum field theory. This leads to the idea of universality: the Ising models on different regular lattices or even more general planar graphs belong to the same renormalization space, with a unique critical point, and so at criticality the scaling limit of the Ising model should always be the same (it should be independent of the lattice while the critical temperature depends on it).

Being unique, the scaling limit at the critical point must satisfy translation, rotation and scale invariance, which allows us to deduce some information about correlations [PP66, Kad66].

In seminal papers [BPZ84b, BPZ84a] Belavin, Polyakov and Zamolodchikov, suggested a much stronger invariance of the model. Since the scaling-limit quantum field theory is a local field, it should be invariant by any map which is *locally* a composition of translation, rotation and homothety. Thus it becomes natural to postulate full conformal invariance (under all conformal transformations<sup>13</sup> of subregions). This prediction generated an explosion of activity in conformal field theory, allowing for non rigorous explanations of many phenomena, see [ISZ88] for a collection of the original papers of the subject.

Note that planarity enters into consideration through the fact that conformal maps form a rich family of operators. Indeed, the category of conformal maps in dimension two is composed of many elements, while it restricts in higher dimensions to compositions of rotations, translations and inversions.

**Where are we now?** The above exposition shows two different approaches to the same problem relying heavily on two-dimensionality:

- The exact solvability of the (discrete) planar Ising model, which allows rigorous derivation of important quantities, yet at the same time provides a poor geometric understanding.
- The non-rigorous conformal field theory approach, with the postulate of a 'continuum limit', invariant under many geometric transformations, which allows a deep geometric understanding of the model.

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<sup>13</sup>Conformal maps are maps on open sets of  $\mathbb{C}$  conserving the angles. Equivalently, they are the one-to-one holomorphic maps.

### 3.2 A mathematical setting for conformal invariance of lattice models

To summarize, Conformal Field Theory asserts that a planar statistical model, such as percolation, Ising or the self-avoiding walk, admits a 'scaling limit' at criticality, and that this scaling limit is a conformally invariant object.

From a mathematical perspective, this notion of conformal invariance of a model is ill-posed, since the meaning of scaling limit is not even clear. The following solution to this problem can be implemented: the scaling limit of the model could simply be less rich and retain the information given by 'interfaces' only<sup>14</sup>. The advantage of this approach is that there exists a mathematical setting for families of continuous curves.

Let us first start with the study of one curve. There is a number of ways to isolate an 'interface'<sup>15</sup> in a model. For pedagogical reasons, we simplify the presentation as much as possible by providing two examples in elementary cases. Fix a simply connected domain  $(\Omega, a, b)$  with two points on the boundary and consider discretizations  $(\Omega_\delta, a_\delta, b_\delta)$  of  $(\Omega, a, b)$  by an hexagonal lattice of meshsize  $\delta$ . A particularly simple model to start with is the critical self-avoiding walk. The model of random polymer between  $a_\delta$  and  $b_\delta$  contains by definition only one interface (the walk itself), denoted  $\gamma_\delta^{\text{self-avoidingwalk}}$ . The parameter  $x$  being critical, Conformal Field Theory predicts that  $\gamma_\delta^{\text{self-avoidingwalk}}$  should converge when  $\delta$  goes to 0 to a random continuous curve between  $a$  and  $b$  in  $\Omega$ . A second model of interest is the critical Ising model on the triangular lattice. Assume now that we fix the spins to be +1 on the arc  $\partial_{ab}$  and -1 on the arc  $\partial_{ba}$ . Thus, there exists a unique interface between +1 and -1 going from  $a$  to  $b$ . We call this interface  $\gamma_\delta^{\text{Ising}}$ . Conformal Field Theory once again predicts that  $\gamma_\delta^{\text{Ising}}$  converges when  $\delta$  goes to 0 to a random continuous non-selfcrossing curve between  $a$  and  $b$  in  $\Omega$ . By the way, how would you proceed for site percolation on the triangular lattice (the answer hides in Fig. 1.11)? In fact, Conformal Field Theory also predicts that the limits of  $(\gamma_\delta^{\text{self-avoidingwalk}})_{\delta>0}$  and  $(\gamma_\delta^{\text{Ising}})_{\delta>0}$  must be conformally invariant, where now conformal invariance has a precise meaning:

*A family of random continuous curves  $\gamma_{(\Omega, a, b)}$  indexed by simply connected domains with two marked points on the boundary  $(\Omega, a, b)$  is conformally invariant if for any  $(\Omega, a, b)$  and any conformal map<sup>16</sup>  $\psi : \Omega \rightarrow \mathbb{C}$ ,*

$$\psi \circ \gamma_{(\Omega, a, b)} \text{ has the same law as } \gamma_{(\psi(\Omega), \psi(a), \psi(b))}.$$

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<sup>14</sup>There is no reason why all the information of a model should be encoded into information on interfaces, yet one can hope that most of the relevant quantities can be recovered from it.

<sup>15</sup>*i.e.* a boundary between two different regions determined by the model

<sup>16</sup>conformal means holomorphic and one-to-one. Via Riemann mapping theorem, we know that many such maps exist

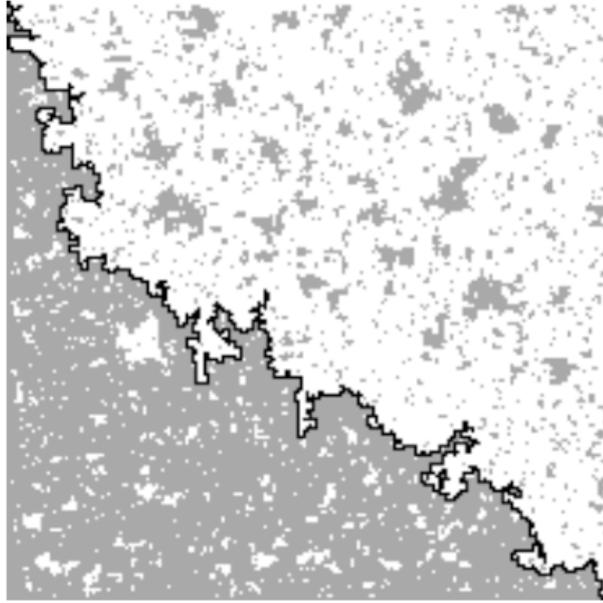


Figure 1.9: The interface of an Ising model at critical temperature (© Stanislav Smirnov).

In words, the random curve obtained by taking the scaling limit of self-avoiding walks on  $(\psi(\Omega), \psi(a), \psi(b))$  has the same law as the image by  $\psi$  of the scaling limit of self-avoiding walks on  $(\Omega, a, b)$ . It is clear when working on the hexagonal lattice, that rotations by an angle  $\pi/3$  are preserving the model. Conformal Field Theory predicts that the model possesses much more symmetries as soon as we allow ourselves to go to the scaling limit.

In 1999, Schramm proposed a natural candidate for the possible conformally invariant families of continuous non-selfcrossing curves. He noticed that interfaces of models further satisfy the *domain Markov property* (see Chapter 11), which, together with the assumption of conformal invariance, determine the possible families of curves. In [Sch00], he introduced the *Schramm-Loewner Evolution*<sup>17</sup> – SLE for short. The  $SLE(\kappa)$ , for  $\kappa > 0$ , is the random Loewner Evolution with driving process  $\sqrt{\kappa}B_t$ , where  $(B_t)$  is a standard Brownian motion (the precise definition of SLE is presented in Chapter 11). By construction, the process is conformally invariant, random and fractal. In addition, it is possible to study quite precisely the behavior of SLEs using stochastic calculus and to derive path properties such as the Hausdorff dimension, intersection exponents, etc... Depending on  $\kappa$ , the behavior of the process is very different, as one can see on Fig. 1.10. The prediction of Conformal Field Theory then translates into the following predictions for models.

*The limit of  $(\gamma_\delta^{self-avoidingwalk})_{\delta>0}$  and  $(\gamma_\delta^{Ising})_{\delta>0}$  in  $(\Omega, a, b)$  is a Schramm-Loewner Evolution.*

<sup>17</sup>In the original paper, the process is called Stochastic-Loewner Evolution.

Naturally, the parameter  $\kappa$  depends on the model, yet, it is usually possible to guess which one it should be. For instance, self-avoiding walks should converge to  $\text{SLE}(8/3)$ , while Ising interfaces should converge to  $\text{SLE}(3)$ .

In order to finish this chapter, let us deal with families of interfaces. In the case of self-avoiding walks, the problem does not make sense, yet for the Ising model, there are many interfaces. More precisely, consider the Ising model without boundary conditions in an approximation of  $\Omega$ . Interfaces now form a family of loops. By consistency, each loop should look like a  $\text{SLE}(3)$ . Sheffield and Werner [SW10a, SW10b] introduced a one-parameter family of processes of non-intersecting loops which are conformally invariant – called the Conformal Loop Ensembles  $\text{CLE}(\kappa)$  for  $\kappa > 8/3$ . Non-surprisingly, loops of  $\text{CLE}(\kappa)$  are locally similar to  $\text{SLE}(\kappa)$ . In the case of the Ising model, the limits of interfaces all-together should be a  $\text{CLE}(3)$ .

Interestingly, path properties of SLEs and CLEs allow us to derive some critical exponents governing the scaling limit at criticality. Then, scaling relations allow us to obtain a complete understanding of the phase transition.

### 3.3 Conformal invariance of percolation and Ising models

Even though we now have a mathematical framework for conformal invariance, it remains an extremely hard task to prove convergence of interfaces to SLEs. Observe that working with interfaces offers a further simplification: properties of these interfaces should also be conformally invariant. Therefore, we could simply look at an *observable* of the model, *i.e.* something that we can measure by looking at the configuration. Of course, it is not clear that this observable would tell us anything about critical exponents, yet it already represents a significant step toward conformal invariance.

In 1994, Langlands, Pouliot and Saint-Aubin [LPSA94] published a number of numerical values in support of conformal invariance (in the scaling limit) of crossing probabilities in the percolation model. More precisely, they checked that taking different topological rectangles, the probability  $C_\delta(\Omega, A, B, C, D)$  of having a path of adjacent open edges from  $AB$  to  $CD$  converges when  $\delta$  goes to 0 towards a limit which is the same for  $(\Omega, A, B, C, D)$  and  $(\Omega', A', B', C', D')$  if they are image of each other by a conformal map<sup>18</sup>. The paper [LPSA94], while only numerical, attracted many mathematicians to the domain. The same year, Cardy [Car92] proposed an explicit formula for the limit of crossing probabilities. In 2001, Smirnov proved Cardy's formula rigorously for critical site percolation on the triangular lattice, hence rigorously providing a concrete example of a conformally invariant property of the model. A somewhat incredible consequence of this theorem is that the mechanism can be reversed: even though Cardy's formula seems much weaker than convergence to SLE, they are actually equivalent. In other words, conformal invariance of one well-chosen observable of the model can be sufficient to prove conformal invariance of interfaces, which in turn is sufficient to determine critical exponents. We are now in a

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<sup>18</sup>This property can be expressed in terms of properties of an interface, thus keeping this discussion in the framework proposed earlier.

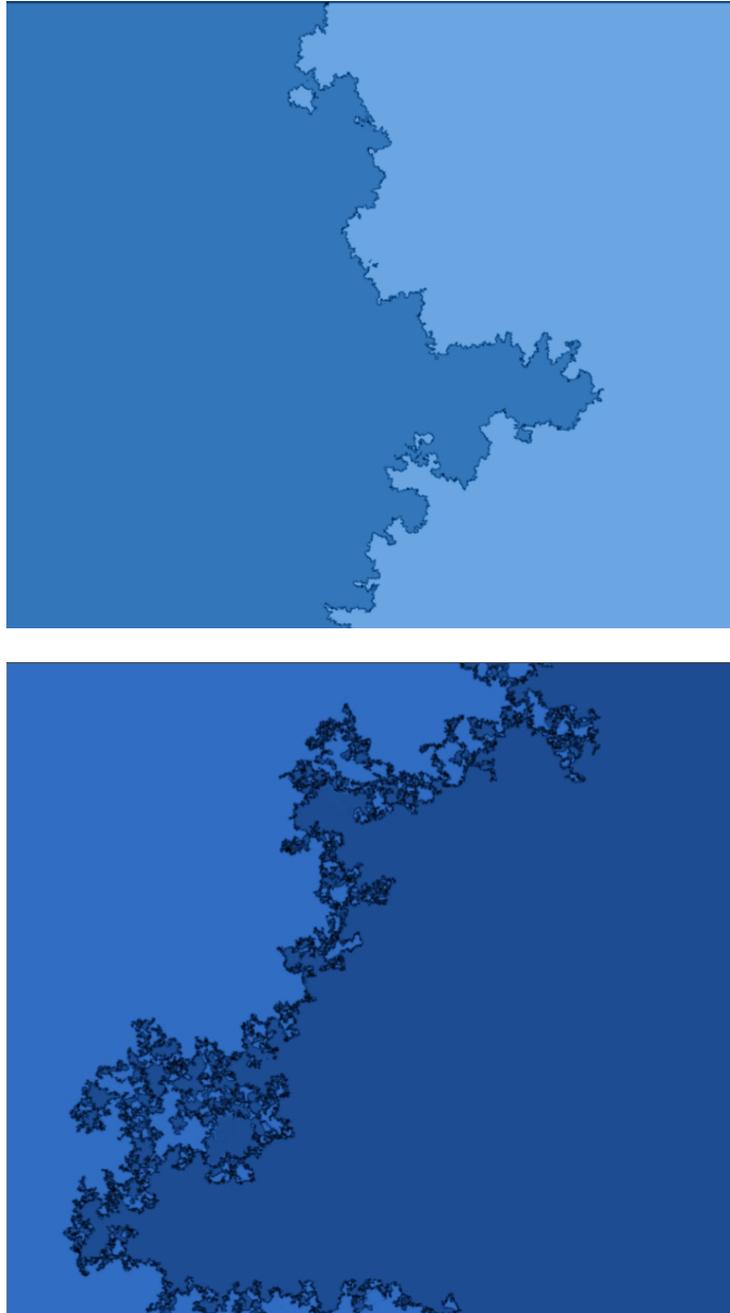


Figure 1.10: Two examples of Schramm-Loewner Evolution ( $SLE(8/3)$  and  $SLE(6)$ ). The behavior is very different: the first one is almost surely a simple curve while the second one has self-touching points, the hausdorff dimensions are different, etc... (© V. Beffara).

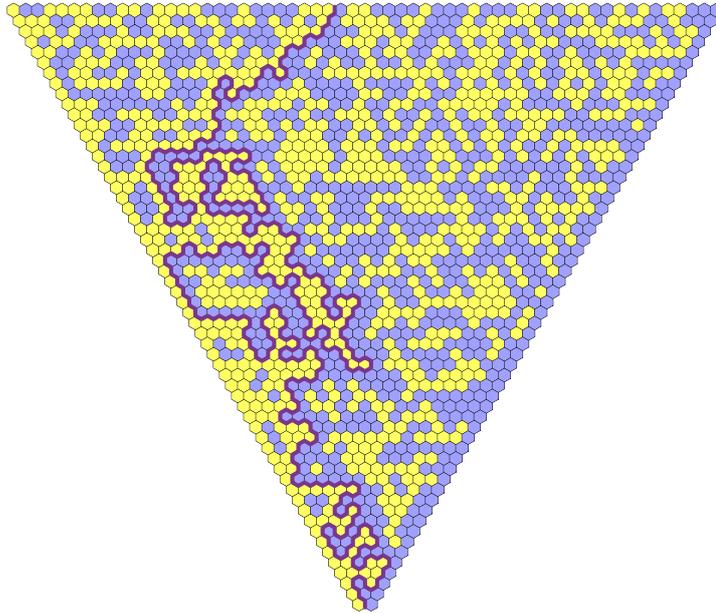


Figure 1.11: An exploration path for percolation. It converges to SLE(6) in the scaling limit (© V. Beffara).

much better position in order to understand conformal invariance of a model: it suffices to show that an observable of the discrete model converges to a conformally invariant (in fact a conformally covariant) family of functions.

In 2010, Smirnov struck a second time by exhibiting conformally covariant observables for the so-called FK-Ising [Smi10a] and Ising [CS09] models. Nonetheless, in this case the study of the critical regime is harder than in the percolation case. Indeed, long-range dependence at criticality makes the mathematical understanding more involved and even convergence of interfaces to SLE is difficult. Anyway, the philosophy remains the same and full conformal invariance should follow from conformal covariance of these observables.

We conclude this paragraph with a warning (or a touch of hope, depending on personal opinion): there are very few models which have been proved to be conformally invariant. For instance, the self-avoiding walk does not belong to this restricted club and it remains a very important open problem to prove convergence of self-avoiding walks to SLE(8/3).

### 3.4 Discrete holomorphicity and statistical models

The previous discussion (especially in the Ising case) sheds a new light on both approaches described in Subsection 3.1: combinatorial properties of the discrete Ising model allow us to prove the convergence of discrete observables to conformally covariant objects. In other words, exact integrability and Conformal Field Theory are connected via the proof of the conformal invariance of the Ising model.

Archetypical examples of conformally covariant objects are holomorphic solutions to boundary value problems such as Dirichlet or Riemann problems. It becomes natural that discrete observables which are conformally covariant in the scaling limit are naturally preharmonic or preholomorphic functions, *i.e.* relevant discretizations of harmonic and holomorphic functions. Therefore, proofs of conformal invariance harness *discrete complex analysis* in a substantial way. The use of discrete holomorphicity appeared first in the case of dimers [Ken00] and has been extended to several statistical physics models since then. Other than being interesting in themselves, preholomorphic functions found several applications in geometry, analysis, combinatorics, probability, and we refer the interested reader to the expositions by Lovász [Lov04], Stephenson [Ste05], Mercat [Mer01], Bobenko and Suris [BS08].

To conclude this section, we are now in a possession of a natural mathematical framework to prove conformal invariance of a model: one needs to prove conformal invariance of an observable. Proving this requires a deep understanding of discrete complex analysis, and of its connections to the model. Very often, the integrability properties of the model are at the heart of the proof, thus showing a new connection between exactly solvable models and Conformal Field Theory.

## 4 Unifying families of models

Percolation, Ising and self-avoiding walks provide us with three examples of models which are conformally invariant in the scaling limit (only conjecturally for the self-avoiding walk). They correspond to three values of the Schramm-Loewner Evolution ( $\kappa$  equals 6, 3 and  $8/3$  respectively). But what about other values of  $\kappa$ ? Is it always possible to find a conformally invariant model which interfaces converge to SLE( $\kappa$ )? More importantly, can these seemingly very different models be related? At last, can this relation explain the similarities between the different models? The answer to these questions come from the existence of two grand families of models. These models will be at the heart of the theory, we would like to present them now.

### 4.1 Random-cluster model

Fortuin and Kasteleyn introduced the random-cluster model in 1969. Roughly speaking, the *random-cluster model* on a graph  $G$  is also a percolation model, in the sense that the output is a random subgraph of  $G$  with the same set of vertices and a subset of its edges, but not longer independent. More precisely, an edge of a finite graph  $G$  is either *open* or *closed*. The random-cluster configuration  $\omega$  is the graph obtained by keeping only the open edges. Let  $p \in [0, 1]$  and  $q \in (0, \infty)$ . The probability of  $\omega$  for the random-cluster model on  $G$  with parameters  $p, q$  is given by

$$\phi_{p,q}(\omega) := \frac{1}{Z_{G,p,q}} p^{\#\text{ open edges}} (1-p)^{\#\text{ closed edges}} q^{\#\text{ connected components}}$$

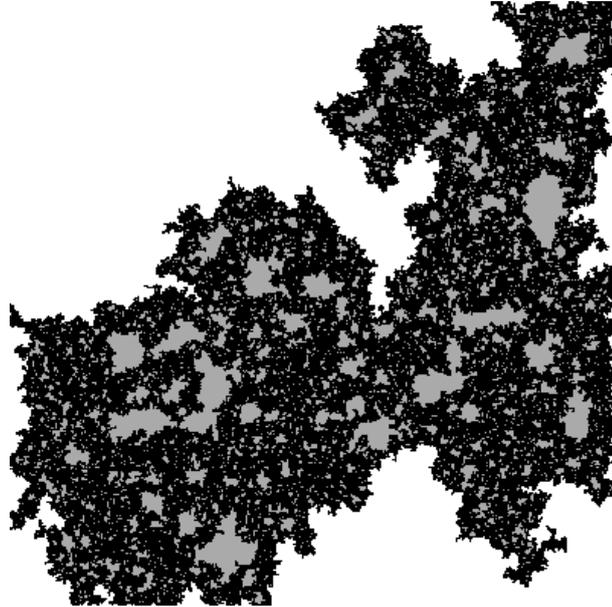


Figure 1.12: A macroscopic cluster in a critical percolation configuration with  $p = 1/2$ .

where  $Z_{G,p,q}$  is once again a normalizing factor called the partition function of the model. When  $q = 1$ , the model is simply edge percolation (a model very similar to the site percolation described earlier). When  $q \neq 1$ , the model is different and exhibits long range dependence.

It is possible (yet non-trivial) to define the model on  $\mathbb{Z}^2$ . As for percolation, the random-cluster model with fixed  $q > 0$  should encounter a phase transition in  $p$ . Below some critical parameter  $p_c(q)$ , there is no infinite cluster, while above it, there exists a unique infinite cluster. The phase transition is different when  $q$  varies, and the richness of this behavior is one of the success of random-cluster models. More precisely,

- when  $q \in (0, 4]$ , the transition is expected to be continuous, in the sense that the infinite-density cluster  $\theta(p, q)$  converges to 0 when  $p \searrow p_c(q)$ . The critical phase should also be conformally invariant, and the collection of interfaces at criticality<sup>19</sup> should converge to  $\text{CLE}(\kappa)$ , where  $\kappa = 4\pi/\arccos(-\sqrt{q}/2)$ .
- when  $q > 4$ , the phase transition becomes *first order*. More precisely,  $\theta$  does not converge to 0 when  $p$  goes down to  $p_c(q)$ .

Another important advantage of the random-cluster model is its connection to other models. When  $p \rightarrow 0$  with  $q/p \rightarrow 0$ , we obtain a model of a random connected graph, called the uniform spanning tree, see Fig. 1.13. When  $q$  is an integer, one can play the following game. Color independently each connected component of a  $(p, q)$ -random-cluster

<sup>19</sup>We did not describe interfaces in percolation or the random-cluster model, yet one can consider boundary of connected components for instance.

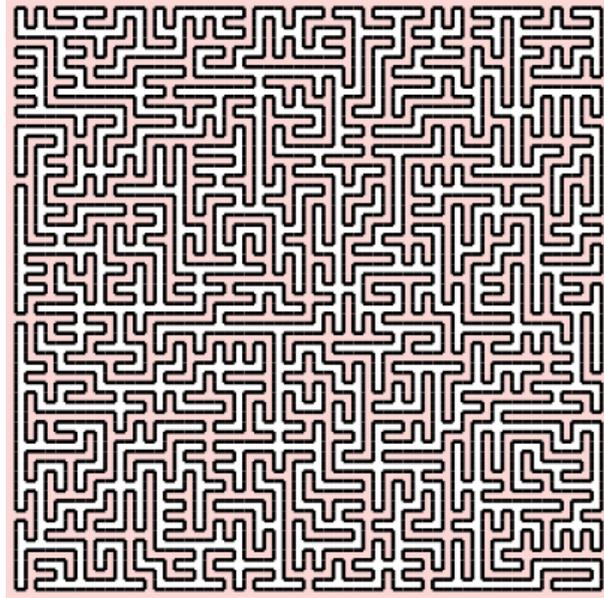


Figure 1.13: The pink part forms a spanning tree (a tree passing through every vertex). The black path is a space-filling curve bordering the spanning tree. It is also possible to consider the scaling limit of the black path: it converges to SLE(8) [LSW04a] (© O. Schramm).

configuration  $\omega$  with one of  $q$  fixed colors chosen uniformly. We obtain a random coloring  $\sigma \in \{1, \dots, q\}^\Lambda$  of  $\Lambda$ . The probability measure  $P$  is a Boltzman measure with energy given by

$$H_{q,\Lambda}(\sigma) := - \sum_{x \sim y} 1_{\sigma_x = \sigma_y}.$$

The random coloring of the lattice with law  $P$  is called the *Potts* model with  $q$  colors at a temperature  $T$ . When  $q = 2$ , it corresponds to the Ising model (simply call one color +1 and the other -1). Therefore, there exists a *coupling* of the Ising model with the  $q = 2$  random-cluster model. This property links the Ising model to random-cluster models and thus to percolation.

## 4.2 $O(n)$ -models

We would like to finish this first chapter by introducing another class of models. Ising's conjecture of the absence of phase transition led Heisenberg to introduce his famous model. In the classical version of this quantum model, spins are 3-dimensional unit vectors. The energy of a configuration is then

$$H(\sigma) := - \sum_{x \sim y} \langle \sigma_x, \sigma_y \rangle.$$

In [Sta68], this model was generalized by taking spins to be  $n$ -dimensional unit vectors, and called *spin  $O(n)$ -models*. The  $n = 1$  model is the Ising model yet again. Ironically, while spin  $O(n)$ -models were introduced in order to create relevant models for magnetism with a phase transition, it appears that only the  $n = 1$  model (the Ising model) exhibits one.

Now take the Ising model on the triangular lattice. Interfaces between  $+1$  and  $-1$  define a *loop model* on the dual hexagonal lattice. The statistics of this model is easy to compute: the probability of a configuration is proportional to  $e^{-2\beta \# \text{ edges}}$ . In fact, this model gives rise to a family of models, called loop  $O(n)$ -models. Consider a finite subgraph  $\Lambda$  of the hexagonal lattice and set  $n \geq 0$  and  $x > 0$ . Choose a configuration  $\omega$  of loops with one self-avoiding path starting from the origin with probability

$$\mathbb{P}_{x,n}(\omega) := \frac{x^{\# \text{ edges}} n^{\# \text{ loops}}}{Z_{\Lambda,x,n}}.$$

When  $n = 0$ , we obtain the self-avoiding walk and the model undergoes a phase transition when  $x$  varies from 0 to  $\infty$ . The  $n = 1$  and  $x = 1$  model is exactly the interfaces of site percolation with parameter  $p = 1/2$  on the triangular lattice. For integer values of  $n$ , one can relate the spin  $O(n)$ -model to the loop  $O(n)$ -model, see Chapter 13 for details. Therefore, spin and loop  $O(n)$ -models form two families of models that relate the self-avoiding walk, the percolation and the Ising models.

### Conclusion

We presented several aspect of statistical physics, in particular when the models are planar. We sketched deep links between physics and mathematics. Nevertheless, most of what we presented is still conjectural. In this thesis, we make some of the connections between physics and mathematics rigorous by studying random-cluster and  $O(n)$ -models.



# Chapter 2

## Discrete complex analysis on graphs

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**Abstract:** This chapter must be understood as a toolbox. It gathers several theorems concerning the theory of discrete holomorphic maps on discretizations of domains of the plane. These theorems will be extensively used in the whole manuscript. The two first parts are classical and can be found in any textbook on discrete harmonic functions. The third part is extracted from [Smi10a].

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Complex analysis is the study of harmonic and holomorphic functions in complex domains. In this section, we shall discuss how to discretize harmonic and holomorphic functions, and what are the properties of these discretizations.

There are many ways to introduce discrete structures on graphs which can be developed in parallel to the usual complex analysis. We will consider scaling limits (as mesh of the lattice tends to zero), therefore we wish to deal with discrete structures which converge to the continuous complex analysis as graphs become finer and finer.

The chapter is organized as follows. The first section (perhaps the most important one) defines the notion of discrete approximation of a continuous domain. The second section deals with discrete harmonic functions. While the theory is fairly classical, we chose to expose it anyway, in particular because specific properties are needed later, and that they are not necessarily known to everyone. Section 3 introduces discrete holomorphic functions. Section 4 is devoted to  $s$ -holomorphic maps while the last section discusses other possible approaches.

### 1 Lattices and approximation of domains

#### 1.1 Primal, dual and medial graphs

The (**rotated**) square lattice  $\mathbb{L} = (\mathbb{V}, \mathbb{E})$  is the graph with vertex set  $\mathbb{V} := e^{i\pi/4}\mathbb{Z}^2$  and edge set  $\mathbb{E}$  given by edges between nearest neighbors, see Fig. 2.1. An edge with endpoints  $x$  and  $y$  will be denoted by  $[xy]$ . If there exists an edge  $e$  such that  $e = [xy]$ , write  $x \sim y$ .

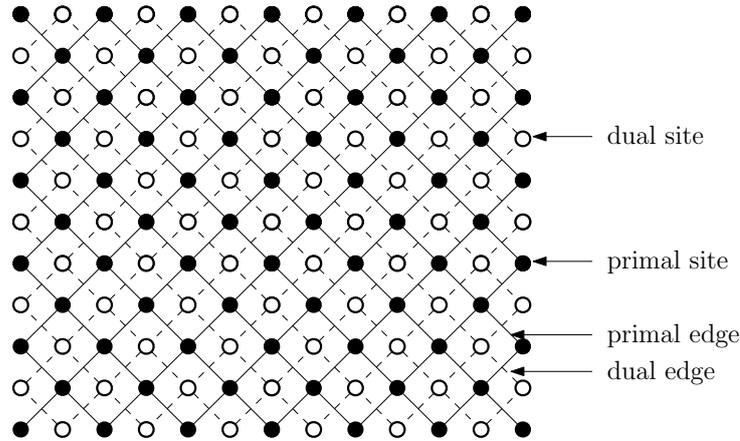


Figure 2.1: The black sites together with the plain edges constitute the primal lattice  $\mathbb{L}$ . The white sites, together with the dashed edges constitute the dual lattice  $\mathbb{L}^*$ .

A finite graph  $G = (V, E)$  is always a subgraph of  $\mathbb{L}$  and is called a **primal graph**. The **boundary** of  $G$ , denoted by  $\partial G$ , is the set of sites of  $G$  with fewer than four neighbors in  $G$ .

The **dual graph**  $G^*$  of a planar graph  $G$  is defined as follows: sites of  $G^*$  correspond to faces of  $G$  (for convenience, the infinite face will not correspond to a dual site), edges of  $G^*$  connect sites corresponding to two adjacent faces of  $G$ . The **dual lattice** of  $\mathbb{L}$  is denoted by  $\mathbb{L}^*$ .

The **medial lattice**  $\mathbb{L}^\diamond$  is the graph with the centers of edges of  $\mathbb{L}$  as vertex set, and edges connecting nearest vertices, see Fig. 2.2. The **medial graph**  $G^\diamond$  is the subgraph of  $\mathbb{L}^\diamond$  composed of all the vertices of  $\mathbb{L}^\diamond$  corresponding to edges of  $G$ . Note that  $\mathbb{L}^\diamond$  is a rotate and rescaled (by a factor  $1/\sqrt{2}$ ) version of  $\mathbb{L}$ . We will often use the connection between the faces of  $\mathbb{L}^\diamond$  and the sites of  $\mathbb{L}$  and  $\mathbb{L}^*$ . A face of the medial lattice is said to be *black* if it corresponds to a site of  $\mathbb{L}$ , and *white* otherwise. Faces are sometimes called *diamonds* and a color is associated to them in a unequivocal fashion. Edges of  $\mathbb{L}^\diamond$  are oriented counterclockwise around black faces.

## 1.2 Approximations of domains

We will be interested in finer and finer graphs approximating continuous domains. For  $\delta > 0$ , the square lattice  $\sqrt{2}\delta\mathbb{L}$  of mesh-size  $\sqrt{2}\delta$  will be denoted by  $\mathbb{L}_\delta$ . The definitions of dual and medial lattices extend to this context. Note that the medial lattice  $\mathbb{L}_\delta^\diamond$  has mesh-size  $\delta$ <sup>1</sup>.

For a simply connected domain  $\Omega$  in the plane, set  $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$ . The edges connecting sites of  $\Omega_\delta$  are those included in  $\Omega$ . The graph  $\Omega_\delta$  should be thought of as a discretization

<sup>1</sup>The convention is convenient since the medial lattice will be used more frequently than the primal one.

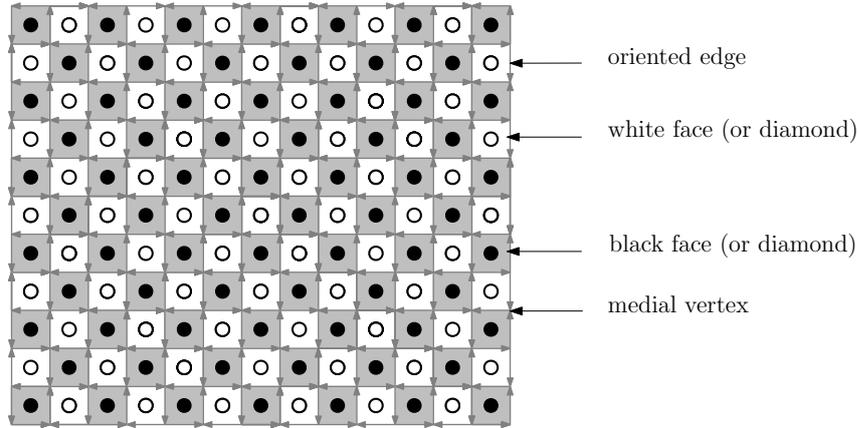


Figure 2.2: The medial lattice associated to  $\mathbb{L}$  and  $\mathbb{L}^*$ . Each face corresponds to either a black site or a white site.

of  $\Omega$ . We will always make the assumption that the graph is simply connected<sup>2</sup>. Under mild hypothesis, this assumption is always fulfilled when  $\delta$  is small enough.

More generally, when no continuous domain  $\Omega$  is specified,  $\Omega_\delta$  stands for a finite simply connected subgraph of  $\mathbb{L}_\delta$ .

We will be considering sequences of functions on  $\Omega_\delta$  for  $\delta$  going to 0. In order to make functions live in the same space, we implicitly perform the following operation: for a function  $f$  on  $\Omega_\delta$ , choose a diagonal for every square and extend the function to  $\Omega$  in a piecewise linear way on every triangle. Since no confusion will be possible, the extension is denoted by  $f$  as well.

## 2 Preharmonic functions

### 2.1 Definition and connection with random walks

Introduce the (non-normalized) discretization of the Laplacian operator  $\Delta := \frac{1}{4}(\partial_{xx}^2 + \partial_{yy}^2)$  in the case of the square lattice  $\mathbb{L}_\delta$ . For  $u \in \mathbb{L}_\delta$  and  $f : \mathbb{L}_\delta \rightarrow \mathbb{C}$ , define

$$\Delta_\delta f(u) = \frac{1}{4} \sum_{v \sim u} (f(v) - f(u)).$$

The definition extends to rescaled square lattices in a straightforward way (for instance to  $\mathbb{L}_\delta^\circ$ ).

**Definition 2.1.** *A function  $h : \Omega_\delta \rightarrow \mathbb{C}$  is preharmonic (resp. pre-superharmonic, pre-subharmonic) if  $\Delta_\delta h(x) = 0$  (resp.  $\leq 0, \geq 0$ ) for every  $x \in \Omega_\delta$ .*

One fundamental tool in the study of preharmonic functions is the classical relation between preharmonic functions and simple random walks:

<sup>2</sup>Meaning that it is connected and the complement in  $\mathbb{L}$  is connected.

Let  $(X_n)$  be a simple random walk killed at the first time it exits  $\Omega_\delta$ , then  $h$  is preharmonic on  $\Omega_\delta$  if and only if  $(h(X_n))$  is a martingale.

Using this fact, one can prove that harmonic functions are determined by their value on  $\partial\Omega_\delta$ , that they satisfy Harnack's principle, etc. We refer to [Law91] for a deeper study on preharmonic functions and their link to random walks. Also note that the set of preharmonic functions is a complex vector space. As in the continuum, it is easy to see that preharmonic functions satisfy the maximum and minimum principles.

## 2.2 The discrete harmonic measure

The discrete harmonic measure  $H_{\Omega_\delta}(\cdot, y)$  of  $y \in \partial\Omega_\delta$  is the unique harmonic function on  $\Omega_\delta \setminus \partial\Omega_\delta$  vanishing on the boundary  $\partial\Omega_\delta$ , except at  $y$ , where it equals 1. Equivalently,  $H_{\Omega_\delta}(x, y)$  is the probability that a simple random walk starting from  $x$  exits  $\Omega_\delta \setminus \partial\Omega_\delta$  through  $y$ .

**Proposition 2.2.** *For any harmonic function  $h : \Omega_\delta \rightarrow \mathbb{C}$ ,*

$$h = \sum_{y \in \partial\Omega_\delta} h(y) H_{\Omega_\delta}(\cdot, y).$$

**Proof** Note that  $\sum_{y \in \partial\Omega_\delta} h(y) H_{\Omega_\delta}(\cdot, y)$  is harmonic in  $\Omega_\delta$  with same boundary conditions as  $h$ . Since a harmonic function is determined by its boundary conditions, the result follows.  $\square$

We recall two results on discrete harmonic measures. The first one is asserting that the exiting distribution of a random-walk starting at the center of a cube is more or less uniform.

**Proposition 2.3.** *There exists  $C > 0$  such that  $H_{Q_\delta}(0, y) \leq C\delta$  for every  $\delta > 0$  and  $y \in \partial Q_\delta$ , where  $Q = [-1, 1]^2$ .*

We omit the easy proof of this statement. The second result is a discrete (weak) Beurling estimate.

**Proposition 2.4** (Beurling's estimate). *There exists  $\alpha > 0$  such that for any  $1 \gg r > \delta > 0$  and any curve  $\gamma$  inside  $\mathbb{D} := \{z : |z| < 1\}$  from  $C = \{z : |z| = 1\}$  to  $\{z : |z| = r\}$ , the probability for a random walk on  $\mathbb{D}_\delta$  starting at 0 to exit  $(\mathbb{D} \setminus \gamma)_\delta$  through  $C$  is smaller than  $r^\alpha$ .*

**Proof** For any annulus  $A_x := \{z : x \leq |z| \leq 2x\}$ , with  $r \leq x \leq 1$ , the random walk trajectory has a uniformly positive probability  $c > 0$  to close a loop around the origin while crossing this annulus. In this case, the trajectory necessarily intersects  $\gamma$ . Since the random walk trajectory must cross roughly  $\log_2 r$  annuli  $A_{2^{-n}}$ , and that at each step it has a probability at least  $c > 0$  to close a circuit, the result follows with  $\alpha = -1/\log_2 c$ .  $\square$

### 2.3 Derivative estimates and compactness criteria

For general functions, a control on the gradient provides regularity estimates on the function itself. It is a well-known fact that harmonic functions satisfy the reverse property: controlling the function allows us to control the gradient. The following lemma shows that the same is true for preharmonic functions.

**Proposition 2.5.** *There exists  $C > 0$  such that, for any preharmonic function  $h : \Omega_\delta \rightarrow \mathbb{C}$  and any two neighboring sites  $x, y \in \Omega_\delta$ ,*

$$|h(x) - h(y)| \leq C\delta \frac{\sup_{z \in \Omega_\delta} |h(z)|}{d(x, \Omega^c)}. \quad (2.1)$$

**Proof** Let  $x, y \in \Omega_\delta$ . The preharmonicity of  $h$  translates to the fact that  $h(X_n)$  is a martingale (where  $X_n$  is a simple random walk killed at the first time it exits  $\Omega_\delta$ ). Therefore, for  $x, y$  two neighboring sites of  $\Omega_\delta$ , we have

$$h(x) - h(y) = \mathbb{E}[h(X_\tau) - h(Y_{\tau'})] \quad (2.2)$$

where under  $\mathbb{E}$ ,  $X$  and  $Y$  are two simple random walks starting respectively at  $x$  and  $y$ , and  $\tau, \tau'$  are any stopping times. Let  $2r = d(x, \Omega^c) > 0$ , so that  $U = x + [-r, r]^2$  is included in  $\Omega_\delta$ . Fix  $\tau$  and  $\tau'$  to be the hitting times of  $\partial U_\delta$  and consider the following coupling of  $X$  and  $Y$  (one has complete freedom in the choice of the joint law in (2.2)):  $(X_n)$  is a simple random walk and  $Y_n$  is constructed as follows,

- if  $X_1 = y$ , then  $Y_n = X_{n+1}$  for  $n \geq 0$ ,
- if  $X_1 \neq y$ , then  $Y_n = \sigma(X_{n+1})$ , where  $\sigma$  is the orthogonal symmetry with respect to the perpendicular bisector  $\ell$  of  $[X_1, y]$ , whenever  $X_{n+1}$  does not reach  $\ell$ . As soon as it does, set  $Y_n = X_{n+1}$ .

It is easy to check that  $Y$  is also a simple random walk. Moreover, we have

$$|h(x) - h(y)| \leq \mathbb{E}[|h(X_\tau) - h(Y_{\tau'})| 1_{X_\tau \neq Y_{\tau'}}] \leq 2 \left( \sup_{z \in \partial U_\delta} |h(z)| \right) \mathbb{P}(X_\tau \neq Y_{\tau'})$$

Using the definition of the coupling, the probability on the right is known: it is equal to the probability that  $X$  does not touch  $\ell$  before exiting the ball and is smaller than  $\frac{C'}{r}\delta$  (with  $C'$  a universal constant), since  $U_\delta$  is of radius  $r/\delta$  for the graph distance. We deduce that

$$|h(x) - h(y)| \leq 2 \left( \sup_{z \in \partial U_\delta} |h(z)| \right) \frac{C'}{r} \delta \leq 2 \left( \sup_{z \in \Omega_\delta} |h(z)| \right) \frac{C'}{r} \delta$$

□

Recall that functions on  $\Omega_\delta$  are implicitly extended to  $\Omega$ .

**Proposition 2.6.** *A family  $(h_\delta)_{\delta>0}$  of preharmonic functions on the graphs  $\Omega_\delta$  is precompact for the uniform topology on compact subsets of  $\Omega$  if one of the following properties holds:*

(1)  $(h_\delta)_{\delta>0}$  is uniformly bounded on any compact subset of  $\Omega$ ,

or

(2) for any compact subset  $K$  of  $\Omega$ , there exists  $M = M(K) > 0$  such that for any  $\delta > 0$

$$\delta^2 \sum_{x \in K_\delta} |h_\delta(x)|^2 \leq M.$$

**Proof** Let us prove that the proposition holds under the first hypothesis and then that the second hypothesis implies the first one.

We are faced with a family of continuous maps  $h_\delta : \Omega \rightarrow \mathbb{C}$  and we aim to apply the Arzelà-Ascoli theorem. It is sufficient to prove that functions  $h_\delta$  are uniformly Lipschitz on any compact subset since they are uniformly bounded on any compact subset of  $\Omega$ . Let  $K$  be a compact subset of  $\Omega$ . Proposition 2.5 shows that  $|h_\delta(x) - h_\delta(y)| \leq C_K \delta$  for any two neighbors  $x, y \in K_\delta$ , where

$$C_K = C \frac{\sup_{\delta>0} \sup_{x \in \Omega: d(x, K) \leq r/2} |h_\delta(x)|}{d(K, \Omega^c)},$$

implying that  $|h_\delta(x) - h_\delta(y)| \leq 2C_K |x - y|$  for any  $x, y \in K_\delta$  (not necessarily neighbors). The Arzelà-Ascoli theorem concludes the proof.

Now assume that the second hypothesis holds, and let us prove that  $(h_\delta)_{\delta>0}$  is bounded on any compact subset of  $\Omega$ . Take  $K \subset \Omega$  compact, let  $2r = d(K, \Omega^c) > 0$  and consider  $x \in K_\delta$ . Using the second hypothesis, there exists  $k := k(x)$  such that  $\frac{r}{2\delta} \leq k \leq \frac{r}{\delta}$  and

$$\delta \sum_{y \in \partial U_{k\delta}} |h_\delta(y)|^2 \leq 2M/r, \quad (2.3)$$

where  $U_{k\delta} = x + [-\delta k, \delta k]^2$  is the box of size  $k$  (for the graph distance) around  $x$  and  $M = M(y + [-r, r]^2)$ . Proposition 2.2 implies

$$h_\delta(x) = \sum_{y \in \partial U_{k\delta}} h_\delta(y) H_{U_{k\delta}}(x, y) \quad (2.4)$$

for every  $x \in U_{\delta k}$ . Using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} h_\delta(x)^2 &= \left( \sum_{y \in \partial U_{k\delta}} h_\delta(y) H_{U_{k\delta}}(x, y) \right)^2 \\ &\leq \left( \delta \cdot \sum_{y \in \partial U_{k\delta}} |h_\delta(y)|^2 \right) \left( \frac{1}{\delta} \cdot \sum_{y \in \partial U_{k\delta}} H_{U_{k\delta}}(x, y)^2 \right) \leq 2M/r \cdot C \end{aligned}$$

where  $C$  is a uniform constant. The last inequality used Proposition 2.3 to affirm that  $H_{U_{k\delta}}(x, y) \leq C\delta$  for some  $C = C(r) > 0$ .  $\square$

## 2.4 Discrete Dirichlet problem and convergence in the scaling limit

Preharmonic functions on square lattices of smaller and smaller mesh size were studied in a number of papers in the early twentieth century (see e.g. [PW23, Bou26, Lus26]), culminating in the seminal work of Courant, Friedrichs and Lewy. It was shown in [CFL28] that solutions to the Dirichlet problem for a discretization of an elliptic operator converge to the solution of the analogous continuous problem as the mesh of the lattice tends to zero. A first interesting fact is that the limit of preharmonic functions is indeed harmonic.

**Proposition 2.7.** *Any limit of a sequence of preharmonic functions on  $\Omega_\delta$  converging uniformly on any compact subset of  $\Omega$  is harmonic in  $\Omega$ .*

**Proof** Let  $(h_\delta)$  be a sequence of preharmonic functions on  $\Omega_\delta$  converging to  $h$ . Via Propositions 2.5 and 2.6,  $(\frac{1}{\delta}[h_\delta(\cdot + \delta) - h_\delta])_{\delta > 0}$  is precompact. Since  $\partial_x h$  is the only possible sub-sequential limit of the sequence,  $(\frac{1}{\sqrt{2}\delta}[h_\delta(\cdot + \delta) - h_\delta])_{\delta > 0}$  converges (indeed its discrete primitive converges to  $h$ ). Similarly, one can prove convergence of discrete derivatives of any order. In particular,  $0 = \frac{1}{2\delta^2}\Delta_\delta h_\delta$  converges to  $\frac{1}{4}[\partial_{xx}h + \partial_{yy}h]$ . Therefore,  $h$  is harmonic.  $\square$

In particular, preharmonic functions with a given boundary value problem converge in the scaling limit to a harmonic function with the same boundary value problem in a rather strong sense, including convergence of all partial derivatives. The finest result of convergence of discrete Dirichlet problems to the continuous ones will not be necessary in our setting and we state the minimal required result:

**Theorem 2.8.** *Let  $\Omega$  be a simply connected domain with two marked points  $a$  and  $b$  on the boundary, and  $f$  a bounded continuous function on the boundary of  $\Omega$ . Let  $f_\delta : \partial\Omega_\delta \rightarrow \mathbb{C}$  be a sequence of uniformly bounded functions converging uniformly away from  $a$  and  $b$  to  $f$ . Let  $h_\delta$  be the unique preharmonic map on  $\Omega_\delta$  such that  $(h_\delta)|_{\partial\Omega_\delta} = f_\delta$ . Then*

$$h_\delta \longrightarrow h \quad \text{when } \delta \rightarrow 0$$

*uniformly on compact subsets of  $\Omega$ , where  $h$  is the unique harmonic function on  $\Omega$ , continuous on  $\bar{\Omega}$ , satisfying  $h|_{\partial\Omega} = f$ .*

**Proof** Since  $(f_\delta)_{\delta > 0}$  is uniformly bounded by some constant  $M$ , the minimum and maximum principles imply that  $(h_\delta)_{\delta > 0}$  is bounded by  $M$ . Therefore, the family  $(h_\delta)$  is precompact (Proposition 2.6). Let  $\tilde{h}$  be a sub-sequential limit. Necessarily,  $\tilde{h}$  is harmonic inside the domain (Proposition 2.7) and bounded. To prove that  $\tilde{h} = h$ , it suffices to show that  $\tilde{h}$  can be continuously extended to the boundary by  $f$ .

Let  $x \in \partial\Omega \setminus \{a, b\}$  and  $\varepsilon > 0$ . There exists  $R > 0$  such that for  $\delta$  small enough,

$$|f_\delta(x') - f_\delta(x)| < \varepsilon \quad \text{for every } x' \in \partial\Omega \cap Q(x, R),$$

where  $Q(x, R) = x + [-R, R]^2$ . For  $r < R$  and  $y \in Q(x, r)$ , we have

$$|h_\delta(y) - f_\delta(x)| = \mathbb{E}_y[f_\delta(X_\tau) - f_\delta(x)]$$

for  $X$  a random walk starting at  $y$ , and  $\tau$  its hitting time of the boundary. Decomposing between walks exiting the domain inside  $Q(x, R)$  and others, we find

$$|h_\delta(y) - f_\delta(x)| \leq \varepsilon + 2M\mathbb{P}_y[X_\tau \notin Q(x, R)]$$

Proposition 2.4 guarantees that  $\mathbb{P}_y[X_\tau \notin Q(x, R)] \leq (r/R)^\alpha$  for some independent constant  $\alpha > 0$ . Taking  $r = R(\varepsilon/2M)^{1/\alpha}$  and letting  $\delta$  go to 0, we obtain  $|h(y) - f(x)| \leq 2\varepsilon$  for every  $y \in Q(x, r)$ .  $\square$

## 2.5 Discrete Green functions

This paragraph concludes the section by mentioning the important example of discrete Green functions. For  $y \in \Omega_\delta \setminus \partial\Omega_\delta$ , let  $G_{\Omega_\delta}(\cdot, y)$  be the *discrete Green function* in the domain  $\Omega_\delta$  with singularity at  $y$ , *i.e.* the unique function on  $\Omega_\delta$  such that

- its Laplacian on  $\Omega_\delta \setminus \partial\Omega_\delta$  equals 0 except at  $y$ , where it equals 1,
- $G_{\Omega_\delta}(\cdot, y)$  vanishes on the boundary  $\partial\Omega_\delta$ .

The quantity  $-G_{\Omega_\delta}(x, y)$  is the number of visits at  $x$  of a random walk started at  $y$  and stopped at the first time it reaches the boundary. Equivalently, it is also the number of visits at  $y$  of a random walk started at  $x$  stopped at the first time it reaches the boundary. Green functions are very convenient, in particular because of the Riesz representation formula for (not-necessarily harmonic) functions:

**Proposition 2.9** (Riesz representation formula). *Let  $f : \Omega_\delta \rightarrow \mathbb{C}$  be a function vanishing on  $\partial\Omega_\delta$ . We have*

$$f = \sum_{y \in \Omega_\delta} \Delta_\delta f(y) G_{\Omega_\delta}(\cdot, y).$$

**Proof** Note that  $f - \sum_{y \in \Omega_\delta} \Delta_\delta f(y) G_{\Omega_\delta}(\cdot, y)$  is harmonic and vanishes on the boundary. Hence, it equals 0 everywhere.  $\square$

Finally, a regularity estimate on discrete Green functions will be needed. This proposition is slightly technical. In the following,  $aQ_\delta = [-a, a]^2 \cap \mathbb{L}_\delta$  and  $\nabla_x f(x) = (f(x + \delta) - f(x), f(x + i\delta) - f(x))$ .

**Proposition 2.10.** *There exists  $C > 0$  such that for any  $\delta > 0$  and  $y \in 9Q_\delta$ ,*

$$\sum_{x \in Q_\delta} |\nabla_x G_{9Q_\delta}(x, y)| \leq C\delta \sum_{x \in Q_\delta} G_{9Q_\delta}(x, y).$$

**Proof** In the proof,  $C_1, \dots, C_6$  denote universal constants. First assume  $y \in 9Q_\delta \setminus 3Q_\delta$ . Using random walks, one can easily show that there exists  $C_1 > 0$  such that

$$\frac{1}{C_1} G_{9Q_\delta}(x, y) \leq G_{9Q_\delta}(x', y) \leq C_1 G_{9Q_\delta}(x, y)$$

for every  $x, x' \in 2Q_\delta$  (this is a special application of Harnack's principle). Using Proposition 2.5, we deduce

$$\sum_{x \in Q_\delta} |\nabla_x G_{9Q_\delta}(x, y)| \leq \sum_{x \in Q_\delta} C_2 \delta \max_{x \in 2Q_\delta} G_{9Q_\delta}(x, y) \leq C_1 C_2 \delta \sum_{x \in Q_\delta} G_{9Q_\delta}(x, y)$$

which is the claim for  $y \in 9Q_\delta \setminus 3Q_\delta$ .

Assume now that  $y \in 3Q_\delta$ . Using the fact that  $G_{9Q_\delta}(x, y)$  is the number of visits of  $x$  for a random walk starting at  $y$  (and stopped on the boundary), we find

$$\sum_{x \in Q_\delta} G_{9Q_\delta}(x, y) \geq C_3 / \delta^2.$$

Therefore, it suffices to prove  $\sum_{x \in Q_\delta} |\nabla_x G_{9Q_\delta}(x, y)| \leq C_4 / \delta$ . Let  $G_{\mathbb{L}_\delta}$  be the Green function in the whole plane, *i.e.* the function with Laplacian equal to  $\delta_{x,y}$ , normalized so that  $G_{\mathbb{L}_\delta}(y, y) = 0$ , and with sublinear growth. This function has been widely studied, it was proved in [MW40] that

$$G_{\mathbb{L}_\delta}(x, y) = \frac{1}{\pi} \ln \left( \frac{|x - y|}{\delta} \right) + C_5 + o \left( \frac{\delta}{|x - y|} \right).$$

Now,  $G_{\mathbb{L}_\delta}(\cdot, y) - G_{9Q_\delta}(\cdot, y) - \frac{1}{\pi} \ln \left( \frac{1}{\delta} \right)$  is harmonic and has bounded boundary conditions on  $\partial 9Q_\delta$ . Therefore, Proposition 2.5 implies

$$\sum_{x \in Q_\delta} |\nabla_x (G_{\mathbb{L}_\delta}(x, y) - G_{9Q_\delta}(x, y))| \leq C_6 \delta \cdot 1 / \delta^2 = C_6 / \delta.$$

Moreover, the asymptotic of  $G_{\mathbb{L}_\delta}(\cdot, y)$  leads to

$$\sum_{x \in Q_\delta} |\nabla_x G_{\mathbb{L}_\delta}(x, y)| \leq C_7 / \delta.$$

Summing the two inequalities, the result follows readily.  $\square$

## 3 Preholomorphic functions

### 3.1 Historical introduction

Preholomorphic functions appeared implicitly in Kirchhoff's work in 1847 [Kir47], in which a graph is modeled as an electric network. Assume every edge of the graph is a unit resistor and for  $u \sim v$ , let  $F(uv)$  be the current from  $u$  to  $v$ . The first and the second Kirchhoff's laws of electricity can be restated:

- the sum of currents flowing from a vertex is zero:

$$\sum_{v \sim u} F(uv) = 0, \quad (2.5)$$

- the sum of the currents around any oriented closed contour  $\gamma$  is zero:

$$\sum_{[uv] \in \gamma} F(uv) = 0. \quad (2.6)$$

Different resistances amount to putting weights into (2.5) and (2.6). The second law is equivalent to saying that  $F$  is given by the gradient of a potential function  $H$ , and the first equivalent to  $H$  being preharmonic.

Besides the original work of Kirchhoff, the first notable application of preholomorphic functions is perhaps the famous article [BSST40] of Brooks, Smith, Stone and Tutte, where preholomorphic functions were used to construct tilings of rectangles by squares.

Preholomorphic functions distinctively appeared for the first time in the papers [Isa41, Isa52] of Isaacs, where he proposed two definitions (and called such functions 'monodiffic'). Both definitions ask for a discrete version of the Cauchy-Riemann equations  $\partial_{i\alpha} F = i\partial_{\alpha} F$  or equivalently that the  $\bar{z}$ -derivative is 0. In the first definition, the equation that the function must satisfy is

$$i[f(E) - f(S)] = f(W) - f(S)$$

while in the second, it is

$$i[f(E) - f(W)] = f(N) - f(S),$$

where  $N$ ,  $E$ ,  $S$  and  $W$  are the four corners of a face. A few papers of his and others mathematicians followed, studying the first definition, which is asymmetric on the square lattice. The second (symmetric) definition was reintroduced by Ferrand, who also discussed the passage to the scaling limit and gave new proofs of Riemann uniformization and the Courant-Friedrichs-Lewy theorems [Fer44, LF55]. This was followed by extensive studies of Duffin and others, starting with [Duf56].

### 3.2 Isaacs's definition of preholomorphic functions

We will be working with Isaacs's second definition (although the theories based on both definitions are almost the same). The definition involves the following discretization of the  $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$  operator. For a complex valued function  $f$  on  $\mathbb{L}_\delta$  (or on a finite subgraph of it), and  $x \in \mathbb{L}_\delta^*$ , define

$$\bar{\partial}_\delta f(x) = \frac{1}{2}[f(E) - f(W)] + \frac{i}{2}[f(N) - f(S)]$$

where  $N$ ,  $E$ ,  $S$  and  $W$  denote the four vertices adjacent to the dual vertex  $x$  indexed in the obvious way.

**Remark 2.11.** *When defining derivation, one uses duality between a graph and its dual. Quantities related to the derivative of a function on  $G$  are defined on the dual graph  $G^*$ . Similarly, notions related to the second derivative are defined on the graph  $G$  again, whereas a primitive would be defined on  $G^*$ .*

**Definition 2.12.** *A function  $f : \Omega_\delta \rightarrow \mathbb{C}$  is called preholomorphic if  $\bar{\partial}_\delta f(x) = 0$  for every  $x \in \Omega_\delta^*$ . For  $x \in \Omega_\delta^*$ ,  $\bar{\partial}_\delta f(x) = 0$  is called the discrete Cauchy-Riemann equation at  $x$ .*

The theory of preholomorphic functions starts much like the usual complex analysis. Preholomorphic functions are preharmonic. Moreover, sums of preholomorphic functions are also preholomorphic, discrete contour integrals vanish, primitive (in a simply-connected domain) and derivative are well-defined and are preholomorphic functions on the dual square lattice, etc... In particular, the (discrete) gradient of a preharmonic function is preholomorphic (this property has been proposed as a relevant generalization in higher dimensions).

**Proposition 2.13.** *Preholomorphic functions are preharmonic for a slightly modified Laplacian (the average over edges at distance  $\sqrt{2}\delta$  minus the value at the point).*

Unfortunately, the product of two preholomorphic functions is no longer preholomorphic in general: *e.g.*, while restrictions of 1,  $z$ , and  $z^2$  to the square lattice are preholomorphic, the higher powers are only approximately so. The restriction of a continuous holomorphic function to  $\mathbb{L}_\delta$  satisfies discrete Cauchy-Riemann equations up to  $O(\delta^3)$ . This makes the theory of preholomorphic functions significantly harder than the usual complex analysis, since one cannot transpose proofs from continuum to discrete in a straightforward way.

### 3.3 Discrete contours and weak discrete-holomorphicity

In the continuum, many definitions of holomorphicity are equivalent. Most of these definitions have natural counterparts in the discrete. The previous section was dealing with discretization of Cauchy-Riemann equations. This definition is the most suitable to study integrable systems. Nevertheless, we can be interested in weaker versions of discrete holomorphicity.

Morera's theorem asserts that continuous functions with integrals around any contour vanishing are exactly holomorphic functions. Therefore, it can be interesting to have an equivalent of this definition in the discrete.

**Definition 2.14** (Discrete contours). *A discrete contour in a graph  $G$  is a one-to-one circuit of oriented edges in  $G^*$ .*

Recall that every edge  $e$  of  $G$  is in correspondence with an edge of  $G^*$  denoted  $e^*$  (and *vice versa*). For a function  $f : E[G] \rightarrow \mathbb{C}$  and a discrete contour  $\gamma$ , the *discrete integral* of  $f$  on  $\gamma$  is defined by

$$I_\gamma(f) := \sum_{e^*=[uv] \in \gamma} f(e) \cdot (v - u).$$

As mentioned in Remark 2.11, discrete integrals are thus defined on the dual graph.

**Definition 2.15** (Weak discrete holomorphicity). *A function  $h : G \rightarrow \mathbb{C}$  is weakly discrete holomorphic if its integral on any discrete contour vanishes.*

This definition corresponds to assuming only the second Kirchhoff's law. Note that the definition is much weaker than the previous one. For instance, a weakly discrete holomorphic function is not determined by its boundary conditions.

A sequence of weakly discrete holomorphic functions on approximations of a given domain converging uniformly on any compact subset tends to a continuous function with vanishing integrals along contours. Morera's theorem then implies that the limit is holomorphic. Hence, this notion is sufficient to imply holomorphicity in the scaling limit, *when such scaling limit is known to exist.*

Let us mention that Smirnov used this notion of weak discrete holomorphicity in order to prove Cardy's formula [Smi01]. Many statistical models will be shown to possess weakly discrete holomorphic observables. The difficulty is that convergence (or simply precompactness) of these observables is out of reach for now.

## 4 $s$ -holomorphic functions

As explained in the previous sections, there are difficulties when dealing with the square of a preholomorphic function. In order to partially overcome this difficulty, we introduce  $s$ -holomorphic functions (for *spin*-holomorphic), a notion that will be central in the study of the spin and FK fermionic observables. This notion was developed in [Smi10a] and we refer to it for additional information.

### 4.1 Definition of $s$ -holomorphic functions

In this section,  $s$ -holomorphic functions are defined on the medial lattice  $\mathbb{L}_\delta^\circ$  only. For any edge of the medial lattice  $e$ <sup>3</sup>, the complex line passing through the origin and  $\sqrt{e}$  (the choice of the square root is not important) is denoted by  $\ell(e)$ . The different lines associated with medial edges on  $\mathbb{L}_\delta^\circ$  are  $\mathbb{R}$ ,  $e^{i\pi/4}\mathbb{R}$ ,  $i\mathbb{R}$  and  $e^{-i\pi/4}\mathbb{R}$ , see Fig. 2.3.

**Definition 2.16.** *A function  $f : \Omega_\delta^\circ \rightarrow \mathbb{C}$  is  $s$ -holomorphic if for any edge  $e$  of  $\Omega_\delta^\circ$ , we have*

$$P_{\ell(e)}[f(x)] = P_{\ell(e)}[f(y)]$$

where  $x, y$  are the endpoints of  $e$  and  $P_\ell$  is the orthogonal projection on  $\ell$ .

The definition of  $s$ -holomorphicity is not rotationally invariant. Nevertheless,  $f$  is  $s$ -holomorphic if and only if  $e^{i\pi/4}f(i\cdot)$  (resp.  $if(-\cdot)$ ) is  $s$ -holomorphic.

**Proposition 2.17.** *Any  $s$ -holomorphic function  $f : \Omega_\delta^\circ \rightarrow \mathbb{C}$  is preholomorphic on  $\Omega_\delta^\circ$ .*

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<sup>3</sup>The edge  $e$  being oriented, it can be thought of as a complex number.

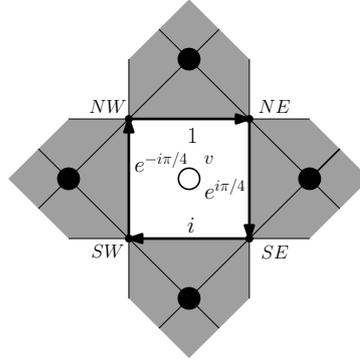


Figure 2.3: Lines  $\ell(e)$  for medial edges around a white face.

**Proof** Let  $f : \Omega_\delta^\circ \rightarrow \mathbb{C}$  be a  $s$ -holomorphic function. Let  $v$  be a vertex of  $\mathbb{L}_\delta \cup \mathbb{L}_\delta^*$  (this is the vertex set of the dual of the medial lattice). Assume that  $v \in \Omega_\delta^*$ , the other case is similar. We aim to show that  $\bar{\partial}_\delta f(v) = 0$ . Let  $NW$ ,  $NE$ ,  $SE$  and  $SW$  be the four vertices around  $v$  as illustrated in Fig. 2.3. Next, let us write relations provided by the  $s$ -holomorphicity, for instance

$$P_{\mathbb{R}}[f(NW)] = P_{\mathbb{R}}[f(NE)].$$

Expressed in terms of  $f$  and its complex conjugate  $\bar{f}$  only, we obtain

$$f(NW) + \overline{f(NW)} = f(NE) + \overline{f(NE)}.$$

Doing the same with the other edges, we find

$$\begin{aligned} f(NE) + i\overline{f(NE)} &= f(SE) + i\overline{f(SE)} \\ f(SE) - \overline{f(SE)} &= f(SW) - \overline{f(SW)} \\ f(SW) - i\overline{f(SW)} &= f(NW) - i\overline{f(NW)} \end{aligned}$$

Multiplying the second identity by  $-i$ , the third by  $-1$ , the fourth by  $i$ , and then summing the four identities, we obtain

$$0 = (1 - i)[f(NW) - f(SE) + if(SW) - if(NE)] = 2(1 - i)\bar{\partial}_\delta f(v)$$

which is exactly the discrete Cauchy-Riemann equation in the medial lattice.  $\square$

## 4.2 Discrete primitive of $F^2$

One might wonder why  $s$ -holomorphicity is an interesting concept, since it is more restrictive than preholomorphicity. The answer comes from the fact that a relevant discretization of  $\frac{1}{2}\text{Im}(f^z f^2)$  can be defined for  $s$ -holomorphic functions  $f$ .

**Theorem 2.18.** *Let  $f : \Omega_\delta^\circ \rightarrow \mathbb{C}$  be an  $s$ -holomorphic function on the discrete simply connected domain  $\Omega_\delta^\circ$ , and  $b_0 \in \Omega_\delta$ , then there exists a unique function  $H : \Omega_\delta \cup \Omega_\delta^* \rightarrow \mathbb{C}$  such that*

$$\begin{aligned} H(b_0) &= 1 \quad \text{and} \\ H(b) - H(w) &= \delta |P_{\ell(e)}[f(x)]|^2 \left( = \delta |P_{\ell(e)}[f(y)]|^2 \right) \end{aligned}$$

for every edge  $e = [xy]$  of  $\Omega_\delta^\circ$  bordered by a black face  $b \in \Omega_\delta$  and a white face  $w \in \Omega_\delta^*$ .

An elementary computation shows that for two neighboring sites  $b_1, b_2 \in \Omega_\delta$ , with  $v$  being the medial vertex at the center of  $[b_1 b_2]$ ,

$$H(b_1) - H(b_2) = \frac{1}{2} \operatorname{Im} [f(v)^2 \cdot (b_1 - b_2)],$$

the same relation holding for sites of  $\Omega_\delta^*$ . This legitimizes the fact that  $H$  is an analogue of  $\frac{1}{2} \operatorname{Im} (f^z f^2)$ .

**Proof** The uniqueness of  $H$  is straightforward since  $\Omega_\delta^\circ$  is simply connected. To obtain the existence, construct the value at some point by summing increments along an arbitrary path from  $b_0$  to this point. The only thing to check is that the value obtained does not depend on the path chosen to define it. Equivalently, we must check the second Kirchhoff's law. Since the domain is simply connected, it is sufficient to check it for elementary 'square' contours around each medial vertex  $v$  (these are the simplest closed contours). Therefore, we need to prove that

$$|P_{\ell(n)}[f(v)]|^2 - |P_{\ell(e)}[f(v)]|^2 + |P_{\ell(s)}[f(v)]|^2 - |P_{\ell(w)}[f(v)]|^2 = 0, \quad (2.7)$$

where  $n, e, s$  and  $w$  are the four medial edges with end-point  $v$ , indexed in the obvious way. Note that  $\ell(n)$  and  $\ell(s)$  (resp.  $\ell(e)$  and  $\ell(w)$ ) are orthogonal. Hence, (2.7) follows from

$$|P_{\ell(n)}[f(v)]|^2 + |P_{\ell(s)}[f(v)]|^2 = |f(v)|^2 = |P_{\ell(e)}[f(v)]|^2 + |P_{\ell(w)}[f(v)]|^2. \quad (2.8)$$

□

Even if the primitive of  $f$  is preholomorphic and thus preharmonic, this is not the case for  $H$  in general<sup>4</sup>. Nonetheless,  $H$  satisfies subharmonic and superharmonic properties. Denote by  $H^\bullet$  and  $H^\circ$  the restrictions of  $H : \Omega_\delta \cup \Omega_\delta^* \rightarrow \mathbb{C}$  to  $\Omega_\delta$  (black faces) and  $\Omega_\delta^*$  (white faces).

**Proposition 2.19.** *If  $f : \Omega_\delta^\circ \rightarrow \mathbb{C}$  is  $s$ -holomorphic, then  $H^\bullet$  and  $H^\circ$  are respectively subharmonic and superharmonic.*

<sup>4</sup> $H$  is roughly (the imaginary part of) the primitive of the square of  $f$ .

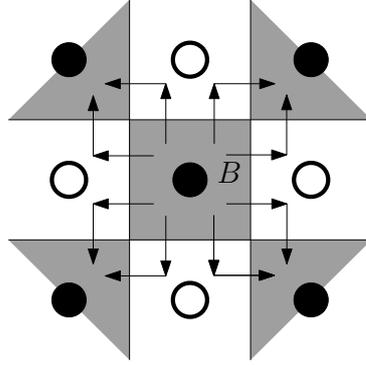


Figure 2.4: Arrows corresponding to contributions to  $2\Delta H^\bullet$ . Note that arrows from black to white contribute negatively, those from white to black positively.

**Proof** Let  $B$  be a vertex of  $\Omega_\delta \setminus \partial\Omega_\delta$ . We aim to show that the sum of increments of  $H^\bullet$  between  $B$  and its four neighbors is positive. In other words, we need to prove that the sum of increments along the sixteen arrows drawn in Fig. 2.4 is positive. Let  $a, b, c$  and  $d$  be the four values of  $\sqrt{\delta}P_{\ell(e)}[f(y)]$  for every vertex  $y \in \Omega_\delta^\circ$  around  $B$  and any edge  $e = [yz]$  bordering  $B$  (there are only four different values thanks to the definition of  $s$ -holomorphicity). An easy computation shows that the eight 'interior' increments are thus  $-a^2, -b^2, -c^2, -d^2$  (each appearing twice). Using the  $s$ -holomorphicity of  $f$  on vertices of  $\Omega_\delta^\circ$  around  $B$ , we can compute the eight 'exterior' increments in terms of  $a, b, c$  and  $d$ : we obtain  $(a\sqrt{2}-b)^2, (b\sqrt{2}-a)^2, (b\sqrt{2}-c)^2, (c\sqrt{2}-b)^2, (c\sqrt{2}-d)^2, (d\sqrt{2}-c)^2, (d\sqrt{2}+a)^2, (a\sqrt{2}+d)^2$ . Hence, the sum  $S$  of increments equals

$$S = 4(a^2 + b^2 + c^2 + d^2) - 4\sqrt{2}(ab + bc + cd - da) \quad (2.9)$$

$$= 4|e^{-i\pi/4}a - b + e^{i3\pi/4}c - id|^2 \geq 0. \quad (2.10)$$

The proof for  $H^\circ$  follows along the same lines.  $\square$

**Remark 2.20.** *A subharmonic function in a domain is smaller than the harmonic function with the same boundary conditions. Therefore,  $H^\bullet$  is smaller than the harmonic function solving the same boundary value problem while  $H^\circ$  is bigger than the harmonic function solving the same boundary value problem. Moreover,  $H^\bullet(b)$  is larger than  $H^\circ(w)$  for two neighboring faces. Hence, if  $H^\bullet$  and  $H^\circ$  are close to each other on the boundary, then they are 'sandwiched between two harmonic functions with roughly the same boundary conditions'. In this case, they are almost harmonic. This fact will be central in the proof of conformal invariance.*

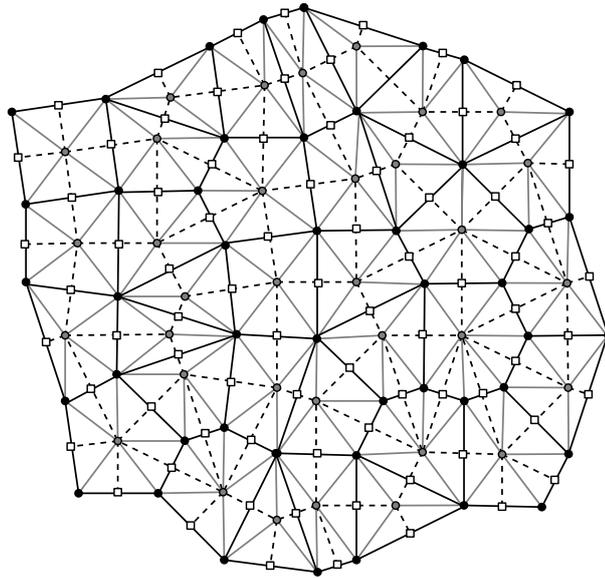


Figure 2.5: The black graph is the isoradial graph. Grey vertices are the vertices on the dual graph. There exists a radius  $r > 0$  such that all faces can be put into an incircle of radius  $r$ . Dual vertices have been drawn in such a way that they are the centers of these circles.

## 5 Isoradial graphs and circle packings

Duffin [Duf68] extended the definition of preholomorphic functions to isoradial graphs. *Isoradial graphs* are planar graphs that can be embedded in such a way that there exists  $r > 0$  so that each face has a circumcircle of same radius  $r > 0$ , see Fig. 2.5. When the embedding satisfies this property, it is said to be an isoradial embedding. We would like to point out that isoradial graphs form a rather large family of graphs. While not every topological quadrangulation (graph all of whose faces are quadrangles) admits a isoradial embedding, Kenyon and Schlenker [KS05] gave a simple necessary and sufficient topological condition for its existence. It seems that the first appearance of a related family of graphs in the probabilistic context was in the work of Baxter [Bax89], where the eight vertex model and the Ising model were considered on  $Z$ -invariant graphs, arising from planar line arrangements. These graphs are topologically the same as the isoradial ones, and though they are embedded differently into the plane, by [KS05] they always admit isoradial embeddings. In [Bax89], Baxter was not considering scaling limits, and so the actual choice of embedding was immaterial for his results. However, weights in his models would suggest an isoradial embedding, and the Ising model was so considered by Mercat [Mer01], Boutilier and de Tilière [BdT10, BdT11], Chelkak and Smirnov [CS08] (see Chapter 17 for more details). Additionally, the dimer and the uniform spanning tree models on such graphs also have nice properties, see e.g. [Ken02]. Today, isoradial graphs seem to be the largest family of graphs for which certain lattice models, including

the Ising model, have nice integrability properties (for instance, the star-triangle relation works nicely). A second reason to study isoradial graphs is that it is perhaps the largest family of graphs for which the Cauchy-Riemann operator admits a nice discretization. In particular, restrictions of holomorphic functions to such graphs are preholomorphic to higher orders. The fact that isoradial graphs are natural graphs both for discrete analysis and statistical physics sheds yet more light on the connection between the two domains.

In [Thu86], Thurston proposed circle packings as another discretization of complex analysis. Some beautiful applications were found, including yet another proof of the Riemann uniformization theorem by Rodin and Sullivan [RS87]. More interestingly, circle packings were used by He and Schramm [HS93] in the best result so far on the Koebe uniformization conjecture, stating that any domain can be conformally uniformized to a domain bounded by circles and points. In particular, they established the conjecture for domains with countably many boundary components. More about circle packings can be learned from Stephenson's book [Ste05]. Note that unlike the discretizations discussed above, the circle packings lead to non-linear versions of the Cauchy-Riemann equations, see *e.g.* the discussion in [BMS05].



# Part I

## The random-cluster models



# Chapter 3

## Two-dimensional random-cluster models

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**Abstract:** The family of random-cluster models is presented mathematically. The chapter gathers several properties (some non-standard) on this model and we refer to the extensive literature on the subject for additional information. The presentation is deliberately not general and is focused on crucial properties for this text.

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### 1 The family of random-cluster models

#### 1.1 Definition of the model

The random-cluster model can be defined on any graph. However, we restrict ourselves to the square lattice  $\mathbb{L} = (\mathbb{V}, \mathbb{E})$  of mesh size 1 (recall that it is a version rotated by an angle  $\pi/4$  of  $\mathbb{Z}^2$ ).

A *configuration*  $\omega$  on  $G$  is a subgraph of  $G$ , composed of the same sites and a subset of its edges. The edges belonging to  $\omega$  are called *open*, the others *closed*. Two sites  $a$  and  $b$  are said to be *connected* if there is an *open path*, *i.e.* a path composed of open edges only, connecting them (this event will be denoted by  $a \leftrightarrow b$ ). Two sets  $A$  and  $B$  are *connected* if there exists an open path connecting them (denoted  $A \leftrightarrow B$ ). The maximal connected components will be called *clusters*.

*Boundary conditions*  $\xi$  are given by a partition of  $\partial G$ . The graph obtained from the configuration  $\omega$  by identifying (or *wiring*) the edges in  $\xi$  that belong to the same component of  $\xi$  is denoted by  $\omega \cup \xi$ . Boundary conditions should be understood as encoding how sites are connected outside of  $G$ . Let  $o(\omega)$  (resp.  $c(\omega)$ ) denote the number of open (resp. closed) edges of  $\omega$  and  $k(\omega, \xi)$  the number of connected components of  $\omega \cup \xi$ .

**Definition 3.1.** The probability measure  $\phi_{G,p,q}^\xi$  of the random-cluster model on  $G$  with parameters  $p$  and  $q$  and boundary conditions  $\xi$  is defined by

$$\phi_{G,p,q}^\xi(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z_{G,p,q}^\xi} \quad (3.1)$$

for every configuration  $\omega$  on  $G$ , where  $Z_{G,p,q}^\xi$  is a normalizing constant referred to as the partition function.

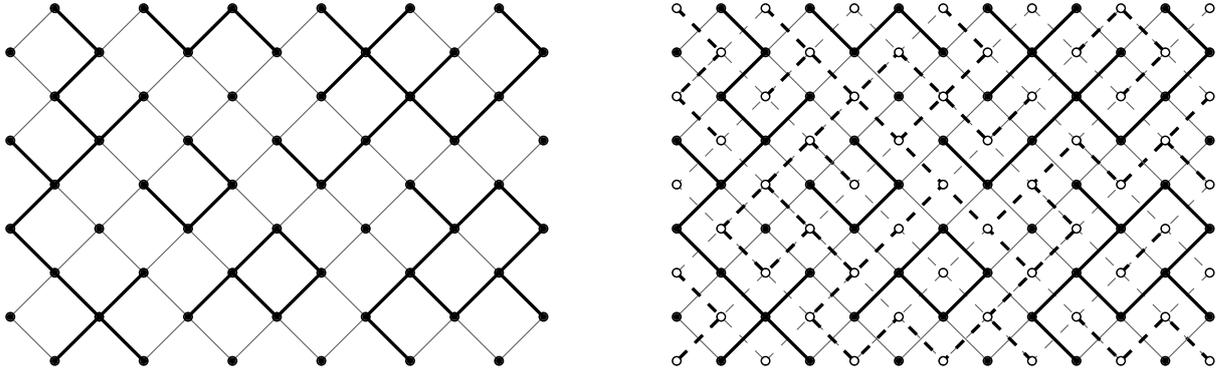


Figure 3.1: **Left:** Example of a configuration on the rotated lattice. **Right:** A configuration together with its dual configuration.

## 1.2 Special boundary conditions

Four boundary conditions play a special role in the study of the random-cluster model:

- The *wired* boundary conditions, denoted by  $\phi_{G,p,q}^1$ , is specified by the fact that all the vertices on the boundary are pairwise wired (only one set in the partition).
- The *free* boundary conditions, denoted by  $\phi_{G,p,q}^0$ , is specified by no wiring between sites.
- The *periodic* boundary conditions: for  $n \geq 1$ , the torus of size  $n$  can be seen as the box  $[0, n]^2$  with the boundary conditions obtained by imposing that  $(i, 0)$  is wired to  $(i, n)$  for every  $i \in [0, n]$  and that  $(0, j)$  is connected to  $(n, j)$  for every  $j \in [0, n]$ . The random-cluster measure on the torus of size  $n$  is denoted by  $\phi_{p,q,[0,n]^2}^P$  or more concisely  $\phi_{p,q,n}^P$ . Note that this realization of the torus provides us with a natural embedding in the plane (although of course the boundary conditions cannot be realized using disjoint paths outside the square  $[0, n]^2$  because the torus itself is not a planar graph).

- The *Dobrushin* boundary conditions: assume that  $\partial G$  is a self-avoiding polygon in  $\mathbb{L}$ , let  $a$  and  $b$  be two sites of  $\partial G$ . Orienting its boundary counterclockwise defines two oriented boundary arcs  $\partial_{ab}$  and  $\partial_{ba}$ ; the Dobrushin boundary conditions are defined to be free on  $\partial_{ab}$  (there are no wirings between boundary sites) and wired on  $\partial_{ba}$  (all the boundary sites are pairwise connected). These arcs are referred to as the *free arc* and the *wired arc*, respectively. The measure associated to these boundary conditions will be denoted by  $\phi_{G,p,q}^{a,b}$ .

## 2 Finite energy and Domain Markov properties

### 2.1 The domain Markov property

Consider a graph  $G = (V, E)$  and  $F \subset E$ . One can encode, using appropriate boundary conditions  $\xi$ , the influence of the configuration outside  $F$  on the measure within it. In other words, given the state of edges outside a graph, the conditional measure inside  $F$  is a random-cluster measure with boundary conditions given by the wiring outside  $F$ . More formally,

**Theorem 3.2.** *Let  $G = (V, E)$  be a graph,  $(p, q) \in [0, 1] \times (0, \infty)$  and  $\xi$  boundary conditions. Fix  $F \subset E$ . Let  $X$  be a random variable measurable in terms of edges in  $F$  (call  $\mathcal{F}_{E \setminus F}$  the  $\sigma$ -algebra generated by edges of  $E \setminus F$ ). Then,*

$$\phi_{G,p,q}^{\xi}(X|\mathcal{F}_{E \setminus F})(\psi) = \phi_{F,p,q}^{\xi \cup \psi}(X),$$

where  $\psi$  is a configuration outside  $F$  and  $\xi \cup \psi$  is the wiring inherited by  $\xi$  and the edges in  $\psi$ .

This property allows us to decorrelate events in disjoint areas even though they are not independent.

**Proof** Let us deal with the case  $F = E \setminus \{e\}$ . Let  $\omega$  a configuration on  $F$  and define  $\omega^e$  to be the configuration on  $E$  coinciding with  $\omega$  on  $F$  and with  $e$  open. Then for any configuration  $\omega$ ,

$$\begin{aligned} \phi_{G,p,q}^{\xi}(\omega|\mathcal{F}_{\{e\}})(e \text{ open}) &:= \phi_{G,p,q}^{\xi}(\omega|e \text{ open}) = \phi_{G,p,q}^{\xi}(\omega^e) / \phi_{G,p,q}^{\xi}(e \text{ open}) \\ &= \frac{p^{o(\omega)+1}(1-p)^{c(\omega)}q^{k(\omega^e, \xi)}}{\sum_{\tilde{\omega}} p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})}q^{k(\tilde{\omega}, \xi)}} / \frac{\sum_{\tilde{\omega}: e \text{ is open}} p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})}q^{k(\tilde{\omega}, \xi)}}{\sum_{\tilde{\omega}} p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})}q^{k(\tilde{\omega}, \xi)}} \\ &= \frac{p^{o(\omega)+1}(1-p)^{c(\omega)}q^{k(\omega, \psi)}}{\sum_{\tilde{\omega}|_F} p^{o(\tilde{\omega}|_F)+1}(1-p)^{c(\tilde{\omega}|_F)}q^{k(\tilde{\omega}|_F, \psi)}} \\ &= \phi_{G,p,q}^{\psi}(\omega) \end{aligned}$$

where  $\psi$  is given by the boundary conditions  $\xi$  with the two end-points of  $e$  wired together. Similarly

$$\phi_{G,p,q}^{\xi}(\omega|\mathcal{F}_{\{e\}})(e \text{ closed}) = \phi_{G \setminus e, p, q}^{\xi}(\tilde{\omega})$$

and the claim follows easily for  $F = E \setminus \{e\}$ . The result can be deduced for every random variable  $X$  by linearity. Now, one can repeat the previous reasoning recursively and the result follows for any arbitrary subset of edges  $F$ .  $\square$

## 2.2 Finite energy property

This is a very simple property of random-cluster models. Let  $\epsilon \in (0, 1/2)$ . The conditional probability for an edge to be open, knowing the states of all the other edges, is bounded away from 0 and 1 uniformly in  $p \in (\epsilon, 1 - \epsilon)$  and in the configuration away from this edge. This property extends to any finite family of edges. Via the domain Markov property:

**Proposition 3.3.** *Let  $p, q > 0$  and  $G$  a graph. There exists  $c = c(G, p, q)$  such that for any configuration  $\omega$ ,  $\phi_{G, p, q}^\xi(\omega) \geq c$  for any boundary conditions  $\xi$ .*

The proof is extremely easy and is not included here. A typical example of a model not satisfying the finite energy property is the uniform (or any decent) measure on spanning trees. Indeed, knowing the whole configuration outside an edge  $e$ , it is not necessarily possible for  $e$  to be open (for instance if there is a cycle once  $e$  is open).

## 3 Strong positive association when $q \geq 1$

An event is called *increasing* if it is preserved by addition of open edges. A typical increasing event is the existence of a path between two sets  $A$  and  $B$ . The class of increasing events is central in the study of random-cluster models, due to the so-called positive association of the model.

### 3.1 Holley criterion

A measure  $\mu_1$  *stochastically dominates*  $\mu_2$  if for every increasing event  $A$ ,  $\mu_1(A) \geq \mu_2(A)$ . We first present a sufficient condition for two measures to be stochastically ordered.

Let  $\Omega$  be the space of subgraphs  $(V, F)$  of  $G = (V, E)$  with  $F \subset E$  (edges in  $F$  are called open). We restrict ourselves to positive<sup>1</sup> probability measures on  $\Omega$ . For  $\omega_1, \omega_2 \in \Omega$ ,  $\omega_1 \vee \omega_2$  (resp.  $\omega_1 \wedge \omega_2$ ) is the configuration with set of open edges being the union (resp. the intersection) of the sets of open edges of  $\omega_1$  and  $\omega_2$ .

**Theorem 3.4** (Holley inequality [Hol74]). *Let  $\mu_1, \mu_2$  be two measures such that*

$$\mu_1(\omega_1 \vee \omega_2)\mu_2(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \Omega, \quad (3.2)$$

*then  $\mu_1(A) \geq \mu_2(A)$  for any increasing event  $A$ .*

<sup>1</sup>*i.e.* measures  $\mu$  which satisfies  $\mu(\omega) > 0$  for every  $\omega \in \Omega$ .

The proof of this statement is a construction via Markov chains of a coupling  $(\omega_1, \omega_2)$  between the two measures ( $\omega_1$  is chosen according to the measure  $\mu_1$ , and  $\omega_2$  according to the measure  $\mu_2$ ), in a way that every open edge in  $\omega_2$  is open in  $\omega_1$ . The proof is omitted here (see Theorem (2.1) of [Gri06] for details). Let us mention that Theorem 3.4 possesses an elegant simplification: (3.2) does not need to be checked for every configurations  $\omega_1, \omega_2$ . Define  $\omega^e$  (resp.  $\omega_e$ ) to be the configurations coinciding with  $\omega$  on  $E \setminus \{e\}$ , and with  $e$  open (resp.  $e$  closed). Define  $\omega_f^e$  (resp.  $\omega_e^f, \omega^{ef}$  and  $\omega_{ef}$ ) to be the configurations coinciding with  $\omega$  on  $E \setminus \{e, f\}$  and with  $e$  open and  $f$  closed (resp.  $e$  closed and  $f$  open,  $e, f$  open and  $e, f$  closed).

**Theorem 3.5.** *Let  $\mu_1, \mu_2$  be two measures such that for any  $\omega$  and  $e, f$ ,*

$$\mu_1(\omega^e)\mu_2(\omega_e) \geq \mu_1(\omega_e)\mu_2(\omega^e) \quad (3.3)$$

$$\mu_1(\omega^{ef})\mu_2(\omega_{ef}) \geq \mu_1(\omega_e^f)\mu_2(\omega_f^e), \quad (3.4)$$

*then  $\mu_1$  stochastically dominates  $\mu_2$ .*

Holley criterion is particularly suitable to prove the Fortuin-Kasteleyn-Ginibre inequality [FKG71]. First proved by Harris in the case of product measures (in this case, it is called Harris inequality), the inequality relates the probability of the intersection of two events to the product of the probabilities. It belongs to the class of *correlation inequalities* (several other examples will be provided in this manuscript).

**Theorem 3.6** (FKG lattice condition). *Let  $G = (V, E)$  be a finite graph and  $\mu$  be a positive measure on  $\Omega$ . If for any configuration  $\omega$  and  $e, f \in E$*

$$\mu(\omega^{ef})\mu(\omega_{ef}) \geq \mu(\omega_e^f)\mu(\omega_f^e), \quad (3.5)$$

*then for any increasing events  $A, B$ ,*

$$\mu(A \cap B) \geq \mu(A)\mu(B). \quad (3.6)$$

The previous inequality immediately implies

$$\mu(XY) \geq \mu(X)\mu(Y) \quad (3.7)$$

for any increasing random variables  $X, Y$ . By taking the complement, one can also work with decreasing events or decreasing random variables.

**Proof** Equation (3.6) can be understood as  $\mu(\cdot|B)$  stochastically dominates  $\mu(\cdot)$ . Let us check Holley inequalities (3.3) and (3.4). We do it only for (3.4) ((3.3) is even easier). Fix  $\omega$  as well as  $e$  and  $f$ ,

$$\mu(\omega^{ef}|B)\mu(\omega_{ef}) \geq \mu(\omega_e^f|B)\mu(\omega_f^e)$$

is equivalent to (multiplying by  $\mu(B)$ )

$$1_{\omega^{ef} \in B} \mu(\omega^{ef})\mu(\omega_{ef}) \geq 1_{\omega_e^f \in B} \mu(\omega_e^f)\mu(\omega_f^e).$$

The indicator function on the left is equal to 1 if the one on the right is equal to 1, therefore, the previous inequality is a consequence of (3.5).  $\square$

### 3.2 Strong positive association for random-cluster models

**Theorem 3.7** (Fortuin-Kasteleyn-Ginibre inequality [FKG71]). *Fix a finite graph  $G$ , boundary conditions  $\xi$  and two parameters  $p \in [0, 1]$  and  $q \geq 1$ . For any two increasing events  $A$  and  $B$ , we have*

$$\phi_{G,p,q}^{\xi}(A \cap B) \geq \phi_{G,p,q}^{\xi}(A) \phi_{G,p,q}^{\xi}(B). \quad (3.8)$$

**Proof** Let us check criterion (3.5). Fix  $\omega$  a configuration and two edges  $e, f$ . We need to prove

$$[p/(1-p)]^{o(\omega^{ef})+o(\omega_{ef})} q^{k(\omega^{ef})+k(\omega_{ef})} \geq [p/(1-p)]^{o(\omega_e^f)+o(\omega_f^e)} q^{k(\omega_e^f)+k(\omega_f^e)}.$$

Since  $o(\omega^{ef}) + o(\omega_{ef}) = o(\omega_e^f) + o(\omega_f^e)$ , the only property to check is that  $k(\omega^{ef}) + k(\omega_{ef}) \geq k(\omega_e^f) + k(\omega_f^e)$  (recall  $q \geq 1$ ). Yet this inequality is obvious if we study whether both end-points of  $f$  are already connected or not in  $\omega|_{G \setminus \{e\}}$ .  $\square$

**Corollary 3.8.** *Fix a finite graph  $G$ , boundary conditions  $\xi$  and  $q \geq 1$ . For any  $p_1 \leq p_2$  and any increasing event  $A$ ,*

$$\phi_{p_1,q,G}^{\xi}(A) \leq \phi_{p_2,q,G}^{\xi}(A). \quad (3.9)$$

**Proof** For a random variable  $X$ , an easy computation implies

$$\phi_{p_2,q,G}^{\xi}(X) = \phi_{p_1,q,G}^{\xi}(XY)/K$$

where  $K$  is a normalizing constant and

$$Y(\omega) = \left( \frac{p_2/(1-p_2)}{p_1/(1-p_1)} \right)^{o(\omega)}.$$

Plugging  $X = 1$ , we find  $K = \phi_{p_1,q,G}^{\xi}(Y)$ . Now,  $X$  and  $Y$  are increasing, therefore (3.7) implies

$$\phi_{p_2,q,G}^{\xi}(X) = \phi_{p_1,q,G}^{\xi}(XY)/\phi_{p_1,q,G}^{\xi}(Y) \geq \phi_{p_1,q,G}^{\xi}(X).$$

$\square$

**Theorem 3.9.** *Fix a finite graph  $G$  and two parameters  $p \in [0, 1]$  and  $q \geq 1$ . For any boundary conditions  $\psi \leq \xi$  (i.e. sites wired in  $\psi$  are wired in  $\xi$ ), we have*

$$\phi_{G,p,q}^{\psi}(A) \leq \phi_{G,p,q}^{\xi}(A) \quad (3.10)$$

for any increasing event  $A$ .

**Proof** Consider  $\psi$  as being the partition  $(E_1, \dots, E_k)$  of boundary vertices and construct a new graph by adding edges between vertices of  $E_i$  for every  $i$ . Call this new graph  $G_0$  and  $E_0$  the set of additional edges. Now, the domain Markov property implies

$$\begin{aligned}\phi_{G,p,q}^\xi(\cdot) &= \phi_{G_0,p,q}^\xi(\cdot \mid \text{all the edges of } E_0 \text{ are closed}) \\ \phi_{G,p,q}^\psi(\cdot) &= \phi_{G_0,p,q}^\xi(\cdot \mid \text{all the edges of } E_0 \text{ are open}).\end{aligned}$$

Using the FKG inequality twice, we obtain

$$\phi_{G,p,q}^\xi(A) \leq \phi_{G_0,p,q}^\xi(A) \leq \phi_{G,p,q}^\psi(A)$$

for any increasing event  $A$  depending on edges in  $G$ . □

For stochastic ordering, the free and the wired boundary conditions are thus extremal. More formally, for any increasing event  $A$  and any boundary conditions  $\xi$ ,

$$\phi_{G,p,q}^0(A) \leq \phi_{G,p,q}^\xi(A) \leq \phi_{G,p,q}^1(A). \quad (3.11)$$

Combined with the domain Markov property, the comparison between boundary conditions allows us to bound conditional probabilities.

## 4 Planar duality

### 4.1 Statement and self-dual point

In two dimensions, one can associate with any random-cluster model on a graph  $G$  a dual model on  $G^*$ . Given a subgraph configuration  $\omega$ , construct a model on  $G^*$  by declaring any edge of the dual graph to be open (resp. closed) if the corresponding edge of the primal lattice is closed (resp. open) for the initial configuration. The new configuration is called the *dual configuration* of  $\omega$  and is denoted  $\omega^*$ , see Fig. 3.1.

Two sites  $u$  and  $v$  in  $G^*$  are said to be *dual-connected* if there is a *dual-open path*, i.e. an open path in the dual model between  $u$  and  $v$  (this event will be denoted by  $a \overset{*}{\leftrightarrow} b$ ). Two sets  $U$  and  $V$  are *dual connected* if there exists a dual-open path connecting them (denoted  $U \overset{*}{\leftrightarrow} V$ ). The maximal dual-connected components will be called *dual-(open) clusters*.

So far, nothing depends on the model and the construction of the dual configuration is not especially interesting. The miracle of this duality is that the dual configuration is also a random-cluster configuration, however with other parameters. In crude words, the *duality* could be describe as:

*If  $\omega$  is sampled according to a random-cluster measure with parameters  $(p, q)$ , the law of the dual configuration  $\omega^*$  is the random-cluster measure on  $G^*$  with parameters  $(p^*, q)$  where*

$$p^* = p^*(p, q) := \frac{(1-p)q}{(1-p)q + p}. \quad (3.12)$$

When defining the dual of a random-cluster model, one must be careful about boundary conditions or the previous statement remains too vague. Before describing in more detail how boundary conditions should be handled, let us introduce the self-dual point  $p_{sd}(q)$ :

**Definition 3.10.** *The self-dual point  $p_{sd} = p_{sd}(q)$  is the unique solution of the equation  $p^*(p_{sd}, q) = p_{sd}$ , i.e.*

$$p_{sd}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}. \quad (3.13)$$

## 4.2 Planar duality and boundary conditions

As mentioned above, the previous statement is very rough, and one needs to be careful about boundary conditions. We now treat three crucial examples to us.

**Free-wired boundary conditions** The dual of the wired boundary conditions are the free boundary conditions. Similarly, the dual of a random-cluster model with free boundary conditions is a random-cluster model with wired boundary conditions. Formally,

**Proposition 3.11.** *The dual model of the random-cluster on  $G$  with parameters  $(p, q)$  and wired boundary conditions is the random-cluster with parameters  $(p^*, q)$  and free boundary conditions on  $G^*$ .*

**Proof** Note that the state of edges between two sites of  $\partial G$  is not relevant when boundary conditions are wired. Indeed, sites on the boundary are connected via boundary conditions anyway, so that the state of each boundary edge does not alter the connectivity properties of the subgraph, and is independent of other edges. For this reason, forget about edges between boundary sites and consider only inner edges (which correspond to edges of  $G^*$ ):  $o(\omega)$  and  $c(\omega)$  then denote the number of open and closed inner edges.

Set  $e^*$  for the dual edge of  $G^*$  associated to the (inner) edge  $e$ . From the definition of the dual configuration  $\omega^*$  of  $\omega$ , we have  $o(\omega^*) = a - o(\omega)$  where  $a$  is the number of edges in  $G^*$  and  $o(\omega^*)$  is the number of open dual edges. Moreover, connected components of  $\omega^*$  correspond exactly to faces of  $\omega$ , so that  $f(\omega) = k(\omega^*)$ , where  $f(\omega)$  is the number of faces (counting the infinite face). Using Euler's formula

$$\# \text{ edges} + \# \text{ connected components} + 1 = \# \text{ sites} + \# \text{ faces},$$

which is valid for any planar graph, we obtain, with  $s$  being the number of sites in  $G$ ,

$$k(\omega) = s - 1 + f(\omega) - o(\omega) = s - 1 + k(\omega^*) - a + o(\omega^*).$$

The probability of  $\omega^*$  is equal to the probability of  $\omega$  under  $\phi_{G,p,q}^1$ , *i.e.*

$$\begin{aligned} \phi_{G,p,q}^1(\omega) &= \frac{1}{Z_{G,p,q}^1} p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)} \\ &= \frac{(1-p)^a}{Z_{G,p,q}^1} [p/(1-p)]^{o(\omega)} q^{k(\omega)} \\ &= \frac{(1-p)^a}{Z_{G,p,q}^1} [p/(1-p)]^{a-o(\omega^*)} q^{s-1-a+k(\omega^*)+o(\omega^*)} \\ &= \frac{p^a q^{s-1-a}}{Z_{G,p,q}^1} [q(1-p)/p]^{o(\omega^*)} q^{k(\omega^*)} = \phi_{p^*,q,G^*}^0(\omega^*) \end{aligned}$$

since  $q(1-p)/p = p^*/(1-p^*)$ , which is exactly the statement.  $\square$

**Dobrushin boundary conditions** The same reasoning as before (using Euler's formula) shows that the dual of  $\phi_{G,p,q}^{a,b}$  is  $\phi_{G^*,p^*,q}^{b^*,a^*}$ . In words, the dual of a random-cluster model with parameters  $(p, q)$ , free boundary conditions on  $\partial_{ab}$  and wired boundary conditions on  $\partial_{ba}$  is the random-cluster model with parameters  $(p^*, q)$  with wired boundary conditions on  $\partial_{ab}^*$  and free boundary conditions on  $\partial_{ba}^*$ , where  $\partial_{ab}^*$  is the inner dual arc adjacent to  $\partial_{ab}$  and  $\partial_{ba}^*$  is the outer arc adjacent to  $\partial_{ba}$ <sup>2</sup>.

**Periodic boundary conditions** The case of periodic boundary conditions, or equivalently the case of the random-cluster model defined on a torus (with no boundary conditions) is a little more involved: indeed, its dual is *not* a random-cluster model; yet it is not very different from one, and that will be enough for our purposes. In order to state duality in this case, additional notations are required. Let  $f(\omega)$  be the number of faces delimited by  $\omega$ , *i.e.* the number of connected components of the complement of the set of open edges, and  $s(\omega)$  be the number of vertices in the underlying graph (it does not depend on  $\omega$ ). We will now define an additional parameter  $\delta(\omega)$ .

Call a (maximal) connected component of  $\omega$  a *net* if it contains two non-contractible simple loops of different homotopy classes, and a *cycle* if it is non-contractible but is not a net. Notice that every configuration  $\omega$  can be of one of three types:

- One of the clusters of  $\omega$  is a net. Then no other cluster of  $\omega$  can be a net or a cycle. In that case, let  $\delta(\omega) = 2$ ;
- One of the clusters of  $\omega$  is a cycle. Then no other cluster can be a net, but other clusters can be cycles as well (in which case all the involved, simple loops are in the same homotopy class). Then let  $\delta(\omega) = 1$ ;
- None of the clusters of  $\omega$  is a net or a cycle. Let  $\delta(\omega) = 0$ .

---

<sup>2</sup>One should be careful when defining these arcs. In the next chapters, we will take care of this technical issue.

With this additional notation, Euler's formula becomes

$$s(\omega) - o(\omega) + f(\omega) = k(\omega) + 1 - \delta(\omega). \quad (3.14)$$

Besides, these terms transform in a simple way under duality:  $o(\omega) + o(\omega^*)$  is a constant,  $f(\omega) = k(\omega^*)$  and  $\delta(\omega) = 2 - \delta(\omega^*)$ . The same proof as that of usual duality, taking the additional topology into account, then leads to the relation

$$(\phi_{p,q,n}^p)^*(\{\omega\}) \propto q^{1-\delta(\omega)} \phi_{p^*,q,n}^p(\{\omega\}). \quad (3.15)$$

This means that even though the dual model of the periodic boundary conditions random-cluster model is not exactly a random-cluster model at the dual parameter, it is absolutely continuous with respect to it and the Radon-Nikodym derivative is bounded above and below by constants depending only on  $q$ . Another way of stating the same result would be to define a *balanced random-cluster model* with weights

$$\tilde{\phi}_{p,q,n}^p(\{\omega\}) = \frac{(\sqrt{q})^{1-\delta(\omega)}}{Z} \phi_{p,q,n}^p(\{\omega\}) :$$

this one is absolutely continuous with respect to the usual random-cluster model and does satisfy exact duality.

## 5 Infinite-volume measures and phase transition.

### 5.1 Definition of infinite-volume measures

The definition of an infinite-volume random-cluster measure is not direct. Indeed, one cannot count the number of open or closed edges on  $\mathbb{L} = (\mathbb{V}, \mathbb{E})$  since they could be (and would be) infinite. We thus define infinite-volume measures indirectly: they are the measures which coincide, when restricted to a finite box, with random-cluster measures in finite volume.

So far, the problem of the  $\sigma$ -field was eluded since every set of configurations was measurable in finite volume. In infinite-volume, we must be careful and proceed as follows:  $\Omega$  is the space of configurations on the whole lattice and  $\mathcal{F}$  is the smallest  $\sigma$ -algebra containing every events depending on a finite number of edges.

**Definition 3.12.** *Let  $p \in [0, 1]$  and  $q \in (0, \infty)$ . A probability measure  $\phi$  on  $(\Omega, \mathcal{F})$  is called an infinite-volume random-cluster measure with parameters  $p$  and  $q$  if for every event  $A \in \mathcal{F}$  and any box  $\Lambda$ ,*

$$\phi(A|\mathcal{F}_{\mathbb{L} \setminus \Lambda})(\xi) = \phi_{\Lambda,p,q}^\xi(A),$$

for  $\phi$ -almost every  $\xi \in \Omega$ , where  $\mathcal{F}_{\mathbb{E} \setminus \Lambda}$  is the  $\sigma$ -algebra generated by edges in  $\mathbb{E} \setminus \Lambda$ .

The domain Markov property and the comparison between boundary conditions allow us to construct infinite-volume measures. Indeed, consider a sequence of measures on boxes of increasing size with free boundary conditions. This sequence is increasing in the sense of stochastic domination, which implies that it converges weakly to a limiting measure, called the random-cluster measure on  $\mathbb{L}$  with free boundary conditions and denoted by  $\phi_{p,q}^0$ <sup>3</sup>. This construction can be performed with many other sequences of measures, defining several *a priori* different infinite-volume measures on  $\mathbb{L}$ . For instance, one can define the random-cluster measure  $\phi_{p,q}^1$  with wired boundary conditions by considering the decreasing sequence of random-cluster measures on finite boxes with wired boundary conditions. It could also be possible to see infinite-volume measures existing intrinsiquely, in the sense that they are not limits of random-cluster measures in finite volume.

The question of uniqueness of infinite measures is very difficult in general. The following powerful theorem answers partially this question and will be useful in the next paragraphs.

**Theorem 3.13** (see Theorem (4.60) of [Gri06]). *For  $q \geq 1$ , the set  $\mathcal{D}_q$  of edge-weight  $p$  for which uniqueness fails is at most countable.*

There is an easy criterion, due to the positive association, to decide whether or not the infinite-volume measures are unique for some parameters  $p$  and  $q$ :

**Proposition 3.14.** *Let  $p \in [0, 1]$  and  $q \in (0, \infty)$ . If  $\phi_{p,q}^1 = \phi_{p,q}^0$ , then there exists a unique infinite-volume measure with parameters  $p$  and  $q$ , denoted  $\phi_{p,q}$ .*

## 5.2 Ergodicity of infinite-volume random-cluster measures

A property that will be used implicitly in many arguments in this chapter and the next ones is the *ergodicity* of the measures. More precisely,

**Theorem 3.15** (Corollary (4.23) of [Gri06]). *Fix  $p \in [0, 1]$  and  $q \in (0, \infty)$ . Any translational-invariant event  $A \in \mathcal{F}$  has probability 0 or 1 under the measures  $\phi_{p,q}^1$  and  $\phi_{p,q}^0$ .*

## 5.3 Critical point

We are now in a position to discuss the phase transition of the random-cluster model.

**Theorem 3.16.** *There exists a critical point  $p_c \in (0, 1)$  such that:*

- For  $p < p_c$ , any infinite-volume measure has no infinite cluster almost surely.
- For  $p > p_c$ , any infinite-volume measure has a unique infinite cluster almost surely.

---

<sup>3</sup>More precisely, the restrictions to a box  $\Lambda_N$  of measures with free boundary conditions on boxes  $\Lambda_n$ ,  $n \geq N$ , form an increasing sequence of measures, allowing us to construct a limiting measure  $\phi$  on  $\Lambda_N$  by the formula  $\phi(A) := \lim_{N \rightarrow \infty} \phi_{p,q,N}(A)$ . Since these limits are compatible for different  $N$ , it defines a measure on  $\mathbb{Z}^2$  (with  $\sigma$ -algebra  $\mathcal{F}$ ).

Note that several parts of the previous statement are not straightforward:

- It is natural to define the parameter  $p_c$  as the infimum of edge-weights  $p$  for which there is an infinite-volume measure possessing an infinite cluster with positive probability. Yet, non-uniqueness of infinite-volume measures can compromise this strategy. Fortunately, Theorem 3.8 guarantees that the set of edge-weights such that uniqueness fails is discrete, which is enough to legitimate the definition of  $p_c$ .

- The fact that the infinite cluster exists with probability one or zero is a consequence of ergodicity (Theorem 3.15).

- The uniqueness of the infinite cluster is a consequence of an argument of Burton-Keane [BK89] (note that this uniqueness can fail when considering random-cluster models on more general graphs such as non-amenable Cayley graphs).

- The fact that  $p_c$  lies strictly between 0 and 1 is not obvious (and false in one dimension). A counting argument similar to Peierls's argument [Pei36] allows us to rule out these two possibilities. Since Peierls's argument will be presented in the case of the Ising model, we do not spend more time on it now.

Overall, the existence of a critical point is not completely direct. Nevertheless, it remains a well understood problem. Its computation is a much harder task and the existence of a nice formula for  $p_c(q)$  is not even obvious.

On the square lattice, it is natural to conjecture that the critical point satisfies  $p_c = p_{sd}$ .

**Conjecture 3.17.** *The critical parameter  $p_c(q)$  of the random-cluster model on the square lattice equals  $p_{sd}(q) = \sqrt{q}/(1 + \sqrt{q})$  for every  $q \geq 1$ .*

Indeed, if one assumes  $p_c \neq p_{sd}$ , there would be two phase transitions, one at  $p_c$ , due to the change of behavior in the primal model, and one at  $p_c^*$ , due to the change of behavior in the dual model. Hence, the natural assumption that only one phase transition occurs implies  $p_c = p_{sd}$ . Nevertheless, this heuristic argument is not a mathematical proof, and a formal derivation was lacking for many years. Recently, the critical point was finally identified rigorously. This is the subject (among other things) of Chapter 4.

## 6 The inequality $p_c \geq p_{sd}$ .

A lower bound for the critical value can be derived using the uniqueness of the infinite cluster. Indeed, if one assumes that  $p_c < p_{sd}$ , the configuration at  $p_{sd}$  must contain one infinite open cluster and one infinite dual open cluster (since the dual random-cluster model is then supercritical as well). Intuition indicates that such coexistence would imply that there is more than one infinite open cluster; an elegant argument (due to Zhang in the case of percolation) formalizes this idea. We refer to the presentation in Theorem (6.17) of [Gri06] for full detail, but still give a sketch of the argument.

**Proposition 3.18.** *For  $q \geq 1$ , there exists almost surely no infinite cluster at  $p_{sd}(q)$  for the infinite-volume measure with free boundary conditions.*

The proof goes as follows, see Figure 3.2. Assume that  $p_c < p_{sd}$  and consider the random-cluster model with  $p = p_{sd}$ . There is an infinite open cluster, and therefore, one can choose a large box such that the infinite open cluster and the dual infinite open cluster touch the boundary with probability greater than  $1 - \epsilon$ . The FKG inequality (through the so-called “square-root trick”: for two increasing events  $A$  and  $B$  with same probabilities,  $\phi_{G,p,q}^\xi(A \cap B) \geq 1 - (1 - \phi_{G,p,q}^\xi(A))^{1/2}$ ) implies that the infinite open cluster actually touches the top side of the box, using only edges outside the box, with probability greater than  $1 - \epsilon^{1/4}$ . Therefore, with probability at least  $1 - 2\epsilon^{1/4}$ , the infinite open cluster touches both the top and bottom sides, using only edges outside of the box.

A similar argument implies that the infinite dual open cluster touches both the left and right sides of the box with probability at least  $1 - 2\epsilon^{1/4}$ . Therefore, with probability at least  $1 - 4\epsilon^{1/4}$ , the complement of the box contains an infinite open path touching the top of the box, one touching the bottom, and infinite dual open paths touching each of the vertical edges. Enforcing edges in the box to be closed, which brings only a positive multiplicative factor due to the finite energy property of the model, and choosing  $\epsilon$  sufficiently small, there are two infinite open clusters with positive probability. Since the infinite open cluster must be unique, this is a contradiction which implies that  $p_c \geq p_{sd}$ .

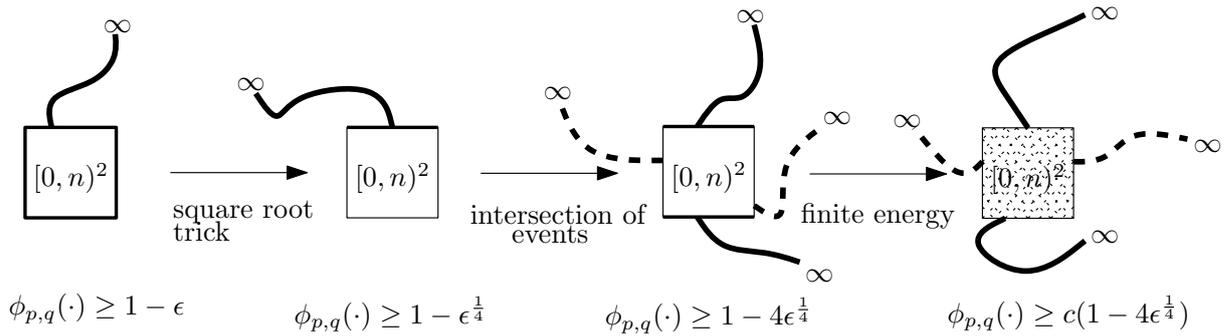


Figure 3.2: A figurative description of the proof of  $p_c \geq p_{sd}$ .

**Uniqueness of the infinite measure for  $p < p_{sd}(q)$**  When  $p < p_{sd} \leq p_c$ , there is no infinite cluster for any infinite-volume measure. The following theorem will be very useful in our study.

**Theorem 3.19.** *Fix  $q \geq 1$ . The unique edge-weight  $p \in [0, 1]$  for which there can exist distinct infinite-volume measures is  $p_{sd}(q)$ .*

From now on, when  $p \neq p_{sd}(q)$ , the unique infinite measure with parameters  $(p, q)$  is denoted by  $\phi_{p,q}$ . This measure can be equivalently thought of as  $\phi_{p,q}^0$  or  $\phi_{p,q}^1$ .

**Proof** First, note that it is sufficient to prove  $\phi_{p,q}^1 = \phi_{p,q}^0$  (or even  $\phi_{p,q}^1 \leq \phi_{p,q}^0$  since the other bound is obvious) for  $p < p_{sd}(q)$ . Indeed, (3.11) implies that every infinite-volume measure is sandwiched between  $\phi_{p,q}^0$  and  $\phi_{p,q}^1$ . Moreover, duality implies that  $\phi_{p^*,q}^0 = \phi_{p^*,q}^1$ , giving uniqueness above  $p_{sd}(q)$  from uniqueness below it.

Fix an increasing event  $A$  depending on a finite number of edges (all included in the box of size  $N$ ). When  $n$  goes to infinity, the probability of the event  $E_n$  that  $[-N, N]^2$  is connected to the exterior of  $[-n, n]^2$  goes to 0 (there is no infinite cluster since  $p < p_c$ ).

On the one hand,  $\phi_{p,q}^1(A \cap E_n)$  goes to 0 when  $n$  goes to infinity. On the other hand, conditioning on the exterior most dual circuit  $\Gamma$  surrounding  $[-N, N]^2$  in  $[-n, n]^2$  to be equal to a deterministic circuit  $\gamma$  in  $\mathbb{L}^*$  implies:

$$\phi_{p,q}^1(A|\Gamma = \gamma) \leq \phi_{p,q,[-n,n]^2}^0(A) \leq \phi_{p,q}^0(A).$$

Indeed, the conditioning boils down to fixing free boundary conditions on  $\gamma$  thanks to the domain Markov property. In addition, comparison between boundary conditions allow us to compare to the case where the free boundary conditions are on  $\partial[-n, n]^2$  (which is further from  $[-N, N]^2$ ). Since the result is uniform on  $\gamma$ ,  $\phi_{p,q}^1(A \cap E_n^c) \leq \phi_{p,q}^0(A)$ . Now,

$$\phi_{p,q}^1(A) = \phi_{p,q}^1(A \cap E_n) + \phi_{p,q}^1(A \cap E_n^c) \leq \phi_{p,q}^1(A \cap E_n) + \phi_{p,q}^0(A),$$

thus implying  $\phi_{p,q}^1(A) \leq \phi_{p,q}^0(A)$  for increasing events depending on a finite number of edges (simply let  $n$  go to infinity). The proof can be concluded by recalling that any increasing events can be approached by increasing events depending only on a finite number of edges.  $\square$

# Chapter 4

## The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$

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**Abstract:** This chapter is devoted to the determination of the critical point of the random-cluster models with  $q \geq 1$  for the square, the hexagonal and the triangular lattices. It is inspired by the article *The self-dual random-cluster model is critical above  $q = 1$*  [BDC10], written with V. Beffara and published in Probability Theory and Related Fields.

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There are no conjectures for the value of the critical point for general infinite graphs. However, in the case of the square lattice, planar duality hints that the critical point is the same as the so-called *self-dual point* satisfying  $p_{sd} = p^*(p_{sd})$ , which has a known value

$$p_{sd}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

In this chapter, we prove this result for all  $q \geq 1$ :

**Theorem 4.1.** *Let  $q \geq 1$ . The critical point  $p_c = p_c(q)$  for the random-cluster model with cluster-weight  $q$  on the square lattice satisfies*

$$p_c = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

A rigorous derivation of the critical point was previously known in three cases. For  $q = 1$ , the model is simply bond percolation, proved by Kesten in 1980 [Kes80] to be critical at  $p_c(1) = 1/2$ . For  $q = 2$ , the self-dual value corresponds to the critical temperature of the Ising model, as first derived by Onsager in 1944 [Ons44]; one can actually couple realizations of the Ising and random-cluster models to relate their critical points, see

Chapter 6. For modern proofs in that case, see [ABF87] or the short proof of Chapter 8 [BDC11]. Finally, for sufficiently large  $q$ , a proof is known based on the fact that the random-cluster model exhibits a first order phase transition (see [LMMS<sup>+</sup>91, LMR86], the proofs are valid for  $q$  larger than 25.72). Let us mention that physicists derived the critical temperature for the Potts models with  $q \geq 4$  in 1978, using non-geometric arguments based on analytic properties of the Hamiltonian [HKW78].

In the subcritical phase, the probability for two points  $x$  and  $y$  to be connected by a path is proved to decay exponentially fast with respect to the distance between  $x$  and  $y$ . In the supercritical phase, the same behavior holds in the dual model. This phenomenon is known as the *sharp phase transition*:

**Theorem 4.2.** *Let  $q \geq 1$ . For any  $p < p_c(q)$ , there exist  $0 < C(p, q), c(p, q) < \infty$  such that for any  $x, y \in \mathbb{Z}^2$ ,*

$$\phi_{p,q}(x \leftrightarrow y) \leq C(p, q)\varepsilon^{-c(p,q)|x-y|}, \quad (4.1)$$

where  $|\cdot|$  denotes the Euclidean norm.

The proof involves two main ingredients. The first one is an estimate on crossing probabilities at the self-dual point  $p = p_{sd} = \sqrt{q}/(1 + \sqrt{q})$ : the probability of crossing a rectangle with aspect ratio  $(\alpha, 1)$  — meaning that the ratio between the width and the height is of order  $\alpha$  — in the horizontal direction is bounded away from 0 and 1 uniformly in the size of the box. It is a generalization of the celebrated Russo-Seymour-Welsh theorem for percolation.

The second ingredient is a collection of sharp threshold theorems, which were originally introduced for product measures. They have been used in many contexts, and are a powerful tool for the study of phase transitions, see Bollobás and Riordan [BR06a, BR06b]. These theorems were later extended to positively associated measures by Graham and Grimmett [GG11, GG06, Gri06]. In our case, they may be used to show that the probability of crossings goes to 1 when  $p > \sqrt{q}/(1 + \sqrt{q})$ .

Actually, the situation is complicated: the dependence inherent in the model makes boundary conditions difficult to handle. More precisely, one can use a classic sharp threshold argument for symmetric increasing events in order to deduce that the crossing probabilities of larger and larger domains, under *wired boundary conditions*, converge to 1 whenever  $p > \sqrt{q}/(1 + \sqrt{q})$ . Moreover, the theorem provides us with bounds on the speed of convergence for rectangles with *wired boundary conditions*. A new way of combining long paths allows us to create an infinite cluster. We emphasize that classical arguments, used by Kesten [Kes80] in the case of percolation, do not seem to work in our case.

This approach allows the determination of the critical value, yet it provides us with a rather weak estimate on the speed of convergence for crossing probabilities. Nevertheless, combining the fact that the crossing probabilities go to 0 when  $p < p_{sd}$  with a very general threshold theorem, we deduce that the cluster-size at the origin has finite moments of any order. It is then an easy step to derive the exponential decay of the two-point function in the subcritical case.

Theorem 4.2 has several notable consequences. First, it extends up to the critical point results that are known for the subcritical random-cluster models under the exponential decay condition (for instance, Ornstein-Zernike estimates [CIV08] or strong mixing properties). Second, it identifies the critical value of the Potts models via the classical coupling between random-cluster models with cluster-weight  $q \in \mathbb{N}$  and the  $q$ -state Potts models (see Chapter 6).

The methods of this chapter harness symmetries of the graph, together with the self-dual property of the square lattice. In the case of the hexagonal and triangular lattices, the symmetries of the graphs, the duality property between the hexagonal and the triangular lattices and the star-triangle relation allow us to extend the crossing estimate proved in Section 1, at the price of additional technical difficulties. The rest of the proof can be carried over to the triangular and the hexagonal lattices as well, yielding the following result:

**Theorem 4.3.** *The critical value  $p_c = p_c(q)$  for the random-cluster model with cluster-weight  $q \geq 1$  satisfies*

$$\begin{aligned} y_c^3 + 3y_c^2 - q &= 0 && \text{on the triangular lattice and} \\ y_c^3 - 3qy_c - q^2 &= 0 && \text{on the hexagonal lattice,} \end{aligned}$$

where  $y_c := p_c/(1 - p_c)$ . Moreover, there is exponential decay in the subcritical phase.

The technology developed in the present chapter relies heavily on the positive association property of the random-cluster measures with  $q \geq 1$ . Our strategy does not extend to random-cluster models with  $q < 1$ . Understanding these models is a challenging open question.

The chapter is organized as follows. Section 1 is devoted to the statement and the proof of the crossing estimates. In Section 2, we briefly present the theory of sharp threshold that will be employed in the next section. Section 3 contains the proofs of Theorems 4.1 and 4.2. Section 4 is devoted to extensions to other lattices and contains the proof of Theorem 4.3.

## 1 Crossing probabilities for rectangles at the self-dual point

In this section, we prove crossing estimates for rectangles of prescribed aspect ratio. This is an extension of the Russo-Seymour-Welsh theory for percolation. We will work with  $p = p_{sd}(q)$  and the measures  $\phi_{p_{sd},q}^1$  and  $\phi_{p_{sd},q,n}^p$ ; we present the proof in the periodic case. The case of the (bulk) wired boundary condition can be derived from this case (see Corollary 4.9).

For a rectangle  $R$ , let  $\mathcal{C}_v(R)$  denote the event that there exists a path between the top and the bottom sides which stays inside the rectangle. Such a path is called a *vertical (open) crossing* of the rectangle. Similarly, define  $\mathcal{C}_h$  to be the event that there exists an *horizontal open crossing* between the left and the right sides. Finally,  $\mathcal{C}_v^*(R^*)$  denotes the event that there exists a dual-open crossing from top to bottom in the dual graph  $R^*$  of  $R$ .

The following theorem states that, at the self-dual point, the probability of crossing a rectangle horizontally is bounded away from 0 uniformly in the sizes of both the rectangle and the torus provided that the aspect ratio of the rectangles remains constant. The size of the ambient torus is denoted by  $m$ . Note that  $p = p^*$  when  $p = p_{sd}$ , and hence the balanced random-cluster measure on the torus is self-dual.

**Theorem 4.4.** *Let  $\alpha > 1$  and  $q \geq 1$ . There exists  $c(\alpha) > 0$  such that for every  $m > \alpha n > 0$ ,*

$$\phi_{p_{sd},q,m}^p(\mathcal{C}_h([0, \alpha n] \times [0, n])) \geq c(\alpha). \quad (4.2)$$

The proof begins with a lemma, which corresponds to the existence of  $c(1)$  and is based on the self-duality of random-cluster measures on the torus. This lemma is classic and is the natural starting point for any attempt to prove RSW-like estimates.

**Lemma 4.5.** *Let  $q \geq 1$ , there exists  $c(1) > 0$  (depending only on the parameter  $q$ ) such that for every  $m > n \geq 1$ ,  $\phi_{p_{sd},q,m}^p(\mathcal{C}_h([0, n]^2)) \geq c(1)$ .*

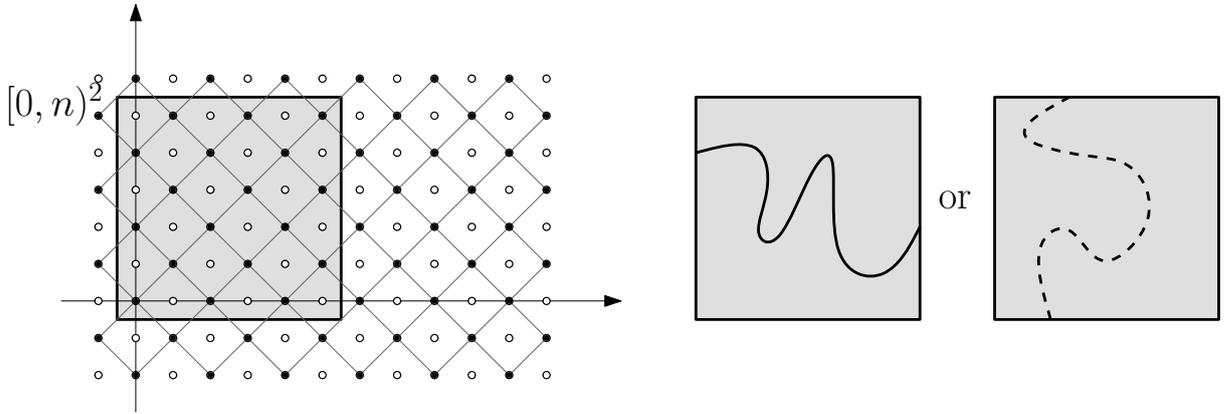


Figure 4.1: **Left:** The square  $[0, n]^2$  (all the sites in the shaded region) and its dual have the same graph structure. **Right:** The events  $\mathcal{C}_h([0, n]^2)$  and  $\mathcal{C}_v^*([0, n]^2)$ .

**Proof** Note that the dual of  $[0, n]^2$  is  $[0, n]^2$  (meaning the sites of the dual torus inside  $[0, n]^2$ ), see Figure 4.1. If there is no open crossing from left to right in  $[0, n]^2$ , there exists necessarily a dual-open crossing from top to bottom in the dual configuration. Hence, the complement of  $\mathcal{C}_h([0, n]^2)$  is  $\mathcal{C}_v^*([0, n]^2)$ , thus yielding

$$\phi_{p_{sd},q,m}^p(\mathcal{C}_h([0, n]^2)) + \phi_{p_{sd},q,m}^p(\mathcal{C}_v^*([0, n]^2)) = 1.$$

Using the duality property for periodic boundary conditions and the symmetry of the lattice, the probability  $\phi_{p_{sd},q,m}^p(\mathcal{C}_v^*([0,n]^2))$  is larger than  $c\phi_{p_{sd},q,m}^p(\mathcal{C}_h([0,n]^2))$  (for some constant  $c$  only depending on  $q$ ), giving

$$1 \leq (1 + c)\phi_{p_{sd},q,m}^p(\mathcal{C}_h([0,n]^2)),$$

which concludes the proof. □

**Remark 4.6.** *This lemma could be stated in terms of the balanced random-cluster measure instead of the usual one. Then, as in the case of percolation, one would obtain that the probability of a horizontal crossing of the square is exactly 1/2. However, because going back and forth between the balanced and standard measure would be a little tedious in what follows, everything is stated in terms of  $\phi_{p_{sd},q,m}^p$  — and  $c(1)$  depends on the value of  $q$ .*

The only major difficulty is to prove that rectangles of aspect ratio  $\alpha$  are crossed in the horizontal direction — with probability uniformly bounded away from 0 — for *some*  $\alpha > 1$ . There are many ways to prove this in the case of percolation. Nevertheless, they always involve independence in a crucial way; in our case, independence fails, so a new argument is needed. The main idea is to invoke self-duality in order to enforce the existence of crossings, even in the case where boundary conditions could look disadvantageous. In order to do that, we introduce the following family of domains, which are in some sense nice symmetric domains.

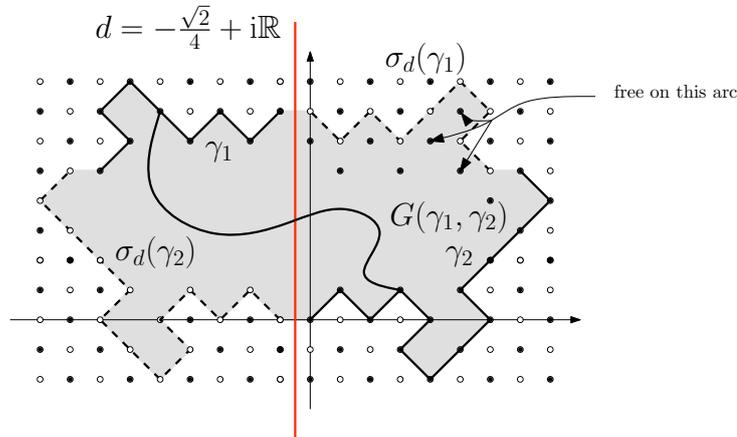


Figure 4.2: Two paths  $\gamma_1$  and  $\gamma_2$  satisfying Hypothesis  $(\star)$  and the graph  $G(\gamma_1, \gamma_2)$ .

Define the line  $d := -\sqrt{2}/4 + i\mathbb{R}$ . The orthogonal symmetry  $\sigma_d$  with respect to this line maps  $\mathbb{L}$  to  $\mathbb{L}^*$ . Let  $\gamma_1$  and  $\gamma_2$  be two paths satisfying the following Hypothesis  $(\star)$ , see Figure 4.2:

- $\gamma_1$  remains on the left of  $d$  and  $\gamma_2$  remains on the right;
- $\gamma_2$  begins at 0 and  $\gamma_1$  begins on a site of  $\mathbb{L} \cap (-\sqrt{2}/2 + i\mathbb{R}_+)$ ;

- $\gamma_1$  and  $\sigma_d(\gamma_2)$  do not intersect (as curves in the plane);
- $\gamma_1$  and  $\sigma_d(\gamma_2)$  end at two sites (one primal and one dual) which are at distance  $\sqrt{2}/2$  from each other.

The definition extends trivially via translation, so that the pair  $(\gamma_1, \gamma_2)$  is said to satisfy Hypothesis  $(\star)$  if one of its translations does.

When following the paths in counter-clockwise order, one can create a circuit by linking the end points of  $\gamma_1$  and  $\sigma_d(\gamma_2)$  by a straight line, the start points of  $\sigma_d(\gamma_2)$  and  $\gamma_2$ , the end points of  $\gamma_2$  and  $\sigma_d(\gamma_1)$ , and the start points of  $\sigma_d(\gamma_1)$  and  $\gamma_1$ . The circuit  $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$  surrounds a set of vertices of  $\mathbb{L}$ . Define the graph  $G(\gamma_1, \gamma_2)$  composed of sites of  $\mathbb{L}$  that are surrounded by the circuit  $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ , and of edges of  $\mathbb{L}$  that remain entirely within the circuit (boundary included).

The *mixed boundary conditions* on this graph are wired on  $\gamma_1$  (all the edges are pairwise connected), wired on  $\gamma_2$ , and free elsewhere. The measure on  $G(\gamma_1, \gamma_2)$  with parameters  $(p_{sd}, q)$  and mixed boundary conditions is denoted by  $\phi_{p_{sd}, q, \gamma_1, \gamma_2}$  or more simply  $\phi_{\gamma_1, \gamma_2}$ .

**Lemma 4.7.** *For any pair  $(\gamma_1, \gamma_2)$  satisfying Hypothesis  $(\star)$ , the following estimate holds:*

$$\phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geq \frac{1}{1 + q^2}.$$

**Proof** On the one hand, if  $\gamma_1$  and  $\gamma_2$  are not connected,  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$  must be connected by a dual path in the dual model (event corresponding to  $\sigma_d(\gamma_1) \leftrightarrow \sigma_d(\gamma_2)$  in the dual model). Hence,

$$1 = \phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + \sigma_d * \phi_{\gamma_1, \gamma_2}^*(\gamma_1 \leftrightarrow \gamma_2), \quad (4.3)$$

where  $\sigma_d * (\phi_{\gamma_1, \gamma_2}^*)$  denotes the image under  $\sigma_d$  of the dual measure of  $\phi_{\gamma_1, \gamma_2}$ . This measure lies on  $G(\gamma_1, \gamma_2)$  as well and has parameters  $(p_{sd}, q)$ .

When looking at the dual measure of a random-cluster model, the boundary conditions are transposed into new boundary conditions for the dual measure. In the case of the periodic boundary conditions, the boundary conditions for the dual measure are the same. Here, the boundary conditions become wired on  $\gamma_1 \cup \gamma_2$  and free elsewhere (this is easy to check using Euler's formula).

It is very important to notice that the boundary conditions are *not* exactly the mixed one, since  $\gamma_1$  and  $\gamma_2$  are wired *together*. Nevertheless, the Radon-Nikodym derivative of  $\sigma_d * \phi_{\gamma_1, \gamma_2}^*$  with respect to  $\phi_{\gamma_1, \gamma_2}$  is easy to bound. Indeed, for any configuration  $\omega$ , the number of cluster can differ only by 1 when counted in  $\sigma_d * \phi_{\gamma_1, \gamma_2}^*$  or  $\phi_{\gamma_1, \gamma_2}$  so that the ratio of partition functions belongs to  $[1/q, q]$ . Therefore, the ratio of probabilities of the configuration  $\omega$  remains between  $1/q^2$  and  $q^2$ . This estimate extends to events by summing over all configurations. Therefore,

$$\sigma_d * \phi_{\gamma_1, \gamma_2}^*(\gamma_1 \leftrightarrow \gamma_2) \leq q^2 \phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2).$$

When plugging this inequality into (4.3), we obtain

$$\phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + q^2 \phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geq 1$$

which implies the claim.  $\square$

We are now in a position to prove the key result of this section.

**Proposition 4.8.** *For all  $m > 3n/2 > 0$ , the following holds:*

$$\phi_{p_{sd}, q, m}^p[\mathcal{C}_v([0, n] \times [0, 3/2n])] \geq \frac{c(1)^3}{2(1+q^2)}.$$

Before proving this proposition, let us show how it implies Theorem 4.4. The strategy is straightforward and classic: crossings can be combined together using the FKG inequality only.

**Proof of Theorem 4.4** If  $\alpha < 3/2$ , Proposition 4.8 implies the claim so we can assume  $\alpha > 3/2$ . Define the following rectangles, see Figure 4.3:

$$R_j^h = [jn/2, jn/2 + 3n/2) \times [0, n) \quad \text{and} \quad R_j^v = [jn/2, jn/2 + n) \times [0, n)$$

for  $j \in [0, \lfloor 2\alpha \rfloor - 1]$ , where  $\lfloor x \rfloor$  denotes the integer part of  $x$ . If every rectangle  $R_j^h$  is crossed horizontally, and every rectangle  $R_j^v$  is crossed vertically, then  $[0, \alpha n) \times [0, n)$  is crossed horizontally. This event is denoted by  $B$ . The rectangle  $R_j^h$  is crossed horizontally with probability greater than  $c(1)^3/[2(1+q^2)]$  (Proposition 4.8), the rectangle  $R_j^v$  is crossed vertically with probability greater than  $c(1)$  (Lemma 4.5) and so, using the FKG inequality,

$$\phi_{p_{sd}, q, m}^p(\mathcal{C}_h([0, \alpha n) \times [0, n))) \geq \phi_{p_{sd}, q, m}^p(B) \geq \left( \frac{c(1)^4}{2(1+q^2)} \right)^{\lfloor 2\alpha \rfloor}.$$

The claim follows with  $c(\alpha) := [c(1)^4/(2+2q^2)]^{\lfloor 2\alpha \rfloor}$ .  $\square$

**Proof of Proposition 4.8** The proof goes as follows: we start with creating two paths crossing square boxes, and we then prove that they are connected with good probability.

**Setting of the proof.** Consider the rectangle  $R = [0, 3n/2) \times [0, n)$  which is the union of the rectangles  $R_1 = [0, n) \times [0, n)$  and  $R_2 = [n/2, 3n/2) \times [0, n)$ , see Figure 4.3. Let  $A$  be the event defined by the following conditions:

- $R_1$  and  $R_2$  are both crossed horizontally (these events have probability at least  $c(1)$  to occur, using Lemma 4.5);



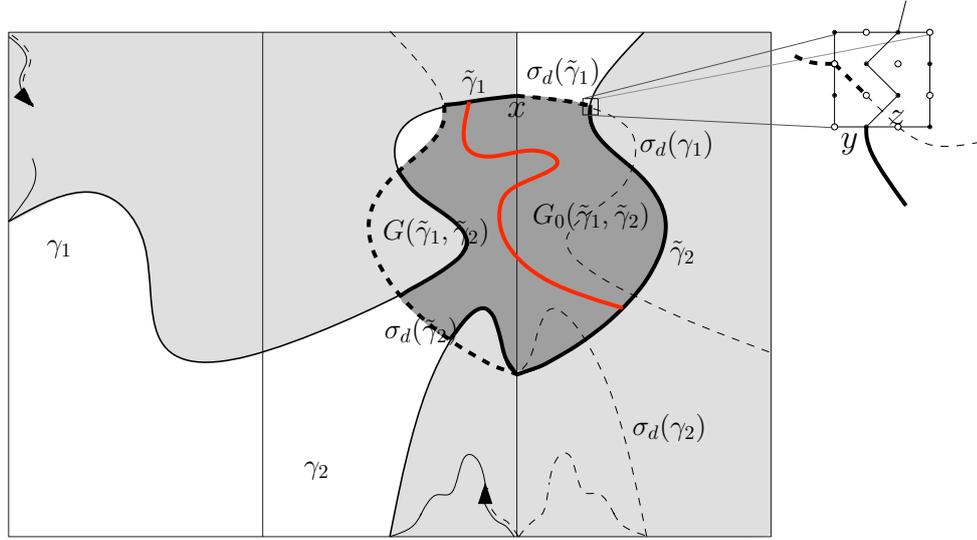


Figure 4.4: The light gray area is the part of  $R$  that is *a priori* discovered by the exploration processes (note that this area can be much smaller). The dark gray is the domain  $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$ . All the paths involved in the construction are depicted. Note that dashed curves are “virtual paths” of the dual lattice obtained by the reflection  $\sigma_d$ : they are not necessarily dual open.

- First, define the symmetric paths  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$  of  $\gamma_1$  and  $\gamma_2$  with respect to the line  $d := (n - \sqrt{2}/4) + i\mathbb{R}$ ;
- Then, parametrize the path  $\sigma_d(\gamma_1)$  by the distance (along the path) to its starting point  $\sigma_d(x)$  and define  $\tilde{\gamma}_1 \subset \sigma_d(\gamma_1)$  so that  $\sigma_d(\tilde{\gamma}_1)$  is the part of  $\sigma_d(\gamma_1)$  between the start of the path and the first time it intersects  $\gamma_2$ . As before, the paths are considered as curves of the plane; denote the intersection point of the two curves by  $z$ . Note that  $\gamma_1$  and  $\gamma_2$  are not intersecting, which forces  $\sigma_d(\gamma_1)$  and  $\gamma_2$  to be;
- From this, parametrize the path  $\gamma_2$  by the distance to its starting point  $(n, 0)$  and set  $y$  to be the last visited site in  $\mathbb{L}$  before the intersection  $z$ . Define  $\tilde{\gamma}_2$  to be the part of  $\gamma_2$  between the last point intersecting  $n + i\mathbb{R}$  before  $y$  and  $y$  itself;
- Paths  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  satisfy Hypothesis  $(\star)$  so that the graph  $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$  can be defined;
- Construct a sub-graph  $G_0(\gamma_1, \gamma_2)$  of  $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$  as follows: the edges are given by the edges of  $\mathbb{L}$  included in the connected component of  $G(\tilde{\gamma}_1, \tilde{\gamma}_2) \setminus (\gamma_1 \cup \gamma_2)$  (i.e.  $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$  minus the set  $\gamma_1 \cup \gamma_2$ ) containing  $d$  (it is the connected component which contains  $x - \varepsilon i$ , where  $\varepsilon$  is a very small number), and the sites are given by their endpoints.

The graph  $G_0(\gamma_1, \gamma_2)$  has a very useful property: none of its edges has been discovered by the previous exploration paths. Indeed,  $\sigma_d(\gamma_1)$  and  $\sigma_d(x)$  are included in the

unexplored connected component of  $R \setminus R_1$ , and so does  $G_0(\gamma_1, \gamma_2) \cap (R \setminus R_1)$ . Edges of  $G_0(\gamma_1, \gamma_2)$  in  $R_1$  are in the same connected component of  $R \setminus (\gamma_1 \cup \gamma_2)$  as  $x - \epsilon i$ , and thus lie ‘below’  $\gamma_1$ .

**Conditional probability estimate.** Still assuming that  $\gamma_1$  and  $\gamma_2$  do not intersect, we would like to estimate the probability of  $\gamma_1$  and  $\gamma_2$  being connected by a path knowing that  $\Gamma_1 = \gamma_1$  and  $\Gamma_2 = \gamma_2$ . Following the exploration procedure described above,  $\gamma_1$  and  $\gamma_2$  can be discovered without touching any edge in the interior of  $G_0(\gamma_1, \gamma_2)$ . Therefore, the process in the domain is a random-cluster model with specific boundary conditions.

The boundary of  $G_0(\gamma_1, \gamma_2)$  can be split into several sub-arcs of various types (see Figure 4.4): some are sub-arcs of  $\gamma_1$  or  $\gamma_2$ , while the others are (adjacent to) sub-arcs of their symmetric images  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$ . The conditioning on  $\Gamma_1 = \gamma_1$  and  $\Gamma_2 = \gamma_2$  ensures that the edges along the sub-arcs of the first type are open; the connections along the others depend on the exact explored configuration in a much more intricate way, but in any case the boundary conditions imposed on the configuration inside  $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$  are larger than the mixed boundary conditions. Notice that *any* boundary conditions dominate the free one and that  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  are two sub-arcs of the first type (they are then wired). Thus, the measure restricted to  $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$  stochastically dominates the restriction of  $\phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}$  to  $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$ .

From these observations, we deduce that for any increasing event  $B$  depending only on edges in  $G_0(\gamma_1, \gamma_2)$ ,

$$\phi_{p_{sd}, q, m}^p(B | \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) \geq \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(B). \quad (4.5)$$

In particular, this inequality can be applied to  $\{\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1, \gamma_2)\}$ . Note that if  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  are connected in  $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ ,  $\gamma_1$  and  $\gamma_2$  are connected in  $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$ . The first event is of  $\phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}$ -probability at least  $1/(1 + q^2)$ , implying

$$\begin{aligned} \phi_{p_{sd}, q, m}^p(\gamma_1 \leftrightarrow \gamma_2 | \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2) &\geq \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1, \gamma_2)) \\ &\geq \phi_{\tilde{\gamma}_1, \tilde{\gamma}_2}(\tilde{\gamma}_1 \leftrightarrow \tilde{\gamma}_2) \geq \frac{1}{1 + q^2}. \end{aligned} \quad (4.6)$$

**Conclusion of the proof.** Note the following obvious fact: if  $\gamma_1$  and  $\gamma_2$  intersect, the conditional probability that  $\Gamma_1$  and  $\Gamma_2$  intersect, knowing  $\Gamma_1 = \gamma_1$  and  $\Gamma_2 = \gamma_2$  is equal to 1 — in particular, it is greater than  $1/(1 + q^2)$ . Now,

$$\begin{aligned} \phi_{p_{sd}, q, m}^p(\mathcal{C}_h(R)) &\geq \phi_{p_{sd}, q, m}^p(\mathcal{C}_h(R) \cap A) \\ &\geq \phi_{p_{sd}, q, m}^p(\{\Gamma_1 \leftrightarrow \Gamma_2\} \cap A) \\ &= \phi_{p_{sd}, q, m}^p(\phi_{p_{sd}, q, m}^p(\Gamma_1 \leftrightarrow \Gamma_2 | \Gamma_1, \Gamma_2) \mathbb{1}_A) \\ &\geq \frac{1}{1 + q^2} \phi_{p, q, m}^p(A) \geq \frac{c(1)^3}{2(1 + q)^2} \end{aligned}$$

where the first two inequalities are due to inclusion of events, the third one to the definition of conditional expectation, and the fourth and fifth ones, to (4.6) and (4.4).  $\square$

An equivalent of Theorem 4.4 holds in the case of the infinite-volume random-cluster measure with wired boundary conditions.

**Corollary 4.9.** *Let  $\alpha > 1$  and  $q \geq 1$ ; there exists  $c(\alpha) > 0$  such that for every  $n \geq 1$ ,*

$$\phi_{p_{sd},q}^1[\mathcal{C}_h([0, \alpha n] \times [0, n])] \geq c(\alpha). \quad (4.7)$$

**Proof** Let  $\alpha > 1$  and  $m > 2\alpha n > 0$ . Using the invariance under translations of  $\phi_{p_{sd},q,m}^p$  and comparison between boundary conditions, we have

$$\phi_{p_{sd},q,[-\frac{m}{2}, \frac{m}{2}]^2}^1[\mathcal{C}_h([0, \alpha n] \times [0, n])] \geq \phi_{p_{sd},q,m}^p[\mathcal{C}_h([0, \alpha n] \times [0, n])] \geq c(\alpha).$$

When  $m$  goes to infinity, the left hand side converges to the probability in infinite volume, so that

$$\phi_{p_{sd},q}^1[\mathcal{C}_h([0, \alpha n] \times [0, n])] \geq c(\alpha).$$

□

**Remark 4.10.** *The only place where periodic and (bulk) wired boundary conditions are used is in the estimate of Lemma 4.5. For instance, if one could prove that the probability for a square box to be crossed from top to bottom with free boundary conditions stays bounded away from 0 when  $n$  goes to infinity, then an equivalent of Theorem 4.4 would follow with free boundary conditions.*

*Uniform estimates with respect to boundary conditions should be true for  $q \in [1, 4)$ ; the random-cluster model is expected to be conformally invariant in the scaling limit. It should be false for  $q \geq 4$ . Indeed, for  $q > 4$ , the phase transition is (conjecturally) of first order in the sense that there should not be uniqueness of the infinite-volume measure. At  $q = 4$ , the random-cluster model should be conformally invariant, but the probability of a crossing with free boundary conditions should converge to 0. Nevertheless, the probability that there is an open circuit surrounding the box of size  $n$  in the box of size  $2n$  with free boundary conditions should stay bounded away from 0.*

*Proving an equivalent of Theorem 4.4 with uniform estimates with respect to boundary conditions is an important question, since it would allow us to study the critical phase. The special case  $q = 2$  will be derived in Chapter 9.*

## 2 A sharp threshold theorem for crossing probabilities

The aim of this section is to understand the behavior of the function  $p \mapsto \phi_{p,q,n}^\xi(A)$  for a non-trivial increasing event  $A$ . This increasing function is equal to 0 at  $p = 0$  and to 1 at  $p = 1$ , and we are interested in the range of  $p$  for which its value is between  $\varepsilon$  and  $1 - \varepsilon$  for some positive  $\varepsilon$  (this range is usually referred to as a *window*). Under mild conditions on  $A$ , the window will be narrow for large graphs, and its width can be bounded above in terms of the size of the underlying graph, which is known as a *sharp threshold* behavior.

Historically, the general theory of sharp thresholds was first developed by Kahn, Kalai and Linial [KKL88] (see also [Fri04, FK96, KS06]) in the case of product measures. In lattice models such as percolation, these results are used via a differential equality known as Russo's formula, see [Gri99, Rus81]. Both sharp threshold theory and Russo's formula were later extended to random-cluster measures with  $q \geq 1$ , see references below. These arguments being not totally standard, we remind the readers of the classical results and refer them to [Gri06] for general results. Except for Theorem 4.13, the proofs are quite short so that it is natural to include them. The proofs are directly extracted from the Grimmett's monograph [Gri06].

In the whole section,  $G$  denotes a finite graph; if  $e$  is an edge of  $G$ , let  $J_e$  be the random variable equal to 1 if the edge  $e$  is open, and 0 otherwise. Let us start with an example of a differential inequality, which will be useful in the proof of Theorem 4.2.

**Proposition 4.11** (see [Gri06, GP97]). *Let  $q \geq 1$ ; for any random-cluster measure  $\phi_{p,q,G}^\xi$  with  $p \in (0, 1)$  and any increasing event  $A$ ,*

$$\frac{d}{dp} \phi_{p,q,G}^\xi(A) \geq 4 \phi_{p,q,G}^\xi(A) \phi_{p,q,G}^\xi(H_A),$$

where  $H_A(\omega)$  is the Hamming distance between  $\omega$  and  $A$ .

**Proof** Let  $A$  be an increasing event. The key step is the following inequality, see [BGK93, Gri06], which can be obtained by differentiating with respect to  $p$  (for details of the computation, see Theorem (2.46) of [Gri06]):

$$\frac{d}{dp} \phi_{p,q,G}^\xi(A) = \frac{1}{p(1-p)} \sum_{e \in E} [\phi_{p,q,G}^\xi(\mathbb{1}_A J_e) - \phi_{p,q,G}^\xi(J_e) \phi_{p,q,G}^\xi(A)]. \quad (4.8)$$

A similar differential formula is actually true for any random variable  $X$ , but this fact will not be used in the proof. Define  $|\eta|$  to be the number of open edges in the configuration, it is simply the sum of the random variables  $J_e$ ,  $e \in E$ . With this notation, one can rewrite (4.8) as

$$\begin{aligned} \frac{d}{dp} \phi_{p,q,G}^\xi(A) &= \frac{1}{p(1-p)} [\phi_{p,q,G}^\xi(|\eta| \mathbb{1}_A) - \phi_{p,q,G}^\xi(|\eta|) \phi_{p,q,G}^\xi(A)] \\ &= \frac{1}{p(1-p)} [\phi_{p,q,G}^\xi((|\eta| + H_A) \mathbb{1}_A) - \phi_{p,q,G}^\xi(|\eta| + H_A) \phi_{p,q,G}^\xi(A) \\ &\quad - \phi_{p,q,G}^\xi(H_A \mathbb{1}_A) + \phi_{p,q,G}^\xi(H_A) \phi_{p,q,G}^\xi(A)] \\ &\geq \frac{1}{p(1-p)} \phi_{p,q,G}^\xi(H_A) \phi_{p,q,G}^\xi(A). \end{aligned}$$

To obtain the second line, simply add and subtract the same quantity. In order to go from the second line to the third, remark two things: in the second line, the third term equals 0 (when  $A$  occurs, the Hamming distance to  $A$  is 0), and the sum of the first two

terms is positive thanks to the FKG inequality (indeed, it is easy to check that  $|\eta| + H_A$  is increasing). The claim follows since  $p(1-p) \leq 1/4$ .  $\square$

This proposition has an interesting reformulation: integrating the formula between  $p_1$  and  $p_2 > p_1$ , we obtain

$$\phi_{p_1, q, G}^\xi(A) \leq \phi_{p_2, q, G}^\xi(A) e^{-4(p_2 - p_1)\phi_{p_2, q, G}^\xi(H_A)} \quad (4.9)$$

(note that  $H_A$  is a decreasing random variable). If one can prove that the typical value of  $H_A$  is sufficiently large, for instance because  $A$  occurs with small probability, then one can obtain bounds for the probability of  $A$ . This kind of differential formula is very useful in order to prove the existence of a sharp threshold. The next example presents a sharper estimate of the derivative.

Intuitively, the derivative of  $\phi_{p, q, G}^\xi(A)$  with respect to  $p$  is governed by the influence of one single edge, switching from closed to open (roughly speaking, considering the increasing coupling between  $p$  and  $p + dp$ , it is unlikely that two edges switch their state). The following definition is therefore natural in this setting. The *(conditional) influence* on  $A$  of the edge  $e \in E$ , denoted by  $I_A(e)$ , is defined as

$$I_A(e) := \phi_{p, q, G}^\xi(A | J_e = 1) - \phi_{p, q, G}^\xi(A | J_e = 0).$$

**Proposition 4.12.** *Let  $q \geq 1$  and  $\varepsilon > 0$ ; there exists  $c = c(q, \varepsilon) > 0$  such that for any random-cluster measure  $\phi_{p, q, G}^\xi$  with  $p \in [\varepsilon, 1 - \varepsilon]$  and any increasing event  $A$ ,*

$$\frac{d}{dp} \phi_{p, q, G}^\xi(A) \geq c \sum_{e \in E} I_A(e).$$

**Proof** Note that, by definition of  $I_A(e)$ ,

$$\phi_{p, q, G}^\xi(\mathbb{1}_A J_e) - \phi_{p, q, G}^\xi(A) \phi_{p, q, G}^\xi(J_e) = I_A(e) \phi_{p, q, G}^\xi(J_e) (1 - \phi_{p, q, G}^\xi(J_e))$$

so that (4.8) becomes

$$\begin{aligned} \frac{d}{dp} \phi_{p, q, G}^\xi(A) &= \frac{1}{p(1-p)} \sum_{e \in E} \phi_{p, q, G}^\xi(J_e) (1 - \phi_{p, q, G}^\xi(J_e)) I_A(e) \\ &= \sum_{e \in E} \frac{\phi_{p, q, G}^\xi(J_e) (1 - \phi_{p, q, G}^\xi(J_e))}{p(1-p)} I_A(e) \end{aligned}$$

from which the claim follows since the term

$$\frac{\phi_{p, q, G}^\xi(J_e) (1 - \phi_{p, q, G}^\xi(J_e))}{p(1-p)}$$

is bounded away from 0 uniformly in  $p \in [\varepsilon, 1 - \varepsilon]$  and  $e \in E$  when  $q$  is fixed.  $\square$

There has been an extensive study of the largest influence in the case of product measures. It was initiated in [KKL88] and recently lead to important consequences in statistical models, see *e.g.* [BR06a, BR06b]. The following theorem is a special case of the generalization to positively-correlated measures.

**Theorem 4.13** (see [GG11]). *Let  $q \geq 1$  and  $\varepsilon > 0$ ; there exists a constant  $c = c(q, \varepsilon) \in (0, \infty)$  such that the following holds. Consider a random-cluster model on a graph  $G$  with  $|E|$  denoting the number of edges of  $G$ . For every  $p \in [\varepsilon, 1 - \varepsilon]$  and every increasing event  $A$ , there exists  $e \in E$  such that*

$$I_A(e) \geq c \phi_{p,q,G}^\xi(A) (1 - \phi_{p,q,G}^\xi(A)) \frac{\log |E|}{|E|}.$$

There is a particularly efficient way of using Proposition 4.12 together with Theorem 4.13. In the case of a translation-invariant event on a torus of size  $n$ , horizontal (resp. vertical) edges play a symmetric role, so that the influence is the same for all the edges of a given orientation. In particular, Proposition 4.12 together with Theorem 4.13 provide us with the following differential inequality:

**Theorem 4.14.** *Let  $q \geq 1$  and  $\varepsilon > 0$ . There exists a constant  $c = c(q, \varepsilon) \in (0, \infty)$  such that the following holds. Let  $n \geq 1$  and let  $A$  be a translation-invariant event on the torus of size  $n$ : for any  $p \in [\varepsilon, 1 - \varepsilon]$ ,*

$$\frac{d}{dp} \phi_{p,q,n}^p(A) \geq c (\phi_{p,q,n}^p(A) (1 - \phi_{p,q,n}^p(A))) \log n.$$

For a non-empty increasing event  $A$ , the previous inequality can be integrated between two parameters  $p_1 < p_2$  (we recognize the derivative of  $\log(x/(1-x))$ ) to obtain

$$\frac{1 - \phi_{p_1,q,n}^p(A)}{\phi_{p_1,q,n}^p(A)} \geq \frac{1 - \phi_{p_2,q,n}^p(A)}{\phi_{p_2,q,n}^p(A)} n^{c(p_2-p_1)}.$$

If  $\phi_{p_1,q,n}^\xi(A)$  is assume to stay bounded away from 0 uniformly in  $n \geq 1$ , there exists  $c' > 0$  such that

$$\phi_{p_2,q,n}^p(A) \geq 1 - c' n^{-c(p_2-p_1)}. \tag{4.10}$$

This inequality will be instrumental in the next section.

### 3 The proofs of Theorems 4.1 and 4.2

The previous two sections combine in order to provide estimates on crossing probabilities (see [BR06a, BR06b] for applications in the case of percolation). Indeed, one can consider the event that *some* long rectangle is crossed in a torus. At  $p = p_{sd}$ , the probability of this event is known to be bounded away from 0 uniformly in the size of the torus (thanks to Theorem 4.4). Therefore, Theorem 4.14 can be applied to conclude that the probability

goes to 1 when  $p > p_{sd}$  (there is also an explicit estimate on the probability). It is then an easy step to deduce a lower bound for the probability of crossing a *particular* rectangle.

Theorem 4.1 is proved by constructing a path from 0 to infinity when  $p > p_{sd}$ , which is usually done by combining crossings of rectangles. There is a major difficulty in doing such a construction: one needs to transform estimates in the torus into estimates in the whole plane. One solution is to replace the periodic boundary conditions by wired boundary conditions. The path construction is a little tricky since it must propagate wired boundary conditions through the construction (see Proposition 4.17); it does not follow the standard lines.

Theorem 4.2 follows from a refinement of the previous construction in order to estimate the Hamming distance of a typical configuration to the event  $\{0 \leftrightarrow \mathbb{L} \setminus [-n, n]^2\}$ . It allows the use of Proposition 4.11, which improves bounds on the probability that the origin is connected to distance  $n$ . With these estimates, the cluster size at the origin can be shown to have finite moments of any order, whenever  $p < p_{sd}$ . Then, it is a standard step to obtain exponential decay in the subcritical phase.

The following two lemmas will be useful in the proofs of both theorems. We start with proving that crossings of long rectangles exist with very high probability when  $p > p_{sd}$ .

**Lemma 4.15.** *Let  $\alpha > 1$ ,  $q \geq 1$  and  $p > p_{sd}$ ; there exists  $\varepsilon_0 = \varepsilon_0(p, q, \alpha) > 0$  and  $c_0 = c_0(p, q, \alpha) > 0$  such that*

$$\phi_{p,q,\alpha^2 n}^{\mathbb{P}}(\mathcal{C}_v([0, n] \times [0, \alpha n])) \geq 1 - c_0 n^{-\varepsilon_0} \quad (4.11)$$

for every  $n \geq 1$ .

**Proof** The proof will make it clear that it is sufficient to treat the case of integer  $\alpha$ , we therefore assume that  $\alpha$  is a positive integer (not equal to 1). Let  $B$  be the event that there exists a vertical crossing of a rectangle with dimensions  $(n/2, \alpha^2 n)$  in the torus of size  $\alpha^2 n$ . This event is invariant under translations and satisfies

$$\phi_{p_{sd},q,\alpha^2 n}^{\mathbb{P}}(B) \geq \phi_{p_{sd},q,\alpha^2 n}^{\mathbb{P}}(\mathcal{C}_v([0, n/2] \times [0, \alpha^2 n])) \geq c(2\alpha^2)$$

uniformly in  $n$ .

Let  $p > p_{sd}$ . Since  $B$  is increasing, Theorem 4.14 (more precisely (4.10)) can be applied to deduce that there exist  $\varepsilon = \varepsilon(p, q, \alpha)$  and  $c = c(p, q, \alpha)$  such that

$$\phi_{p,q,\alpha^2 n}^{\mathbb{P}}(B) \geq 1 - cn^{-\varepsilon}. \quad (4.12)$$

If  $B$  holds, one of the  $2\alpha^3$  rectangles

$$[in/2, in/2 + n] \times [j\alpha n, (j+1)\alpha n], \quad (i, j) \in \{0, \dots, 2\alpha^2 - 1\} \times \{0, \dots, \alpha - 1\}$$

must be crossed from top to bottom. Denote these events by  $A_{ij}$  — they are translates of  $\mathcal{C}_v([0, n] \times [0, \alpha n])$ . Using the FKG inequality in the second line (this is another instance

of the “square-root trick” mentioned earlier), we find

$$\begin{aligned} \phi_{p,q,\alpha^{2n}}^p(B) &\leq 1 - \phi_{p,q,\alpha^{2n}}^p(B^c) \leq 1 - \phi_{p,q,\alpha^{2n}}^p(\cap_{i,j} A_{ij}^c) \\ &\leq 1 - \prod_{i,j} \phi_{p,q,\alpha^{2n}}^p(A_{ij}^c) = 1 - \left[1 - \phi_{p,q,\alpha^{2n}}^p(\mathcal{C}_v([0, n] \times [0, \alpha n]))\right]^{2\alpha^3}. \end{aligned}$$

Plugging (4.12) into the previous inequality, we deduce

$$\phi_{p,q,\alpha^{2n}}^p(\mathcal{C}_v([0, n] \times [0, \alpha n])) \geq 1 - (cn^{-\varepsilon})^{1/(2\alpha^3)}.$$

The claim follows by setting  $c_0 := c^{1/(2\alpha)^3}$  and  $\varepsilon_0 := \varepsilon/(2\alpha^3)$ .  $\square$

Let  $\alpha > 1$  and  $n \geq 1$ ; define the annulus

$$A_n^\alpha := [-\alpha^{n+1}, \alpha^{n+1}]^2 \setminus [-\alpha^n, \alpha^n]^2.$$

An *open circuit* in an annulus is an open path which surrounds the origin. Denote by  $\mathcal{A}_n^\alpha$  the event that there exists an open circuit surrounding the origin and contained in  $A_n^\alpha$ , together with an open path from this circuit to the boundary of  $[-\alpha^{n+2}, \alpha^{n+2}]^2$ , see Figure 4.5. The following lemma shows that the probability of  $\mathcal{A}_n^\alpha$  goes to 1, provided that  $p > p_{sd}$  and that boundary conditions are wired on  $[-\alpha^{n+2}, \alpha^{n+2}]^2$ .

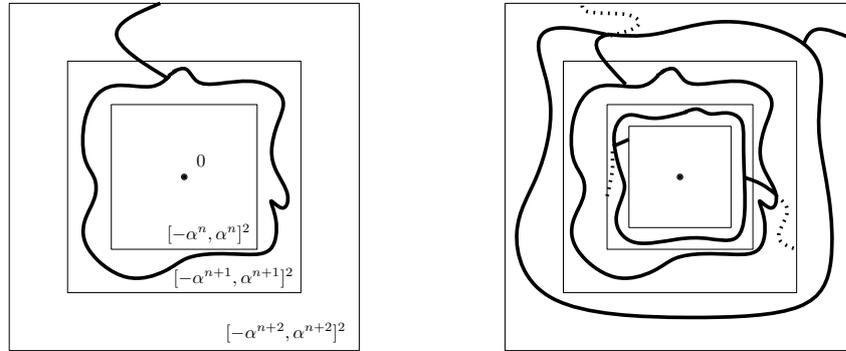


Figure 4.5: **Left:** The event  $\mathcal{A}_n^\alpha$ . **Right:** The combination of events  $\mathcal{A}_n^\alpha$ : it indeed constructs a path from the origin to infinity.

**Lemma 4.16.** *Let  $\delta > 1$ ,  $q \geq 1$  and  $p > p_{sd}$ ; there exists  $c_1 = c_1(p, q, \delta)$  and  $\varepsilon_1 = \varepsilon_1(p, q, \delta)$  such that for every  $n \geq 1$ ,*

$$\phi_{p,q,\delta^{n+2}}^1(\mathcal{A}_n^\delta) \geq 1 - c_1 e^{-\varepsilon_1 n}.$$

**Proof** First, observe that  $\mathcal{A}_n^\delta$  occurs whenever the following events occur simultaneously:

- The following rectangles are crossed vertically:

$$\begin{aligned} R_1 &:= [\delta^n, \delta^{n+1}] \times [-\delta^{n+1}, \delta^{n+1}], \\ R_2 &:= [-\delta^{n+1}, -\delta^n] \times [-\delta^{n+1}, \delta^{n+1}]; \end{aligned}$$

- The following rectangles are crossed horizontally:

$$\begin{aligned} R_3 &:= [-\delta^{n+1}, \delta^{n+1}] \times [\delta^n, \delta^{n+1}], \\ R_4 &:= [-\delta^{n+1}, \delta^{n+1}] \times [-\delta^{n+1}, -\delta^n], \\ R_5 &:= [-\delta^{n+2}, \delta^{n+2}] \times [-\delta^n, \delta^n]. \end{aligned}$$

For the measure in the torus, these events have probability greater than  $1 - c(\delta^n)^{-\varepsilon}$  with  $c = c_0(p, q, 2\delta/(\delta - 1))$  and  $\varepsilon = \varepsilon_0(p, q, 2\delta/(\delta - 1))$ . Using the FKG inequality, we obtain

$$\phi_{p,q,\alpha^{n+2}}^P(\mathcal{A}_n^\delta) \geq (1 - c(\delta^n)^{-\varepsilon})^5$$

from which the following estimate can be deduced, harnessing the comparison between boundary conditions,

$$\phi_{p,q,\delta^{n+2}}^1(\mathcal{A}_n^\delta) \geq (1 - c(\delta^n)^{-\varepsilon})^5.$$

The claim follows by setting  $c_1 := 5c$  and  $\varepsilon_1 := \varepsilon \log \delta$ . □

The following proposition readily implies Theorem 4.1; It will also be useful in the proof of Theorem 4.2. We wish to prove that the probability of the intersection of events  $\mathcal{A}_n^\delta$  is of positive probability when  $p > p_{sd}$ . So far, we know that there is an open circuit with very high probability when we consider the random-cluster measure with wired boundary conditions in a slightly larger box. In order to prove the result, assume the existence of a large circuit. Then, we iteratively condition on events  $\mathcal{A}_{n-k}^\delta$ ,  $k \geq 0$ . When conditioning ‘from the outside to the inside’, there exists an open circuit in  $A_{n-k+1}^\delta$  that surrounds  $A_{n-k}^\delta$  at every step  $k$ . Using comparison between boundary conditions, the measure in  $A_{n-k}^\delta$  stochastically dominates the measure in  $A_{n-k+1}^\delta$  with wired boundary conditions. In other words, we keep track of advantageous boundary conditions. Note that the reasoning, while reminiscent of Kesten’s construction of an infinite path for percolation, is not standard.

**Proposition 4.17.** *Let  $\delta > 1$ ,  $q \geq 1$  and  $p > p_{sd}$ ; there exist  $c, c_1, \varepsilon_1 > 0$  (depending on  $p$ ,  $q$  and  $\delta$ ) such that for every  $N \geq 1$ ,*

$$\phi_{p,q} \left( \bigcap_{n \geq N} \mathcal{A}_n^\delta \right) \geq c \prod_{n=N}^{\infty} (1 - c_1 e^{-\varepsilon_1 n}) > 0.$$

**Proof** Let  $\delta > 1$ ,  $q \geq 1$ ,  $p > p_{sd}$ ,  $N \geq 1$  and recall that there is a unique infinite-volume measure  $\phi_{p,q}$ . For every  $n \geq 1$ , we know that

$$\phi_{p,q} \left( \bigcap_{k=N}^n \mathcal{A}_k^\delta \right) = \phi_{p,q}(\mathcal{A}_n^\delta) \prod_{k=N}^{n-1} \phi_{p,q}(\mathcal{A}_k^\delta | \mathcal{A}_j^\delta, k+1 \leq j \leq n). \quad (4.13)$$

On the one hand, let  $k \in [N, n-1]$ . Conditionally on  $\mathcal{A}_j^\delta$ ,  $k+1 \leq j \leq n$ , there exists a circuit in the annulus  $A_{k+1}^\delta$ . Consider the exterior-most such circuit, denoted by  $\Gamma$ , by exploring from the outside. Conditionally on  $\Gamma = \gamma$ , the unexplored part of the box  $[-\delta^{k+2}, \delta^{k+2}]^2$  follows the law of a random-cluster configuration with wired boundary condition. In particular, the conditional probability that there exists a circuit in  $A_k^\delta$  connected to  $\gamma$  is greater than the probability that there exists a circuit in  $A_k^\delta$  connected to the boundary of  $[-\delta^{k+2}, \delta^{k+2}]^2$  with wired boundary conditions. Therefore, we obtain that almost surely

$$\begin{aligned} \phi_{p,q}(\mathcal{A}_k^\delta | \mathcal{A}_j^\delta, k+1 \leq j \leq n) &= \phi_{p,q}(\phi_{p,q}(\mathcal{A}_k^\delta | \Gamma = \gamma)) \\ &\geq \phi_{p,q}(\phi_{p,q,\delta^{k+2}}^1(\mathcal{A}_k^\delta)) \\ &\geq 1 - c_1 e^{-\varepsilon_1 k} \end{aligned}$$

where Lemma 4.16 was harnessed in the last inequality.

On the other hand, for  $p = p_{sd}$ , consider the event  $\mathcal{A}_n^\delta$  in the bulk. Thanks to Corollary 4.9, its probability is bounded away from 0 uniformly in  $n$ . Since the event is increasing, there exists  $c = c(\delta) > 0$  such that

$$\phi_{p,q}(\mathcal{A}_n^\delta) = \phi_{p,q}^1(\mathcal{A}_n^\delta) \geq c$$

for any  $n \geq N$  and  $p > p_{sd}$ . Plugging the two estimates into (4.13), we obtain

$$\phi_{p,q} \left( \bigcap_{k=N}^n \mathcal{A}_k^\delta \right) \geq c \prod_{k=N}^{n-1} (1 - c_1 e^{-\varepsilon_1 k}) \geq c \prod_{k=N}^{\infty} (1 - c_1 e^{-\varepsilon_1 k}).$$

Letting  $n$  go to infinity concludes the proof.  $\square$

**Proof of Theorem 4.1** The bound  $p_c \geq p_{sd}$  is provided by Proposition 3.18. For  $p > p_{sd}$ , fix  $\delta > 1$ . Applying Proposition 4.17 with  $N = 1$ , we find

$$\phi_{p,q}(0 \leftrightarrow \infty) \geq c \phi_{p,q} \left( \bigcap_{n \geq 1} \mathcal{A}_n^\delta \right) > 0$$

so that  $p$  is supercritical. The constant  $c > 0$  is due to the fact that  $[-\delta^2, \delta^2]^2$  is required to contain open edges only ( $c > 0$  exists using the finite energy property). Since  $p$  is supercritical for every  $p > p_{sd}$ , we deduce  $p_c \leq p_{sd}$ .  $\square$

**Proof of Theorem 4.2** Let  $x$  be a site of  $\mathbb{Z}^2$ , and let  $\mathcal{C}_x$  be the cluster of  $x$ , *i.e.* the maximal connected component containing the site  $x$ . Its cardinality is denoted by  $|\mathcal{C}_x|$ . We first prove that  $|\mathcal{C}_x|$  has finite moments of any order. Then we deduce that the probability of  $\{|\mathcal{C}_x| \geq n\}$  decays exponentially fast in  $n$ . The proof of the Step 2 is extracted from [Gri06].

**Step 1: finite moments for  $|\mathcal{C}_x|$ .** Let  $d > 0$  and  $p < p_{sd}$ ; we wish to prove that

$$\phi_{p,q}(|\mathcal{C}_x|^d) < \infty. \quad (4.14)$$

In order to do so, let  $p_1 := (p + p_{sd})/2$  and define  $D_n := \{x \leftrightarrow \mathbb{Z}^2 \setminus (x + [-n, n]^2)\}$ ; denote by  $H_n$  the Hamming distance to  $D_n$ . Note that  $H_n$  is the minimal number of closed edges that one must cross in order to go from  $x$  to the boundary of the box of size  $n$  centered at  $x$ . Let

$$\alpha := \exp\left[\frac{p_1 - p}{2d + 3}\right] > 1.$$

From Proposition 4.17 applied to the (supercritical) dual model, the probability of  $\bigcap_{n > N} (\mathcal{A}_n^\alpha)^*$  is larger than  $c \prod_N^\infty (1 - c_1 e^{-\varepsilon_1 n}) > 0$  ( $(\mathcal{A}_n^\alpha)^*$  is the occurrence of  $\mathcal{A}_n^\alpha$  in the dual model). Hence, there exists  $N = N(p_1, q, \alpha)$  sufficiently large such that

$$\phi_{p_1,q}\left(\bigcap_{n \geq N}^\infty (\mathcal{A}_n^\alpha)^*\right) \geq \frac{1}{2}.$$

On this event,  $H_n$  is greater than  $(\log n / \log \alpha) - N$  since there is at least one closed circuit in each annulus  $A_k^\alpha$  with  $k \geq N$  (thus increasing the Hamming distance by 1). We obtain

$$\phi_{p_1,q}(H_n) \geq \left(\frac{\log n}{\log \alpha} - N\right) \phi_{p_1,q}\left(\bigcap_{n \geq N}^\infty (\mathcal{A}_n^\alpha)^*\right) \geq \frac{\log n}{4 \log \alpha}$$

for  $n$  sufficiently large. Then, (4.9) implies

$$\phi_{p,q}(D_n) \leq \phi_{p_1,q}(D_n) \exp\left[-4(p_1 - p)\phi_{p_1,q}(H_n)\right] \leq n^{-(2d+3)} \quad (4.15)$$

for  $n$  sufficiently large, from which (4.14) follows readily.

**Step 2: exponential decay.** Note that, from the first inequality of (4.15), it is sufficient to prove that for some constant  $c > 0$ ,

$$\liminf_{n \rightarrow \infty} H_n/n \geq c \quad \text{a.s.}$$

in order to show that  $\phi_{p,q}(D_n)$  decays exponentially fast.

Consider a (not necessarily open) self-avoiding path  $\gamma$  going from the origin to the boundary of the box of size  $n$ . The number  $T(\gamma)$  of closed edges along this path can be bounded from below by the following quantity:

$$\frac{T(\gamma)}{n} \geq \frac{1}{|\gamma|} T(\gamma) \geq \frac{1}{|\gamma|} \sum_{z \in \gamma} \frac{1}{|\mathcal{C}_z|} \geq \left(\frac{1}{|\gamma|} \sum_{z \in \gamma} |\mathcal{C}_z|\right)^{-1}.$$

Indeed, the number of closed edges in  $\gamma$  is larger than the number of distinct clusters intersecting  $\gamma$ . Moreover, if  $\mathcal{C}$  denotes such a cluster, we have that  $1 \geq \sum_{z \in \gamma} |\mathcal{C}|^{-1} \mathbb{1}_{z \in \mathcal{C}}$ . The last inequality is due to Jensen's inequality. Since  $H_n$  can be rewritten as the infimum of  $T(\gamma)$  on paths going from 0 to the boundary of the box, we obtain

$$\frac{H_n}{n} \geq \inf_{\gamma: 0 \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{B}_n} \left( \frac{1}{|\gamma|} \sum_{z \in \gamma} |\mathcal{C}_z| \right)^{-1}. \quad (4.16)$$

The goal of the end of the proof is to give an almost sure lower bound of the right-hand side. We will harness a two-dimensional analogue of the strong law of large number. In order to do that, the random variables  $|\mathcal{C}_z|$  need to be transformed to obtain independent variables. We start with the following domination.

Let  $(\tilde{\mathcal{C}}_z)_{z \in \mathcal{B}_n}$  be a family of independent subsets of  $\mathbb{Z}^2$  distributed as  $\mathcal{C}_z$ . We claim that  $(|\mathcal{C}_z|)_{z \in \mathcal{B}_n}$  is stochastically dominated by the family  $(M_z)_{z \in \mathcal{B}_n}$  defined as

$$M_z := \sup_{y \in \mathbb{Z}^2: z \in \tilde{\mathcal{C}}_y} |\tilde{\mathcal{C}}_y|.$$

Let  $v_1, v_2, \dots$  be a deterministic ordering of  $\mathbb{Z}^2$ . Given the random family  $(\tilde{\mathcal{C}}_z)_{z \in \mathcal{B}_n}$ , we shall construct a family  $(D_z)_{z \in \mathcal{B}_n}$  having the same joint law as  $(\mathcal{C}_z)_{z \in \mathcal{B}_n}$  and satisfying the following condition: for each  $z$ , there exists  $y$  such that  $D_z \subset \tilde{\mathcal{C}}_y$ . First, set  $D_{v_1} = \tilde{\mathcal{C}}_{v_1}$ . Given  $D_{v_1}, D_{v_2}, \dots, D_{v_n}$ , define  $E = \bigcup_{i=1}^n D_{v_i}$ . If  $v_{n+1} \in E$ , set  $D_{v_{n+1}} = D_{v_j}$  for some  $j$  such that  $v_{n+1} \in D_{v_j}$ . If  $v_{n+1} \notin E$ , proceed as follows. Let  $\Delta_e E$  be the set of edges of  $\mathbb{Z}^2$  having exactly one end-vertex in  $E$ . A (random) subset  $F$  of  $\tilde{\mathcal{C}}_{v_{n+1}}$  may be found in such a way that  $F$  has the conditional law of  $\mathcal{C}_{v_{n+1}}$  given that all edges in  $\Delta_e E$  are closed; now set  $D_{v_{n+1}} = F$ . The domain Markov property and the positive association can be used to show that the law of  $\mathcal{C}_{v_{n+1}}$  depends only on  $\Delta_e E$ , and is stochastically dominated by the law of the cluster in the bulk without any conditioning. The required stochastic domination follows accordingly. In particular,  $|\mathcal{C}_z| \leq M_z$  and  $M_z$  has finite moments.

From (4.16) and the previous stochastic domination, we get

$$\liminf_{n \rightarrow \infty} \frac{H_n}{n} \geq \liminf_{n \rightarrow \infty} \inf_{\gamma: 0 \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{B}_n} \left( \frac{1}{|\gamma|} \sum_{z \in \gamma} |\mathcal{C}_z| \right)^{-1} \geq \left( \limsup_{n \rightarrow \infty} \sup_{\gamma: 0 \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{B}_n} \frac{1}{|\gamma|} \sum_{z \in \gamma} M_z \right)^{-1}.$$

The second step is now to replace  $M_z$  by random variables that are independent. Lemma 2 of [FN93] can be harnessed to show that

$$\left( \limsup_{n \rightarrow \infty} \sup_{\gamma: 0 \leftrightarrow \mathbb{Z}^2 \setminus \mathcal{B}_n} \frac{1}{|\gamma|} \sum_{z \in \gamma} M_z \right)^{-1} \geq \left( 2 \limsup_{n \rightarrow \infty} \sup_{|\Gamma| \geq n} \frac{1}{|\Gamma|} \sum_{z \in \Gamma} |\tilde{\mathcal{C}}_z|^2 \right)^{-1}$$

where the supremum is over all finite connected graphs  $\Gamma$  of cardinality larger than  $n$  that contain the origin (also called lattice animals).

Since the  $|\tilde{\mathcal{C}}_z|^2$  are independent and have finite moments of any order, the main result of [CGGK93, GK94] guarantees that

$$2 \limsup_{n \rightarrow \infty} \sup_{|\Gamma| \geq n} \frac{1}{|\Gamma|} \sum_{z \in \gamma} |\tilde{\mathcal{C}}_z|^2 \leq C \quad a.s.$$

for some  $C > 0$ . Therefore, with positive probability,  $\liminf H_n/n$  is greater than a given constant, which concludes the proof.  $\square$

## 4 The critical point for the triangular and hexagonal lattices

Let  $\mathbb{T}$  be the triangular lattice of mesh size 1, embedded in the plane in such a way that the origin is a vertex and the edges of  $\mathbb{T}$  are parallel to the lines of equations  $y = 0$ ,  $y = \sqrt{3}x/2$  and  $y = -\sqrt{3}x/2$ . The dual graph of this lattice is a hexagonal lattice, denoted by  $\mathbb{H}$ , see Figure 4.6. Via planar duality, it is sufficient to handle the case of the triangular lattice in order to prove Theorem 4.3. Define  $p_{\mathbb{T}}$  as being the unique  $p \in (0, 1)$  such that  $y^3 + 3y^2 - q = 0$ , where  $y := p_{\mathbb{T}}/(1 - p_{\mathbb{T}})$ . The goal is to prove that  $p_c(\mathbb{T}) = p_{\mathbb{T}}$ .

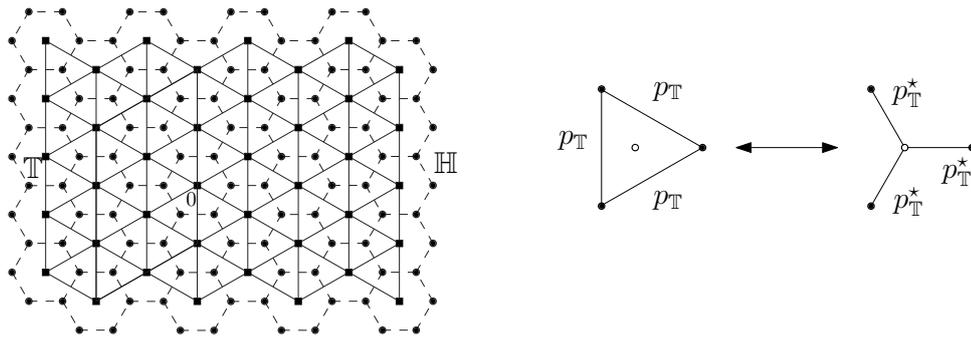


Figure 4.6: **Left:** The triangular lattice  $\mathbb{T}$  with its dual lattice  $\mathbb{H}$ . **Right:** The exchange of the two patterns does not alter the random-cluster connective properties of the black vertices.

The general strategy is the same as in the square lattice case. We prove that at  $p = p_{\mathbb{T}}$ , a crossing estimate similar to Theorem 4.4 holds. Sharp threshold arguments and proofs of Section 3 can be adapted *mutatis mutandis*, replacing square-shaped annuli by hexagonal-shaped annuli. The crossing estimate must be slightly modified, and we present the few changes. It harnesses the planar-duality between the triangular and the hexagonal lattices, and the so-called *star-triangle transformation* (see *e.g.* Section 6.6 of [Gri06] and Figure 4.6). The reader is assumed to be already familiar with the star-triangle transformation.

Let  $e_1 = \sqrt{3}/2 + i/2$  and  $e_2 = i$ ; whenever coordinates are written, they are understood as referring to the basis  $(e_1, e_2)$ . A ‘rectangle’  $[a, b] \times [c, d]$  is the set of points in  $z \in \mathbb{T}$  such

that  $z = \lambda e_1 + \mu e_2$  with  $\lambda \in [a, b)$  and  $\mu \in [c, d)$  (it has a lozenge shape, see *e.g.* Figure 4.8). By analogy with the case of the square lattice,  $\mathcal{C}_v(D)$  denotes the event that there exists a path between the top and the bottom sides of  $D$  which stays inside  $D$ . Such a path is called a *vertical open crossing* of the rectangle. Other quantities are defined similarly. Let  $\mathbb{T}_m$  be the torus of size  $m$  constructed using the “rectangle”  $[0, m] \times [0, m]$  with respect to the basis  $(e_1, e_2)$ . The crossing estimate is presented in the case of the torus  $\mathbb{T}_m$  (deriving the bulk estimate follows the same lines as in the square lattice case);  $\phi_{p_{sd}, q, m}^p$  denotes the random-cluster measure on  $\mathbb{T}_m$ .

**Theorem 4.18.** *Let  $\alpha > 1$  and  $q \geq 1$ . There exists  $c(\alpha) > 0$  such that for every  $m > \alpha n > 0$ ,*

$$\phi_{p_{\mathbb{T}, q, m}^p}(\mathcal{C}_h([0, n] \times [0, \alpha n])) \geq c(\alpha). \tag{4.17}$$

The main difficulty is the adaptation of Lemma 4.7. Define the line  $d := -\sqrt{3}/3 + i\mathbb{R}$ . The orthogonal symmetry  $\sigma_d$  with respect to  $d$  maps  $\mathbb{T}$  to another triangular lattice. Note that this lattice is a sub-lattice of  $\mathbb{H}$  (in the sense that its vertices are also vertices of  $\mathbb{H}$ ). Let  $\gamma_1$  and  $\gamma_2$  be two paths satisfying the following Hypothesis  $(\star)$ , see Figure 4.7:

- $\gamma_1$  remains on the left of  $d$  and  $\gamma_2$  remains on the right,
- $\gamma_2$  begins at 0 and  $\gamma_1$  begins on a site of  $\mathbb{T} \cap (-\sqrt{3}/2 + i\mathbb{R}_+)$ ,
- $\gamma_1$  and  $\sigma_d(\gamma_2)$  do not intersect (as curves in the plane),
- $\gamma_1$  and  $\sigma_d(\gamma_2)$  end at two sites (one primal and one dual) which are at distance  $\sqrt{3}/3$  from one another.

When following the paths in counter-clockwise order, a circuit can be created by linking the end points of  $\gamma_1$  and  $\sigma_d(\gamma_2)$  by a straight line, the start points of  $\sigma_d(\gamma_2)$  and  $\gamma_2$ , the end points of  $\gamma_2$  and  $\sigma_d(\gamma_1)$ , and the start points of  $\sigma_d(\gamma_1)$  and  $\gamma_1$ . The circuit  $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$  surrounds a set of vertices of  $\mathbb{T}$ . Define the graph  $G(\gamma_1, \gamma_2)$  with sites being site of  $\mathbb{T}$  that are surrounded by the circuit  $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ , and with edges of  $\mathbb{T}$  that remain entirely inside the circuit (boundary included).

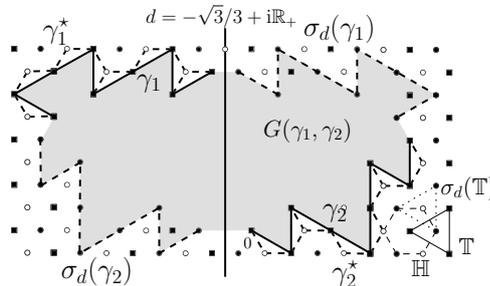


Figure 4.7: The graph  $G(\gamma_1, \gamma_2)$  with the two solid arcs  $\gamma_1$  and  $\gamma_2$  and the dashed arcs  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$ . The dual arcs  $\gamma_1^*$  and  $\gamma_2^*$  are dotted.

An additional technical condition will be needed, which we present now. Note that for any edge of  $\sigma_d(\mathbb{T})$  there is one vertex of  $\mathbb{T}$  and one vertex of  $\mathbb{H}$  at distance  $\sqrt{3}/6$  from its midpoint. For any edge of  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$ , the associated vertex of  $\mathbb{T}$  is assumed to be in the interior of the domain  $G(\gamma_1, \gamma_2)$  (therefore, the associated vertex of  $\mathbb{H}$  is outside the domain, see white vertices in Fig 4.7). This condition will be referred to as Hypothesis  $(\star\star)$ .

The *mixed boundary conditions* on this graph are wired on  $\gamma_1$  (all the edges are pairwise connected), wired on  $\gamma_2$ , and free elsewhere. The measure on  $G(\gamma_1, \gamma_2)$  with parameters  $(p_{\mathbb{T}}, q)$  and mixed boundary conditions is denoted by  $\phi_{p_{\mathbb{T}}, q, \gamma_1, \gamma_2}$  or more simply  $\phi_{\gamma_1, \gamma_2}$ . With these definitions, we find an equivalent of Lemma 4.7:

**Lemma 4.19.** *For any  $\gamma_1, \gamma_2$  satisfying Hypotheses  $(\star)$  and  $(\star\star)$ , we have*

$$\phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geq \frac{1}{1 + q^2}.$$

**Proof** As previously, if  $\gamma_1$  and  $\gamma_2$  are not connected,  $\gamma_1^*$  and  $\gamma_2^*$  are connected in the dual model, where  $\gamma_1^*, \gamma_2^* \subset \mathbb{H}$  are the dual arcs bordering  $G(\gamma_1, \gamma_2)$  close to  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$ . Thanks to Hypothesis  $(\star\star)$  and the mixed boundary conditions, this event is equivalent to the event that  $\sigma_d(\gamma_1)$  and  $\sigma_d(\gamma_2)$  are dual connected. Using Hypothesis  $(\star)$  and the symmetry, we deduce

$$\phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + \sigma_d * \phi_{\gamma_1, \gamma_2}^*(\gamma_1 \leftrightarrow \gamma_2) = 1,$$

where as before  $\sigma_d * \phi_{\gamma_1, \gamma_2}^*$  denotes the push-forward under the symmetry  $\sigma_d$  of the dual measure of  $\phi_{\gamma_1, \gamma_2}$  — in particular, it lies on  $\sigma_d(\mathbb{H})$  and the edge-weight is  $p_{\mathbb{T}}^*$ . This lattice contains the sites of  $\mathbb{T}$  and those of another copy of the triangular lattice which is denoted by  $\mathbb{T}'$ . Since  $\gamma_1$  and  $\gamma_2$  are two paths of  $\mathbb{T}$ , one can use the star-triangle transformation for any triangle of  $\mathbb{T}$  included in  $G(\gamma_1, \gamma_2)$  that contains a vertex of  $\mathbb{T}'$ : one obtains that  $\sigma_d * \phi_{\gamma_1, \gamma_2}^*(\gamma_1 \leftrightarrow \gamma_2)$  is equal to the probability of  $\gamma_1$  and  $\gamma_2$  being connected, in a model on  $\mathbb{T}$  with edge-weight  $p_{\mathbb{T}}$ . Here, Hypothesis  $(\star\star)$  is needed again in order to ensure that all the triangles containing a vertex of  $\mathbb{T}'$  have no edges on the boundary (which would have forbidden the use of the star-triangle transformation). The same observation as in the case of the square lattice shows that the boundary conditions are the same as for  $\phi_{\gamma_1, \gamma_2}$ , except that arcs  $\gamma_1$  and  $\gamma_2$  are wired together. The same reasoning as in Lemma 4.7 implies that

$$\sigma_d * \phi_{\gamma_1, \gamma_2}^*(\gamma_1 \leftrightarrow \gamma_2) \leq q^2 \phi_{\gamma_1, \gamma_2}(\gamma_1 \leftrightarrow \gamma_2),$$

and the claim follows readily. □

The existence of  $c(1)$  is obtained in the same way as in the case of the square lattice, with only the obvious modifications needed; the details are left as an “exercise for the reader”. Theorem 4.18 is derived exactly as in Section 1, as soon as an equivalent of Proposition 4.8 holds:

**Proposition 4.20.** *There exists a constant  $c(3/2) > 0$  such that, for all  $m > 3n/2 > 0$ ,*

$$\phi_{p_{\mathbb{T}}, q, m}^p(\mathcal{C}_v([0, 3n/2] \times [0, n])) \geq c(3/2).$$

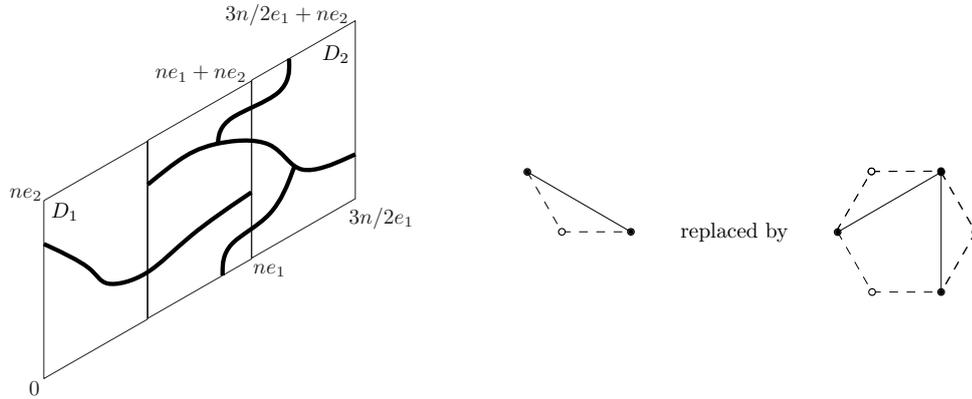


Figure 4.8: **Left:** The set  $[0, 3n/2) \times [0, n)$  and the event  $A$ . **Right:** One can obtain the path  $\Gamma'_1$  from  $\Gamma_1$  by replacing any bad edge with two edges. Since  $\Gamma_1$  is the top-most crossing, it contains no double edges and this construction can be done.

**Proof** The general framework of the proof is the same as before, but some technicalities occur because the underlying lattice is not self-dual. Consider the rectangle  $D = [0, 3n/2) \times [0, n)$ , which is the union of rectangles  $D_1 = [0, n) \times [0, n)$  and  $D_2 = [n/2, 3n/2) \times [0, n)$ , see Figure 4.8. Let  $A$  be the event that:

- $D_1$  and  $D_2$  are both crossed horizontally (each crossing has probability at least  $c(1)$  to occur);
- $[n/2, n) \times \{0\}$  (resp.  $[n, 3n/2) \times \{n\}$ ) is connected inside  $D_2$  to the top side (resp. to the bottom). Using the FKG inequality and symmetries of the lattice, this event occurs with probability larger than  $c(1)^2/4$ .

Therefore,  $A$  has probability larger than  $c(1)^4/4$ .

When  $A$  occurs, define  $\Gamma_1$  to be the top-most crossing of the rectangle  $D_1$ , and  $\Gamma_2$  the right-most crossing in  $D_2$  between  $[n/2, n) \times \{0\}$  and the top side of  $D_2$ . Note that  $\Gamma_2$  is automatically connecting  $[n/2, n) \times \{0\}$  to the right edge and to  $[n, 3n/2) \times \{n\}$ . In order to conclude, it is sufficient to prove that  $\Gamma_1$  and  $\Gamma_2$  are connected with probability larger than some positive constant.

Consider the lowest path  $\Gamma'_1$  above  $\Gamma_1$  which satisfies the following property: for any edge  $e$  in  $\Gamma'_1$ , the associated site of  $\sigma_d(\mathbb{H})$  (see the definition of Hypothesis (\*\*)) is in the connected component of  $D_1 \setminus \Gamma'_1$  above  $\Gamma'_1$ . Such a path can be obtained from  $\Gamma_1$  by replacing every ‘bad’ edge with the other two edges of a triangle, as shown in Figure 4.8. Since  $\Gamma_1$  is the top-most crossing, it cannot have double edges and the path  $\Gamma'_1$  can be constructed. In particular it ends at the same point as  $\Gamma_1$ , and it goes from left to right. Note that it is not necessarily open. Define  $\Gamma'_2$  similarly in the obvious way (the left-most path on the right of  $\Gamma_2$  such that for any edge of  $\Gamma'_2$ , the associate site of  $\sigma_d(\mathbb{H})$  is on the right of  $\Gamma'_2$ ).

We now sketch the end of the proof. Apply a construction similar to the proof of Proposition 4.8 in order to create a domain  $G(\Gamma'_1, \Gamma'_2)$ . With mixed boundary conditions, the probability of connecting  $\Gamma'_1$  to  $\Gamma'_2$  in  $G(\Gamma'_1, \Gamma'_2)$  is larger than  $1/(1+q^2)$  ( $\Gamma'_1$  and  $\Gamma'_2$  have been constructed in such a way that Hypothesis (\*\*\*) is fulfilled). But  $\Gamma_1$  disconnects  $\Gamma'_1$  from  $\Gamma'_2$ , and  $\Gamma_2$  disconnects  $\Gamma'_2$  from  $\Gamma_1$ . Using boundary conditions inherited from the fact that  $\Gamma_1$  and  $\Gamma_2$  are crossings, one can prove that  $\Gamma_1$  is connected to  $\Gamma_2$  in  $G(\Gamma'_1, \Gamma'_2)$  with probability larger than  $1/(1+q^2)$ . The end of the proof follows exactly the same lines as in the case of the square lattice.  $\square$



# Chapter 5

## Parafermions in the random-cluster model

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**Abstract:** Parafermionic observables were introduced in order to study the critical phase  $(p, q) = (p_{sd}(q), q)$ . The main results of this chapter include the weak discrete holomorphicity of parafermionic observables, an alternative proof of Theorem 4.1 for  $q \geq 4$ , and the divergence of the correlation length when approaching the critical point for random-cluster models with  $q \leq 4$  (this shows that the transition is of second order). This chapter is inspired by the article *Parafermions in the planar random-cluster model* [BDCS11] written with V. Beffara and S. Smirnov.

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Critical random-cluster models exhibit a very rich behavior depending on the value of  $q$ . Exact computations can be performed (see [Bax89]), and despite the fact that they do not lead to fully rigorous mathematical proofs, they do provide insight and further conjectures on the behavior of these models at and near criticality. For a wide range of values of  $q$ , the so-called *scaling limit* is expected to be *conformally invariant* (see the second part of the manuscript for additional details). Nevertheless, very little of the behavior of the model is rigorously understood. In particular, the question of the order of the phase transition is far from being solved. The random-cluster phase transition is conjectured to be of *first order* for  $q > 4$  and *second order* for  $q < 4$ .

Definitions of the order of a phase transition differ from one field to the other. In physics, Ehrenfest classified phase transitions based on the behavior of the thermodynamical free energy viewed as a function of other thermodynamical quantities. He defined the order of the phase transition as the lowest derivative of the free energy which is discontinuous at the phase transition. For instance, the partition function is continuous yet non-differentiable when the transition is of first order. Even though physics predict that all the notions of order of a phase transition are the same, probabilistic definitions are slightly different and involve uniqueness of infinite-volume measures or the so-called correlation length. Let us describe these two points of view in the special case of random-cluster models.

The first point of view invokes Gibbs measures. The random-cluster model is then said to exhibit a first order phase transition if there are several critical infinite-volume measures. This boils down to  $\phi_{p_{sd},q}^1$  being different from  $\phi_{p_{sd},q}^0$ , or equivalently to the  $\phi_{p_{sd},q}^1$ -almost sure existence of an infinite cluster. On the contrary, the transition is of second order if  $\phi_{p_{sd},q}^1 = \phi_{p_{sd},q}^0$ .

The second point of view uses the *correlation length*  $\xi(p)$  defined by:

$$\xi(p)^{-1} = -\inf_{n>0} \frac{1}{n} \log \phi_{p,q}^0(0 \leftrightarrow n).$$

The transition is of second order if the correlation length goes to infinity when  $p$  goes to  $p_c$ . It is of first order otherwise. In physics, the two definitions are believed to be equivalent in natural cases. Nevertheless, the second definition of first order phase transition is *a priori* stronger than the first one. Indeed, exponential decay for  $\phi_{p_{sd},q}^0$  is implied, thanks to a submultiplicativity argument<sup>1</sup>, by the fact that the correlation length does not diverge near criticality. A classical application of the Borel-Cantelli lemma gives that exponential decay of correlations for  $\phi_{p_{sd},q}^0$  implies the existence of an infinite cluster in its dual  $\phi_{p_{sd},q}^1$ .

In order to understand the phase transition in random-cluster models, the so-called *parafermionic observables* are studied in depth. These observables were first introduced in [Smi10a] for random-cluster models with parameter  $q \in [0, 4]$ , as (anti)-holomorphic parafermions of fractional spin  $\sigma \in [0, 1]$ , given by certain vertex operators. So far discrete holomorphicity was rigorously proved only for  $q = 2$ , and probably holds exactly only for this value. In this chapter, these vertex operators are generalized to random-cluster models with arbitrary  $q > 0$ .

Using the parafermionic observable, we are able to prove that the correlation length diverges when  $1 < q < 4$ , which proves that the phase transition is of second order (in the weak sense):

**Theorem 5.1.** *When  $1 \leq q < 4$ ,  $\xi(p)$  tends to infinity when  $p \nearrow p_c(q)$ .*

In fact, the following stronger result can be proved:

**Theorem 5.2.** *When  $1 \leq q \leq 3$ , we have*

$$\sum_{x \in \mathbb{Z}^2} \phi_{p_c,q}^0[0 \leftrightarrow x] = \infty.$$

In the physics literature, the mean-size of the cluster is called *susceptibility*. Note that for  $q = 1$ , this result implies that there is no dual infinite cluster for  $\phi_{1/2,1} = \phi_{p_c(1),1}^0$ . It would be interesting to generalize this argument<sup>2</sup> to other values of  $q$ , for instance when  $q = 3$ , in order to obtain the stronger characterization of a second order phase transition: uniqueness of infinite-volume measures at criticality.

<sup>1</sup>See the proof of Theorem 5.1 for details.

<sup>2</sup>The absence of infinite cluster for the infinite-volume measure with wired boundary conditions implies that  $\phi_{p_{sd},q}^1 = \phi_{p_{sd},q}^0$ . Proposition 3.14 then implies uniqueness of the infinite-volume measure.

Let us now deal with the  $q > 4$  case. When  $q \geq 25.72$ , first order phase transition was proved in [LMMS<sup>+</sup>91, LMR86]. We are presently unable to prove that a first order phase transition occurs in the whole regime  $q > 4$ , even though a fairly close result can be proved: consider the graph  $\mathbb{U}$  with vertex set  $\mathbb{Z}^3$  and edges given by

- $[(x, y, z), (x + 1, y, z)]$  for every  $x, y, z \in \mathbb{Z}$ ,
- $[(x, y, z), (x, y + 1, z)]$  for every  $x, y, z \in \mathbb{Z}$  such that ' $y \neq 0$ ' or ' $y = 0$  and  $x \geq 0$ ',
- $[(x, 0, z), (x, 1, z - 1)]$  for every  $x < 0$  and  $z \in \mathbb{Z}$ .

**Theorem 5.3.** *When  $p = p_{sd}$  and  $q > 4$ , there exists an infinite cluster (not using boundary sites) almost surely for the measure  $\phi_{\mathbb{U}, p_{sd}, q}^1$ .*

We hope it is possible to bootstrap the result on  $\mathbb{U}$  to  $\mathbb{Z}^2$ , thus proving the weak characterization of first order phase transition.

We conclude this chapter by providing an alternative derivation of the critical point when  $q > 4$ . While this result also follows from the previous chapter, the technique gives (a little) more information on the critical phase and is probably more robust. Note that comparison between random-cluster models allow us to extend the next theorem to  $q = 4$ .

**Theorem 5.4.** *Let  $q > 4$ . The critical point  $p_c = p_c(q)$  for the random-cluster model with parameter  $q$  on the square lattice satisfies*

$$p_c = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

The chapter is organized as follows. In the next section, the loop representation of the random-cluster model is introduced, and the parafermionic observable is defined. Two very important properties are also proved. They will be used extensively in the second part of this manuscript. The second section deals with critical random-cluster models on  $\mathbb{U}$ . Theorems 5.1, 5.2 and 5.3 are proved in the third section. We also introduce a parafermionic observable in the degenerated case  $q = 4$ . The last section is devoted to the proof of Theorem 5.4.

## 1 The loop model representation and parafermionic observables

### 1.1 The loop representation of the planar random-cluster model

Let  $(G, a, b)$  be a Dobrushin domain. In this paragraph, we aim for the construction of the loop representation of the random-cluster model, defined on the medial graph  $(G^\circ, a^\circ, b^\circ)$  of the Dobrushin domain. The medial graph  $G^\circ$  is defined in a slightly non-classical way, see Fig. 5.1 for an explanatory picture: consider  $G$  together with its dual  $G^*$  and add all the sites of  $\mathbb{L}^*$  adjacent to the free arc  $\partial_{ab}$  (call this set  $\partial_{ab}^*$ ), the medial graph  $G^\circ$  is the subgraph of  $\mathbb{L}^\circ$  given by:

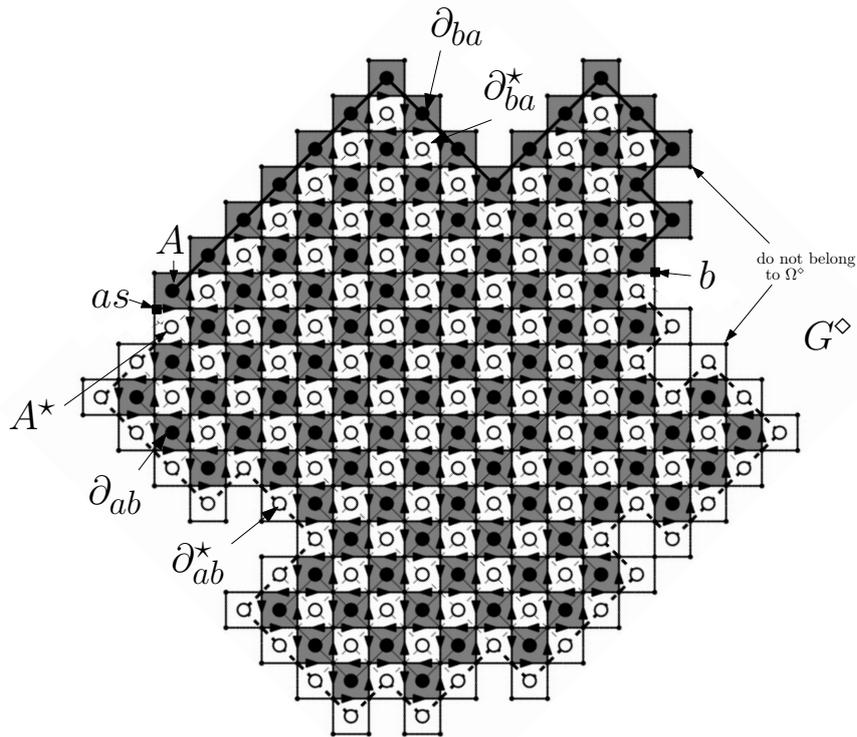


Figure 5.1: A domain  $G$  with Dobrushin boundary conditions: the vertices of the primal graph are black, the vertices of the dual graph  $G^*$  are white, and between them lies the medial graph  $G^\circ$ . The arcs  $\partial_{ba}$  and  $\partial_{ab}^*$  are the two outermost arcs. The arcs  $\partial_{ba}^*$  and  $\partial_{ab}$  are the arcs bordering  $\partial_{ba}$  and  $\partial_{ab}^*$  from the inside. The arcs  $\partial_{ab}$  and  $\partial_{ba}$  (resp.  $\partial_{ab}^*$  and  $\partial_{ba}^*$ ) are drawn in solid lines (resp. dashed lines)

- $E[G^\diamond]$  is the set of edges bordering faces of  $\mathbb{L}^\diamond$  corresponding to  $V[G] \cup V[G^*] \cup \partial_{ab}^*$
- $V[G^\diamond]$  is the set of end-points of edges in  $E[G^\diamond]$ .

The medial vertices  $a^\diamond$  and  $b^\diamond$  are the two medial vertices of  $G^\diamond$  having three adjacent medial edges.

The random-cluster measure on  $(G, a, b)$  with Dobrushin boundary conditions has a rather convenient representation in this setting. Consider a configuration  $\omega$ , it defines clusters in  $G$  and dual clusters in  $G^*$  (note that the arc  $\partial_{ab}$  being free, the arc  $\partial_{ab}^*$  must be dual-wired thanks to planar duality). Through every vertex of the medial graph passes either an open bond of  $G$  or a dual open bond of  $G^*$ , hence there is a unique way to draw an Eulerian (*i.e.* using every edge exactly once) collection of loops on the medial lattice. These loops are the *interfaces*, separating clusters from dual clusters. Namely, a loop arriving at a vertex of the medial lattice, always makes a  $\pm\pi/2$  turn so as not to cross the open or dual open bond through this vertex, see Fig. 5.1. Besides loops, the configuration will have a single curve joining the vertices adjacent to  $a$  and  $b$ , which are the only vertices in  $V^\diamond$  with three adjacent edges (the edges entering  $a$  and  $b$  are denoted by  $e_a$  and  $e_b$  respectively). This curve is called the *exploration path* and is denoted by  $\gamma$ . It corresponds to the interface between the cluster connected to the wired arc  $\partial_{ba}$  and the dual cluster connected to the free arc  $\partial_{ab}^*$ .

This gives a bijection between random-cluster configurations on  $G$  and Eulerian loop configurations on  $G^\diamond$ . The probability measure can be nicely rewritten (using Euler's formula) in terms of the loop picture:

**Proposition 5.5.** *Let  $p \in (0, 1)$  and  $q > 0$  and let  $(G, a, b)$  be a Dobrushin domain, then for any configuration  $\omega$ ,*

$$\phi_{G,p,q}^{a,b}(\omega) = \frac{1}{Z} x^{o(\omega)} \sqrt{q}^{-\ell(\omega)} \quad (5.1)$$

where  $x = p/[\sqrt{q}(1-p)]$ ,  $\ell(\omega)$  is the number of loops in the loop configuration associated to  $\omega$ ,  $o(\omega)$  is the number of open edges, and  $Z$  is the normalization constant.

**Proof** Recall that

$$\phi_{G,p,q}^{a,b}(\omega) = \frac{1}{Z} [p/(1-p)]^{o(\omega)} q^{k(\omega)}.$$

The dual of  $\phi_{G,p,q}^{a,b}$  is  $\phi_{G^*,p^*,q}^{b,a}$ . With  $\omega^*$  being the dual configuration of  $\omega$ , we find

$$\begin{aligned} \phi_{G,p,q}^{a,b}(\omega) &= \sqrt{\phi_{G,p,q}^{a,b}(\omega) \phi_{G^*,p^*,q}^{b,a}(\omega^*)} \\ &= \frac{1}{\sqrt{ZZ^*}} \sqrt{p/(1-p)}^{o(\omega)} \sqrt{q}^{k(\omega)} \sqrt{p^*/(1-p^*)}^{o(\omega^*)} \sqrt{q}^{k(\omega^*)} \\ &= \frac{1}{\sqrt{ZZ^*}} \sqrt{\frac{p(1-p^*)}{(1-p)p^*}}^{o(\omega)} \sqrt{p^*/(1-p^*)}^{o(\omega^*)+o(\omega)} \sqrt{q}^{k(\omega)+k(\omega^*)} \\ &= \frac{\sqrt{q} \sqrt{p^*/(1-p^*)}^{o(\omega)+o(\omega^*)}}{\sqrt{ZZ^*}} x^{o(\omega)} \sqrt{q}^{k(\omega)+k(\omega^*)-1} \end{aligned}$$

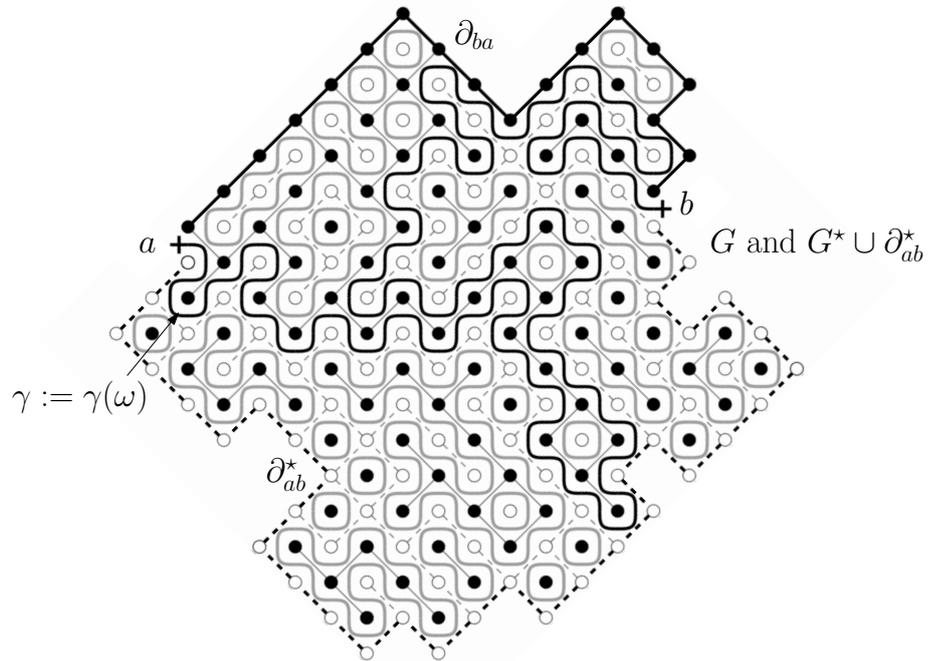


Figure 5.2: A random-cluster configuration in the Dobrushin domain  $(G, a, b)$ , together with the corresponding interfaces on the medial lattice: the loops are grey, and the exploration path  $\gamma$  from  $a^\circ$  to  $b^\circ$  is black. Note that the exploration path is the interface between the open cluster connected to the wired arc and the dual-open cluster connected to the white faces of the free arc.

where the definition of  $p^*$  was used to prove that  $\frac{p(1-p^*)}{(1-p)p^*} = x^2$ . Note that  $\ell(\omega) = k(\omega) + k(\omega^*) - 1$  and

$$\tilde{Z} = \frac{\sqrt{ZZ^*}}{\sqrt{q}\sqrt{p^*/(1-p^*)}^{o(\omega)+o(\omega^*)}}$$

does not depend on the configuration (the sum  $o(\omega) + o(\omega^*)$  being equal to the total number of edges). Altogether, this implies the claim.  $\square$

## 1.2 Observables for Dobrushin domains.

Fix a Dobrushin domain  $(G, a, b)$ . Following [Smi10a], an observable  $F : E_\diamond \rightarrow \mathbb{C}$  is now defined on the edges of the medial graph. Roughly speaking,  $F$  is a modification of the probability that the exploration path passes through an edge. First, introduce the following definition

**Definition 5.6.** *The winding  $W_\Gamma(z, z')$  of a curve  $\Gamma$  between two edges  $z$  and  $z'$  of the medial graph is the total (signed) rotation (in radians) that the curve makes from the mid-point of the edge  $z$  to that of the edge  $z'$  (see Fig. 5.5).*

We are now in a position to define Smirnov's edge-observable

**Definition 5.7** (Smirnov's observable). *Consider a Dobrushin domain  $(G, a, b)$  and two parameters  $p \in (0, 1)$ ,  $q > 0$ . Define the (FK) parafermionic observable  $F$  for any edge  $e \in E_\diamond$  by*

$$F(e) := \phi_{G,p,q}^{a,b} \left( e^{i\sigma W_\gamma(e, e_b)} \mathbb{1}_{e \in \gamma} \right), \quad (5.2)$$

where  $\gamma$  is the exploration path and  $\sigma$  is given by the relation

$$\sin(\sigma\pi/2) = \frac{\sqrt{q}}{2} \quad (5.3)$$

For  $q \in [0, 4]$ , the observable  $F$  is a holomorphic parafermion of spin  $\sigma$ , which is a real number in  $[0, 1]$ . For  $q \geq 4$ ,  $\sigma \in 1 + i\mathbb{R}$  and does not have an obvious physical meaning; it would nonetheless be amusing to find one.

Sometimes, we use  $\tilde{\sigma} := \sigma - 1$  and the observable  $\tilde{F}$  defined for any edge  $e \in E_\diamond$  by

$$\tilde{F}(e) := \phi_{G,p,q}^{a,b} \left( e^{i\tilde{\sigma} W_\gamma(e, e_b)} \mathbb{1}_{e \in \gamma} \right), \quad (5.4)$$

## 1.3 Two fundamental properties of the observable

Let  $\alpha = \alpha(p, q) \in [0, 2\pi)$  be given by the relation

$$e^{i\alpha(p)} := \frac{e^{i\sigma\pi/2} + x(p)}{e^{i\sigma\pi/2}x(p) + 1}. \quad (5.5)$$

**Proposition 5.8.** *Consider a medial vertex  $v$  in  $G^\diamond \setminus \partial G^\diamond$ . The two edges pointing towards  $v$  are indexed by  $N$  and  $S$ , and the other two by  $E$  and  $W$  in the obvious way. Then,*

$$F(N) - F(S) = e^{i\alpha(p,q)} i[F(E) - F(W)]. \quad (5.6)$$

When  $p = p_{sd}(q)$ ,  $\alpha(p, q) = 0$  and the previous relation becomes a discretization of the Cauchy-Riemann equation (see Chapter 2). Note that (5.6) immediately translates into the relation:

$$\tilde{F}(N) + \tilde{F}(S) = e^{i\alpha(p,q)} [\tilde{F}(E) + \tilde{F}(W)]. \quad (5.7)$$

Even though this relation has no natural interpretation in terms of discrete complex analysis, it would sometimes be more convenient to handle than (5.6).

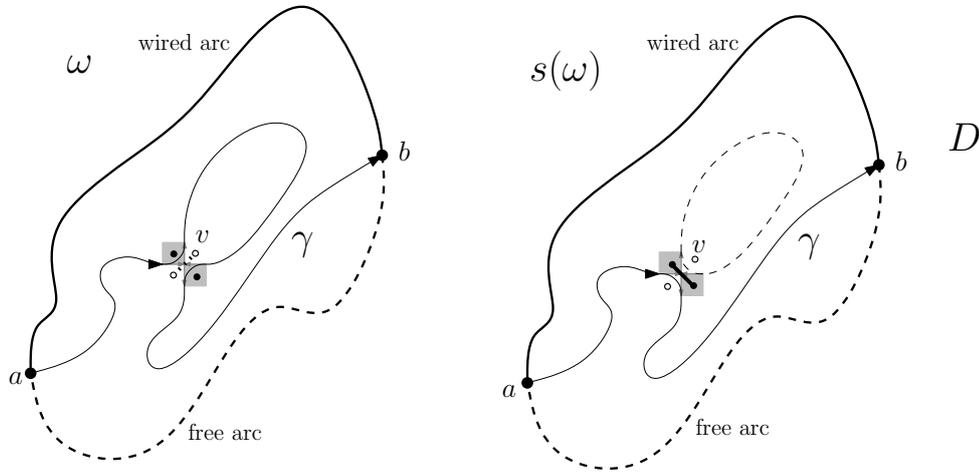


Figure 5.3: Two associated configurations  $\omega$  and  $s(\omega)$

**Proof** Let us assume that  $v$  corresponds to a primal edge pointing  $SE$  to  $NW$ , see Fig. 5.4. The case  $NE$  to  $SW$  is similar.

We consider the involution  $s$  (on the space of configurations) which switches the state (open or closed) of the edge of the primal lattice corresponding to  $v$ . Let  $e$  be an edge of the medial graph and denote by

$$e_\omega := \phi_{G, p_{sd}, q}^{a,b}(\omega) e^{i\sigma W_\gamma(e, e_b)} 1_{e \in \gamma}$$

the contribution of the configuration  $\omega$  to  $F_\delta(e)$ . Since  $s$  is an involution, the following relation holds:

$$F_\delta(e) = \sum_\omega e_\omega = \frac{1}{2} \sum_\omega [e_\omega + e_{s(\omega)}].$$

In order to prove (5.6), it suffices to prove the following for any configuration  $\omega$ :

$$N_\omega + N_{s(\omega)} - S_\omega - S_{s(\omega)} = e^{i\alpha(p,q)} i[E_\omega + E_{s(\omega)} - W_\omega - W_{s(\omega)}]. \quad (5.8)$$

There are three possibilities:

**Case 1:** the exploration path  $\gamma(\omega)$  does not go through any of the edges adjacent to  $v$ . It is easy to see that neither does  $\gamma(s(\omega))$ . All the terms then vanish and (5.8) trivially holds.

**Case 2:**  $\gamma(\omega)$  goes through two edges around  $v$ . Note that it follows the orientation of the medial graph, and thus enters  $v$  through either  $W$  or  $E$  and leaves through  $N$  or  $S$ . Assume that  $\gamma(\omega)$  enters through the edge  $W$  and leaves through the edge  $S$  (*i.e.* that the primal edge corresponding to  $v$  is open). The other cases are treated similarly. It is then possible to compute the contributions of all the edges adjacent to  $v$  of  $\omega$  and  $s(\omega)$  in terms of  $W_\omega$ . Indeed,

- The probability of  $s(\omega)$  is equal to  $1/(x\sqrt{q})$  times the probability of  $\omega$  (due to the fact that there is one less open edge of weight  $x$  and one less loop of weight  $\sqrt{q}$ , see Proposition 5.5);
- Windings of the curve can be expressed using the winding at  $W$ . For instance, the winding of  $N$  in the configuration  $\omega$  is equal to the winding of  $W$  plus an additional  $\pi/2$  turn.

The contributions are given as:

| configuration | $W$                    | $E$                              | $N$                               | $S$                              |
|---------------|------------------------|----------------------------------|-----------------------------------|----------------------------------|
| $\omega$      | $W_\omega$             | 0                                | 0                                 | $e^{i\pi/4}W_\omega$             |
| $s(\omega)$   | $W_\omega/(x\sqrt{q})$ | $e^{i\pi/2}W_\omega/(x\sqrt{q})$ | $e^{-i\pi/4}W_\omega/(x\sqrt{q})$ | $e^{i\pi/4}W_\omega/(x\sqrt{q})$ |

Using the identity  $e^{i\sigma\pi/2} - e^{-i\sigma\pi/2} = i\sqrt{q}$ , we deduce (5.8) by summing (with the right weight) the contributions of all the edges around  $v$ .

**Case 3:**  $\gamma(\omega)$  goes through the four medial edges around  $v$ . Then the exploration path of  $s(\omega)$  goes through only two, and the computation is the same as in the second case.

In conclusion, (5.8) is always satisfied and the claim is proved.  $\square$

So far, we have shown that the (partial) integrability of the random-cluster model implies local properties of the observable yet the observable was not related to connectivity properties of the model. On the boundary, it is in fact possible to connect the observable to the probability to be connected to the boundary.

**Lemma 5.9.** *Let  $u \in G$  be a site on the free arc  $\partial_{ab}$ , and  $e$  be a side of the black diamond associated to  $u$  which borders a white diamond of  $\partial_{ab}^*$ , see Figure 5.5. Then,*

$$F(e) = e^{i\sigma W(e, e_b)} \cdot \phi_{G,p,q}^{a,b}(u \leftrightarrow \text{wired arc}),$$

where  $W(e, e_b)$  is the winding of an arbitrary curve on the medial lattice from  $e$  to  $e_b$ .

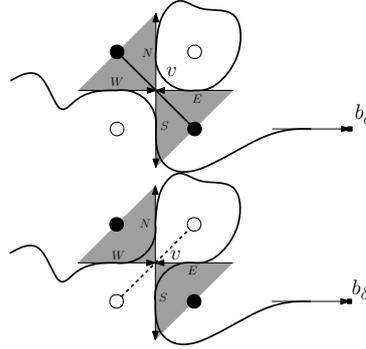


Figure 5.4: A zoom on the consequence of switching the state of one bond in terms of loops.

**Proof** Let  $u$  be a site of the free arc and recall that the exploration path is the interface between the open cluster connected to the wired arc and the dual open cluster connected to the free arc. Since  $u$  belongs to the free arc,  $u$  is connected to the wired arc if and only if  $e$  is on the exploration path, so that

$$\phi_{G,p,q}^{a,b}(u \leftrightarrow \text{wired arc}) = \phi_{G,p,q}^{a,b}(e \in \gamma).$$

The edge  $e$  being on the boundary, the exploration path cannot wind around it, so that the winding of the curve is deterministic. Call it  $W(e, e_b)$ . We deduce from this remark that

$$\begin{aligned} F(e) &= \phi_{G,p,q}^{a,b}(e^{i\sigma W(e, e_b)} \mathbb{1}_{e \in \gamma}) = e^{i\sigma W(e, e_b)} \phi_{G,p,q}^{a,b}(e \in \gamma) \\ &= e^{i\sigma W(e, e_b)} \phi_{G,p,q}^{a,b}(u \leftrightarrow \text{wired arc}). \end{aligned}$$

□

**The observable in infinite Dobrushin domains.** The definition of  $F$  can be extended to the case of infinite Dobrushin domains. Consider two non-intersecting doubly-infinite arcs  $\partial$  on  $\mathbb{L}$  and  $\partial^*$  on  $\mathbb{L}^*$  defining an infinite simply-connected domain  $G$  of  $\mathbb{L}^\circ$ . This domain has two ends, denoted  $-\infty$  and  $\infty$ , where  $\infty$  is found at the end of  $\partial$ <sup>3</sup>. Set  $\phi_{G,p,q}^{\infty, -\infty}$  to be the random-cluster measure<sup>4</sup> of parameters  $(p, q)$  on  $G$  with wired boundary conditions on  $\partial$  and dual-wired boundary conditions on  $\partial^*$  (corresponding to free boundary conditions on the adjacent primal arc). The arc  $\partial$  is the wired arc and  $\partial^*$  is the free one. For instance, the strip  $\mathcal{S}_\ell = \mathbb{Z} \times [0, \ell]$  with wired boundary conditions on the bottom and free boundary conditions on the top enters into this framework.

<sup>3</sup>When going along  $\partial$  in the clockwise direction

<sup>4</sup>Here, the measure is not necessarily unique. We thus assume that it is constructed using nested boxes with free boundary conditions on the intersection of their boundary with  $\partial^*$ , and wired elsewhere. When there is no infinite-cluster in infinite volume, the measure is unique (for instance when  $p \neq p_{sd}$ ).

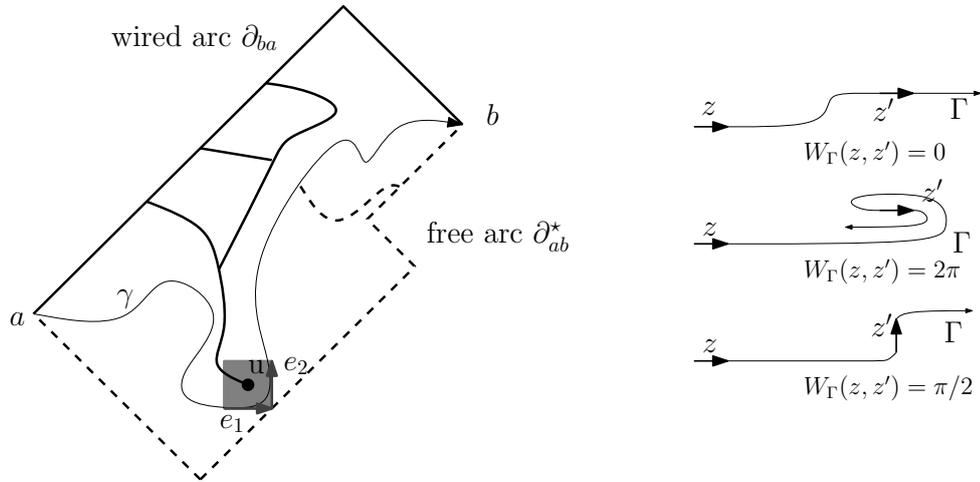


Figure 5.5: **Left:** A schematic picture of the exploration path and a boundary point  $u$ , together with two possible choices  $e_1$  and  $e_2$  for  $e$ . If  $u$  is connected to the wired arc, the exploration path must go through  $e$ . **Right:** The winding of a curve. In the first example, the curve did one quarter-turn on the left and one quarter-turn on the right.

The loop representation also exists in this setting. The  $\phi_{G,p,q}^{\infty,-\infty}$ -probability of having both an infinite cluster and a dual infinite cluster being 0, there is a *unique* interface  $\gamma$  going from  $+\infty$  to  $-\infty$  and separating the primal cluster connected to  $\partial$  and the dual-cluster connected to  $\partial^*$ . We define

$$F(e) := \phi_{G,p,q}^{\infty,-\infty} \left[ e^{i\sigma W_\gamma(e,-\infty)} \mathbb{1}_{e \in \gamma} \right]$$

where  $W_\gamma(e, -\infty)$  is the winding of the curve between  $e$  and  $-\infty$ . This winding is well-defined up to an additive constant (since  $-\infty$  does not really make sense as a medial edge) that we fix on a case by case basis. It is easy to see that  $F$  is the limit of observables in finite boxes, so that properties of fermionic observables in finite Dobrushin domains carry over to the infinite-volume case. In particular, the conclusions of the previous lemmas apply to the infinite case as well.

## 2 The phase transition through parafermionic observables

### 2.1 Random-cluster models on surfaces with a singularity

We introduce a family of domains. Recall the definition of the graph  $\mathbb{U}$ : it is given by the vertex set  $\mathbb{Z}^3$  and the edge set containing

- $[(x, y, z), (x + 1, y, z)]$  for every  $x, y, z \in \mathbb{Z}$ ,

- $[(x, y, z), (x, y + 1, z)]$  for every  $x, y, z \in \mathbb{Z}$  such that ' $y \neq 0$ ' or ' $y = 0$  and  $x \geq 0$ ',
- $[(x, 0, z), (x, 1, z - 1)]$  for every  $x < 0$  and  $z \in \mathbb{Z}$ .

This graph can be seen as a graph on the universal cover of  $\mathbb{R}^2 \setminus \{(-1/2, 1/2)\}$ . Its medial graph is defined similarly to the previous cases and is denoted by  $\mathbb{U}^\circ$ .

A subgraph of  $\mathbb{U}$  is said to be *simply connected* if its complement in  $\mathbb{U}$  is connected. It is a Dobrushin domain if it has two marked points on its boundary. It is possible to define the parafermionic observable on any Dobrushin domain in exactly the same way as for planar simply connected domains  $G^\circ$ . Moreover, the following local relations are still valid at criticality:

**Proposition 5.10.** *Let  $q > 0$  and  $p = p_{sd}(q)$  and fix a Dobrushin domain  $G$  of  $\mathbb{U}$ . Consider a medial vertex  $v$  in  $G^\circ \setminus \partial G^\circ$ . Index the two edges pointing toward  $v$  by  $N$  and  $S$ , and the other twos by  $E$  and  $W$  in the obvious way. Then,*

$$\tilde{F}(N) + \tilde{F}(S) = \tilde{F}(E) + \tilde{F}(W) \quad (5.9)$$

**Proof** We shall not repeat the proof of Proposition 5.8 (which implies this case when  $G$  is planar). The only point which could differ is the winding of possible loops, which could be different of  $2\pi$  on general graphs. Yet, this is not the case for these graphs and the proposition holds true.  $\square$

For  $n \in \mathbb{N}$ , define

$$U_{\infty, n} := \{(x, y, z) \in \mathbb{U} : |x| + |y| \leq n\}$$

For  $\theta \in \frac{\pi}{2}\mathbb{N}$ , define  $U_{\theta, n}$  to be the connected component of the origin in  $U_{\infty, n} \setminus (\rho_\theta \ell \cup \rho_{-\theta} \ell)$ , where

$$\ell := \{(x, y, 0) : x > 0 \text{ and } x = y\} \cup \{(x, y, 0) : x > 0 \text{ and } x = y + 1\}$$

and  $\rho_\theta$  is the rotation in  $\mathbb{U}$  by an angle  $\theta$ .

**Proposition 5.11.** *Fix  $q \neq 4$  and  $p = p_{sd}(q)$ . There exists  $C > 0$  such that for every  $\theta, n$ , there exists  $\delta_x : \partial U_{\theta, n} \rightarrow [0, C]$  such that*

$$\sum_{\partial U_{\theta, n}} \delta_x \cdot e^{-i\tilde{\sigma}\Theta(x)} \cdot \phi_{U_{\theta, n}}^0(0 \leftrightarrow x) = 1, \quad (5.10)$$

where  $\Theta : \partial U_{\theta,n} \rightarrow \mathbb{R}$  is defined by

$$\Theta(x, y, z) := 2\pi z + \begin{cases} \theta + \pi/2 & \text{if } (x, y, z) \in \rho_\theta \ell \\ -\theta - \pi/2 & \text{if } (x, y, z) \in \rho_{-\theta} \ell \\ -3\pi/4 & \text{if } y = 0, x < 0, \\ -\pi/2 & \text{if } x, y < 0, \\ -\pi/4 & \text{if } x = 0, y < 0, \\ 0 & \text{if } x > 0, y < 0, \\ \pi/4 & \text{if } x > 0, y = 0, \\ \pi/2 & \text{if } x, y > 0, \\ 3\pi/4 & \text{if } x = 0, y > 0, \\ \pi & \text{if } x < 0, y > 0. \end{cases}$$

The function  $\Theta(x, y, z)$  is a step function following the usual definition of the angle.

**Proof** Fix  $q \neq 4$ ,  $p = p_{sd}(q)$  and drop them from the notation. Consider the random-cluster model on  $U_{\theta,n}$  with free boundary conditions. This model can be thought of as a random-cluster model in a Dobrushin domain, where the wired arc is empty. In other words, if  $e_0$  denotes the medial edge adjacent to 0 and pointing south-west,  $e_0$  is both  $e_a$  and  $e_b$  and the exploration path  $\gamma$  is the loop passing through it. For technical reasons, it will be more convenient to consider the edge  $e_0$  as being two half-edges  $e_a$  and  $e_b$ . Then, we define the parafermionic observable in this domain as usual.

Summing the relation (5.9) over all vertices in  $U_{\theta,n}^\circ$  containing four adjacent medial edges, we obtain

$$\sum_{e \text{ entering } E} F(e) - \sum_{e \text{ exiting } E} F(e) = 0, \quad (5.11)$$

where  $E$  is the set of medial edges with two endpoints in  $U_{\theta,n}^\circ \setminus \partial U_{\theta,n}^\circ$  and the edges entering (resp. exiting) are the edges appearing not in  $E$  entering (resp. exiting)  $E$ .

Lemma 5.9 shows that for  $e$  on the boundary,

$$F(e) = e^{i\sigma W(e, e_b)} \phi_{U_{\theta,n}}^0(0 \leftrightarrow x)$$

where  $x$  is the site bordered by  $e$ .

Now, one entering and one exiting edge is associated to each boundary site:

- if  $x \neq 0$ , the entering edge is  $e_a$  and the exiting is  $e_b$ , and the associated windings are 0 and  $2\pi$ ,
- for sites in  $\ell$ , the entering edge has winding  $-\Theta(x) + \pi/2$  and the exiting  $-\Theta(x) + 3\pi/2$ ,
- for other sites with two neighbors in  $U_{\theta,n}$ , the entering edge has winding  $-\Theta(x) + \pi/2$  and the exiting  $-\Theta(x) + 3\pi/2$ ,

- for the remaining sites (with one neighbor), the entering edge has winding  $-\Theta(x) + \pi/4$  and the exiting  $-\Theta(x) + 7\pi/4$ .

Plugging this new input into (5.11), we obtain

$$1 - e^{i\tilde{\sigma}2\pi} + \sum_{1 \text{ neighbor}} (e^{i\tilde{\sigma}\pi/4} - e^{7i\tilde{\sigma}\pi/4})e^{-i\tilde{\sigma}\Theta(x)}\phi_{U_{\theta,n}}^0(0 \leftrightarrow x) + \quad (5.12)$$

$$+ \sum_{\text{others}} (e^{i\tilde{\sigma}\pi/2} - e^{3i\tilde{\sigma}\pi/2})e^{-i\tilde{\sigma}\Theta(x)}\phi_{U_{\theta,n}}^0(0 \leftrightarrow x) = 0 \quad (5.13)$$

which gives

$$\sin(3\tilde{\sigma}\pi/4) \sum_{1 \text{ neighbor}} e^{-i\tilde{\sigma}\Theta(x)}\phi_{U_{\theta,n}}^0(0 \leftrightarrow x) + \sin(\tilde{\sigma}\pi/2) \sum_{\text{others}} e^{-i\tilde{\sigma}\Theta(x)}\phi_{U_{\theta,n}}^0(0 \leftrightarrow x) = \sin(\tilde{\sigma}\pi). \quad (5.14)$$

To conclude, note that

$$\begin{aligned} \sin(\tilde{\sigma}\pi) &= \sqrt{q - q^2/4} \\ \sin(\tilde{\sigma}\pi/2) &= \sqrt{1 - q/4} \\ \sin(3\tilde{\sigma}\pi/4) &= \frac{1}{2}\sqrt{1 - q/4}(\sqrt{1 + \sqrt{q}/2} + \sqrt{q}/(2\sqrt{1 + \sqrt{q}/2})). \end{aligned}$$

□

## 2.2 Divergence of the correlation length when $1 < q < 4$

The main subject of this paragraph is the proof of Theorems 5.1 and 5.2.

**Proof of Theorem 5.2** Fix  $1 \leq q \leq 3$  and  $p = p_{sd}(q)$ . Set  $\partial_n^2 = \rho_\pi \ell \cup \rho_{-\pi}$  and call  $\partial_1^n = \partial U_{\pi,n} \setminus \partial_2^n$ . Taking the real part of (5.10), we find

$$\sum_{x \in \partial_1^n} \delta_x \cos(\tilde{\sigma}\Theta(x)) \cdot \phi_{U_{\pi,n}}^0(0 \leftrightarrow x) = 1 - \sum_{x \in \partial_2^n} \delta_x \cos(\tilde{\sigma}\Theta(x)) \cdot \phi_{U_{\pi,n}}^0(0 \leftrightarrow x).$$

Yet,  $\cos(\tilde{\sigma}\Theta(x)) = \cos(\tilde{\sigma}3\pi/2)$  is non-positive on  $\partial_2^n$  since  $\tilde{\sigma} \geq 1/3$  for  $q \in [1, 3]$ . Therefore,

$$\sum_{x \in \partial_1^n} \phi_{U_{\pi,n}}^0(0 \leftrightarrow x) \geq \sum_{x \in \partial_1^n} \frac{\delta_x}{C} \cos(\tilde{\sigma}\Theta(x)) \cdot \phi_{U_{\pi,n}}^0(0 \leftrightarrow x) \geq \frac{1}{C}$$

where  $C$  is defined in Proposition 5.11. Since  $\partial_1^n$  is a subset of  $\partial\Lambda_n$  (where  $\Lambda_n$  is the ball of size  $n$  for the graph distance),

$$\begin{aligned} \sum_{x \in \mathbb{L}} \phi_{p_{sd},q}^0(0 \leftrightarrow x) &\geq \sum_{n>0} \sum_{x \in \partial\Lambda_n} \phi_{p_{sd},q}^0(0 \leftrightarrow x) \\ &\geq \sum_{n>0} \sum_{x \in \partial_1^n} \phi_{U_{\pi,n}}^0(0 \leftrightarrow x) \\ &\geq \sum_{n>0} \frac{1}{C} = \infty. \end{aligned}$$

□

We now prove a stronger result than Theorem 5.1: correlations decay polynomially fast at criticality. This property distinguishes the critical phase from the subcritical one, since correlations decay exponentially fast in the latter. Before proving this stronger result, let us show how it implies Theorem 5.1:

**Proof of Theorem 5.1** We have for every  $n, m > 0$ , using the FKG inequality,

$$\phi_{p,q}^0(0 \leftrightarrow (n+m)) \geq \phi_{p,q}^0(0 \leftrightarrow n, n \leftrightarrow n+m) \geq \phi_{p,q}^0(0 \leftrightarrow n)\phi_{p,q}^0(0 \leftrightarrow m)$$

which implies that

$$\phi_{p,q}^0(0 \leftrightarrow n) \leq e^{-n/\xi(p)},$$

where  $\xi(p)$  is the correlation length. If  $\xi(p)$  does not converge to  $\infty$ , it increases to  $\xi > 0$  when  $p \nearrow p_{sd}$ . We thus obtain at  $p_c$ :

$$\phi_{p_{sd},q}^0(0 \leftrightarrow n) = \lim_{p \nearrow p_{sd}} \phi_{p,q}^0(0 \leftrightarrow n) \leq \lim_{p \nearrow p_{sd}} e^{-n/\xi(p)} = e^{-n/\xi}.$$

In particular, it converges exponentially fast to 0, which is in contradiction with the polynomial decay of correlations (see Proposition 5.12 below), thus proving the claim. □

**Proposition 5.12.** *Let  $q \in [1, 4)$  and  $p = p_{sd}$ . There exists  $c > 0$  such that*

$$\phi_{p_{sd},q}^0(0 \leftrightarrow x) \geq \frac{1}{|x|^c}.$$

We first use the same reasoning as in the proof of Theorem 5.2 to provide lower bounds on the probability for points in  $\mathbb{U}$  to be connected. We then use these lower bounds to prove lower bounds on the probability for two points of a Dobrushin domain to be connected. This finally allows us to conclude the proof by getting rid of boundary conditions.

**Proof** Fix  $1 < q < 4$  and  $p = p_{sd}$  and drop them from the notation. In this proof, the constants  $C_1, C_2, \dots$  will depend only on  $q$ .

**Connection probabilities in a Dobrushin domain** We generalize the argument employed in the previous proof. Fix  $\theta \in \pi/2\mathbb{N}$  such that  $\cos[\tilde{\sigma}(\theta + \pi/2)] < 0$  and  $\cos[\tilde{\sigma}\theta'] \geq 0$  for every  $0 \leq \theta' < \theta + \pi/2$ . Set  $\partial_1^n$  to be the set of points on  $\partial U_{\theta,n}$  such that  $\theta(x) < \theta$ , and  $\partial_2^n = \partial U_{\theta,n} \setminus \partial_1^n$ . The same reasoning as in the previous proof implies

$$\sum_{x \in \partial_1^n} \phi_{U_{\theta,n}}^0(0 \leftrightarrow x) \geq \frac{1}{C}$$

It implies that there exists  $x \in \partial_1^n$  such that

$$\phi_{U_{\theta,n}}^0(0 \leftrightarrow x) \geq \frac{1}{C|\partial_q^n|} = \frac{C_2}{n}.$$

Let us translate and rotate  $U_{\theta,n}$  (by an application  $T$ ) in such a way that  $Tx = 0$  and the domain lies above the origin. Trivially, for every  $n \geq m > 0$

$$\phi_{TU_{\theta,n}}^0(0 \leftrightarrow \partial[-m, m]^2) \geq \phi_{TU_{\theta,n}}^0(0 (= Tx) \leftrightarrow T0) \geq \frac{C_2}{n}. \quad (5.15)$$

Note that  $TU_{\theta,n}$  is not planar, which represents a difficulty. Nevertheless, (5.15) implies two estimates in planar Dobrushin domains.

First, boundary conditions on  $R_n = [-2n, 2n] \times [0, n]$  inherited from those on  $TU_{\theta,n}$  are stochastically dominated by wired boundary conditions on the top and free elsewhere. If  $\phi_{R_n}^{\text{dobr}}$  denotes the measure on  $R_n$  with these boundary conditions, we find

$$\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow \partial[-m, m]^2) \geq \frac{C_2}{n} \quad (5.16)$$

for every  $m \leq n$ .

Second, boundary conditions on  $C_n = [-2n, 2n] \times [0, 2n] \setminus \{n\} \times [n, 2n]$  inherited from those on  $TU_{\theta,n}$  are stochastically dominated by wired boundary conditions on  $\{n\} \times [n, 2n]$  and free elsewhere. If  $\phi_{C_n}^{\text{dobr}}$  denotes the measure on  $C_n$  with these boundary conditions, we obtain

$$\phi_{C_n}^{\text{dobr}}(0 \leftrightarrow \partial[-m, m]^2) \geq \frac{C_2}{n} \quad (5.17)$$

for every  $m \leq n$ .

**Probability of long crossings in a strip** Fix  $\varepsilon < 1/100$ . We aim for the following result:

$$\phi_{S_n}^{\infty, -\infty}([0, 2n] \times [0, 10\varepsilon n] \text{ is horizontally crossed in } S_{10\varepsilon n}) \geq \frac{C_3}{n^c},$$

where  $c = c(\varepsilon)$  and  $\phi_{S_n}^{\infty, -\infty}$  is the random-cluster measure in the strip  $S_n := \mathbb{R} \times [0, n]$  with free boundary conditions on the bottom and wired boundary conditions on the top.

Applying (5.16) for  $m = \varepsilon n$ , we face two cases:

- Case 1:  $\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow \{\varepsilon n\} \times [0, \varepsilon n] \text{ in } [-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \geq \frac{C_2}{4n}$ ,
- Case 2:  $\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow [-\varepsilon n, \varepsilon n] \times \{\varepsilon n\} \text{ in } [-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \geq \frac{C_2}{2n}$ ,

CASE 1: The assumption immediately implies that

$$\phi_{S_n}^{\infty, -\infty}(0 \leftrightarrow \{\varepsilon n\} \times [0, \varepsilon n] \text{ in } [-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \geq \frac{C_2}{4n}.$$

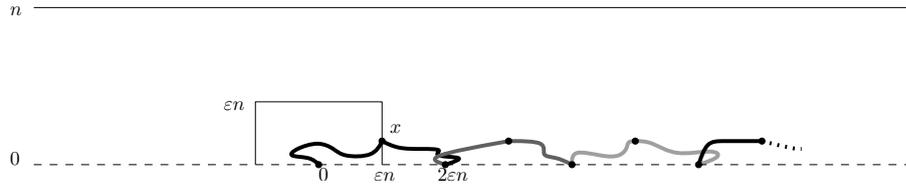


Figure 5.6: Construction of paths to create a long dual-path from left to right.

In particular, there exists  $x \in \{\varepsilon n\} \times [0, \varepsilon n]$  such that

$$\phi_{S_n}^{\infty, -\infty}(0 \leftrightarrow x \text{ in } S_{n/2}) \geq \frac{C_3}{n^2}.$$

Using the FKG inequality and the symmetry under reflexion, see Fig. 5.6, we obtain that

$$\phi_{S_n}^{\infty, -\infty}(0 \leftrightarrow (2\varepsilon n, 0) \text{ in } S_{\varepsilon n}) \geq \frac{C_2^2}{4n^4} \quad (5.18)$$

Using the FKG inequality repeatedly (around  $1/\varepsilon$  times), we find

$$\phi_{S_n}^{\infty, -\infty}(0 \leftrightarrow (2\varepsilon n, 0) \text{ in } S_{\varepsilon n}) \geq \frac{C_3}{n^c} \quad (5.19)$$

for some constants  $C_3 = C_3(\varepsilon)$  and  $c = c(\varepsilon)$ .

CASE 2: There exists  $x \in [-\varepsilon n, \varepsilon n] \times \{\varepsilon n\}$  such that

$$\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow x \text{ in } [-\varepsilon n, \varepsilon n] \times \{\varepsilon n\}) \geq \frac{C_6}{n^3}.$$

Using the FKG inequality yet again and the comparison between boundary conditions, we find

$$\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow (0, 10\varepsilon n) \text{ in } [-2\varepsilon n, 2\varepsilon n] \times [0, 10\varepsilon n]) \geq \frac{C_7}{n^{100}}.$$

Now,

$$\phi_{R_n}^{\text{dobr}}(0 \leftrightarrow 3\varepsilon n \text{ in } S_{10\varepsilon n}) \geq \frac{C_8}{n^{200}}$$

since we can combined the events

- $0 \leftrightarrow (0, 10\varepsilon n) \text{ in } [-2\varepsilon n, 2\varepsilon n] \times [0, 10\varepsilon n]$ ,
- $\varepsilon n \leftrightarrow (\varepsilon n, 10\varepsilon n) \text{ in } [2\varepsilon n, 4\varepsilon n] \times [0, 10\varepsilon n]$ ,
- the two previous vertical paths are connected in  $[-2\varepsilon n, 4\varepsilon n] \times [0, 10\varepsilon n]$

in order to create a path from 0 to  $3\varepsilon n$ . Note that the third event has probability larger than  $1/2$  conditionally on the other two (use duality and crossings in squares with free/wired/free/wired boundary conditions). Now, since  $10\varepsilon < 1/2$ , we are in the same position as in (5.18) and the result follows.

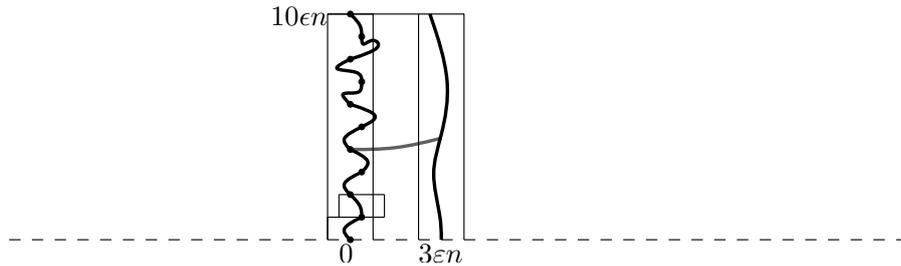


Figure 5.7: Construction of paths to create an arc in the second case of the proof.

**Probability of connection for  $\phi_{p_{sd},q}^0$**  Conditionally on  $0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]$ , the configuration outside in  $C_n \setminus [-\varepsilon n, \varepsilon n]^2$  is stochastically dominated by wired boundary conditions on  $\{n\} \times [n, 2n] \cup \partial[-\varepsilon n, \varepsilon n]^2$  and free elsewhere. Therefore, the probability of having a vertical dual crossing in  $[\varepsilon n, 11\varepsilon n] \times [0, 2n]$  and  $[-11\varepsilon n, -\varepsilon n] \times [0, 2n]$  is larger than  $(C_3/n^c)^2$  (simply wired the arcs  $\{\varepsilon n\} \times [0, 2n]$  and  $\{-\varepsilon n\} \times [0, 2n]$ ). Yet, conditionally on all these events, the two vertical dual crossings are dual connected in  $[-11\varepsilon n, 11\varepsilon n] \times [\varepsilon n, 12\varepsilon n]$  with probability larger than  $1/2$  since the boundary conditions on this square are dominated by wired on the top and bottom and free elsewhere. Now if  $A_n$  denotes the event that  $[-\varepsilon n, \varepsilon n]^2$  is disconnected from the wired arc by a dual crossed path outside of  $[-\varepsilon n, \varepsilon n]^2$ , we find

$$\begin{aligned} \phi_{C_n}^{\text{dobr}}(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n], A_n) &\geq \phi_{C_n}^{\text{dobr}}(A_n \mid 0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \\ &\quad \cdot \phi_{C_n}^{\text{dobr}}(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \\ &\geq \frac{C_2 C_3^2}{2n^{2c+1}}. \end{aligned}$$

In order to conclude, the comparison between boundary conditions implies

$$\begin{aligned} \phi_{p_{sd},q}^0(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n]^2) &\geq \phi_{C_n}^0(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n]) \\ &\geq \phi_{C_n}^{\text{dobr}}(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n] \mid A_n) \\ &\geq \phi_{C_n}^{\text{dobr}}(0 \leftrightarrow \partial[-\varepsilon n, \varepsilon n] \times [0, \varepsilon n], A_n) \geq \frac{C_5}{n^{2c+1}}. \end{aligned}$$

□

### 2.3 Infinite clusters in universal covers when $q > 4$

We now prove Theorem 5.3.

**Proof of Theorem 5.3** Fix  $q > 4$ ,  $p = p_{sd}$  and drop them from the notation. The fact that a cluster exists with probability 0 or 1 is due to the fact that it is a translational invariant event with respect to the vectors  $\{(0, 0, n), n \in \mathbb{Z}\}$ .

Recall that in this case  $i\tilde{\sigma}$  is real and thus the winding term is positive. Fix  $\theta$  and  $n > 0$ , (5.10) implies that

$$\sum_{x \in \partial U_{\theta,n}} e^{i\tilde{\sigma}\Theta(x)} \phi_{U_{\theta,n}}^0(0 \leftrightarrow x) \leq c_q.$$

In fact, the same reasoning can be applied for  $m < n$  to the domain  $V_{\theta,m,n}$  corresponding to the domain  $U_{\theta,n}$  with an additional rectangle of size  $(m, n)$  at the 'end'. We thus obtain

$$\sum_{x \in \partial V_{\theta,m,n}} e^{i\tilde{\sigma}\Theta(x)} \phi_{V_{\theta,m,n}}^0(0 \leftrightarrow x) \leq c_q,$$

which gives, defining  $V_{\theta,m,\infty}$  as the union of  $V_{\theta,m,n}$  for every  $n$ ,

$$\sum_{x \in \partial V_{\theta,m,\infty}} e^{i\tilde{\sigma}\Theta(x)} \phi_{V_{\theta,m,\infty}}^0(0 \leftrightarrow x) \leq c_q,$$

which implies

$$\sum_{x \in \partial^{\theta,m}} \phi_{V_{\theta,m,\infty}}^0(0 \leftrightarrow x) \leq C_1 e^{-|\tilde{\sigma}|\theta},$$

where  $\partial^{\theta,m}$  is the set of boundary points  $x \in V_{\theta,m,\infty}$  with  $\theta(x) \geq \theta$ . Now, assume that  $(0, 0, r)$  and 0 are connected by an open path in  $\mathbb{U}$ . It implies that there exists  $\theta > 2\pi r$ ,  $m > 0$  and a point  $y \in \partial^{\theta,m}$  such that 0 and  $y$  are connected in  $V_{\theta,m,\infty}$ . Using comparison between boundary conditions, we deduce

$$\phi_{\mathbb{U}}^0(0 \leftrightarrow (0, 0, r)) \leq \sum_{\theta > 2\pi r, m > 0} \sum_{x \in \partial^{\theta,m}} \phi_{V_{\theta,m,\infty}}^0(0 \leftrightarrow x) \leq C_2 e^{-2\pi|\tilde{\sigma}|r}.$$

The Borel-Cantelli lemma implies that there is a finite number of couples  $(r, s)$  with  $r < 0$  and  $s > 0$  such that  $(0, 0, r)$  and  $(0, 0, s)$  are connected. In particular, it implies that there exists an infinite cluster in the dual, which is our claim.  $\square$

### 2.4 The case $q = 4$

When  $q = 4$ , Smirnov's parafermionic observable becomes simply

$$\tilde{F}(e) = \phi_{G,p,4}^{a,b}(e \in \gamma).$$

Proposition 5.8 then boils down to the fact that  $\gamma$  enters and exists every vertex the same number of times. Yet this is an easy implication of the fact that  $\gamma$  is a curve. In particular,

the relations are the same for every  $p$  and do not characterize the phase transition. The reason for this loss of information is that we are not looking at the right observable. Somehow, the observable becomes degenerated when  $q \rightarrow 4$  (in particular because the winding term becomes 1), and one should look at an expansion of the observable in powers of  $\tilde{\sigma}$ . When expanding the observable, the second term is

$$G(e) := \phi_{G,p}^{a,b}[W_\gamma(e, e_b)e^{iW_\gamma(e, e_b)}1_{e \in \gamma}].$$

**Proposition 5.13.** *Fix  $q = 4$  and  $p = p_{sd}(4) = 2/3$ . Consider a medial vertex  $v$  in  $G^\circ \setminus \partial G^\circ$ . Index the two edges pointing toward  $v$  by  $N$  and  $S$ , and the other twos by  $E$  and  $W$  in the obvious way. Then,*

$$G(N) - G(S) = i [G(E) - G(W)]. \quad (5.20)$$

**Proof** Consider the parafermionic observable  $F_q$  in  $G^\circ$  for the random-cluster model with parameters  $q$  and  $p_{sd}(q)$ . Proposition 5.8 implies

$$F_q(N) - F_q(S) = i [F_q(E) - F_q(W)]. \quad (5.21)$$

Expanding in  $\sigma - 1$  the winding term in  $F_q$ , we obtain:

$$F_q(e) = \phi_{G,p,q}^{a,b}([1 - \tilde{\sigma}W_\gamma(e, e_b) + O(\tilde{\sigma}^2)]e^{iW_\gamma(e, e_b)}1_{e \in \gamma}).$$

Coming back to (5.21), we deduce

$$\begin{aligned} & \phi_{G,p,q}^{a,b}(e^{iW_\gamma(N, e_b)}1_{N \in \gamma}) - \phi_{G,p,q}^{a,b}(e^{iW_\gamma(S, e_b)}1_{S \in \gamma}) \\ & - i \left( \phi_{G,p,q}^{a,b}(e^{iW_\gamma(E, e_b)}1_{E \in \gamma}) - \phi_{G,p,q}^{a,b}(e^{iW_\gamma(W, e_b)}1_{W \in \gamma}) \right) + O((\sigma - 1)^2) \\ & = (\sigma - 1) \left( \phi_{G,p,q}^{a,b}(W_\gamma(N, e_b)e^{iW_\gamma(N, e_b)}1_{N \in \gamma}) - \phi_{G,p,q}^{a,b}(W_\gamma(S, e_b)e^{iW_\gamma(S, e_b)}1_{S \in \gamma}) \right. \\ & \quad \left. - i \left( \phi_{G,p,q}^{a,b}(W_\gamma(E, e_b)e^{iW_\gamma(E, e_b)}1_{E \in \gamma}) - \phi_{G,p,q}^{a,b}(W_\gamma(W, e_b)e^{iW_\gamma(W, e_b)}1_{W \in \gamma}) \right) \right) \end{aligned}$$

Now, the left hand side of the equality can be rewritten as

$$\phi_{G,p,q}^{a,b}(N \in \gamma) + \phi_{G,p,q}^{a,b}(S \in \gamma) - \phi_{G,p,q}^{a,b}(W \in \gamma) - \phi_{G,p,q}^{a,b}(E \in \gamma) + O((\sigma - 1)^2).$$

Since  $\gamma$  is a curve from  $e_a$  to  $e_b$ , the first four terms cancel each other, and we obtain

$$\begin{aligned} & \phi_{G,p,q}^{a,b}(W_\gamma(N, e_b)e^{iW_\gamma(N, e_b)}1_{N \in \gamma}) - \phi_{G,p,q}^{a,b}(W_\gamma(S, e_b)e^{iW_\gamma(S, e_b)}1_{S \in \gamma}) \\ & = i \left( \phi_{G,p,q}^{a,b}(W_\gamma(W, e_b)e^{iW_\gamma(W, e_b)}1_{W \in \gamma}) - \phi_{G,p,q}^{a,b}(W_\gamma(E, e_b)e^{iW_\gamma(E, e_b)}1_{E \in \gamma}) \right) + O(\sigma - 1). \end{aligned}$$

By letting  $q$  go to 4 from below (in this case  $\sigma$  converges to 1), the result follows readily.

□

**Corollary 5.14.** *There exists  $C > 0$  such that for every  $n$  and  $\theta$ , there exists  $\delta_x : \partial U_{\theta,n} \rightarrow [0, C]$  satisfying*

$$\sum_{x \in \partial U_{\theta,n}} \delta_x \phi_{U_{\theta,n}}^0(0 \leftrightarrow x) = \frac{2}{\pi}.$$

**Proof** The proof of Proposition 5.11 can be adapted to this context with minor changes.  $\square$

**Proposition 5.15.** *Fix  $q = 4$  and  $p = p_{sd}(4) = 2/3$ . There exists  $c > 0$  such that*

$$\phi_{p_{sd},4}^0(0 \leftrightarrow x) \geq \frac{1}{|x|^c}.$$

**Proof** The reasoning is almost the same as in the case  $q < 4$ . First note that the probability to cross a rectangle with wired boundary conditions does not go to 0 as was proved in Theorem 9.1. Therefore,

$$\phi_{p_{sd},4}^1(0 \leftrightarrow \partial[-n, n]^2 \text{ in } [-n, n] \times [0, n]) \geq \frac{C_1}{n}$$

for some constant  $C_1 > 0$ .

Now, consider the domain  $U_{2\pi n^2, n}$ . From Corollary 5.14, there exists  $x \in \partial U_{2\pi n^2, n}$  such that

$$\phi_{U_{2\pi n^2, n}}^0(0 \leftrightarrow x) \geq \frac{C_1}{|\partial U_{2\pi n^2, n}|} = \frac{C_2}{n^3}.$$

Moreover,  $x$  satisfies  $|\theta(x)| \leq n^{3/2}$  since for  $x$ ,

$$\phi_{U_{2\pi n^2, n}}^0(0 \leftrightarrow x) \leq \left(1 - \phi_{U_{2\pi n^2, n}}^0(0 \overset{*}{\leftrightarrow} \partial[-n, n]^2 \text{ in } [-n, n] \times [0, n])\right)^{\theta(x)/2\pi}.$$

Let us translate and rotate  $U_{\theta, n}$  in such a way that  $x \in \{n\} \times [-n, n]$ . We find

$$\phi_{U_{2\pi n^2, n}}^0(x \leftrightarrow \{n/2\} \times [-n, n]) \geq \phi_{U_{2\pi n^2, n}}^0(0 \leftrightarrow x) \geq \frac{C_2}{n^3}.$$

The end of the proof is the same as in the  $q < 4$  case.  $\square$

### 3 An alternative proof that $p_c(q) = p_{sd}(q)$ for $q \geq 4$

When  $q > 4$ , interestingly, the spin variable becomes non-real, therefore it does not have an immediate physical interpretation. However, this allows us to write better estimates even in the absence of exact holomorphicity and relate our observables to the connectivity properties of the model. For  $p \neq p_{sd}$  we prove that observables behave like massive harmonic functions and decay exponentially fast with respect to the distance to the boundary of the domain. Translated into connectivity properties, this implies the sharpness of the phase transition at  $p_{sd}$ . In this section, Theorem 5.4 is proved, as well as the following statement:

**Theorem 5.16.** *Let  $q > 4$ . For every  $p < p_c$ , there exist  $0 < c, C < \infty$  such that*

$$\phi_{p,q}(0 \leftrightarrow a) \leq C e^{-c|a|}$$

for any  $a \in \mathbb{Z}^2$ .

### 3.1 A representation formula for the observable

This section deals with the observable  $F$ . More precisely, the sum of  $\tilde{F}$  over a set  $A \subset E_\diamond$  is bounded in terms of the sum over the boundary edges of  $A$ . Let  $(G, a, b)$  be a Dobrushin domain. For a set  $A$  of edges of  $E_\diamond$ ,  $\partial_e A$  denotes the set of edges of  $E_\diamond \setminus A$  sharing a vertex with an edge of  $A$  (also called the external boundary of the set).

**Proposition 5.17.** *For any  $x \neq 1$  and  $q \neq 4$ , there exists  $C_1 = C_1(p, q) < \infty$  such that for any set of edges  $A \subset E_\diamond$  not containing any edge adjacent to a vertex of  $\partial_v V_\diamond$ , there exists a function  $\delta : \partial_e A \rightarrow [-C_1, C_1]$  such that*

$$(1 - \cos(2\alpha)) \sum_{e \in A} \tilde{F}(e) = \sum_{e \in \partial_e A} \delta_e \tilde{F}(e).$$

**Proof** Recall that  $e^{i\alpha(x)} \neq 1$  since  $x \neq 1$  and  $q \neq 4$ . Sum (5.7) over all vertices adjacent to edges of  $A$ , and divide by  $(1 - e^{i\alpha(x)})$ . It provides a weighted sum of  $\tilde{F}(e)$  (with coefficients denoted by  $c(e)$ ) identical to zero:

$$\sum_{e \in A} c(e) \tilde{F}(e) + \sum_{e \in \partial_e A} c(e) \tilde{F}(e) = 0.$$

For an edge  $e \in A$ ,  $\tilde{F}(e)$  will appear in two identities, corresponding to its endpoints. Since  $e$  is oriented away from one of its ends and towards the other one, the coefficients will be 1 and  $-e^{i\alpha(x)}$ . Thus  $\tilde{F}(e)$  for  $e \in A$  will enter the sum with a coefficient  $c(e) = (1 - e^{i\alpha(x)})/(1 - e^{i\alpha(x)}) = 1$ .

For an edge  $e \in \partial_e A$ ,  $\tilde{F}(e)$  appears in one identity, corresponding to its endpoint belonging to  $A$ . The coefficient is 1 or  $-e^{i\alpha(x)}$ , depending on the orientation of  $e$  with respect to this endpoint. Thus  $\tilde{F}(e)$  enters the sum with a coefficient  $c(e)$  equal to either  $1/(1 - e^{i\alpha(x)})$  or  $-e^{i\alpha(x)}/(1 - e^{i\alpha(x)})$ . The proposition follows immediately by setting  $\alpha_e := -c(e)$  and  $C_1 := \max\{1, |e^{i\alpha(x)}|\}/|1 - e^{i\alpha(x)}|$ .  $\square$

### 3.2 Proof of Theorem 5.4

The main step of the proof is to show that, whenever  $p < p_{sd}$ , there is a very low probability of having vertical crossings of an extremely large rectangle. This statement is sufficient to prove Theorem 5.4, as was already seen in the previous chapter.

For  $L \geq 0$ , consider an infinite horizontal strip  $\mathcal{S}_L = \mathbb{Z} \times \llbracket 0, L \rrbracket$  together with its medial lattice. We define two families of sets, the former ones being subsets of the strip and the latter of the set  $E_\diamond$  of edges of its medial graph. More precisely, write  $e \sim a$  if the edge  $e \in E_\diamond$  is adjacent to the site  $a \in \mathbb{L}$ . For every  $n \geq 0$ , define the following (possibly empty) sets, as depicted in Figure 5.8:

$$\begin{aligned} \mathcal{R}(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket n, L - n \rrbracket, & \mathcal{R}_\diamond(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}(m, n), e \sim a\}, \\ \mathcal{R}^-(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket 0, n - 1 \rrbracket, & \mathcal{R}_\diamond^-(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}^-(m, n), e \sim a\}, \\ \mathcal{R}^+(m, n) &:= \llbracket 0, m \rrbracket \times \llbracket L - n + 1, L \rrbracket, & \mathcal{R}_\diamond^+(m, n) &:= \{e \in E_\diamond : \exists a \in \mathcal{R}^+(m, n), e \sim a\}. \end{aligned}$$

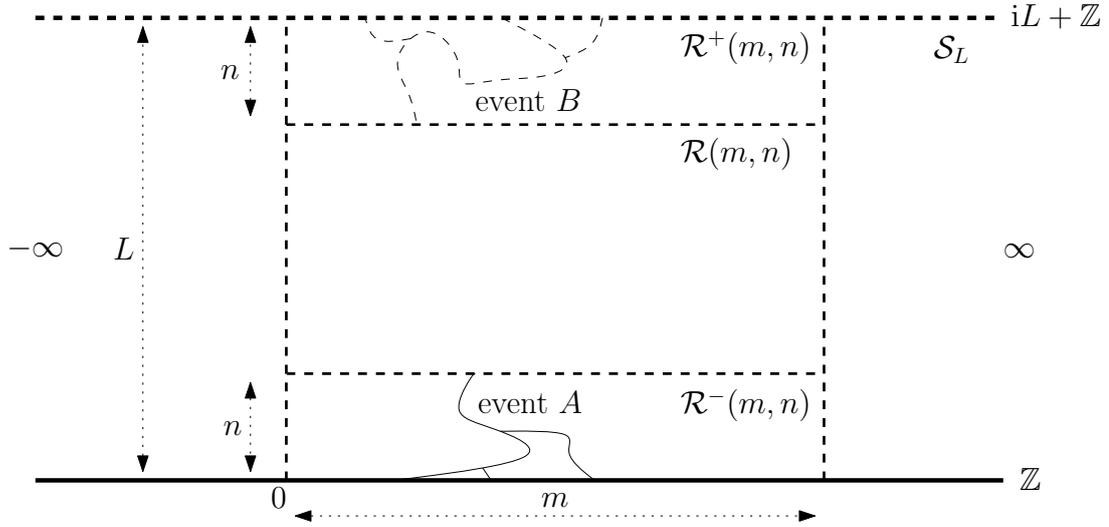


Figure 5.8: Definition of the different rectangles and events  $A$  and  $B$ .

Recall that in a Dobrushin domain,  $\gamma$  denotes the exploration path, *i.e.* the interface between the open cluster connected to the wired arc and the dual open cluster connected to the free arc. The following lemma bounds the probability that the exploration path passes through the rectangle  $\mathcal{R}(m, n)$ :

**Lemma 5.18.** *Let  $q > 4$  and  $p \neq p_{sd}$ , then there exist positive constants  $c_2 = c_2(p, q)$  and  $C_2 = C_2(p, q)$  such that for any  $n < L/2$  and  $m \geq C_2$ ,*

$$\phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(\gamma \cap \mathcal{R}_\circ(m, n) \neq \emptyset) \leq C_2 m e^{-c_2 n}.$$

**Proof** Consider the observable  $\tilde{F}$  defined in the strip  $\mathcal{S}_L$ . Recall that in our setting  $x \neq 1$  and  $\tilde{F}$  is non-negative. Set  $c_2 := -\log(2C_1/(2C_1 + 1))$  and  $C_2 := \max\{4C_1, 8 \exp(|\sigma|2\pi)\}$  where  $C_1$  is defined in Proposition 5.17.

Fix  $m \geq C_2$  and consider some  $n < L/2$ . Denote

$$U_n := \sum_{e \in \mathcal{R}_\circ(m, n)} \tilde{F}(e).$$

Proposition 5.17 along with the non-negativity of  $\tilde{F}$  implies the following estimate:

$$U_n = \sum_{e \in \partial_e \mathcal{R}_\circ(m, n)} \delta_e \tilde{F}(e) \leq C_1 \sum_{e \in \partial_e \mathcal{R}_\circ(m, n)} \tilde{F}(e). \quad (5.22)$$

Divide the boundary  $\partial_e \mathcal{R}_\circ(m, n)$  into four parts: the bottom  $A_{\text{bot}}$ , the top  $A_{\text{top}}$  and both sides  $A_{\text{left}}$  and  $A_{\text{right}}$ .

On the one hand, since  $\tilde{F}$  is invariant under horizontal translations, the sums over the left and right sides are the same as over any vertical cross-section of  $\mathcal{R}_\circ(m, n)$  and we conclude that

$$\sum_{e \in A_{\text{left}} \cup A_{\text{right}}} \tilde{F}(e) = \frac{2}{m} U_n. \quad (5.23)$$

On the other hand, the top and the bottom are contained inside  $U_{n-1} \setminus U_n$ , and therefore

$$\sum_{e \in A_{\text{top}} \cup A_{\text{bottom}}} \tilde{F}(e) \leq U_{n-1} - U_n. \quad (5.24)$$

Combining Equations (5.22), (5.23) and (5.24) and using the inequality  $m \geq C_2 \geq 4C_1$ , we obtain that

$$U_n \leq \frac{2C_1}{m} U_n + C_1(U_{n-1} - U_n) \leq \left(\frac{1}{2} - C_1\right) U_n + C_1 U_{n-1},$$

hence

$$U_n \leq \frac{2C_1}{2C_1 + 1} U_{n-1} = e^{-c_2} U_{n-1}. \quad (5.25)$$

Take now  $n = 0$ ,  $A_{\text{top}}$  and  $A_{\text{bottom}}$  are thus at a distance one to the boundary of the strip: an interface arriving there must have a winding bounded by  $\pm 2\pi$ . Thus for  $e \in A_{\text{top}} \cup A_{\text{bottom}}$  we have

$$\tilde{F}(e) = \phi_{\mathcal{S}_{L,p,q}}^{\infty,-\infty} [e^{-i\tilde{\sigma}W(e)} \mathbb{1}_{e \in \gamma}] \leq e^{|\tilde{\sigma}|2\pi} \phi_{\mathcal{S}_{L,p,q}}^{\infty,-\infty}(e \in \gamma) \leq e^{|\tilde{\sigma}|2\pi}$$

where the last equality is due to Lemma 5.9. Summing this over all  $4m$  edges in the top and bottom sides,

$$\sum_{e \in A_{\text{top}} \cup A_{\text{bottom}}} \tilde{F}(e) \leq 4me^{|\tilde{\sigma}|2\pi} \leq \frac{1}{2} C_2 m. \quad (5.26)$$

Combining (5.22), (5.23) and (5.26) for  $n = 0$  we deduce that

$$U_0 \leq \frac{2C_1}{m} U_0 + \frac{1}{2} C_2 m \leq \frac{1}{2} U_0 + \frac{1}{2} C_2 m,$$

therefore

$$U_0 \leq C_2 m. \quad (5.27)$$

(5.27) along with the iterated (5.25) imply

$$U_n \leq U_0 e^{-c_2 n} \leq C_2 m e^{-c_2 n}. \quad (5.28)$$

Similar reasoning applies to

$$\hat{F}(e) := \phi_{\mathcal{S}_{L,p,q}}^{\infty,-\infty}(e^{-i\tilde{\sigma}W_\gamma(e,e_b)} \mathbb{1}_{e \in \gamma}),$$

yielding the same inequality for

$$V_n := \sum_{e \in \mathcal{R}_\diamond(m,n)} \hat{F}(e).$$

Combining the two inequalities with (5.29), we obtain

$$\begin{aligned} \phi_{\mathcal{S}_{L,p,q}}^{\infty,-\infty}(\gamma \cap \mathcal{R}_\diamond(m,n) \neq \emptyset) &\leq \sum_{e \in \mathcal{R}_\diamond(m,n)} \phi_{\mathcal{S}_{L,p,q}}^{\infty,-\infty}(e \in \gamma) \\ &\leq \frac{1}{2} \sum_{e \in \mathcal{R}_\diamond(m,n)} (\tilde{F}(e) + \hat{F}(e)) \leq \frac{1}{2} (U_n + V_n) \leq C_2 m e^{-c_2 n}, \end{aligned}$$

where we used the fact that

$$\tilde{F}(e) + \hat{F}(e) = 2\phi_{G,p,q}^{a,b}[\cos(\tilde{\sigma}W_\gamma(e_a, e))\mathbb{1}_{e \in \gamma}] \geq 2\phi_{G,p,q}^{a,b}(e \in \gamma) \quad (5.29)$$

(recall that  $\tilde{\sigma}$  is purely imaginary).  $\square$

For a rectangle  $\mathcal{R}$ , define the event  $\mathcal{C}_h(\mathcal{R})$  (resp.  $\mathcal{C}_v(\mathcal{R})$ ) to be the existence of an open path from the left-hand to the right-hand side (resp. from the top to the bottom) of  $\mathcal{R}$ . Similarly, we define  $\mathcal{C}_v^*(\mathcal{R})$  and  $\mathcal{C}_h^*(\mathcal{R})$  in terms of the dual open paths through the rectangle  $\mathcal{R}$  shifted by  $\frac{1}{2} + \frac{i}{2}$ , so that they belong to the dual lattice. In the next lemma, probabilities of such events for rectangles of aspect ratio  $1/3$  whenever  $p < p_{sd}$  are bounded (duality provides estimates for  $p > p_{sd}$  as well). Recall that  $\phi_{p,q}$  is the unique infinite-volume measure.

**Lemma 5.19.** *Let  $q > 4$  and  $p < p_{sd}$ , there exist  $0 < c_3, C_3 < \infty$  such that for every  $m > 0$ ,*

$$\phi_{p,q}(\mathcal{C}_v(\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket)) \leq C_3 e^{-c_3 \sqrt{m}} \text{ a.s.}$$

This result is not surprising when looking at typical (not formerly proved) subcritical behaviors. Indeed, the probability for two points to be connected by an open path in the subcritical phase should decay exponentially fast with respect to the distance between them. It implies that the probability for large rectangles to be crossed from bottom to top is extremely low.

**Proof** Throughout the proof, the side lengths of rectangles involved are implicitly rounded up (for instance,  $\sqrt{n}$  will actually mean  $\lceil \sqrt{n} \rceil$  in that context).

Fix  $p < p_{sd}$  and take  $m$  large enough satisfying

$$C_2 m e^{-c_2 \sqrt{m}} < \frac{1}{3}.$$

We will work with  $L > 2n$ ,  $n = \sqrt{m}$  and the following events, depicted in Figure 5.8:

$$A = \mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m})), \quad B = \mathcal{C}_v^*(\mathcal{R}^+(m, \sqrt{m})).$$

Recall that the exploration path is an interface between the open cluster connected to the (wired) bottom side and the dual open cluster connected to the (free) top side. Therefore, if both  $A$  and  $B$  occur, the exploration path is forced to pass through  $\mathcal{R}_\diamond(m, \sqrt{m})$ , thus Lemma 5.18 implies the estimate

$$\phi_{\mathcal{S}_{L,p,q}}^{\infty, -\infty}(A \cap B) \leq C_2 m e^{-c_2 \sqrt{m}} < \frac{1}{3}. \quad (5.30)$$

Consider the symmetry of the strip exchanging its sides and add  $\frac{1+i}{2}$  so that the lattice is mapped to its dual. Note that it preserves Dobrushin boundary conditions, *e.g.* the wired boundary conditions on the bottom part are sent to the dual wired (= free) boundary conditions on the top part. Therefore, by duality, the random-cluster measure  $\phi_{\mathcal{S}_{L,p,q}}^{\infty, -\infty}$  with

parameters  $p^*(p, q)$  and  $q$  gets mapped to the random-cluster measure on the dual strip with the same boundary conditions and parameters  $p$  and  $q$ . This symmetry also maps the event  $A$  to the event  $B$ , so that

$$\phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(B) = \phi_{\mathcal{S}_L, p^*, q}^{\infty, -\infty}(A) \geq \phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(A),$$

since  $A$  is an increasing event and  $p^*(p, q) > p$  (since  $p < p_{sd}$ ).

Hence (let us return to the fixed parameters  $p$  and  $q$ ), event  $A$  has smaller probability than  $B$ , and (5.30) implies

$$2\phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(A) - 1 \leq \phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(A) + \phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(B) - 1 \leq \phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(A \cap B) < \frac{1}{3},$$

concluding that

$$\phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(\mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m}))) = \phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}(A) < \frac{2}{3}.$$

Letting  $L$  go to infinity, the measure  $\phi_{\mathcal{S}_L, p, q}^{\infty, -\infty}$  converges to the random-cluster measure  $\phi_{p, q}$  in the upper-half plane with wired boundary conditions on  $\mathbb{Z}$ . Therefore, for  $m$  large enough, the probability of the event  $\mathcal{C}_v(\mathcal{R}^-(m, \sqrt{m}))$  given that the bonds of  $\mathbb{Z}$  are open is bounded from above by  $2/3$ .

Since these boundary conditions stochastically dominate all the others and  $A$  is an increasing event, (3.10) implies that the probability of  $A$  is always smaller than  $2/3$ , uniformly with respect to the boundary conditions on  $\mathbb{Z}$  – in other words, uniformly on what happens below the rectangle. Consider  $m$  large enough and divide the rectangle  $\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket$  into rectangles  $R_i$  (where  $i = 1 \dots \sqrt{m/3}$ ) with height  $\sqrt{3m}$  and width  $3m$ . Let  $A_i$  be the event that  $R_i$  is crossed vertically. Notice that for every  $i$ ,  $A_i$  is a translate of the event  $A$ . If there is a vertical crossing of  $\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket$ , there must exist a vertical crossing for each of these  $\sqrt{m/3}$  rectangles so that

$$\phi_{p, q}(\mathcal{C}_v(\llbracket 0, 3m \rrbracket \times \llbracket 0, m \rrbracket)) \leq \phi_{p, q}\left(\bigcap_{i=1}^{\sqrt{m/3}} A_i\right) = \prod_{i=1}^{\sqrt{m/3}} \phi_{p, q}(A_i | A_j, j < i).$$

Estimating the conditional probabilities of events  $A_i$  one by one, using the domain Markov property and the uniform bound on boundary conditions, the claim follows.  $\square$

**Proof of Theorem 5.4** Let  $p < p_{sd}$ . Lemma 5.19 implies that for  $n \geq 1$ ,

$$\phi_{p, q}(0 \leftrightarrow \mathcal{B}_n^c) \leq 4C_3 e^{-c_3 \sqrt{n}}. \quad (5.31)$$

In particular, there is no infinite cluster almost surely and  $p_c \geq p_{sd}$ .

In addition to this, an easy application of Borel-Cantelli Lemma implies that there is almost surely finitely many open circuits surrounding the origin. Hence, there is almost surely a dual infinite cluster which gives  $p^* \geq p_c^*$ . Since it is true for any  $p < p_{sd}$ ,  $p_{sd} \geq p_c^*$ , or equivalently  $p_c \leq p_{sd}$ .  $\square$

**Proof of Theorem 5.16** The rate of decay of the one arm event given in (5.31) is strong enough to harness Theorems (5.64) and (5.66) of [Gri06] (see also the argument in the previous chapter). These theorems prove that the probability decays exponentially fast.  $\square$



## Part II

### The Ising and FK-Ising models



# Chapter 6

## Two-dimensional Ising model

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**Abstract:** This chapter depicts general facts on the planar Ising model which are sometimes hard to find in the literature. One very important section is the section dealing with the low and high temperature expansions along with the definition of the spin fermionic observable.

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### 1 Definition of the Ising model

#### 1.1 Definition on the square lattice

The (spin) Ising model can be defined on any graph. However, we will once more restrict ourselves to the square lattice. Let  $G$  be a finite subgraph of  $\mathbb{L}$ , and  $b \in \{-1, +1\}^{\partial G}$ . The Ising model with *boundary conditions*  $b$  is a random assignment  $\sigma \in \{-1, 1\}^G$  of spins  $\sigma_x \in \{-1, +1\}$  (or simply  $-/+$ ) to vertices of  $G$  such that  $\sigma_x = b_x$  on  $\partial G$ , where  $\sigma_x$  denotes the spin at site  $x$ .

The *Hamiltonian* of the model is defined by

$$H_G^b(\sigma) := - \sum_{x \sim y} \sigma_x \sigma_y.$$

where the summation is over all pairs of neighboring sites  $x, y$  in  $G$ . The *partition function* of the model is

$$Z_{\beta, G}^b = \sum_{\sigma \in \{-1, 1\}^G: \sigma = b \text{ on } \partial G} \exp[-\beta H_G^b(\sigma)], \quad (6.1)$$

where  $\beta$  is the inverse temperature of the model. The Ising measure is simply a Boltzman measure with hamiltonian  $H_G^b$ . More precisely, the probability of a configuration  $\sigma$  is equal to

$$\mu_{\beta, G}^b(\sigma) = \frac{1}{Z_{\beta, G}^b} \exp[-\beta H_G^b(\sigma)]. \quad (6.2)$$

## 1.2 Special boundary conditions

similarly to the random-cluster case, several boundary conditions will be of particular importance in our study:

- all plus (resp. all minus) boundary conditions: the measure with all + (resp. all -) boundary conditions is denoted by  $\mu_{\beta,G}^+$  (resp.  $\mu_{\beta,G}^-$ ),
- free boundary conditions: the measure without any boundary conditions is called the measure with *free boundary conditions* and is denoted by  $\mu_{\beta,G}^f$ ,
- *Dobrushin* boundary conditions: assume that  $\partial G$  is a self-avoiding polygon in  $\mathbb{L}$ , and let  $a$  and  $b$  be two sites of  $\partial G$ . Orienting  $\partial G$  counterclockwise defines two oriented boundary arcs  $\partial_{ab}$  and  $\partial_{ba}$ ; the Dobrushin boundary conditions are defined to be - on  $\partial_{ab}$  and + on  $\partial_{ba}$ . We will refer to those arcs as the *arc minus* and the *arc plus* respectively. The measure associated to these boundary conditions is denoted by  $\mu_{\beta,G}^{a,b}$ . Note that the Dobrushin boundary conditions possess a useful property: they force the existence of a macroscopic interface in the model between the - cluster connected to  $\partial_{ab}$  and the + cluster connected to  $\partial_{ba}$ .

## 2 General properties

### 2.1 DLR condition

Similarly to the random-cluster case, the Ising model satisfies a strong form of domain Markov property. In words, the Ising measure conditioned on the configuration outside of a set  $V$  is equal to the Ising measure with random boundary conditions on the exterior boundary  $\partial_e V$ , *i.e.* the set of sites outside of  $V$  connected by an edge to a site in  $V$ . In particular, the Ising measure only keeps memory of the nearest neighbors (which is in some sense even stronger than the domain Markov property for the random-cluster models).

**Proposition 6.1.** *Let  $V \subset V'$  two finite sets of vertices of  $\mathbb{Z}^2$ . Let  $\sigma$  be a spin-configuration on  $V' \setminus V$ . Then*

$$\mu_{\beta}^b(\cdot|_V \mid \sigma) = \mu_{\beta}^{\sigma_{\partial_e V}}(\cdot).$$

### 2.2 Positive association of the Ising model

An event  $A$  is *increasing* if it is stable by switching of minuses to pluses. A typical example is the existence of a path of pluses between two sets of the space.

**Theorem 6.2** (FKG inequality). *Let  $G$  be a finite graph,  $b$  be boundary conditions and  $\beta > 0$ . For any two increasing events  $A, B$ ,*

$$\mu_{\beta,G}^b(A \cap B) \geq \mu_{\beta,G}^b(A) \mu_{\beta,G}^b(B).$$

**Proof** We use the FKG lattice condition (3.5) once again. Let  $\sigma$  a configuration and  $e, f$  two sites. The partial ordering is the usual ordering of  $\{-, +\}^V$ , set  $\sigma^{ef}$  (respectively  $\sigma_{ef}$ ,  $\sigma_f^e$  and  $\sigma_e^f$ ) to be the configuration agreeing with  $\sigma$  away from  $e$  and  $f$ , and with  $(\sigma_e, \sigma_f) = (+, +)$  (resp.  $(-, -)$ ,  $(+, -)$  and  $(-, +)$ ). The criterion (3.5) translates into the following claim to prove

$$H(\sigma^{ef}) + H(\sigma_{ef}) \leq H(\sigma_e^f) + H(\sigma_f^e), \quad (6.3)$$

When  $e$  and  $f$  are not adjacent, the two sides of (6.3) are equal. When  $e$  and  $f$  are adjacent, we see that the left-hand term of (6.3) corresponds to configurations with  $\sigma_e = \sigma_f$  and  $f$  agreeing, while the right-hand term corresponds to configurations with  $\sigma_e \neq \sigma_f$ . In particular, the left-hand side is indeed smaller than the right-hand one.  $\square$

**Theorem 6.3.** *Let  $G$  be a finite graph and  $\beta > 0$ . For boundary conditions  $b_1 \leq b_2$  and an increasing event  $A$ ,*

$$\mu_{\beta, G}^{b_1}(A) \leq \mu_{\beta, G}^{b_2}(A). \quad (6.4)$$

**Proof** The proof follows the same lines as the previous proof of positive association.  $\square$

Like in the random-cluster model, we say that  $\mu_{\beta, G}^{b_2}$  *stochastically dominates*  $\mu_{\beta, G}^{b_1}$ . Note that the  $+$  boundary conditions are the largest ones in the sense of stochastic ordering, while  $-$  are the smallest.

**Remark 6.4.** *There are however some differences between the Ising and the random-cluster models. On the one hand, there does not exist any increasing coupling between Ising measures at different temperatures. On the other hand, other correlation inequalities are available. Even though it is not used in this document, let us mention one of them: the Griffith-Kelly-Sherman inequality [Gri67, KS68]. For any graph  $G$ , any  $\beta > 0$  and any two sets  $A, B$  of vertices of  $G$ ,*

$$\begin{aligned} \mu_{\beta, G}^+[\sigma_A] &\geq 0, \\ \mu_{\beta, G}^+[\sigma_A \sigma_B] &\geq \mu_{\beta, G}^+[\sigma_A] \mu_{\beta, G}^+[\sigma_B], \end{aligned}$$

where  $\sigma_A = \prod_{v \in A} \sigma_v$ . These inequalities can be used to prove that the derivative with respect to  $\beta$  of  $\mu_{\beta, G}^+[\sigma_0]$  is positive.

### 3 FK-Ising model and Edwards-Sokal coupling

The Ising model can be coupled to the random-cluster model with cluster-weight  $q = 2$  [ES88]. For this reason, the  $q = 2$  random-cluster model will be called FK-Ising. We now present this coupling, called the Edwards-Sokal coupling, along with some consequences for the Ising model.

Let  $G$  be a finite graph and let  $\omega$  be a configuration of open and closed edges on  $G$ . A spin configuration  $\sigma$  can be constructed on the graph  $G$  by assigning independently to each cluster of  $\omega$  a + or - spin with probability 1/2 (more precisely all the sites of a cluster receive the same spin).

**Proposition 6.5.** *Let  $p \in (0, 1)$  and  $G$  a finite graph. If the configuration  $\omega$  is distributed according to a random-cluster measure with parameters  $(p, 2)$  and free boundary conditions, then the spin configuration  $\sigma$  is distributed according to an Ising measure with inverse temperature  $\beta = -\frac{1}{2} \ln(1 - p)$  and free boundary conditions.*

**Proof** Consider a finite graph  $G$ , let  $p \in (0, 1)$ . Consider a measure  $P$  on pairs  $(\omega, \sigma)$ , where  $\omega$  is a random-cluster configuration with free boundary conditions and  $\sigma$  is the corresponding random spin configuration, constructed as explained above. Then, for  $(\omega, \sigma)$ , we have:

$$P[(\omega, \sigma)] = \frac{1}{Z_{p,2,G}^0} p^{o(\omega)} (1-p)^{c(\omega)} 2^{k(\omega)} \cdot 2^{-k(\omega)} = \frac{1}{Z_{p,2,G}^0} p^{o(\omega)} (1-p)^{c(\omega)}.$$

Now, we construct another measure  $\tilde{P}$  on pairs of percolation configurations and spin configurations as follows. Let  $\tilde{\sigma}$  be a spin configuration distributed according to an Ising model with inverse temperature  $\beta$  satisfying  $e^{-2\beta} = 1 - p$  and free boundary conditions. We deduce  $\tilde{\omega}$  from  $\tilde{\sigma}$  by closing all edges between neighboring sites with different spins, and by independently opening with probability  $p$  edges between neighboring sites with same spins. Then, for any  $(\tilde{\omega}, \tilde{\sigma})$ ,

$$\tilde{P}[(\tilde{\omega}, \tilde{\sigma})] = \frac{e^{-2\beta r(\tilde{\sigma})} p^{o(\tilde{\omega})} (1-p)^{a-o(\tilde{\omega})-r(\tilde{\sigma})}}{Z_{\beta,p}^f} = \frac{p^{o(\tilde{\omega})} (1-p)^{c(\tilde{\omega})}}{Z_{\beta,p}^f}$$

where  $a$  is the number of edges of  $G$  and  $r(\tilde{\sigma})$  the number of edges between sites with different spins.

Note that the two previous measures are in fact defined on the same set of 'compatible' pairs of configurations: if  $\sigma$  has been obtained from  $\omega$ , then  $\omega$  can be obtained from  $\sigma$  via the second procedure described above, and the same is true in the reverse direction for  $\tilde{\omega}$  and  $\tilde{\sigma}$ . Therefore,  $P = \tilde{P}$  and the marginals of  $P$  are the random-cluster with parameters  $(p, 2)$  and the Ising model at inverse temperature  $\beta$ , which is the claim.  $\square$

The coupling gives a randomized procedure to obtain a spin-Ising configuration from an FK-Ising configuration (it suffices to assign random spins). The proof of Proposition 6.5 provides a randomized procedure to obtain an FK-Ising configuration from a spin-Ising configuration.

If one considers wired boundary conditions for the random-cluster, the Edwards-Sokal coupling provides us with an Ising configuration with + boundary conditions (or -, the two cases being symmetric). We omit the details, since the generalization is straightforward.

An important consequence of the Edwards-Sokal coupling is the relation between Ising correlations and random-cluster connectivity properties. Indeed, two sites which are connected in the random-cluster configuration must have the same spin, while sites which are not have independent spins. This implies

**Proposition 6.6.** *For  $p \in (0, 1)$ ,  $G$  a finite graph and  $\beta = -\frac{1}{2} \ln(1 - p)$ ,*

$$\begin{aligned}\mu_{\beta, G}^f[\sigma_x \sigma_y] &= \phi_{p, 2, G}^0(x \leftrightarrow y), \\ \mu_{\beta, G}^+[\sigma_x] &= \phi_{p, 2, G}^1(x \leftrightarrow \partial G).\end{aligned}$$

## 4 Infinite-volume measures and phase transition

### 4.1 Definition of infinite-volume measures

Theorem 6.3 allows us to define infinite-volume measures as follows. Consider the nested sequence of boxes  $\Lambda_n = [-n, n]^2$ . For any  $N > 0$  and any increasing event  $A$  depending only on edges in  $\Lambda_N$ , the sequence  $(\mu_{\beta, \Lambda_n}^+(A))_{n \geq N}$  is decreasing. Indeed, any configuration of spins in  $\partial\Lambda_n$  being smaller than all  $+$ , the restriction of  $\mu_{\beta, \Lambda_{n+1}}^+$  to  $\Lambda_n$  is stochastically dominated by  $\mu_{\beta, \Lambda_n}^+$ . One can then define a limit, denoted by  $\mu_{\beta}^+(A)$ , which does not depend on  $N$ . In this way,  $\mu_{\beta}^+$  is defined for increasing events depending on a finite number of sites. It can be further extended into a probability measure on the  $\sigma$ -algebra spanned by cylindrical events (events measurable in terms of a finite number of spins). The resulting measure  $\mu_{\beta}^+$  is called the infinite-volume Ising model with  $+$  boundary conditions.

Observe that, similarly to the random-cluster model, one could construct (a priori) different infinite-volume measures, for instance with  $-$  boundary conditions (the corresponding measure is denoted by  $\mu_{\beta}^-$ ). If infinite-volume measures are defined from a property of compatibility with finite volume measures, then  $\mu_{\beta}^+$  and  $\mu_{\beta}^-$  are extremal among infinite-volume measures of parameter  $\beta$ . In particular, if  $\mu_{\beta}^+ = \mu_{\beta}^-$ , there exists a unique infinite volume measure.

### 4.2 Phase transition

The Ising model in infinite-volume exhibits a phase transition at some critical inverse temperature  $\beta_c$ , above which a spontaneous magnetization appears.

**Theorem 6.7.** *There exists  $\beta_c \in (0, \infty)$  such that:*

- for any  $\beta < \beta_c$ ,  $\mu_{\beta}^+[\sigma_0] = 0$ ,
- for any  $\beta > \beta_c$ ,  $\mu_{\beta}^+[\sigma_0] > 0$ .

Furthermore,  $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$ .

**Proof** Proposition 6.6 immediately implies that  $\beta_c = -\frac{1}{2} \ln[1 - p_c(2)]$  by passing to the infinite-volume. Then, Theorem 4.1 concludes the proof.  $\square$

The proof of the existence of the phase transition on general graphs harnesses the Edwards-Sokal only. Without the help of the FK-Ising model, one can use the GKS inequality (see Remark 6.4) to show directly that  $\mu_\beta^+[\sigma_0]$  is increasing and thus deduce the existence of  $\beta_c$ . Let us mention that the inverse critical temperature was identified (without proof) by Kramers and Wannier [KW41a, KW41b], using the duality between low and high temperature expansions of the Ising model that we present in the next section. Its first rigorous derivation is due to Yang [Yan52]. He uses the exact formula for the (infinite-volume) partition function to compute the spontaneous magnetization of the model (it was previously computed non-rigorously by Onsager). This quantity provides one criterion for localizing the critical point. The first probabilistic computation of the critical inverse temperature is due to Aizenman, Barsky and Fernández [ABF87]. This manuscript contains two alternative proofs of this result, the one mentioned earlier, which harnesses Chapter 4, and the other presented in Chapter 8.

### 4.3 Classification of Gibbs measures

Infinite-volume measures for the Ising models are typical *Gibbs measures* (see Section 2.3.2. of [VEFS93] for details on Gibbs measures). Their classification is thus an important task. While the question is difficult in high-dimension, it is understood in dimension two.

**Proposition 6.8.** *When  $\beta < \beta_c$ , there is a unique infinite-volume measure.*

**Proof** It is sufficient to prove that  $\mu_\beta^+ = \mu_\beta^-$ . Note that we already know  $\mu_\beta^+ \geq \mu_\beta^-$ . Define  $n_A = \frac{1}{2}(1 + \sigma_A)$ , where  $\sigma_A = \prod_{x \in A} \sigma_x$ . Since  $\sum_{x \in A} n_x - n_A$  is increasing, the FKG inequality implies

$$\mu_\beta^+(\sum_{x \in A} n_x - n_A) \geq \mu_\beta^-(\sum_{x \in A} n_x - n_A)$$

which becomes

$$\sum_{x \in A} \mu_\beta^+(n_x) - \mu_\beta^-(n_x) \geq \mu_\beta^+(n_A) - \mu_\beta^-(n_A).$$

Since  $\beta < \beta_c$ , we have  $\mu_\beta^+(\sigma_0) = \mu_\beta^-(\sigma_0) = 0$ , we find  $\mu_\beta^+(n_A) = \mu_\beta^-(n_A)$  for any finite set  $A$ . Yet, the space of functions  $n_A$  spans all measurable functions, so that  $\mu_\beta^+$  and  $\mu_\beta^-$  coincide.  $\square$

The classification when  $\beta > \beta_c$  is more interesting. The space of infinite-volume measures is a simplex.

**Theorem 6.9** (Aizenman, Higushi [Aiz80, Hig81], recent proof in [CV10]). *Fix  $\beta > \beta_c$ . The only two extremal Gibbs measures are  $\mu_\beta^+$  and  $\mu_\beta^-$ .*

This result is no longer true in higher dimensions: non translational-invariant Gibbs measures can be constructed using 3D Dobrushin domains [Dob72]. For instance, one can consider boxes with + boundary conditions on the upper half-space and – boundary conditions on the lower half-space. These boundary conditions imply the existence of a surface between + and –. In dimensions 3 and higher and at very high  $\beta$ , this surface does not fluctuate much and it is possible to prove that the infinite measure constructed by nested sequences of such boxes is not translationally invariant in the vertical direction.

The classification at criticality is in general much more difficult. For the Ising model, this is not the case and it turns out that there exists a unique infinite-volume measure at criticality. Since this fact plays a role in the proof of conformal invariance, we now sketch an elementary proof due to W. Werner (the complete proof can be found in [Wer09b]).

**Proposition 6.10.** *There exists a unique infinite-volume FK-Ising measure with parameter  $p_c$  and there is almost surely no infinite cluster under this measure. Correspondingly, there exists a unique infinite-volume spin Ising measure at  $\beta_c$ .*

**Proof** As described above, it is sufficient to prove that  $\phi_{p_{sd},2}^0 = \phi_{p_{sd},2}^1$ . First note that there is no infinite cluster for  $\phi_{p_{sd},2}^0$  thanks to Proposition 3.18. Via the Edwards-Sokal coupling, the infinite-volume Ising measure with free boundary conditions, denoted by  $\mu_{\beta_c}^f$ , can be constructed by coloring clusters of the measure  $\phi_{p_{sd},2}^0$ . Since there is no infinite cluster, this measure is obviously symmetric by global exchange of +/–. In particular, the argument of Proposition 3.18 can be applied to prove that there are neither + nor – infinite clusters. Therefore, fixing a box, there exists a + star-connected circuit surrounding the box with probability one (two vertices  $x$  and  $y$  are said to be *star-connected* if  $y$  is one of the eight closest neighbors to  $x$ ).

One can then argue that the configuration inside the box stochastically dominates the Ising configuration for the infinite-volume measure with + boundary conditions (the circuit of spin + behaves like + boundary conditions). Thus,  $\mu_{\beta_c}^f$  restricted to the box (in fact to any box) stochastically dominates  $\mu_{\beta_c}^+$ . It implies that  $\mu_{\beta_c}^f \geq \mu_{\beta_c}^+$ . Since the other inequality is obvious,  $\mu_{\beta_c}^f$  and  $\mu_{\beta_c}^+$  are equal.

Via Edwards-Sokal's coupling again,  $\phi_{p_{sd},2}^0 = \phi_{p_{sd},2}^1$  and there is no infinite cluster at criticality. Moreover,  $\mu_{\beta_c}^- = \mu_{\beta_c}^f = \mu_{\beta_c}^+$  and there is a unique infinite-volume Ising measure at criticality.  $\square$

## 5 High and low temperature expansions and Kramers-Wannier duality

### 5.1 The low temperature expansion

The *low temperature expansion* of the Ising model is a graphical representation on the dual lattice. Fix a spin configuration  $\sigma$  for the Ising model on  $G$  with + boundary conditions.

The *collection of contours* of a spin configuration  $\sigma$  is the set of interfaces (edges of the dual graph) separating + and - clusters. In a collection of contours, an even number of dual edges automatically emanates from each dual vertex. Reciprocally, any family of dual edges with an even number of edges emanating from each dual vertex is the collection of contours of exactly one spin configuration (since we fix + boundary conditions).

The interesting feature of the low temperature expansion is that properties of the Ising model can be restated in terms of this graphical representation. We only give the example of the partition function on  $G$  but other quantities can be computed similarly. Let  $\mathcal{E}_{G^*}$  be the set of possible collections of contours, and let  $|\omega|$  be the number of edges of a collection of contours  $\omega$ , then

$$Z_{\beta,G}^+ = e^{\beta \# \text{ edges in } G^*} \sum_{\omega \in \mathcal{E}_{G^*}} (e^{-2\beta})^{|\omega|}. \quad (6.5)$$

## 5.2 High temperature expansion

The *high temperature expansion* of the Ising model is a graphical representation on the primal lattice itself. It is not a geometric representation since one cannot map a spin configuration  $\sigma$  to a subset of configurations in the graphical representation, but a rather convenient way to represent correlations between spins using statistics of contours. It is based on the following identity:

$$e^{\beta \sigma_x \sigma_y} = \cosh(\beta) + \sigma_x \sigma_y \sinh(\beta) = \cosh(\beta) [1 + \tanh(\beta) \sigma_x \sigma_y] \quad (6.6)$$

**Proposition 6.11.** *Let  $G$  be a finite graph and  $a, b$  be two sites of  $G$ . At inverse temperature  $\beta > 0$ ,*

$$Z_{\beta,G}^f = 2^{\# \text{ vertices } G} \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \in \mathcal{E}_G} \tanh(\beta)^{|\omega|} \quad (6.7)$$

$$\mu_{\beta,G}^f[\sigma_a \sigma_b] = \frac{\sum_{\omega \in \mathcal{E}_G(a,b)} \tanh(\beta)^{|\omega|}}{\sum_{\omega \in \mathcal{E}_G} \tanh(\beta)^{|\omega|}}, \quad (6.8)$$

where  $\mathcal{E}_G$  (resp.  $\mathcal{E}_G(a,b)$ ) is the set of families of edges of  $G$  such that an even number of edges emanates from each vertex (resp. except at  $a$  and  $b$ , where an odd number of edges emanates).

The notation  $\mathcal{E}_G$  coincides with the definition  $\mathcal{E}_{G^*}$  in the low temperature expansion for the dual lattice.

**Proof** Let us start with the partition function (6.7). Let  $E$  be the set of edges of  $G$ . We know

$$\begin{aligned}
 Z_{\beta,G}^f &= \sum_{\sigma} \prod_{[xy] \in E} e^{\beta \sigma_x \sigma_y} \\
 &= \cosh(\beta)^{\# \text{ edges in } G} \sum_{\sigma} \prod_{[xy] \in E} [1 + \tanh(\beta) \sigma_x \sigma_y] \\
 &= \cosh(\beta)^{\# \text{ edges in } G} \sum_{\sigma} \sum_{\omega \subset E} \tanh(\beta)^{|\omega|} \prod_{e=[xy] \in \omega} \sigma_x \sigma_y \\
 &= \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \subset E} \tanh(\beta)^{|\omega|} \sum_{\sigma} \prod_{e=[xy] \in \omega} \sigma_x \sigma_y
 \end{aligned}$$

where we used (6.6) in the second equality. Notice that  $\sum_{\sigma} \prod_{e=[xy] \in \omega} \sigma_x \sigma_y$  equals  $2^{\# \text{ vertices } G}$  if  $\omega$  is in  $\mathcal{E}_G$ , and 0 otherwise, hence proving (6.7).

Fix  $a, b \in G$ . By definition,

$$\mu_{\beta,G}^f[\sigma_a \sigma_b] = \frac{\sum_{\sigma} \sigma_a \sigma_b e^{-\beta H(\sigma)}}{\sum_{\sigma} e^{-\beta H(\sigma)}} = \frac{\sum_{\sigma} \sigma_a \sigma_b e^{-\beta H(\sigma)}}{Z_{\beta,G}^f}. \quad (6.9)$$

The second identity boils down to proving that the right hand terms of (6.8) and (6.9) are equal, *i.e.*

$$\sum_{\sigma} \sigma_a \sigma_b e^{-\beta H(\sigma)} = 2^{\# \text{ vertices } G} \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \in \mathcal{E}_G(a,b)} \tanh(\beta)^{|\omega|}.$$

The first lines of the computation for the partition function are the same, and we end up with:

$$\begin{aligned}
 \sum_{\sigma} \sigma_a \sigma_b e^{-\beta H(\sigma)} &= \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \subset E} \tanh(\beta)^{|\omega|} \sum_{\sigma} \sigma_a \sigma_b \prod_{e=[xy] \in \omega} \sigma_x \sigma_y \\
 &= 2^{\# \text{ vertices } G} \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \in \mathcal{E}_G(a,b)} \tanh(\beta)^{|\omega|}
 \end{aligned}$$

since  $\sum_{\sigma} \sigma_a \sigma_b \prod_{e=[xy] \in \omega} \sigma_x \sigma_y$  equals  $2^{\# \text{ vertices } G}$  if  $\omega \in \mathcal{E}_G(a,b)$ , and 0 otherwise.  $\square$

The set  $\mathcal{E}_G$  is the set of collections of loops on  $G$  when forgetting the way we draw loops (since some elements of  $\mathcal{E}_G$ , like a 'figure in eight', can be decomposed into loops in several ways), while  $\mathcal{E}_G(a,b)$  is the set of collections of loops on  $G$  together with one curve from 0 to  $a$ .

Let us mention that the high-temperature expansion can be extended to other Ising models. For instance, the partition function of the Ising model on  $(G, a, b)$  with free boundary conditions conditioned on the event that  $a$  and  $b$  have the same spin is given by

$$Z_{\beta,G}^{a,b} = 2^{\# \text{ vertices } G} \cosh(\beta)^{\# \text{ edges in } G} \sum_{\omega \in \mathcal{E}_G(a,b)} \tanh(\beta)^{|\omega|}. \quad (6.10)$$

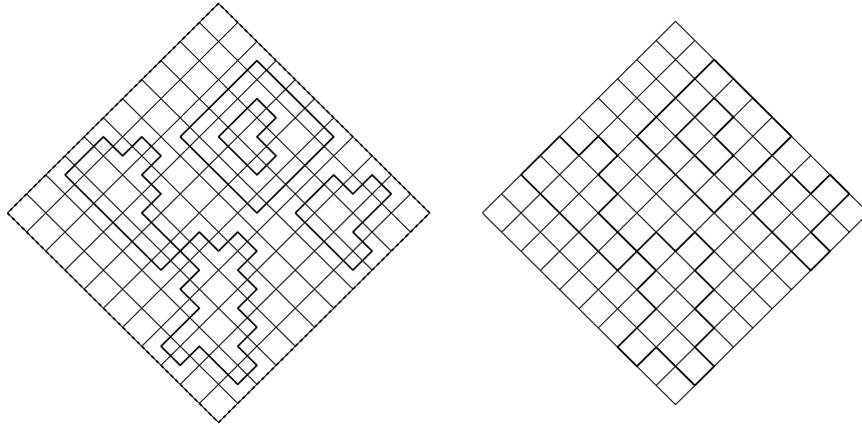


Figure 6.1: The possible collections of contours for + boundary conditions in the low-temperature expansion do not contain edges between boundary sites of  $G$ . Therefore, they correspond to collections of contours in  $\mathcal{E}_{G^*}$ , which are exactly the collection of contours involved in the high-temperature expansion of the Ising model on  $G^*$  with free boundary conditions.

### 5.3 Two applications: Kramers-Wannier duality and Peierls's argument

**Proposition 6.12** (Kramers-Wannier duality). *Let  $\beta > 0$  and define  $\beta^* \in (0, \infty)$  such that  $\tanh(\beta^*) = e^{-2\beta}$ , then for every graph  $G$ ,*

$$2^{\# \text{ vertices } G^*} \cosh(\beta^*)^{\# \text{ edges in } G^*} Z_{\beta, G}^+ = (e^\beta)^{\# \text{ edges in } G^*} Z_{\beta^*, G^*}^f. \quad (6.11)$$

**Proof** When writing the contour of connected components for Ising with + boundary conditions, the only edges of  $\mathbb{L}^*$  used are those of  $G^*$ . Indeed, edges between boundary sites cannot be present since boundary spins are +. Thus, the right and left hand side terms of (6.11) both correspond to the sum over  $\mathcal{E}_{G^*}$  of  $(e^{-2\beta})^{|\omega|}$  or equivalently of  $\tanh(\beta^*)^{|\omega|}$ , implying the equality (see Fig. 6.1).  $\square$

We are now in a position to expose Kramers-Wannier argumentation. Physicists expect the partition function to exhibit only one singularity, localized at the critical point. If  $\beta_c^* \neq \beta_c$ , there would be at least two singularities, at  $\beta_c$  and  $\beta_c^*$ , thanks to the previous relation between partitions functions at these two temperatures. Thus,  $\beta_c$  should be equal to  $\beta_c^*$ , which implies  $\beta_c = \frac{1}{2} \ln(1 + \sqrt{2})$ . Of course, the assumption that there exists a unique singularity is hard to justify.

For completeness, let us mention Peierls's argument, which rigorously proves that  $\beta_c \in (0, \infty)$ . It harnesses the low and high temperature expansions and is of great historical significance. Interestingly, this argument has been generalized to many models, including the random-cluster model. In particular, the (omitted) proof that the critical value of

the random-cluster model  $p_c(q)$  is not equal to 0 or 1 (Theorem 3.16) follows a similar argument.

**Proposition 6.13** (Peierls argument [Pei36]). *The critical inverse temperature  $\beta_c$  on the square lattice is strictly positive and finite.*

**Proof** Let us prove that  $\beta_c$  is finite. We wish to estimate  $\mu_{\beta,G}^+[\sigma_0]$  when  $\beta$  is very large. Since

$$\mu_{\beta,G}^+[\sigma_0] = 2\mu_{\beta,G}^+[\sigma_0 = 1] - 1,$$

it is sufficient to show that  $\mu_{\beta,G}^+[\sigma_0 = -1] < 1/2$  uniformly in the graph  $G$ . The observation is that  $\{\sigma_0 = -1\}$  is included in the event that there exists a circuit in the low-temperature expansion surrounding 0. Thus,

$$\begin{aligned} \mu_{\beta,G}^+[\sigma_0 = -1] &\leq \frac{\sum_{\omega \in \mathcal{E}_{G^*} : \gamma \text{ surrounding } 0} e^{-2\beta|\omega|}}{\sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta|\omega|}} \\ &\leq \frac{\sum_{\gamma \text{ surrounding } 0} e^{-2\beta|\gamma|} \sum_{\omega \in \mathcal{E}_{G^* \setminus \gamma}} e^{-2\beta|\omega|}}{\sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta|\omega|}} \\ &\leq \sum_{\gamma \text{ surrounding } 0} e^{-2\beta|\gamma|} \leq \sum_{n=1}^{\infty} n4^n e^{-2\beta n} < 1/2 \end{aligned}$$

for  $\beta$  large enough. In the second line, we used the fact that

$$\sum_{\omega \in \mathcal{E}_{G^* \setminus \gamma}} e^{-2\beta|\omega|} \leq \sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta|\omega|}$$

and in the third the fact that the number of paths of length  $n$  surrounding the origin is smaller than  $n4^n$ .

The inequality  $0 < \beta_c$  can be obtained using the high-temperature expansion instead of the low-temperature.  $\square$

## 5.4 Fermionic observable in Dobrushin domains

Let  $(\Omega, a, b)$  be a simply connected domain with two marked points on the boundary. Let  $\Omega_\delta^\circ$  be the medial graph of  $\Omega_\delta$  composed of all the vertices of  $\mathbb{L}_\delta^\circ$  bordering a black face associated to  $\Omega_\delta$ , see Fig 6.3. This definition is non-standard since we include medial vertices not associated to edges of  $\Omega_\delta$ . Let  $a_\delta$  and  $b_\delta$  be two vertices of  $\partial\Omega_\delta^\circ$  close to  $a$  and  $b$ . We call the triplet  $(\Omega_\delta^\circ, a_\delta, b_\delta)$  a *spin-Dobrushin domain*.

Let  $\mathcal{E}(a_\delta, z_\delta) = \mathcal{E}_{\Omega_\delta}(a_\delta, z_\delta)$  be the set of collections of contours on  $\Omega_\delta$  composed of loops and one interface starting at  $a_\delta$  and finishing at  $z_\delta$ . Recall that there is an ambiguity in the way loops are drawn. In order to solve this issue, every  $\omega \in \mathcal{E}_{\Omega_\delta}(a_\delta, z_\delta)$  is associated to a unique family of loops with one interface by forcing every loop and interface to take

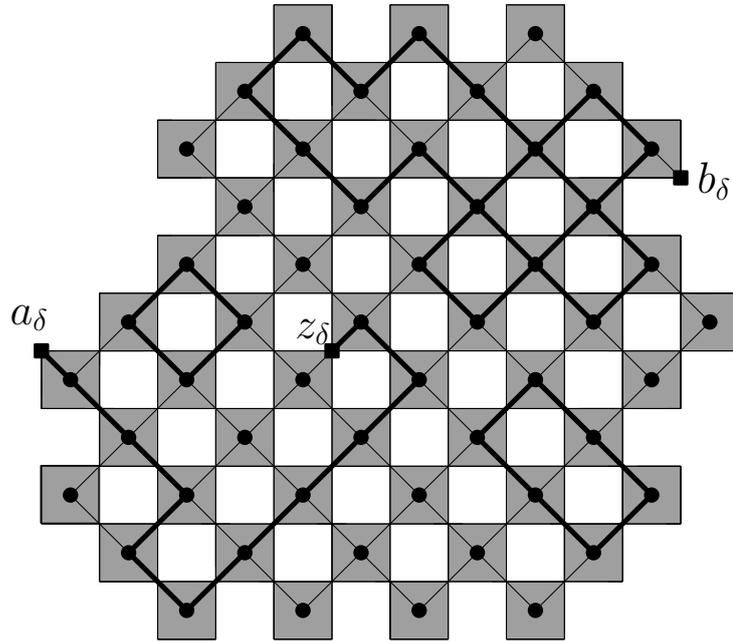


Figure 6.2: An example of a collection of contours in  $\mathcal{E}(a_\delta, z_\delta)$  on the lattice  $\Omega_\delta$ .

a turn to the left whenever there is an ambiguity<sup>1</sup>. The unique interface from  $a_\delta$  to  $z_\delta$  is called  $\gamma = \gamma(\omega)$ .

The *winding*  $W_\Gamma(z, z')$  of a curve  $\Gamma$  between two sites  $z$  and  $z'$  of the medial graph is the total (signed) rotation (in radians) that the curve makes from  $z$  to  $z'$ . With these notations, we can define the spin-Ising fermionic observable.

**Definition 6.14.** *On a spin Dobrushin domain  $(\Omega_\delta^\circ, a_\delta, b_\delta)$ , the spin-Ising fermionic observable at  $z_\delta \in \Omega_\delta^\circ$  is defined by*

$$F_{\Omega_\delta, a_\delta, b_\delta}(z_\delta) = \frac{\sum_{\omega \in \mathcal{E}(a_\delta, z_\delta)} e^{-\frac{1}{2}iW_{\gamma(\omega)}(a_\delta, z_\delta)} (\sqrt{2} - 1)^{|\omega|}}{\sum_{\omega \in \mathcal{E}(a_\delta, b_\delta)} e^{-\frac{1}{2}iW_{\gamma(\omega)}(a_\delta, b_\delta)} (\sqrt{2} - 1)^{|\omega|}}.$$

The complex modulus of the denominator of the fermionic observable is connected to the partition function of a conditioned critical Ising model. Indeed, fix  $b_\delta \in \partial\Omega_\delta^\circ$ . Even though  $\mathcal{E}(a_\delta, b_\delta)$  is not exactly a high-temperature expansion (since there are two half-edges starting from  $a_\delta$  and  $b_\delta$  respectively), it is in bijection with the set  $\mathcal{E}(a, b)$ . Therefore, (6.10) can be used to relate the denominator of the fermionic observable to the partition function of the Ising model on the primal graph with free boundary conditions conditioned on the fact that  $a$  and  $b$  have the same spin.

The weights of edges are critical (since  $\sqrt{2} - 1 = e^{-2\beta_c}$ ). Therefore, the Kramers-Wannier duality has an enlightning interpretation here. The high-temperature expansion

<sup>1</sup>This arbitrary choice is physically irrelevant. We could have chosen any other rule.

can be thought of as the low-temperature expansion of an Ising model on the dual graph, where the dual graph is constructed by adding one layer of dual vertices around  $\partial G$ , see Fig. 6.2. Now, the existence of an interface between  $a_\delta$  and  $b_\delta$  is equivalent to the existence of an interface between pluses and minuses in this new Ising model. Therefore, it corresponds to a model with Dobrushin boundary conditions on the dual graph. This fact is not surprising since the dual boundary conditions of the free boundary conditions conditioned on  $\sigma_a = \sigma_b$  are the Dobrushin ones. More importantly, it suggests a connection between the fermionic observable and the interface in Dobrushin domains.

Let us mention that the numerator of the observable has also an interpretation using high-temperature expansions. In fact, it can be shown that it corresponds to the high-temperature expansion of the partition function of an Ising model with a disorder operator at  $z_\delta$ . More precisely, this operator introduces a monodromy at  $z_\delta$ . Every time one turns around  $z_\delta$ , the spins are reversed. Equivalently, it boils down to reverse the correlation constants along an arbitrary simple curve from  $z_\delta$  to the boundary of the domain.

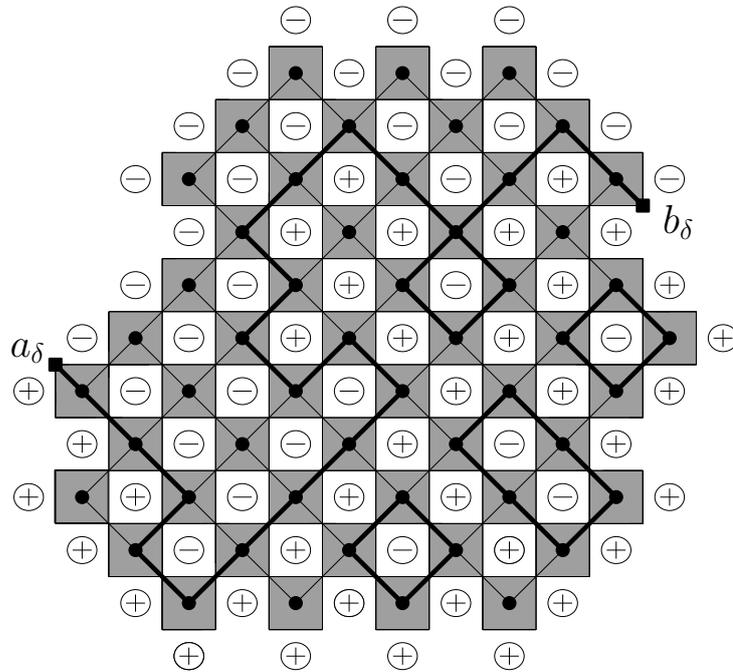


Figure 6.3: A high temperature expansion of an Ising model on the primal lattice together with the corresponding configuration on the dual lattice. The constraint that  $a_\delta$  is connected to  $b_\delta$  corresponds to the partition function of the Ising model with  $+/-$  boundary conditions on the domain.

## 6 Potts models and random-cluster models

The Edwards-Sokal coupling is not specific to the Ising model. More generally, the random-cluster with integer parameter  $q \geq 2$  can be coupled with the Potts model. The Potts model with  $q$  colors is a random  $q$ -coloring of a finite graph  $G$ . Let us restrict to the free boundary conditions case. The energy of a configuration  $\sigma$  is given by

$$H_{q,G}(\sigma)^f := -2 \sum_{x \sim y} 1_{\sigma_x = \sigma_y}$$

and the probability at inverse temperature  $\beta$  by

$$\frac{e^{-\beta H_{q,G}^f(\sigma)}}{\sum_{\tilde{\sigma}} e^{-\beta H_{q,G}^f(\tilde{\sigma})}},$$

where the summation is over any  $q$ -coloring of  $G$ .

Let  $q \geq 2$  and let  $G$  be a finite graph. Assume a configuration  $\omega$  of open and closed edges on  $G$  is given. One can deduce a  $q$ -coloring  $\sigma$  of the graph  $G$  by assigning independently to each cluster of  $\omega$  a color among the  $q$  colors, each with probability  $1/q$ .

**Proposition 6.15.** *Let  $p \in (0,1)$  and  $G$  a finite graph. If the configuration  $\omega$  is distributed according to a random-cluster measure with parameters  $(p,q)$  and free boundary conditions, then the coloring  $\sigma$  is distributed according to a Potts measure with inverse temperature  $\beta = -\frac{1}{2} \ln(1-p)$  and free boundary conditions.*

**Proof** The proof of the Edwards-Sokal coupling works *mutatis mutandis* in this case.  $\square$

The coupling has many important implications. For instance, it allows us to sample efficiently Potts configurations via algorithms on the random-cluster model such as Swendsen-Wang [SW87] (see also Section 8 of [Gri06] and references therein). As before, we also obtain a dictionary between properties of Potts models and their random-cluster representations. Let us recall that one of the principal motivations for geometric or graphical representations is the obtention of additional correlations inequalities. They can take different forms depending on the model under study. In our case, random-cluster measures verify the FKG inequality while Potts models do not. In fact, there is no straightforward notion of increasing events for Potts models and the equivalent of the spin-Ising FKG inequality does not exist. This is one (among many others) reason which prevents mathematicians and physicists from understanding the Potts model in a satisfying fashion.

Potts models also exhibit a phase transition at an inverse temperature  $\beta_c(q)$ . Below this critical inverse temperature, there is a unique Gibbs measure, while above this inverse temperature, there are multiple Gibbs measures. Theorem 4.1 has the following important corollary:

**Theorem 6.16.** *For  $q \geq 2$ , the critical inverse temperature of the  $q$ -color Potts model is  $\beta_c(q) = \frac{1}{2} \log(1 + \sqrt{q})$ . In addition, correlations decays exponentially fast when  $\beta < \beta_c$  and the surface tension is strictly positive when  $\beta > \beta_c$ .*

# Chapter 7

## Conformal invariance of the FK-Ising and Ising models

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**Abstract:** This section is devoted to the proof of conformal invariance of the FK-Ising and Ising models. These two results are due to Smirnov and Chelkak-Smirnov. Proofs are included for self-containedness and since techniques invoked in them are crucial in the next chapters. The proofs are adapted from lecture notes written by the author and S. Smirnov for the Clay Probability Summer School in Buzios, 2010 [DCS11].

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There are many different ways to define conformal invariance of a model. A geometric definition of conformal invariance could be that interfaces in the model are conformally invariant. Alternatively, conformal invariance can also refer to the fact that relevant observables of the model are conformally covariant in the scaling limit. More precisely, that a family of observables in discrete domains converge in the scaling-limit to a conformally covariant family of functions.

**Definition 7.1.** *A family of functions  $F_\Omega : \Omega \rightarrow \mathbb{C}$  indexed by simply-connected domains (sometimes with marked points on the boundary) is conformally covariant if there exists  $\alpha > 0$  such that for any domain  $\Omega$  and any conformal map  $\psi : \Omega \rightarrow \mathbb{C}$  (i.e. holomorphic and one-to-one),*

$$F_\Omega(z) = \psi'(z)^\alpha \cdot F_{\psi(\Omega)}(\psi(z)) \quad \text{for every } z \in \Omega.$$

*If  $\alpha = 0$ , the family is said to be conformally invariant.*

Note that an archetype of a conformally covariant family of functions is the solution to boundary problems such as Dirichlet or Riemann problems.

A family of observables for random-cluster models were introduced in Chapter 5. In fact, these observables are weakly discrete-holomorphic and it is reasonable to expect that their scaling limits are holomorphic. The boundary conditions can be determined and correspond to discrete Riemann-Hilbert boundary problems. It provides a good hint

that the scaling-limit of the observable is conformally covariant. Unfortunately, weakly discrete-holomorphic functions are not determined by their boundary conditions and it is not possible at the moment to prove that parafermionic observables converge in the scaling limit to a conformally covariant family of functions.

When  $q = 2$  (the case of FK-Ising), the observable satisfies specific additional integrability properties that allow us to compute it very explicitly. Shortly, the complex argument of the edge-observable is determined since the spin  $\sigma$  equals  $1/2$  and the winding at an edge takes values in a set of the form  $W_0 + 2\pi\mathbb{Z}$ . This additional information allowed Smirnov to prove that the observables are  $s$ -holomorphic and converge to a conformally covariant family of functions.

In this chapter, discrete Dobrushin domains are discretizations of simply connected domains  $\Omega$  with two marked points  $a$  and  $b$  on the boundary. Furthermore, we assume  $b_\delta$  is the south-east corner of the black face associated to  $b$ .

**Theorem 7.2** (Conformal invariance of FK-Ising, Smirnov [Smi10a]). *Let  $(\Omega, a, b)$  be a simply connected domain with two marked points on its boundary. Let  $F_\delta$  be the vertex fermionic observable in  $(\Omega_\delta^\circ, a_\delta, b_\delta)$  defined by*

$$F_\delta(v) = \frac{1}{2} \sum_{e \sim v} F_\delta(e), \quad (7.1)$$

where  $e \sim v$  means that  $v$  is an endpoint of  $e$ . We have

$$\frac{1}{\sqrt{2\delta}} F_\delta(\cdot) \rightarrow \sqrt{\phi'(\cdot)} \quad \text{when } \delta \rightarrow 0 \quad (7.2)$$

uniformly on any compact subset of  $\Omega$ , where  $\phi$  is any conformal map from  $\Omega$  to the strip  $\mathbb{R} \times (0, 1)$  mapping  $a$  to  $-\infty$  and  $b$  to  $\infty$ <sup>1</sup>.

The Ising model is also conformally invariant in this sense: the conformally covariant observable is the fermionic observable introduced in Chapter 6. We also assume Dobrushin domains are approximation of continuous ones, and that  $b_\delta$  is the south-east corner of the black face associated to  $b$ .

**Theorem 7.3** (Conformal invariance of the Ising model, Chelkak-Smirnov [CS09]). *Let  $(\Omega, a, b)$  be a simply connected domain with two marked points on its boundary, the boundary is assumed to be smooth in a neighborhood of  $b$ . Let  $F_\delta$  be the fermionic spin observable in  $(\Omega_\delta, a_\delta, b_\delta)$ , then*

$$F_\delta(\cdot) \rightarrow \sqrt{\frac{\psi'(\cdot)}{\psi'(b)}} \quad \text{when } \delta \rightarrow 0 \quad (7.3)$$

uniformly on every compact subset of  $\Omega$ , where  $\psi$  is any conformal map from  $\Omega$  to the upper half-plane  $\mathbb{H}$ , mapping  $a$  to  $\infty$  and  $b$  to  $0$ .

<sup>1</sup>The strip is two-ended:  $-\infty$  (resp.  $\infty$ ) is the end on the left (resp. on the right).

Before diving into the proof, let us mention that conformal invariance of these observables is sufficient to prove a much stronger form of conformal invariance, namely conformal invariance of interfaces. This discussion is deferred to Chapter 11.

The proofs of conformal invariance of the FK-Ising (due to Smirnov) and Ising (due to Chelkak-Smirnov) are presented in Sections 1 and 2 respectively. The arguments involved in this proof will be useful in the next chapters. Let us mention that conformal invariance of discrete models is known in a very few other cases (namely random-walks via Lévy’s theorem, loop-erased random walks [LSW04a], dimers [Ken00], site-percolation on the triangular lattice [Smi01] and uniform-spanning trees [LSW04a]).

## 1 Convergence of the FK fermionic observable

In this section, fix a simply connected domain  $(\Omega, a, b)$  with two points on the boundary. For  $\delta > 0$ , always consider a discrete FK Dobrushin domain  $(\Omega_\delta^\circ, a_\delta, b_\delta)$  and the critical FK-Ising model with Dobrushin boundary conditions on it. Since the domain is fixed, set  $F_\delta = F_{\Omega_\delta^\circ, a_\delta, b_\delta, p_{sd}}$  for the FK fermionic observable.

The proof of convergence is in three steps:

- First, prove the  $s$ -holomorphicity of the observable.
- Second, prove the convergence of the function  $H_\delta$  naturally associated to the  $s$ -holomorphic functions  $F_\delta/\sqrt{2\delta}$  (see Section 4 of Chapter 2).
- Third, prove that  $F_\delta/\sqrt{2\delta}$  converges to  $\sqrt{\phi'}$ .

### 1.1 $s$ -holomorphicity of the (vertex) fermionic observable for FK-Ising.

The two next lemmata deal with the edge fermionic observable. They are the key steps of the proof of the  $s$ -holomorphicity of the vertex fermionic observable.

**Lemma 7.4.** *For an edge  $e \in \Omega_\delta^\circ$ ,  $F_\delta(e)$  belongs to  $\ell(e)$ .*

**Proof** The winding at an edge  $e$  can only take its value in the set  $W + 2\pi\mathbb{Z}$  where  $W$  is the winding at  $e$  of an arbitrary interface passing through  $e$ . Therefore, the winding weight involved in the definition of  $F_\delta(e)$  is always proportional to  $e^{iW/2}$  with a real coefficient, ergo  $F_\delta(e)$  is proportional to  $e^{iW/2}$ . In any FK Dobrushin domain,  $b_\delta$  is the south-east corner and the last edge is thus going to the right. Therefore  $e^{iW/2}$  belongs to  $\ell(e)$  for any  $e$  and so does  $F_\delta(e)$ .  $\square$

Even though the proof is finished, we make a short parenthesis: the definition of  $s$ -holomorphicity is not rotationally invariant, nor is the definition of FK Dobrushin domains, since the medial edge pointing to  $b_\delta$  has to be oriented south-east. The latter condition has been introduced in such a way that this lemma holds true. Even though

this condition seems arbitrary, it has no influence on the convergence result, meaning that one could perform a (slightly modified) proof with another orientation.

Proposition 5.8 implies the following result:

**Lemma 7.5.** *Consider a medial vertex  $v$  in  $\Omega_\delta^\circ \setminus \partial\Omega_\delta^\circ$ . We have*

$$F_\delta(N) + F_\delta(S) = F_\delta(E) + F_\delta(W)$$

where  $N$ ,  $E$ ,  $S$  and  $W$  are the adjacent edges indexed in clockwise order.

**Proof** Since  $\sigma = 1/2$ ,  $F$  is the complex conjugate of  $\tilde{F}$  and the lemma follows from (5.7).  $\square$

We are now in a position to prove  $s$ -holomorphicity

**Proposition 7.6.** *The vertex fermionic observable  $F_\delta$  is  $s$ -holomorphic.*

Recall that the FK fermionic observable is defined on medial edges as well as on medial vertices. Convergence of the observable means convergence of the *vertex* observable. The edge observable is just a very convenient tool in the proof.

**Proof** The previous lemma and the definition of the vertex fermionic observable imply

$$F_\delta(v) := \frac{1}{2} \sum_{e \sim v} F_\delta(e) = F_\delta(N) + F_\delta(S) = F_\delta(E) + F_\delta(W).$$

Using Lemma 7.4,  $F_\delta(N)$  and  $F_\delta(S)$  are orthogonal, so that  $F_\delta(N)$  is the projection of  $F_\delta(v)$  on  $\ell(N)$  (and similarly for other edges). Therefore, for a medial edge  $e = [xy]$ ,  $F_\delta(e)$  is the projection of  $F_\delta(x)$  and  $F_\delta(y)$  with respect to  $\ell(e)$ , which proves that the vertex fermionic observable is  $s$ -holomorphic.  $\square$

The function  $F_\delta/\sqrt{2\delta}$  is preholomorphic for every  $\delta > 0$ . Moreover, Lemma 5.9 identifies the boundary conditions of  $F_\delta/\sqrt{2\delta}$  (its argument is determined) so that this function solves a discrete Riemann-Hilbert boundary value problem. These problems are significantly harder to handle than the Dirichlet problems. Therefore, it is more convenient to work with a discrete analogue of  $\text{Im} \left( \int^z [F_\delta(z)/\sqrt{2\delta}]^2 dz \right)$ , which should solve an approximate Dirichlet problem.

## 1.2 Convergence of $(H_\delta)_{\delta>0}$ .

Since the FK fermionic observable  $F_\delta/\sqrt{2\delta}$  is  $s$ -holomorphic, Theorem 2.18 defines a function  $H_\delta$ .

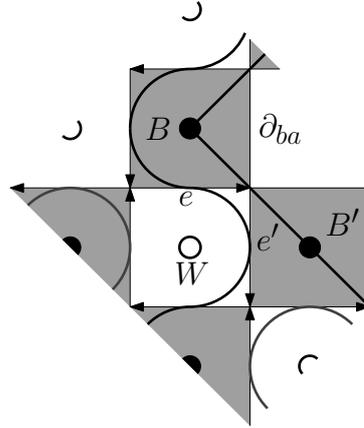


Figure 7.1: Two adjacent sites  $B$  and  $B'$  on  $\partial_{ba}$  together with the notations used in the proof of Lemma 7.8.

**Corollary 7.7.** *Let  $A$  be the black face (vertex of  $\Omega_\delta$ ) bordering  $a_\delta$ , see Fig. 5.1. There exists a unique function  $H_\delta : \Omega_\delta \cup \Omega_\delta^* \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} H_\delta(A) &= 1 \quad \text{and} \\ H_\delta(B) - H_\delta(W) &= |P_{\ell(e)}[F_\delta(x)]|^2 = |P_{\ell(e)}[F_\delta(y)]|^2 \end{aligned} \quad (7.4)$$

for the edge  $e = [xy]$  of  $\Omega_\delta^\circ$  bordered by a black face  $B \in \Omega_\delta$  and a white face  $W \in \Omega_\delta^*$ . Moreover, its restriction  $H^\bullet$  to  $\Omega_\delta$  is subharmonic and its restriction  $H_\delta^\circ$  to  $\Omega_\delta^*$  is superharmonic.

Let us start with two lemmata addressing the question of boundary conditions for  $H_\delta$ .

**Lemma 7.8.** *The function  $H_\delta^\bullet$  is equal to 1 on the arc  $\partial_{ba}$ ,  $H_\delta^\circ$  is equal to 0 on the arc  $\partial_{ab}^*$ .*

**Proof** We first prove that  $H_\delta^\bullet$  is constant on  $\partial_{ba}$ . Let  $B$  and  $B'$  be two adjacent consecutive sites of  $\partial_{ba}$ . They are both adjacent to the same dual vertex  $W \in \Omega_\delta^*$ , see Fig. 7.1. Let  $e$  (resp.  $e'$ ) be the edge of the medial lattice between  $W$  and  $B$  (resp.  $B'$ ). We deduce

$$H_\delta^\bullet(B) - H_\delta^\bullet(B') = |F_\delta(e)|^2 - |F_\delta(e')|^2 = 0 \quad (7.5)$$

The second equality is due to  $|F_\delta(e)| = \phi_{\Omega_\delta^\circ, p_{sd}}^{a_\delta, b_\delta}(W \overset{*}{\leftrightarrow} \partial_{ab}^*)$  (Lemma 5.9). Hence,  $H_\delta^\bullet$  is constant along the arc. Since  $H_\delta^\bullet(A) = 1$ , the result follows readily.

Similarly,  $H_\delta^\circ$  is constant on the arc  $\partial_{ab}^*$ . Moreover, the dual white face  $A^* \in \partial_{ab}^*$  bordering  $a_\delta$  (see Fig. 5.1) satisfies

$$H_\delta^\circ(A^*) = H_\delta^\bullet(A) - |F_\delta(e_a)|^2 = 1 - 1 = 0 \quad (7.6)$$

( $e_a$  necessarily belongs to  $\gamma$ ). Therefore  $H_\delta^\circ = 0$  on  $\partial_{ab}^*$ .  $\square$

**Lemma 7.9.** *The function  $H_\delta^\bullet$  converges to 0 on the arc  $\partial_{ab}$  uniformly away from  $a$  and  $b$ ,  $H_\delta^\circ$  converges to 1 on the arc  $\partial_{ba}^*$  uniformly away from  $a$  and  $b$ .*

**Proof** Once again, we prove the result for  $H_\delta^\bullet$ , the same reasoning then holds for  $H_\delta^\circ$ . Let  $B$  be a site of  $\partial_{ab}$  at distance  $r$  of  $\partial_{ba}$  (and therefore at graph distance  $r/\delta$  of  $\partial_{ba}$  in  $\Omega_\delta$ ). Let  $W$  be an adjacent site of  $B$  on  $\partial_{ab}^*$ . Lemma 7.8 implies  $H_\delta^\circ(W) = 0$ . From the definition of  $H_\delta$ , we find

$$H_\delta^\bullet(B) = H_\delta^\circ(W) + |P_{\ell(e)}[F_\delta(e)]|^2 = |P_{\ell(e)}[F_\delta(e)]|^2 = \phi_{\Omega_\delta, p_{sd}}^{a_\delta, b_\delta}(e \in \gamma)^2.$$

Note that  $e \in \gamma$  if and only if  $B$  is connected to the 'wired arc'  $\partial_{ba}$ . Therefore,  $\phi_{\Omega_\delta, p_{sd}}^{a_\delta, b_\delta}(e \in \gamma)$  is equal to the probability that there exists an open path from  $B$  to  $\partial_{ba}$ . Since the boundary conditions on  $\partial_{ab}$  are free, the comparison between boundary conditions shows that the latter probability is smaller than the probability that there exists a path from  $B$  to  $\partial U_\delta$  in the box  $U_\delta = (B + [-r, r]^2) \cap \mathbb{L}_\delta$  with wired boundary conditions. Therefore,

$$H_\delta^\bullet(B) = \phi_{\Omega_\delta, p_{sd}}^{a_\delta, b_\delta}(e \in \gamma)^2 \leq \phi_{U_\delta, p_{sd}}^1(B \leftrightarrow \partial U_\delta)^2.$$

Proposition 6.10 implies that the right hand side converges to 0 (there is no infinite cluster for  $\phi_{p_{sd}, 2}^1$ ), which gives a uniform bound for  $B$  away from  $a$  and  $b$ .  $\square$

The two previous lemmata assert that the boundary conditions for  $H_\delta^\bullet$  and  $H_\delta^\circ$  are roughly 0 on the arc  $\partial_{ab}$  and 1 on the arc  $\partial_{ba}$ . Moreover,  $H_\delta^\bullet$  and  $H_\delta^\circ$  are almost harmonic. This should imply that  $(H_\delta)_{\delta>0}$  converges to the solution of the Dirichlet problem, which is the subject of the next proposition.

**Proposition 7.10.** *Let  $(\Omega, a, b)$  be a simply connected domain with two points on the boundary, then  $(H_\delta)_{\delta>0}$  converges to  $\text{Im}(\phi)$  uniformly on any compact subsets of  $\Omega$  when  $\delta$  goes to 0, where  $\phi$  is any conformal map from  $\Omega$  to  $\mathbb{T} = \mathbb{R} \times (0, 1)$  sending  $a$  to  $-\infty$  and  $b$  to  $\infty$ .*

Before starting, remark that  $\text{Im}(\phi)$  is the solution of the Dirichlet problem on  $(\Omega, a, b)$  with boundary conditions 1 on  $\partial_{ba}$  and 0 on  $\partial_{ab}$ .

**Proof** From Corollary 7.7,  $H_\delta^\bullet$  is subharmonic, let  $h_\delta^\bullet$  be the preharmonic function with same boundary conditions as  $H_\delta^\bullet$  on  $\partial\Omega_\delta$ . Note that  $H_\delta^\bullet \leq h_\delta^\bullet$ . Similarly,  $h_\delta^\circ$  is defined to be the preharmonic function with same boundary conditions as  $H_\delta^\circ$  on  $\partial\Omega_\delta^*$ . If  $K \subset \Omega$  is fixed, where  $K$  is compact, let  $b_\delta \in K_\delta$  and  $w_\delta \in K_\delta^*$  any neighbor of  $b_\delta$ , we have

$$h_\delta^\circ(w_\delta) \leq H_\delta^\circ(w_\delta) \leq H_\delta^\bullet(b_\delta) \leq h_\delta^\bullet(b_\delta). \quad (7.7)$$

Using Lemmata 7.8 and 7.9, boundary conditions for  $H_\delta^\bullet$  (and therefore  $h_\delta^\bullet$ ) are uniformly converging to 0 on  $\partial_{ab}$  and 1 on  $\partial_{ba}$  away from  $a$  and  $b$ . Moreover,  $|h_\delta^\bullet|$  is bounded by 1 everywhere. This is sufficient to apply Theorem 2.8:  $h_\delta^\bullet$  converges to  $\text{Im}(\phi)$  on any compact subset of  $\Omega$  when  $\delta$  goes to 0. The same reasoning applies to  $h_\delta^\circ$ . The convergence for  $H_\delta^\bullet$  and  $H_\delta^\circ$  follows easily since they are sandwiched between  $h_\delta^\bullet$  and  $h_\delta^\circ$ .  $\square$

### 1.3 Convergence of FK fermionic observables $(F_\delta/\sqrt{2\delta})_{\delta>0}$ .

This section contains the proof of Theorem 7.2. The strategy is straightforward:  $(F_\delta/\sqrt{2\delta})_{\delta>0}$  is proved to be a precompact family for the uniform convergence on compact subsets of  $\Omega$ . Then, the possible sub-sequential limits are identified using  $H_\delta$ .

**Proof of Theorem 7.2** First assume that the precompactness of the family  $(F_\delta/\sqrt{2\delta})_{\delta>0}$  has been proved. Let  $(F_{\delta_n}/\sqrt{2\delta_n})_{n\in\mathbb{N}}$  be a convergent subsequence and denote its limit by  $f$ . Note that  $f$  is holomorphic as limit of preholomorphic functions. For two points  $x, y \in \Omega$ , we have:

$$H_{\delta_n}(y) - H_{\delta_n}(x) = \frac{1}{2} \operatorname{Im} \left( \int_x^y \frac{1}{\delta_n} F_{\delta_n}^2(z) dz \right)$$

(for simplicity, also denote the closest points of  $x, y$  in  $\Omega_{\delta_n}$  by  $x, y$ ). On the one hand, the convergence of  $(F_{\delta_n}/\sqrt{2\delta_n})_{n\in\mathbb{N}}$  being uniform on any compact subset of  $\Omega$ , the right hand side converges to  $\operatorname{Im} \left( \int_x^y f(z)^2 dz \right)$ . On the other hand, the left hand side converges to  $\operatorname{Im}(\phi(y) - \phi(x))$ . Since both quantities are holomorphic functions of  $y$ , there exists  $C \in \mathbb{R}$  such that  $\phi(y) - \phi(x) = C + \int_x^y f(z)^2 dz$  for every  $x, y \in \Omega$ . Therefore  $f$  equals  $\sqrt{\phi'}$ . Since this is true for any convergent subsequence, the result follows.

Therefore, the proof boils down to the precompactness of  $(F_\delta/\sqrt{2\delta})_{\delta>0}$ . We will use the second criterion in Proposition 2.6. Note that it is sufficient to prove this result for squares  $Q \subset \Omega$  such that a bigger square  $9Q$  (with same center) is contained in  $\Omega$ .

Fix  $\delta > 0$ . When jumping diagonally over a medial vertex  $v$ , the function  $H_\delta$  changes by  $\operatorname{Re}(F_\delta^2(v))$  or  $\operatorname{Im}(F_\delta^2(v))$  depending on the direction, so that

$$\delta^2 \sum_{v \in Q_\delta^\circ} |F_\delta(v)/\sqrt{2\delta}|^2 = \delta \sum_{x \in Q_\delta} |\nabla H_\delta^\bullet(x)| + \delta \sum_{x \in Q_\delta^*} |\nabla H_\delta^\circ(x)| \tag{7.8}$$

where  $\nabla H_\delta^\bullet(x) = (H_\delta^\bullet(x + \delta) - H_\delta^\bullet(x), H_\delta^\bullet(x + i\delta) - H_\delta^\bullet(x))$ , and  $\nabla H_\delta^\circ$  is defined similarly for  $H_\delta^\circ$ . It follows that it is enough to prove uniform boundedness of the right hand side in (7.8). We only treat the sum involving  $H_\delta^\bullet$ , the other sum can be handled similarly.

Write  $H_\delta^\bullet = S_\delta + R_\delta$  where  $S_\delta$  is an harmonic function with same boundary conditions on  $\partial 9Q_\delta$  as  $H_\delta^\bullet$ . Note that  $R_\delta \leq 0$  is automatically subharmonic. In order to prove that the sum of  $|\nabla H_\delta^\bullet|$  on  $Q_\delta$  is bounded by  $C/\delta$ , we deal separately with  $|\nabla S_\delta|$  and  $|\nabla R_\delta|$ . First,

$$\sum_{x \in Q_\delta} |\nabla S_\delta(x)| \leq \frac{C_1}{\delta^2} \cdot C_2 \delta \left( \sup_{x \in \partial Q_\delta} |S_\delta(x)| \right) \leq \frac{C_3}{\delta} \left( \sup_{x \in 9Q_\delta} |H_\delta^\bullet(x)| \right) \leq \frac{C_4}{\delta},$$

where in the first inequality we used Proposition 2.5 and the maximum principle for  $S_\delta$ , and the second the fact that  $S_\delta$  and  $H_\delta^\bullet$  share the same boundary conditions on  $9Q_\delta$ . The last inequality comes from the fact that  $H_\delta^\bullet$  converges, hence remains bounded uniformly in  $\delta$ .

Second, recall that  $G_{9Q_\delta}(\cdot, y)$  is the Green function in  $9Q_\delta$  with singularity at  $y$ . Since  $R_\delta$  equals 0 on the boundary, Proposition 2.9 implies

$$R_\delta(x) = \sum_{y \in 9Q_\delta} \Delta R_\delta(y) G_{9Q_\delta}(x, y), \tag{7.9}$$

thus giving

$$\nabla R_\delta(x) = \sum_{y \in 9Q_\delta} \Delta R_\delta(y) \nabla_x G_{9Q_\delta}(x, y)$$

Therefore,

$$\begin{aligned} \sum_{x \in Q_\delta} |\nabla R_\delta(x)| &= \sum_{x \in Q_\delta} \left| \sum_{y \in 9Q_\delta} \Delta R_\delta(y) \nabla_x G_{9Q_\delta}(x, y) \right| \\ &\leq \sum_{y \in 9Q_\delta} \Delta R_\delta(y) \sum_{x \in Q_\delta} |\nabla_x G_{9Q_\delta}(x, y)| \\ &\leq \sum_{y \in 9Q_\delta} \Delta R_\delta(y) C_5 \delta \sum_{x \in Q_\delta} G_{9Q_\delta}(x, y) \\ &= C_5 \delta \sum_{x \in Q_\delta} \sum_{y \in 9Q_\delta} \Delta R_\delta(y) G_{9Q_\delta}(x, y) \\ &= C_5 \delta \sum_{x \in Q_\delta} R_\delta(x) = C_6 / \delta \end{aligned}$$

The second line uses the fact that  $\Delta R_\delta \geq 0$ , the third Proposition 2.10, the fifth Proposition 2.9 again, and the last one the fact that  $Q_\delta$  contains of order  $1/\delta^2$  sites and the fact that  $R_\delta$  is bounded uniformly in  $\delta$  (since  $H_\delta$  and  $S_\delta$  are).

Thus,  $\delta \sum_{x \in Q_\delta} |\nabla H_\delta^\bullet|$  is uniformly bounded. Since the same result holds for  $H_\delta^\circ$ ,  $(F_\delta/\sqrt{2\delta})_{\delta>0}$  is precompact on  $Q$  (and more generally on any compact subset of  $\Omega$ ) and the proof is completed.  $\square$

## 2 Convergence of the spin fermionic observable

We now turn to the proof of convergence for the spin fermionic observable. Fix a simply connected domain  $(\Omega, a, b)$  with two points on the boundary. For  $\delta > 0$ , always consider the spin fermionic observable on the discrete spin Dobrushin domain  $(\Omega_\delta^\circ, a_\delta, b_\delta)$ . Since the domain is fixed, we set  $F_\delta = F_{\Omega_\delta^\circ, a_\delta, b_\delta}$ . We follow the same three steps as before, beginning with the  $s$ -holomorphicity. The other two steps are only sketched, since they are more technical than in the FK-Ising case, see [CS09].

**Proposition 7.11.** *For  $\delta > 0$ ,  $F_\delta$  is  $s$ -holomorphic on  $\Omega_\delta^\circ$ .*

**Proof** Let  $x, y$  two adjacent medial vertices connected by the edge  $e = [xy]$ . Let  $v$  be the vertex of  $\Omega_\delta$  bordering the (medial) edge  $e$ . As before, set  $x_\omega$  (resp.  $y_\omega$ ) for the contribution of  $\omega$  to  $F_\delta(x)$  (resp.  $F_\delta(y)$ ). We wish to prove that

$$\sum_{\omega} P_{\ell(e)}(x_\omega) = \sum_{\omega} P_{\ell(e)}(y_\omega). \quad (7.10)$$

Note that the curve  $\gamma(\omega)$  finishes at  $x_\omega$  or at  $y_\omega$  so that  $\omega$  cannot contribute to  $F_\delta(x)$  and  $F_\delta(y)$  at the same time. Thus, it is sufficient to partition the set of configurations

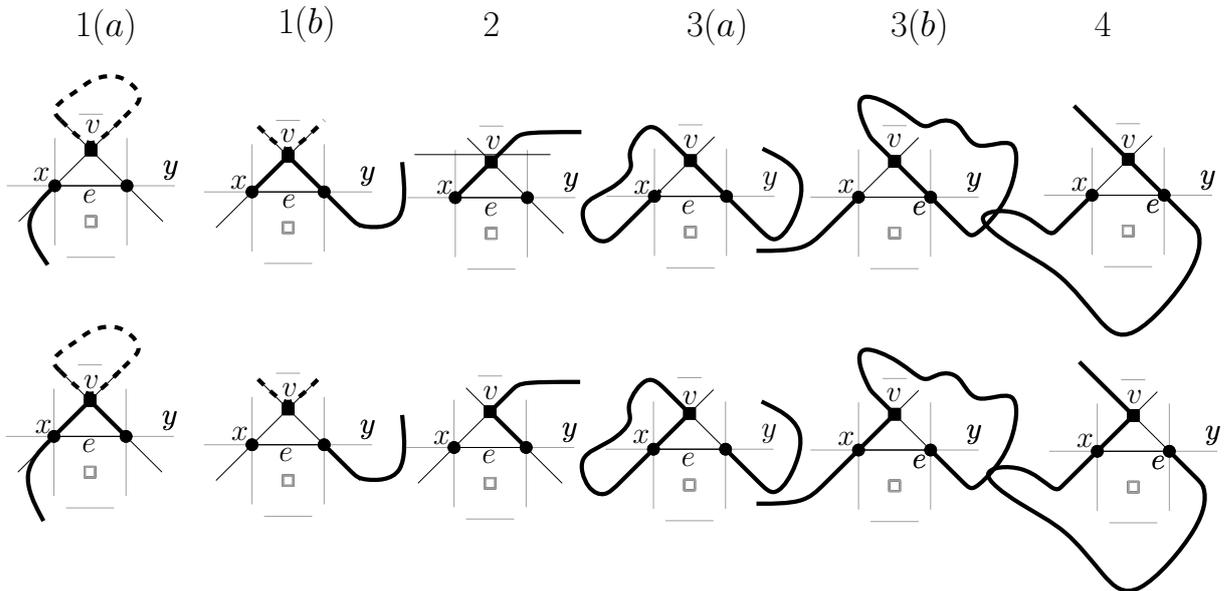


Figure 7.2: The different possible cases in the proof of Proposition 7.11:  $\omega$  is depicted at the top, and  $\omega'$  at the bottom.

into pairs of configurations  $(\omega, \omega')$ , one contributing to  $y$ , the other one to  $x$ , such that  $P_{\ell(e)}(x_\omega) = P_{\ell(e)}(y_{\omega'})$ .

Without loss of generality, assume that  $e$  is pointing south-east, thus  $\ell(e) = \mathbb{R}$  (other cases can be done similarly). First note that

$$x_\omega = \frac{1}{Z} e^{-i\frac{1}{2}[W_{\gamma(\omega)}(a_\delta, x_\delta) - W_{\gamma'}(a_\delta, b_\delta)]} (\sqrt{2} - 1)^{|\omega|},$$

where  $\gamma(\omega)$  is the interface in the configuration  $\omega$ ,  $\gamma'$  is any curve from  $a_\delta$  to  $b_\delta$  (recall that the  $W_{\gamma'}(a_\delta, b_\delta)$  does not depend on  $\gamma'$ ), and  $Z$  is a normalizing real number not depending on the configuration. There are six types of pairs that one can create, see Fig. 7.2 depicting the four main cases. Case 1 corresponds to the case where the interface reaches  $x$  or  $y$  and then extends by one step to reach the other vertex. In Case 2,  $\gamma$  reaches  $v$  before  $x$  and  $y$ , and makes an additional step to  $x$  or  $y$ . In Case 3,  $\gamma$  reaches  $x$  or  $y$  and sees a loop preventing it from being extended to the other vertex (in contrast to Case 1). In Case 4,  $\gamma$  reaches  $x$  or  $y$ , then goes away from  $v$  and comes back to the other vertex. Recall that the curve must always go to the left: in cases 1(a), 1(b), and 2 there can be a loop or even the past of  $\gamma$  passing through  $v$ . However, this does not change the computation.

We obtain the following table for  $x_\omega$  and  $y_{\omega'}$  (we always express  $y_{\omega'}$  in terms of  $x_\omega$ ). Moreover, one can compute the argument modulo  $\pi$  of contributions  $x_\omega$  since the orientation of  $e$  is known. When upon projecting on  $\mathbb{R}$ , the result follows.

| configuration             | Case 1(a)                        | Case 1(b)                               | Case 2                | Case 3(a)             | Case 3(b)             | Case 4                 |
|---------------------------|----------------------------------|---|-----------------------|-----------------------|-----------------------|------------------------|
| $x_\omega$                | $x_\omega$                       | $x_\omega$                              | $x_\omega$            | $x_\omega$            | $x_\omega$            | $x_\omega$             |
| $y_{\omega'}$             | $(\sqrt{2}-1)e^{i\pi/4}x_\omega$ | $\frac{e^{i\pi/4}}{\sqrt{2}-1}x_\omega$ | $e^{-i\pi/4}x_\omega$ | $e^{3i\pi/4}x_\omega$ | $e^{3i\pi/4}x_\omega$ | $e^{-5i\pi/4}x_\omega$ |
| arg. $x_\omega \bmod \pi$ | $5\pi/8$                         | $\pi/8$                                 | $\pi/8$               | $5\pi/8$              | $5\pi/8$              | $5\pi/8$               |

□

**Proof of Theorem 7.3.** The proof is roughly sketched in the following, we refer to [CS09] for a complete proof.

Since  $F_\delta$  is  $s$ -harmonic, one can define the observable  $H_\delta$  as in Theorem 2.18, with the requirement that it is equal to 0 on the white face adjacent to  $b$ . Then,  $H_\delta^\circ$  is constant equal to 0 on the boundary as in the FK-Ising case. Note that  $H_\delta$  should not converge to 0, even if boundary conditions are 0 away from  $a$ . Firstly,  $H_\delta^\circ$  is superharmonic and not harmonic, even though it is expected to be almost harmonic (away from  $a$ ,  $H_\delta^\bullet$  and  $H_\delta^\circ$  are close), it will not be true near  $a$ . Actually,  $H_\delta$  should not remain bounded around  $a$ .

The main difference compared to the previous section is indeed the unboundedness of  $H_\delta$  near  $a_\delta$  which prevents us from the immediate use of Proposition 2.6. It is actually possible to prove that away from  $a$ ,  $H_\delta$  remains bounded, see [CS09]. This uses more sophisticated tools, among which the 'boundary modification trick' (see Chapter 9 for a quick description in the FK-Ising case [DCHN10], and [CS09] for the Ising original case). As before, boundedness implies precompactness (and thus boundedness) of  $(F_\delta)_{\delta>0}$  away from  $a$  via Proposition 2.6. Since  $H_\delta$  can be expressed in terms of  $F_\delta$ , it is easy to deduce that  $H_\delta$  is also precompact.

Now consider a convergent subsequence  $(f_{\delta_n}, H_{\delta_n})$  converging to  $(f, H)$ . One can check that  $H$  is equal to 0 on  $\partial\Omega \setminus \{a\}$ . Moreover, the fact that  $H_\delta^\circ$  equals 0 on the boundary and is superharmonic implies that  $H_\delta^\circ$  is larger or equal to 0 everywhere, implying  $H \geq 0$  in  $\Omega$ . This property of harmonic functions in a domain almost determines them. There is only a one parameter family of positive harmonic functions equal to 0 on the boundary. These functions are exactly the imaginary part of conformal maps from  $\Omega$  to the upper half-plane  $\mathbb{H}$  mapping  $a$  to  $\infty$ . We can further assume that  $b$  is mapped to 0, since we are interested only in the imaginary part of these functions.

Fix one conformal map  $\psi$  from  $\Omega$  to  $\mathbb{H}$ , mapping  $a$  to  $\infty$  and  $b$  to 0. There exists  $\lambda > 0$  such that  $H = \lambda \text{Im}\psi$ . As in the case of the FK-Ising, one can prove that  $\text{Im}(f^z f^2) = H$ , implying that  $f^2 = \lambda\psi'$ . Since  $f(b) = 1$  (it is obvious from the definition that  $F_\delta(b_\delta) = 1$ ),  $\lambda$  equals  $\frac{1}{\psi'(b)}$ . In conclusion,  $f(z) = \sqrt{\psi'(z)/\psi'(b)}$  for every  $z \in \Omega$ .

Note that some regularity hypothesis on the boundary near  $b$  are needed to ensure that the sequence  $(f_{\delta_n}, H_{\delta_n})$  also converges near  $b$ . This is the reason for assuming that the boundary near  $b$  is smooth. We also mention that there is no normalization here. The normalization 'from the point of view of  $b$ ' was already present in the definition of the observable.

□

# Chapter 8

## The fermionic observable away from the critical point

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**Abstract:** The FK fermionic observable (case  $q = 2$ ) is studied away from the self-dual point. An alternative derivation of the fact that the self-dual and critical points coincide is obtained, which implies that the critical inverse temperature of the Ising model equals  $\frac{1}{2} \log(1 + \sqrt{2})$ . Moreover, the correlation length of the model is related to the large deviation behavior of a certain massive random walk (thus confirming an observation by Messikh [Mes06]), which allows us to compute it explicitly. This chapter is inspired by the article *Smirnov's fermionic observable away from the critical point* [BDC11], written with Vincent Beffara and published in *Annals of probabilities*.

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The problem of identifying the critical value of the Ising model is more than fifty years old. The reader is referred to Chapter 6 for details. Summarizing, Kramers and Wannier identified (without proof) the critical temperature where a phase transition occurs, separating an ordered from a disordered phase, using planar duality [KW41a, KW41b]. In 1944, Kaufman and Onsager [KO50] computed the free energy of the model, paving the way to an analytic derivation of its critical temperature. In 1987, Aizenman, Barsky and Fernández [ABF87] found a computation of the critical temperature based on differential inequalities. Recently, a determination of the critical value of the FK-Ising model provides yet another proof of this result, see Chapter 4. All of these strategies are quite involved, and the first goal of this chapter is to propose an alternative method, relying only on Smirnov's fermionic observable:

**Theorem 8.1.** *The critical inverse temperature of the Ising model on the square lattice  $\mathbb{Z}^2$  is equal to*

$$\beta_c = \frac{1}{2} \ln(1 + \sqrt{2}).$$

Beyond the determination of the critical inverse temperature, physicists and mathematicians are interested in estimates for the correlation between two spins,  $\mu_\beta[\sigma(a)\sigma(b)]$ .

McCoy and Wu [MW73] derived a closed formula for the two-point function, and an asymptotic analysis shows that it decays exponentially fast when  $\beta < \beta_c$ . In addition to this, it was noticed by Messikh [Mes06] that the rate of decay is connected to large deviations estimates for the simple random walk. This chapter presents a direct derivation of this link, which provides a quick proof of the following theorem:

**Theorem 8.2.** *Let  $\beta < \beta_c$  and let  $\mu_\beta$  denote the (unique) infinite-volume Ising measure at inverse temperature  $\beta$ ; fix  $a = (a_1, a_2) \in \mathbb{L}$ . Then,*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \ln (\mu_\beta[\sigma(0)\sigma(na)]) = a_1 \operatorname{arcsinh} sa_1 + a_2 \operatorname{arcsinh} sa_2,$$

where  $s$  solves the equation

$$\sqrt{1 + (sa_1)^2} + \sqrt{1 + (sa_2)^2} = \sinh 2\beta + \sinh^{-1} 2\beta.$$

Instead of working with the Ising model, it is once again more convenient to deal with its random-cluster representation. The determination of  $\beta_c$  being equivalent to the determination of the critical point  $p_c$  for the FK-Ising, we aim for the latter.

The idea of the argument is the following. Below the self-dual point, the observable can be defined but discrete holomorphicity fails and the observable decays exponentially fast in the distance to the wired boundary. Along the free boundary, the modulus of the observable can be written exactly as a connection probability, so in the  $p < p_{sd}$  regime the two-point function is exponentially small as well, and that implies that the system is in the subcritical regime and that a dual cluster exists. These two properties show that  $p \leq p_c \leq p^*$  and Theorem 8.1 follows.

In fact, the rate of exponential decay (and therefore Theorem 8.2) can be derived by comparing the observable to the Green function of a massive random walk (Proposition 8.7); the key ingredient is the observation that the observable is massive harmonic in the bulk for  $p < p_{sd}$ . The correspondence between the two-point function of the Ising model and that of the massive random walk was previously noticed by Messikh [Mes06].

Section 1 contains the proof of Theorem 8.1: it is shown that the observable decays exponentially fast. Section 2 is devoted to a refinement of estimates on the observable, which leads to the proof of Theorem 8.2.

**In this chapter, rotate the lattice by an angle  $\pi/4$  and fix  $q = 2$  and drop it from the notations.**

## 1 Proof of Theorem 8.1

The proof consists of three steps:

- We first prove using Proposition 5.8 and Lemma 7.4 that the observable decays exponentially fast when  $p < p_{sd}$  in a well chosen Dobrushin domain (namely a strip with free boundary conditions on the top and wired boundary conditions on the

bottom). Lemma 5.9 then implies that the probability that a point on the top of the strip is connected to the bottom decays exponentially fast in the height of the strip.

- We derive exponential decay of the connectivity function for the infinite-volume measure with free boundary conditions from the first part.
- Finally, we show that exponential decay implies that the random-cluster model is subcritical when  $p < p_{sd}$ , and that its dual is supercritical. This last step concludes the proof of Theorem 8.1 and is classical.

In the proof, points are identified with their complex coordinates.

**Step 1: Exponential decay in the strip.** Let  $p < p_{sd}$  and consider the random-cluster model on the strip  $\mathcal{S}_\ell$  of height  $\ell > 0$  with wired boundary conditions on the bottom and free boundary conditions on the top. Define  $e_k$  and  $e_{k+1}$  to be the north-west-pointing sides of the diamonds associated to the points  $ik$  and  $i(k+1)$ , respectively. Label some of the edges around these two diamonds as  $x, x', x'', y$  and  $y'$  as shown in Figure 8.1.

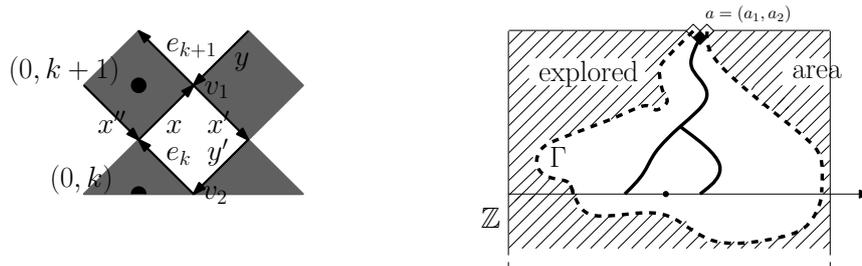


Figure 8.1: **Left:** The labelling of edges around  $e_k$  used in Step 1. **Right:** A dual circuit surrounding an open path in the box  $[-a_2, a_2]^2$ . Conditioning on to the most exterior such circuit gives no information on the state of the edges inside it.

Proposition 5.8 and Lemma 7.4 have a very important consequence: around a vertex  $v$ , the value of the observable on one edge can be expressed in terms of its values on *only two other edges*. This can be done by seeing the relation given by Proposition 5.8 as a linear relation between four vectors in the plane  $\mathbb{R}^2$ , and applying an orthogonal projection to a line orthogonal to one of them (which can be chosen using Lemma 7.4). One then gets a linear relation between three real numbers, but using Lemma 7.4 “in reverse” shows that this is enough to determine any of the corresponding three (complex) values of the observable given the other two.

For instance, (5.6) can be projected around  $v_1$  orthogonally to  $F(y)$ , so that a relation is obtained between projections of  $F(x), F(x')$  and  $F(e_{k+1})$ . Moreover, the complex argument (modulo  $\pi$ ) of  $F$  is known (Lemma 7.4) for each edge so that the relation between

projections can be written as a relation between  $F(x)$ ,  $F(x')$  and  $F(e_{k+1})$  themselves. This leads to

$$e^{-i\pi/4}F(x) = \cos(\pi/4 - \alpha)F(e_{k+1}) - \cos(\pi/4 + \alpha)e^{-i\pi/2}F(x'). \quad (8.1)$$

Applying the same reasoning around  $v_2$ , we obtain

$$e^{-i\pi/4}F(x) = \cos(\pi/4 + \alpha)F(e_k) - \cos(\pi/4 - \alpha)e^{-i\pi/2}F(x''). \quad (8.2)$$

The translation invariance implies

$$F(x') = F(x''). \quad (8.3)$$

Moreover, symmetry with respect to the imaginary axis implies that

$$F(x) = e^{i\pi/4}\overline{F(x')} = e^{-i\pi/4}F(x'). \quad (8.4)$$

Indeed, if for a configuration  $\omega$ ,  $x$  belongs to  $\gamma$  and the winding is equal to  $W$ , in the reflected configuration  $\omega'$ ,  $x'$  belongs to  $\gamma(\omega')$  and the winding is equal to  $\pi/2 - W$ .

Plugging (8.3) and (8.4) into (8.1) and (8.2) leads to

$$F(e_{k+1}) = e^{-i\pi/4} \frac{1 + \cos(\pi/4 + \alpha)}{\cos(\pi/4 - \alpha)} F(x) = \frac{[1 + \cos(\pi/4 + \alpha)] \cos(\pi/4 + \alpha)}{[1 + \cos(\pi/4 - \alpha)] \cos(\pi/4 - \alpha)} F(e_k).$$

Remember that  $\alpha(p) > 0$  since  $p < p_{sd}$ , so that the multiplicative constant is less than 1. Using Lemma 5.9 and the previous equality inductively, there exists  $c_1 = c_1(p) < 1$  such that, for every  $\ell > 0$ ,

$$\phi_{\mathcal{S}_{\ell,p}}^{\infty,-\infty}[\mathbb{i}\ell \leftrightarrow \mathbb{Z}] = |F(e_\ell)| = c_1^\ell |F(e_1)| \leq c_1^\ell,$$

where  $\phi_{\mathcal{S}_{\ell,p}}^{\infty,-\infty}$  is the random cluster measure on the strip  $\mathbb{Z} \times [0, \ell]$  with edge-weight  $p$ , free boundary conditions on the top and wired boundary conditions on the bottom. The last inequality is due to the fact that the observable has complex modulus less than 1.

**Step 2: Exponential decay for  $\phi_p^0$  when  $p < p_{sd}$ .** Fix again  $p < p_{sd}$ . Let  $N \in \mathbb{N}$  and recall that  $\phi_{p,N}^0 := \phi_{p,2,[-N,N]^2}^0$  converges to the infinite-volume measure with free boundary conditions  $\phi_p^0$  when  $N$  goes to infinity.

Consider a configuration in the box  $[-N, N]^2$ , and let  $A_{\max}$  be the site of the cluster of the origin which maximizes the  $\ell^\infty$ -norm  $\max\{|x_1|, |x_2|\}$  (it could be equal to  $N$ ). If there is more than one such site,  $A_{\max}$  is defined to be the greatest one in lexicographical order. Assume that  $A_{\max}$  equals  $a = a_1 + ia_2$  with  $a_2 \geq |a_1|$  (the other cases can be treated the same way by symmetry, using the rotational invariance of the lattice).

By definition, if  $A_{\max}$  equals  $a$ ,  $a$  is connected to 0 in  $[-a_2, a_2]^2$ . In addition to this, because of our choice of the free boundary conditions, there exists a dual circuit starting

from  $a + i/2$  in the dual of  $[-a_2, a_2]^2$  (which is the same as  $\mathbb{L}^* \cap [-a_2 - 1/2, a_2 + 1/2]^2$ ) and surrounding both  $a$  and  $0$ . Let  $\Gamma$  be the outermost such dual circuit: we get

$$\phi_{p,N}^0(A_{\max} = a) = \sum_{\gamma} \phi_{p,N}^0(a \leftrightarrow 0 | \Gamma = \gamma) \phi_{p,N}^0(\Gamma = \gamma), \quad (8.5)$$

where the sum is over contours  $\gamma$  in the dual of  $[-a_2, a_2]^2$  that surround both  $a$  and  $0$ .

The event  $\{\Gamma = \gamma\}$  is measurable in terms of edges outside or on  $\gamma$ . In addition, conditioning on this event implies that the edges of  $\gamma$  are dual-open. Therefore, from the domain Markov property, the conditional distribution of the configuration inside  $\gamma$  is a random-cluster model with free boundary conditions. Comparison between boundary conditions implies that the probability of  $\{a \leftrightarrow 0\}$  conditionally on  $\{\Gamma = \gamma\}$  is smaller than the probability of  $\{a \leftrightarrow 0\}$  in the strip  $\mathcal{S}_{a_2}$  with free boundary conditions on the top and wired boundary conditions on the bottom. Hence, for any such  $\gamma$ ,

$$\phi_{p,N}^0(a \leftrightarrow 0 | \Gamma = \gamma) \leq \phi_{\mathcal{S}_{a_2,p}}^{\infty,-\infty}(a \leftrightarrow 0) = \phi_{\mathcal{S}_{a_2,p}}^{\infty,-\infty}(a \leftrightarrow \mathbb{Z}) \leq c_1^{a_2} = c_1^{|a|/2}$$

(observe that for the second measure,  $\mathbb{Z}$  is wired, so that  $\{a \leftrightarrow 0\}$  and  $\{a \leftrightarrow \mathbb{Z}\}$  have the same probability). Plugging this into (8.5),

$$\phi_{p,N}^0(A_{\max} = a) \leq \sum_{\gamma} c_1^{|a|/2} \phi_{p,N}^0(\Gamma = \gamma) \leq c_1^{|a|/2}.$$

Fix  $n \leq N$ . Since  $c_1 < 1$ , the previous inequality implies there exist two constants  $0 < c_2, C_2 < \infty$  such that

$$\phi_{p,N}^0(0 \leftrightarrow \mathbb{Z}^2 \setminus [-n, n]^2) \leq \sum_{a \in [-N, N]^2 \setminus [-n, n]^2} \phi_{p,N}^0(A_{\max} = a) \leq \sum_{a \in [-n, n]^2} c_1^{|a|/2} \leq C_2 e^{-c_2 n}.$$

Since the estimate is uniform in  $N$ , we deduce that

$$\phi_p^0(0 \leftrightarrow \mathbb{Z}^2 \setminus [-n, n]^2) \leq C_2 e^{-c_2 n}. \quad (8.6)$$

**Step 3: Exploiting exponential decay.** The inequality  $p_c \geq p_{sd}$  follows from (8.6) since exponential decay prevents the existence of an infinite cluster for  $\phi_p^0$  when  $p < p_{sd}$ .

The reasoning to prove  $p_c \leq p_{sd}$  is standard. Let  $A_n$  be the event that the point  $(n, 0)$  is in an open circuit which surrounds the origin. Notice that this event is included in the event that the point  $(n, 0)$  is in a cluster of radius larger than  $n$ . For  $p < p_{sd}$ , (8.6) implies that the probability of  $A_n$  decays exponentially fast. The Borel-Cantelli lemma shows that there is almost surely only a finite number of values of  $n$  such that  $A_n$  occurs. In other words, there is only a finite number of open circuits surrounding the origin, which enforces the existence of an infinite dual cluster. It means that the dual model is supercritical whenever  $p < p_{sd}$ . Equivalently, the primal model is supercritical whenever  $p > p_{sd}$ , which implies  $p_c \leq p_{sd}$ .  $\square$

## 2 Proof of Theorem 8.2

In this section, the correlation length is computed in all directions. In [Mes06], Messikh noticed that this correlation length was connected to large deviations for random walks and asked whether there exists a direct proof of the correspondence. Indeed, large deviations results are easy to obtain for random walks, so that one could deduce Theorem 8.2 easily. In the following, we exhibit what we believe to be the first direct proof of this result.

An equivalent way to deal with large deviations of the simple random walk is to study the *massive Green function*  $G_m$ , defined in the bulk as

$$G_m(x, y) := \mathbb{E}^x \left[ \sum_{n \geq 0} m^n \mathbb{1}_{X_n=y} \right],$$

where  $\mathbb{E}^x$  is the law of a simple random walk starting at  $x$ .

The correlation length of the two-dimensional Ising model is the same as the correlation length for its random-cluster representation so that we will state the result in terms of the random-cluster. The parameters  $p$  and  $\alpha = \alpha(p)$  are used without revealing the connection with  $\beta$  in the notation.

**Proposition 8.3.** *For  $p < p_{sd}$  and any  $a \in \mathbb{L}$ ,*

$$-\lim_{n \rightarrow \infty} \frac{1}{n} \log \phi_p^0(0 \leftrightarrow na) = -\lim_{n \rightarrow \infty} \frac{1}{n} \log G_m(0, na) \quad (8.7)$$

where  $m = \cos[2\alpha(p)]$  — the value of  $\alpha(p)$  is given by (5.5).

In [Mes06], the statement involves Laplace transforms yet it can be translated it into the previous terms. Moreover, the mass is expressed in terms of  $\beta$ , but it is elementary to compute it in terms of  $\alpha$ . Theorem 8.2 follows from this proposition by first relating the two-point functions of the Ising and  $q = 2$  random-cluster models as was mentioned earlier, and then deriving the asymptotics of the massive Green function explicitly — the details can be found for instance in the proof of Proposition 8 in [Mes06].

Before delving into the actual proof, here is a short outline of the strategy. Exponential decay in the strip was already shown: it was an essentially one-dimensional computation. We now aim to refine it into a two-dimensional version for correlations between two points 0 and  $a$  in the bulk, and once again the observable is used to estimate them. The basic step, namely obtaining local linear relations between the values of the observable, is the same, although it is complicated by the lack of translation invariance. The point is that the observable is massive harmonic when  $p \neq p_{sd}$  (see Lemma 8.4 below). Since  $G_m(\cdot, \cdot)$  is massive harmonic in both variables away from the diagonal  $x = y$ , it is possible to compare both quantities.

The main problem is that we are interested in correlations in the bulk. The observable can be defined directly in the bulk (see below) but it provides only a lower bound on the correlations. In order to obtain an upper bound, we have to introduce an “artificial”

domain (that will be  $T(a)$  below), which needs two features: the observable in it can be well estimated, and at the same time correlations inside it have comparable probabilities to correlations in the bulk. For the second one, it is equivalent to impose that the Wulff shape centered at 0 and having  $a$  on its boundary is contained in the domain in the neighborhood of  $a$ ; from convexity, it is then natural to construct  $T(a)$  as the whole plane minus two wedges, one with vertex at 0 and the other with vertex at  $a$ .

The proof is rather technical since one needs to deal with the behavior of the observable on the boundary of the domains. This was also an issue in Smirnov's proof. At criticality, the difficulty was overcome by working with the discrete primitive  $H$  of  $F^2$ . Unfortunately, there is no nice equivalent of  $H$  to work with away from criticality. The solution is to use a representation of  $F$  in terms of a massive random walk. This representation extends to the boundary and allows us to control the behavior of  $F$  everywhere.

**Proof** Let  $p < p_{sd}$ . Without loss of generality, consider  $a = (a_1, a_2) \in \mathbb{L}$  satisfying  $a_2 \geq a_1 \geq 0$ . In the proof, a site  $u \in \mathbb{L}$  is identified with the unique side  $e_u$  of the associated black diamond which points north-west. In other words  $F(u)$  and  $\{u \in \gamma\}$  should be understood as  $F(e_u)$  and  $\{e_u \in \gamma\}$  — notice that this differs from the notation used in [Smi10a].

**The lower bound.** Consider the observable  $F$  in the bulk defined as follows: for every edge  $e$  not equal to  $e_0$ ,

$$F(e) := \phi_p^0 \left( e^{\frac{i}{2} W_\gamma(e, e_0)} \mathbb{1}_{e \in \gamma} \right), \quad (8.8)$$

where  $\gamma$  is the unique loop passing through  $e_0$ . Note that this definition is justified by the fact that  $p$  is subcritical, and that it immediately implies that

$$\phi_p^0(0 \leftrightarrow a) \geq |F(a)|. \quad (8.9)$$

Note that  $F$  is not well defined at  $e_0$ . Indeed,  $e_0$  can be thought of as the start of the loop  $\gamma$  or its end. In other words,  $F$  is multi-valued at  $e_0$ , with value 1 or -1.

Proposition 5.8 can be extended to this context following a very similar proof, but taking into account that  $F$  is multi-valued at  $e_0$ . More precisely, let  $e_0 = xy$ . Around any vertex  $v \notin \{x, y\}$  the relation in Proposition 5.8 still holds; besides,

$$\begin{cases} F(SE) + 1 = e^{i\alpha(p)} [F(SW) + F(NE)] & \text{if } v = y \\ F(SW) + F(NE) = e^{i\alpha(p)} [-1 + F(SE)] & \text{if } v = x \end{cases}$$

where the  $NE$  (resp.  $SE$ ,  $SW$ ) is the edge at  $v$  pointing to the north-east (resp. south-east, south-west). In other words, the statement of Proposition 5.8 still formally holds if the convention becomes  $F(e_0) = 1$  when considering the relation around  $x$ , and  $F(e_0) = -1$  when considering the relation around  $y$ .

One can see that Lemma 7.4 is still valid. In fact, the two lemmas imply that  $F$  is massive harmonic:

**Lemma 8.4.** *Let  $p < p_{sd}$  and consider the observable  $F$  in the bulk. For any site  $X$  not equal to 0, we have*

$$\Delta_\alpha F(X) := \frac{\cos 2\alpha}{4} [F(W) + F(S) + F(E) + F(N)] - F(X) = 0,$$

where  $W, S, E$  and  $N$  are the four neighbors of  $X$ .

**Proof** Consider a site  $X$  inside the domain and recall that  $X$  is identified with the corresponding edge of the medial lattice pointing north-west. Index the edges around  $X$  in the same way as in Case 1 of Figure 8.2. By considering the six equations corresponding to vertices that end one of the edges  $x_1, \dots, x_6$  (being careful to identify the edges  $A, B, C$  and  $D$  correctly for each of the vertices), the following linear system can be obtained:

$$\begin{cases} F(X) + F(y_1) &= e^{i\alpha} [F(x_1) + F(x_6)] \\ F(y_2) + F(x_1) &= e^{i\alpha} [F(x_2) + F(W)] \\ F(S) + F(x_2) &= e^{i\alpha} [F(y_3) + F(x_3)] \\ F(x_3) + F(x_4) &= e^{i\alpha} [F(y_4) + F(X)] \\ F(E) + F(x_5) &= e^{i\alpha} [F(x_4) + F(y_5)] \\ F(x_6) + F(y_6) &= e^{i\alpha} [F(x_5) + F(N)] \end{cases}$$

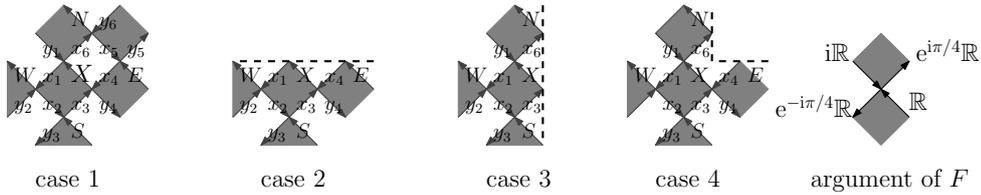


Figure 8.2: Indexation of the edges around vertices in the different cases.

Recall that by definition,  $F(X)$  is real. For an edge  $e$ , denote by  $f(e)$  the projection of  $F(e)$  on the line directed by its argument ( $\mathbb{R}, e^{i\pi/4}\mathbb{R}, i\mathbb{R}$  and  $e^{-i\pi/4}\mathbb{R}$ ). By projecting orthogonally to the  $F(y_i), i = 1 \dots 6$ , the system becomes:

$$\begin{cases} f(X) &= \cos(\pi/4 + \alpha)f(x_1) &+ \cos(\pi/4 - \alpha)f(x_6) & (1) \\ f(x_1) &= \cos(\pi/4 + \alpha)f(x_2) &+ \cos(\pi/4 - \alpha)f(W) & (2) \\ f(x_3) &= \cos(\pi/4 - \alpha)f(S) &+ -\cos(\pi/4 + \alpha)f(x_2) & (3) \\ f(X) &= \cos(\pi/4 + \alpha)f(x_3) &+ \cos(\pi/4 - \alpha)f(x_4) & (4) \\ f(x_4) &= \cos(\pi/4 + \alpha)f(E) &+ \cos(\pi/4 - \alpha)f(x_5) & (5) \\ f(x_6) &= -\cos(\pi/4 - \alpha)f(x_5) &+ \cos(\pi/4 + \alpha)f(N) & (6) \end{cases}$$

By adding (2) to (3), (5) to (6) and (1) to (4), we find

$$\begin{cases} f(x_3) + f(x_1) &= \cos(\pi/4 - \alpha)[f(W) + f(S)] & (7) \\ f(x_6) + f(x_4) &= \cos(\pi/4 + \alpha)[f(E) + f(N)] & (8) \\ 2f(X) &= \cos(\pi/4 + \alpha)[f(x_3) + f(x_1)] + \cos(\pi/4 - \alpha)[f(x_6) + f(x_4)] & (9) \end{cases}$$

Plugging (7) and (8) into (9) leads to

$$2f(X) = \cos(\pi/4 + \alpha) \cos(\pi/4 - \alpha)[f(W) + f(S) + f(E) + f(N)].$$

The edges  $X, \dots, N$  are pointing in the same direction so the previous equality becomes an equality with  $F$  in place of  $f$  (use Lemma 7.4). A simple trigonometric identity then leads to the claim.  $\square$

Define the Markov process with generator  $\Delta_\alpha$ , which one can see either as a branching process or as the random walk of a massive particle. We choose the latter interpretation and write this process  $(X_n, m_n)$  where  $X_n$  is a random walk with jump probabilities defined in terms of  $\Delta_\alpha$  — the proportionality between jump probabilities is the same as the proportionality between coefficients — and  $m_n$  is the mass associated to this random walk. The law of the random walk starting at  $x$  is denoted  $\mathbb{P}^x$ . Note that the mass of the walk decays by a factor  $\cos 2\alpha$  at each step.

Denote by  $\tau$  the hitting time of 0. The last lemma translates into the following formula for any  $a$  and any  $t$ ,

$$F(a) = \mathbb{E}^a[F(X_{t \wedge \tau})m_{t \wedge \tau}]. \quad (8.10)$$

The sequence  $(F(X_t)m_t)_{t \leq \tau}$  is obviously uniformly integrable, so that (8.10) can be improved to

$$F(a) = \mathbb{E}^a[F(X_\tau)m_\tau]. \quad (8.11)$$

Equations (8.9), (8.11) together with Lemma 8.5 below give

$$\phi_p^0(0 \leftrightarrow a) \geq \frac{c}{|a|} G_{\cos 2\alpha}(0, a),$$

which implies the lower bound.

**Lemma 8.5.** *There exists  $c > 0$  such that, for every  $a$  in the upper-right quadrant,*

$$|\mathbb{E}^a[F(X_\tau)m_\tau]| \geq \frac{c}{|a|} G_{\cos 2\alpha}(0, a).$$

**Proof** Recall that  $F(X_\tau)$  is equal to 1 or -1 depending on the last step the walk takes before reaching 0. Let us rewrite  $\mathbb{E}^a[F(X_\tau)m_\tau]$  as

$$\mathbb{E}^a[m^\tau 1_{\{X_{\tau-1}=W \text{ or } S\}}] - \mathbb{E}^a[m^\tau 1_{\{X_{\tau-1}=N \text{ or } E\}}].$$

Now, let  $\Delta_\alpha$  be the line  $y = -x$ , and let  $T$  be the time of the last visit of  $\Delta_\alpha$  by the walk before time  $\tau$  (set  $T = \infty$  if it does not exist). On the event that  $X_{\tau-1} = W$  or  $S$ , this time is finite, and reflecting the part of the path between  $T$  and  $\tau$  across  $\Delta_\alpha$  produces a path from  $a$  to 0 with  $X_{\tau-1} = E$  or  $N$ . This transformation is one-to-one, so summing over all paths, we obtain

$$\mathbb{E}^a[m^\tau 1_{\{X_{\tau-1}=W \text{ or } S\}}] - \mathbb{E}^a[m^\tau 1_{\{X_{\tau-1}=N \text{ or } E\}}] = -\mathbb{E}^a[m^\tau 1_{\{X_{\tau-1}=N \text{ or } E\}} 1_{\{T=\infty\}}]$$

which in turn is equal to  $-\mathbb{E}^a[m^\tau 1_{\{T=\infty\}}]$ . General arguments of large deviation theory imply that  $\mathbb{E}^a[m^\tau 1_{\{T=\infty\}}] \geq \frac{c}{|a|} G_{\cos 2\alpha}(0, a)$  for some universal constant  $c$ .  $\square$

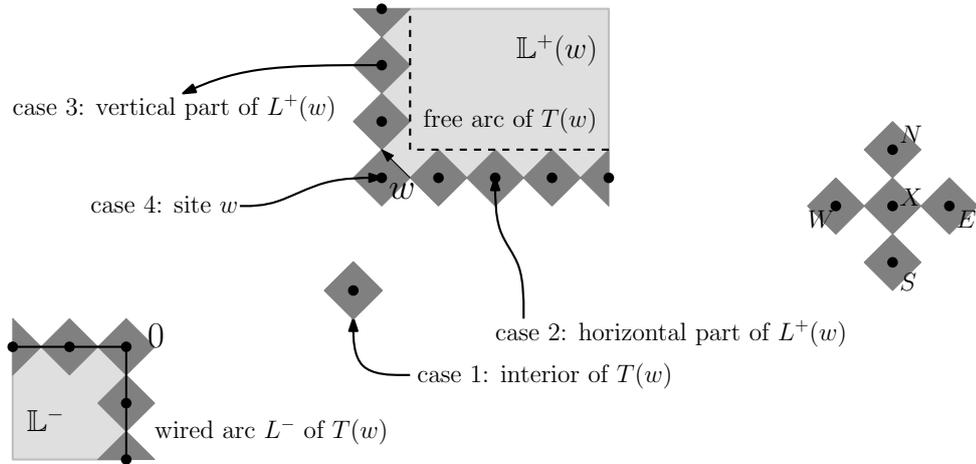


Figure 8.3: The set  $T(w)$ . The different cases listed in the definition of the Laplacian are pictured.

**The upper bound.** Assume that  $0$  is connected to  $a$  in the bulk. We first show how to reduce the problem to estimations of correlations for points on the boundary of a domain.

For every  $u = u_1 + iu_2$  and  $v = v_1 + iv_2$  two sites of  $\mathbb{L}$ , write  $u < v$  if  $u_1 < v_1$  and  $u_2 < v_2$ . This relation is a partial ordering of  $\mathbb{L}$ . Consider the following sets

$$\mathbb{L}^+(u) = \{x \in \mathbb{L} : u < x\} \quad \text{and} \quad \mathbb{L}^- = \{x \in \mathbb{L} : x < 0\};$$

and

$$T(u) = \mathbb{L} \setminus (\mathbb{L}^+(u) \cup \mathbb{L}^-).$$

In the following,  $L^+(u)$  and  $L^-$  will denote the interior boundaries of  $T(u)$  near  $\mathbb{L}^+(u)$  and  $\mathbb{L}^-$  respectively, see Figure 8.3. The measure with wired boundary conditions on  $\mathbb{L}^-$  and free boundary conditions on  $\mathbb{L}^+(u)$  is denoted  $\phi_{T(u)}$ .

Assume that  $a$  is connected to  $0$  in the bulk. By conditioning on  $w$  which maximizes the partial  $>$ -ordering in the cluster of  $0$  (it is the same reasoning as in Section 3), we obtain the following:

$$\phi_p^0(a \leftrightarrow 0) \leq \sum_{w>a} \phi_{T(w)}(w \leftrightarrow \mathbb{L}^-) \leq C_3|a| \max_{w>a, |w| \leq c_3|a|} \phi_{T(w)}(w \leftrightarrow \mathbb{L}^-) \quad (8.12)$$

for  $c_3, C_3$  large enough. The existence of  $c_3$  is given by the fact that the two-point function decays exponentially fast: *a priori* estimates on the correlation length show that the maximum above cannot be reached at any  $w$  which is much further away from the origin than  $a$ , and even that the sum of the corresponding probabilities is actually of a smaller order than the remaining terms. Summarizing, it is sufficient to estimate the probability of the right-hand side of (8.12).

Observe that  $w$  is on the free arc of  $T(w)$ , so that, harnessing Lemma 5.9, we find

$$\phi_{T(w)}(w \leftrightarrow L^-) = |F(w)|, \quad (8.13)$$

where  $F$  is the observable in the infinite Dobrushin domain  $T(w)$  (the winding is fixed in such a way that it equals 0 at  $e_w$ ). Now, similarly to Lemma 8.4,  $F$  satisfies local relations in the domain  $T(w)$ :

**Lemma 8.6.** *The observable  $F$  satisfies  $\Delta_\alpha F = 0$  for every site not on the wired arc, where the massive Laplacian  $\Delta_\alpha$  on  $T(w)$  is defined by the following relations: for all  $g : T(w) \mapsto \mathbb{R}$ ,  $(g + \Delta_\alpha g)(X)$  is equal to:*

$$\begin{aligned} & \frac{\cos 2\alpha}{4} [g(W) + g(S) + g(E) + g(N)] \quad \text{inside the domain;} \\ & \frac{\cos 2\alpha}{2(1 + \cos(\pi/4 - \alpha))} [g(W) + g(S)] + \frac{\cos(\pi/4 + \alpha)}{1 + \cos(\pi/4 - \alpha)} g(E) \quad \text{on the horizontal part of } L^+(w); \\ & \frac{\cos 2\alpha}{2(1 + \cos(\pi/4 - \alpha))} [g(W) + g(S)] + \frac{\cos(\pi/4 + \alpha)}{1 + \cos(\pi/4 - \alpha)} g(N) \quad \text{on the vertical part of } L^+(w); \\ & \frac{\cos 2\alpha}{4} [g(W) + g(S)] + \frac{\cos(\pi/4 - \alpha)}{2} [g(E) + g(N)] \quad \text{at } w, \end{aligned}$$

with  $N$ ,  $E$ ,  $S$  and  $W$  being the four neighbors of  $X$ .

**Proof** When the site is inside the domain, the proof is the same as in Lemma 8.4. For boundary sites, a similar computation can be done. For instance, consider Case 2 in Fig. 8.2. Equations (3) and (7) in the proof of Lemma 8.4 are preserved. Furthermore, Lemma 5.9 implies that

$$f(X) = f(x_1) = \phi_{T(w)}(X \leftrightarrow L^-)$$

and similarly  $f(x_4) = f(E)$  (where  $f$  is still as defined in the proof of Lemma 8.4). Plugging all these equations together, we obtain the second equality. The other cases are handled similarly.  $\square$

Now, we aim to use a representation with massive random walks similar to the proof of the lower bound. One technical point is the fact that the mass at  $w$  is larger than 1. This could *a priori* prevent  $(F(X_t)m_t)_t$  from being uniformly integrable. Therefore, the behavior at  $w$  needs to be treated separately. Denote by  $\tau_1$  the hitting time (for  $t > 0$ ) of  $w$ , and by  $\tau$  the hitting time of  $L^-$ . Since the masses are smaller than 1, excepted at  $w$ ,  $(F(X_t)m_t)_{t \leq \tau \wedge \tau_1}$  is uniformly integrable and we can applying the stopping theorem to obtain:

$$F(w) = \mathbb{E}^w [F(X_{\tau \wedge \tau_1})m_{\tau \wedge \tau_1}] = \mathbb{E}^w [F(X_{\tau_1})m_{\tau_1} \mathbb{1}_{\tau_1 < \tau}] + \mathbb{E}^w [F(X_\tau)m_\tau \mathbb{1}_{\tau < \tau_1}].$$

Since  $X_{\tau_1} = w$ , the previous formula can be rewritten as

$$F(w) = \frac{\mathbb{E}^w [F(X_\tau)m_\tau \mathbb{1}_{\tau < \tau_1}]}{1 - \mathbb{E}^w (m_{\tau_1} \mathbb{1}_{\tau_1 < \tau})}. \quad (8.14)$$

When  $w$  goes to infinity in a prescribed direction,  $[1 - \mathbb{E}^w (m_{\tau_1} \mathbb{1}_{\tau_1 < \tau})]$  converges to the analytic function  $h : [0, 1] \rightarrow \mathbb{R}$ ,  $p \mapsto 1 - \mathbb{E}^w (m_{\tau_1})$  (since the function is translation-invariant). The function  $h$  is not equal to 0 when  $p = 0$ , implying that it is equal to 0 for

a discrete set  $\mathcal{P}$  of points. In particular, for  $p \notin \mathcal{P}$ , the first term in the right hand side stays bounded when  $w$  goes to infinity. Denoted by  $C_4 = C_4(p)$  such a bound. Recalling that  $|F| \leq 1$  and that the mass is smaller than 1 except at  $w$ , (8.14) becomes

$$|F(w)| \leq C_4 \left| \mathbb{E}^w [F(X_\tau) m_\tau \mathbb{1}_{\tau < \tau_1}] \right| \leq \mathbb{E}^w [m_\tau \mathbb{1}_{\tau < \tau_1}] \quad (8.15)$$

$$\leq C_4 \sum_{w < x} \mathbb{E}^x [(\cos 2\alpha)^\tau \mathbb{1}_{\tau < \tau_1} \mathbb{1}_{\{(X_t) \text{ avoids } L^+(w)\}}] \leq C_4 \sum_{w < x} G_{\cos 2\alpha}(0, x) \quad (8.16)$$

where the last inequality is due to the release of the conditioning on avoiding  $L^+(w)$ .

Finally, it only remains to bound the right hand side. From (8.16), we deduce

$$|F(w)| \leq C_5 |w| G_{\cos 2\alpha}(0, w) \quad (8.17)$$

where the existence of  $C_5$  is due to the exponential decay of  $G_{\cos 2\alpha}(\cdot, \cdot)$  and the fact that  $G_{\cos 2\alpha}(0, x) \leq G_{\cos 2\alpha}(0, w)$  whenever  $w < x$ . We deduce from (8.12), (8.13) and (8.17) that

$$\phi_p(0 \leftrightarrow a) \leq C_3 C_5 |a|^2 \max_{w > a, |w|_\infty \leq c_5 |a|_\infty} G_m(0, w) \leq C_6 |a|^2 G_m(0, a). \quad (8.18)$$

Taking the logarithm, the claim is obtained for all  $p < p_{sd}$  not in the discrete set  $\mathcal{P}$ . The result follows for every  $p$  using the fact that the correlation length is increasing in  $p$ .  $\square$

# Chapter 9

## Connection probabilities and RSW-type bounds for the two-dimensional FK-Ising and Ising models

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**Abstract:** This chapter is devoted to bounds on crossing probabilities in the critical FK-Ising model. These bounds are uniform in the size of the rectangles and in the boundary conditions, they are analogues for the FK-Ising model to the celebrated Russo-Seymour-Welsh bounds for percolation [Rus78, SW78]. The chapter is inspired by the article *Connection probabilities and RSW-type bounds for the two-dimensional FK-Ising model* [DCHN10], written with Clément Hongler and Pierre Nolin, and published in *Communications in Pure and Applied mathematics*.

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Consider *rectangles*  $R$  of the form  $[0, n] \times [0, m]$  for  $n, m > 0$ , and translations of them. The event that there exists a *vertical crossing* in  $R$ , i.e. an open path from the bottom side  $[0, n] \times \{0\}$  to the top side  $[0, n] \times \{m\}$ , is denoted by  $\mathcal{C}_v(R)$ . Our main result is the following:

**Theorem 9.1** (RSW-type crossing bounds). *Let  $0 < \beta_1 < \beta_2$ . There exist two constants  $0 < c_- \leq c_+ < 1$  (depending only on  $\beta_1$  and  $\beta_2$ ) such that for any rectangle  $R$  with side lengths  $n$  and  $m \in [\beta_1 n, \beta_2 n]$  (i.e. with aspect ratio bounded away from 0 and  $\infty$  by  $\beta_1$  and  $\beta_2$ ), one has*

$$c_- \leq \phi_{p_{\text{sd}}, 2, R}^\xi(\mathcal{C}_v(R)) \leq c_+$$

for any boundary conditions  $\xi$ , where  $\phi_{p_{\text{sd}}, 2, R}^\xi$  denotes the random-cluster measure on  $R$  with parameters  $(p, q) = (p_{\text{sd}}, 2)$  and boundary conditions  $\xi$ .

Our proof relies mostly on Smirnov's observable. More precisely, it is based on precise estimates on connection probabilities for boundary vertices, they allow us to use a second-moment method on the number of pairs of connected sites. In order to do that, the

fermionic observable is used to reveal some harmonicity on the discrete level, which enables us to express macroscopic quantities such as connection probabilities in terms of discrete harmonic measures. We would like to stress that our argument remains completely in a discrete setting, using essentially elementary combinatorial tools: in particular, it does not make use of continuum limits [Smi10b].

Crossing bounds turned out to be instrumental in the study of the percolation model at and near its phase transition – for instance to derive Kesten’s *scaling relations* [Kes87], that link the main macroscopic observables, such as the density of the infinite cluster and the characteristic length. These bounds are also useful in the study of variations of percolation, in particular for models exhibiting a self-organized critical behavior. Theorem 9.1 is then of particular interest in the study of the FK-Ising model at and near criticality (see Chapter 12 as well).

Theorem 9.1 also appears to be useful in enabling to transfer properties of the scaling limit objects back to the discrete models. It is therefore expected to be helpful to prove the existence of critical exponents, in particular of the *arm exponents*. Connections between discrete models and their continuum counterparts usually involve decorrelation of different scales, and thus use spatial independence between regions which are far enough from each other. In the random cluster model, one usually addresses the lack of spatial independence by successive conditionings, using repeatedly the spatial (or domain) Markov property of random-cluster models. For this reason, proving bounds that are *uniform in the boundary conditions* seems to be important. An example of application of this technique is given in Subsection 3.1.

This theorem allows us to derive easily several noteworthy results. Among the consequences, let us mention power law bounds for magnetization at criticality for the Ising model, first established by Onsager in [Ons44], tightness results for the interfaces coming from the Aizenman-Burchard technology, and the value  $1/2$  of the one-arm half-plane exponent – which describes both the asymptotic probability of large-distance connections starting from a boundary point for the FK-Ising model, and the decay of boundary magnetization in the Ising model. Moreover, Theorem 9.1 is used in [LS10] to establish a polynomial upper bound for the mixing time of the Glauber dynamics at criticality, and in [CN07], such crossing bounds allowed the authors to construct sub-sequential scaling limits for the spin field of the critical Ising model.

We would also like to mention that other proofs of Russo-Seymour-Welsh-type bounds have already been proposed. In [CS09], Chelkak and Smirnov give a direct and elegant argument to explicitly compute certain crossing probabilities in the scaling limit, but their argument only applies for some specific boundary conditions (alternatively wired and free on the four sides). In [CN07], Camia and Newman also propose to obtain RSW as a corollary of a recently announced result [CS09]: the convergence of the full collection of interfaces for the Ising model to the conformal loop ensemble  $CLE(3)$ . The interpretation of  $CLE(3)$  in terms of the Brownian loop soup [Wer03] is also used. However, to the author’s knowledge, the proofs of these two results are quite involved, and moreover, the reasoning proposed only applies for the infinite-volume measure. In these two cases, uniformity with respect to the boundary conditions is not addressed, and there does

not seem to be an easy argument to avoid this difficulty. While weaker forms might be sufficient for some applications, it seems however that this stronger form is needed in many important cases, and that it considerably shortens several existing arguments.

Another application of Theorem 9.1 is crossing formulæ for the critical Ising model. Denote by  $\tilde{A}_{n,m}$  the event that there exists a circuit of pluses surrounding  $\Lambda_n$  in  $\Lambda_m$ .

**Theorem 9.2** (circuits in annuli). *There exists a constant  $c > 0$  such that for all  $n$ ,*

$$\mu_{\beta_c, A_{n/2, 4n}}^-(\tilde{A}_{n, 2n}) \geq c.$$

It is a good point to mention a related result. The high-temperature Ising model on the triangular lattice is expected to have the same scaling limit as critical site-percolation on the triangular lattice. The reason is that correlation between sites decays exponentially fast, and each site has probability  $1/2$  to be either  $+$  or  $-$ . For instance, the infinite temperature limit is exactly site-percolation. This observation makes the model extremely interesting, since it provides a (possibly tractable) model for which universality in the temperature  $T > T_c$  holds. Unfortunately, the mathematical understanding of high-temperature Ising models remains fairly basic. We now prove, using techniques of Chapter 4, that RSW-types estimates hold true for high-temperature Ising models on the triangular lattice. This result should be useful in order to prove conformal invariance of this regime as well.

**Theorem 9.3.** *Let  $\alpha > 1$  and  $\beta < \beta_c$ . There exist  $c = c(\alpha) > 0$  and  $K = K(\alpha) > 0$  such that for every  $n > 0$ ,*

$$\mu_{\beta, R_{\alpha n, n}^K}^\xi(\mathcal{C}_h(R_{\alpha n, n})) \geq c \tag{9.1}$$

*uniformly in the boundary condition  $\xi$ , where, in the coordinate system  $(1, e^{i\pi/3})$ ,*

$$\begin{aligned} R_{\alpha n, n} &:= [0, \alpha n] \times [0, n] \\ R_{\alpha n, n}^K &:= [-K \log n, \alpha n + K \log n] \times [-K \log n, n + K \log n], \end{aligned}$$

*and  $\mathcal{C}_h$  is the existence of a path of adjacent pluses crossing the rectangle horizontally.*

We mention that a slight modification of this result is proved (using different techniques) in [HTZ10].

The chapter is organized as follows. In Section 1, the observable is compared to certain harmonic measures, for which estimates can be proved. These estimates are central in the proof of Theorem 9.1, which is performed in Section 2. Section 3 is devoted to several consequences. Section 4 contains the proof of Theorems 9.2 and 9.3.

**Since  $p = p_{sd}(2)$  and  $q = 2$  are fixed in this chapter, they are dropped from the notation. For technical considerations, all graphs are rotated by an  $\pi/4$ -angle in this chapter.**

## 1 Comparison to harmonic measures

In this section, we obtain a comparison result for the boundary values of the fermionic observable  $F$  introduced in terms of discrete harmonic measures. It will be used to obtain all the quantitative estimates on the observable that are needed for the proof of Theorem 9.1.

### 1.1 Comparison principle

As in the previous chapters, let  $(\Omega, a, b)$  be a discrete Dobrushin domain, with free boundary conditions on the arc  $\partial_{ab}$ , and wired boundary conditions on the other arc  $\partial_{ba}$ . Set  $F$  for the fermionic observable in this domain and  $H$  the imaginary part of the discrete primitive of  $F^2$  (like in previous chapter). Recall that  $H_\bullet$  and  $H_\circ$  are the restrictions of  $H$  to black and white faces respectively.

For our estimates, the medial graph of our discrete domain is extended by adding two extra layers of faces: one layer of white faces adjacent to the black faces of the wired arc, and one layer of black faces adjacent to the white faces of the dual free arc. This extended domain is denoted by  $\bar{\Omega}_\diamond$ .

**Remark 9.4.** *Note that a small technicality arises when adding a new layer of faces: some of these additional faces can overlap faces that were already here. For instance, if the domain has a slit, the free and the wired arc are adjacent along this slit, and the extra layer on the wired arc (resp. on the dual free arc) overlaps the dual free arc (resp. the wired arc). As will be seen,  $H_\bullet$  is equal to 1 on the wired arc, and to 0 on the additional layer along the dual free arc. One should thus remember in the following that the added faces are considered as different from the original ones – it will always be clear from the context which faces are considered.*

For any given black face  $B$ , let us define  $(X_{\bullet t}^B)_{t \geq 0}$  to be the continuous-time random walk on the black faces of  $\bar{\Omega}_\diamond$  starting at  $B$ , that jumps with rate 1 on adjacent black faces, *except* for the black faces on the extra layer of black faces adjacent to the dual free arc onto which it jumps with rate  $\rho := 2/(\sqrt{2} + 1)$ . Similarly, let  $(X_{\circ t}^W)_{t \geq 0}$  denote the continuous-time random walk on the white faces of  $\bar{\Omega}_\diamond$  starting at a white face  $W$  that jumps with rate 1 on adjacent white faces, *except* for the white faces on the extra layer of white faces adjacent to the wired arc onto which it jumps with the same rate  $\rho = 2/(\sqrt{2} + 1)$  as previously.

For a black face  $B$ , let  $\mathbf{HM}_\bullet(B)$  denote the probability that the random walk  $X_{\bullet t}^B$  hits the wired arc from  $b$  to  $a$  before hitting the extra layer adjacent to the free arc. Similarly, for  $W$  a white face, we denote by  $\mathbf{HM}_\circ(W)$  the probability that the random walk  $X_{\circ t}^W$  hits the additional layer adjacent to the wired arc before hitting the free arc. Note that there is no extra difficulty in defining these quantities for infinite discrete domains as well.

With these notations, we obtain the following result:

**Proposition 9.5** (uniform comparability). *Let  $(\Omega, a, b)$  be a discrete Dobrushin domain, and let  $e$  be a medial edge of  $\partial_{ab}$  (thus adjacent to the free arc). Let  $B = B(e)$  be the black*

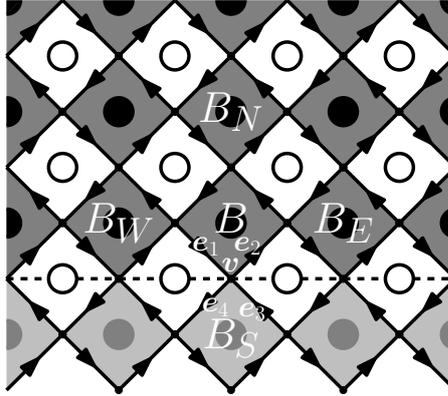


Figure 9.1: Extend  $\Omega_\circ$  by adding two extra layers of medial faces, and extend the functions  $H_\bullet$  and  $H_\circ$  there. Here is represented the extension along the dual free arc.

face bordered by  $e$ , and  $W = W(e)$  be a white face adjacent to  $B$  that does not belong to the dual free arc. Then we have

$$\sqrt{\mathbf{HM}_\circ(W)} \leq |F(e)| \leq \sqrt{\mathbf{HM}_\bullet(B)}. \quad (9.2)$$

**Proof** By (7.4) and the lines following (7.4), we have  $|F(e)|^2 = H(B)$  and  $H(W) = |F(e)|^2 - |F(e')|^2 \leq |F(e)|^2$ , where  $e'$  is the medial edge between  $B$  and  $W$ : it is therefore sufficient to show that  $H(B) \leq \mathbf{HM}_\bullet(B)$  and  $H(W) \geq \mathbf{HM}_\circ(W)$ . We only prove that  $H(B) \leq \mathbf{HM}_\bullet(B)$ , since the other case can be handled in the same way.

For this, we use a variation of a trick introduced in [CS09] and extend the function  $H$  to the extra layer of black faces – added as explained above – by setting  $H$  to be equal to 0 there. It is then sufficient to show that the restriction  $H_\bullet$  of  $H$  to the black faces of  $\bar{\Omega}_\circ$  is subharmonic for the Laplacian that is the generator of the random walk  $X_\bullet$ , since it has the same boundary values as  $\mathbf{HM}_\bullet$  (which is harmonic for this Laplacian). Inside the domain, subharmonicity is given by Corollary 7.7, since there the Laplacian of  $X_\bullet$  is the usual discrete Laplacian (associated with it is just a simple random walk). The only case to check is when a face involved in the computation of the Laplacian belongs to one of the extra layers. For the sake of simplicity, we study the case when only one face belongs to these extra layers.

Denote by  $B_W$ ,  $B_N$ ,  $B_E$  and  $B_S$  the black faces adjacent to  $B$ , and assume that  $B_S$  is on the extra layer (see Figure 9.1). The discrete Laplacian of  $X_\bullet$  at face  $B$  is denoted by  $\Delta_\bullet$ . We claim that

$$\Delta_\bullet H_\bullet(B) = \frac{2 + \sqrt{2}}{6 + 5\sqrt{2}} [H_\bullet(B_W) + H_\bullet(B_N) + H_\bullet(B_E)] + \frac{2\sqrt{2}}{6 + 5\sqrt{2}} H_\bullet(B_S) - H_\bullet(B) \geq 0. \quad (9.3)$$

For that, let us denote by  $e_1, e_2, e_3, e_4$  the four medial edges at the bottom vertex  $v$  between  $B$  and  $B_S$ , in clockwise order, with  $e_1$  and  $e_2$  along  $B$ , and  $e_3$  and  $e_4$  along  $B_S$  (see Figure 9.1) – note that  $e_3$  and  $e_4$  are not edges of  $\Omega_\circ$ , but of  $\mathbb{L}_\circ$ .

Extend  $F$  to  $e_3$  and  $e_4$  by requiring  $F(e_3)$  and  $F(e_1)$  to be orthogonal, as well as  $F(e_4)$  and  $F(e_2)$ , and  $F(e_1) + F(e_3) = F(e_2) + F(e_4)$  to hold true. This defines these two values uniquely: indeed, as noted before,  $F(e_2) = e^{-i\pi/4}F(e_1)$  on the boundary (since  $W_\gamma(e_a, e_1)$  and  $W_\gamma(e_a, e_2)$  are fixed, with  $W_\gamma(e_a, e_2) = W_\gamma(e_a, e_1) + \pi/2$ , and the curve cannot go through one of these edges without going through the other one), which implies, after a small calculation, that

$$|F(e_3)|^2 = \left| \left( \tan \frac{\pi}{8} \right) e^{i\pi/4} F(e_2) \right|^2 = \frac{2 - \sqrt{2}}{2 + \sqrt{2}} |F(e_2)|^2 = \frac{2 - \sqrt{2}}{2 + \sqrt{2}} H_\bullet(B).$$

If  $\tilde{H}_\bullet$  denotes the function defined by  $\tilde{H}_\bullet = H_\bullet$  on  $B$ ,  $B_W$ ,  $B_N$  and  $B_E$ , and by

$$\tilde{H}_\bullet(B_S) = |F(e_3)|^2 = \frac{2 - \sqrt{2}}{2 + \sqrt{2}} H_\bullet(B), \quad (9.4)$$

then  $\tilde{H}_\bullet$  satisfies the same relation (7.4) (definition of  $H$ ) for  $e_3$  and  $e_4$ , as inside the domain. Since the fermionic observable  $F$  verifies the same local equations, the computation performed in Proposition 2.19, Corollary 7.7 applies at  $B$  (with  $\tilde{H}$  instead of  $H$ ), and we deduce

$$\Delta \tilde{H}_\bullet(B) = \frac{1}{4} [\tilde{H}_\bullet(B_W) + \tilde{H}_\bullet(B_N) + \tilde{H}_\bullet(B_E) + \tilde{H}_\bullet(B_S)] - \tilde{H}_\bullet(B) \geq 0. \quad (9.5)$$

Using the definition of  $\tilde{H}_\bullet$ , this inequality can be rewritten as

$$\frac{1}{4} [H_\bullet(B_W) + H_\bullet(B_N) + H_\bullet(B_E)] - \frac{6 + 5\sqrt{2}}{4(2 + \sqrt{2})} H_\bullet(B) \geq 0. \quad (9.6)$$

Now using that  $H_\bullet(B_S) = 0$ , the claim (9.3) follows.  $\square$

## 1.2 Estimates on harmonic measures

In the previous subsection, a comparison principle between the values of  $H$  near the boundary is given, and the harmonic measures associated with two (almost simple) random walks, on the two lattices composed of the black faces and of the white faces respectively. In this subsection, we provide estimates for these two harmonic measures in different domains needed for the proof of Theorem 9.1. We start with giving a lower bound which is useful in the proof of the 1-point estimate.

**Lemma 9.6.** *For  $\beta > 0$  and  $n \geq 0$ , let  $R_n^\beta$  be*

$$R_n^\beta = [-\beta n, \beta n] \times [0, 2n].$$

*Then there exists  $c_1(\beta) > 0$  such that for any  $n \geq 1$ ,*

$$\mathbf{HM}_\circ(W_x) \geq \frac{c_1(\beta)}{n^2} \quad (9.7)$$

*in the Dobrushin domain  $(R_n^\beta, u, u)$  (see Figure 9.2), for all  $x = (x_1, 0)$  and  $u = (u_1, 2n)$  such that  $|x_1|, |u_1| \leq \beta n/2$  (i.e. far enough from the corners),  $W_x$  being any of the two white faces that are adjacent to  $x$  and not on the dual free arc.*

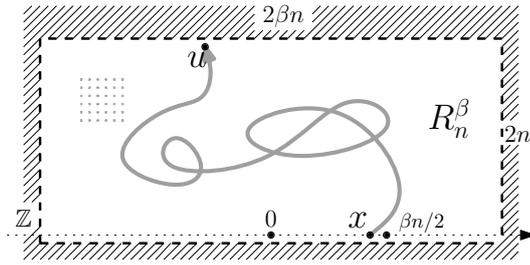


Figure 9.2: Estimate of Lemma 9.6: the dashed line corresponds to the dual free arc.

**Proof** This proposition follows from standard results on simple random walks (gambler’s ruin type estimates). For the sake of conciseness, a detailed proof is not provided.  $\square$

In the remaining part of this section, consider *only* Dobrushin domains  $(\Omega, a, b)$  that contain the origin on the free arc, and are subsets of the medial lattice  $\mathbb{H}_\circ$ , where  $\mathbb{H} = \{(x_1, x_2) \in \mathbb{Z}^2, x_2 \geq 0\}$  denotes the upper half plane – in this case,  $\Omega$  is said to be a Dobrushin  $\mathbb{H}$ -domain. For the following estimates on harmonic measures, the Dobrushin domains that are considered can also be infinite. We are interested in the harmonic measure of the wired arc seen from a given point: without loss of generality, this point is assumed to be the origin. Let  $B_0$  be the corresponding black face of the medial lattice, and  $W_0$  be an adjacent white face which is not on the free arc.

We first prove a lower bound on the harmonic measure. For that, introduce, for  $k \in \mathbb{Z}$  and  $n \geq 0$ , the segments

$$l_n(k) = \{k\} \times [0, n] \quad (= \{(k, j) : 0 \leq j \leq n\}).$$

**Lemma 9.7.** *There exists a constant  $c_2 > 0$  such that for any Dobrushin  $\mathbb{H}$ -domain  $(\Omega, a, b)$ , we have*

$$\mathbf{HM}_\circ(W_0) \geq \frac{c_2}{k}, \tag{9.8}$$

*provided that, in  $\Omega$ , the segment  $l_k(-k)$  disconnects from the origin the intersection of the free arc with the upper half-plane (see Figure 9.3).*

**Proof** The arc  $l_k(-k)$  disconnects the origin from the part of the free arc that lies in the upper half-plane, let us thus consider the connected component of  $\Omega \setminus l_k(-k)$  that contains the origin. In this new domain  $\Omega_0$ , if boundary conditions along  $l_k(-k)$  are free, the harmonic measure of the wired arc is smaller than the harmonic measure of the wired arc in the original domain  $\Omega$ . On the other hand, the harmonic measure of the wired arc in  $\Omega_0$  is larger than the harmonic measure of the wired arc in the slit domain  $(\mathbb{H} \setminus l_k(-k), (-k, k), \infty)$ , which has respectively wired and free boundary conditions to the left and to the right of  $(-k, k)$  (see Figure 9.3). Estimating this harmonic measure is straightforward, using the same arguments as before.  $\square$

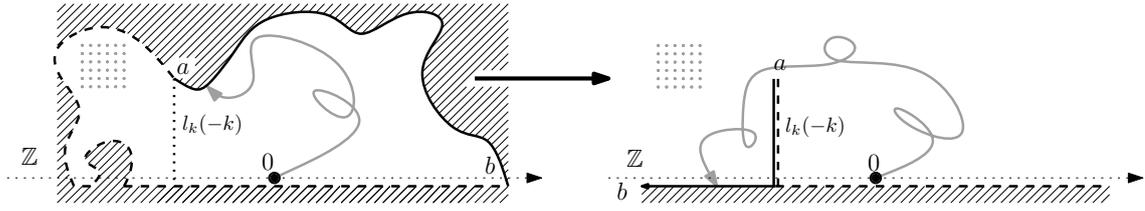


Figure 9.3: The two domains involved in the proof of Lemma 9.7.

Upper bounds on the harmonic measures are now derived. Estimates of two different types will be needed. The first one takes into account the distance between the origin and the wired arc, while the second one requires the existence of a segment  $l_n(k)$  disconnecting the wired arc from the origin (still inside the domain).

**Lemma 9.8.** *There exist constants  $c_3, c_4 > 0$  such that for any Dobrushin  $\mathbb{H}$ -domain  $(\Omega, a, b)$ ,*

- if  $d_1(0)$  denotes the graph distance between the origin and the wired arc,

$$\mathbf{HM}_\bullet(B_0) \leq c_3 \frac{1}{d_1(0)}, \quad (9.9)$$

- and if the segment  $l_n(k)$  disconnects the wired arc from the origin inside  $\Omega$ ,

$$\mathbf{HM}_\bullet(B_0) \leq c_4 \frac{n}{|k|^2}. \quad (9.10)$$

**Proof** Let us first consider item (9.9). For  $d = d_1(0)$ , define the Dobrushin domain  $(\mathcal{B}_d, (-d, 0), (d, 0))$ , where  $\mathcal{B}_d$  is the set of sites in  $\mathbb{H}$  at a graph distance at most  $d$  from the origin (see Figure 9.4). The harmonic measure of the wired arc in  $(\Omega, a, b)$  is smaller than the harmonic measure of the wired arc in this new domain  $\mathcal{B}_d$ , and, as before, this harmonic measure is easy to estimate.

Let us now turn to item (9.10). Since  $l_n(k)$  disconnects the wired arc from the origin, the harmonic measure of the wired arc is smaller than the harmonic measure of  $l_n(k)$  inside  $\Omega$ , and this harmonic measure is smaller than it is in the domain  $\mathbb{H} \setminus l_n(k)$  with wired boundary conditions on the left side of  $l_n(k)$  – right side if  $k < 0$  (see Figure 9.4). Once again, the estimates are easy to perform in this domain.  $\square$

## 2 Proof of Theorem 9.1

We now prove our result, Theorem 9.1. The main step is to prove the uniform lower bound for rectangles of bounded aspect ratio with free boundary conditions. We then use monotonicity to compare boundary conditions and obtain the desired result. In the case

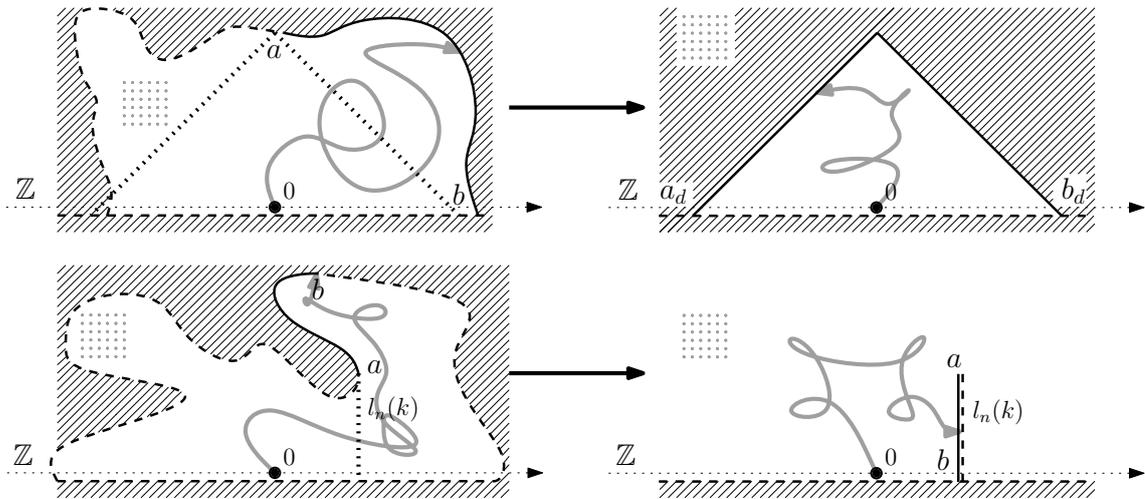


Figure 9.4: The two different upper bounds (9.9) and (9.10) of Lemma 9.8.

of free boundary conditions, the proof relies on a second moment estimate on the number  $N$  of pairs of vertices  $(x, u)$ , on the top and bottom sides of the rectangle respectively, that are connected by an open path.

The organization of this section follows the second-moment estimate strategy. In Proposition 9.10, we first prove a lower bound on the probability of a connection from a given site on the bottom side of a rectangle to a given site on the top side. This estimate gives a lower bound on the expectation of  $N$ . Then, Proposition 9.11 provides an upper bound on the probability that two points on the bottom side of a rectangle are connected to the top side. This proposition is the core of the proof, and it provides the right bound for the second moment of  $N$ . It allows us to conclude the section by using the second moment estimate method, thus proving Theorem 9.1.

In this section, two main tools will be used: the domain Markov property, and probability estimates for connections between the wired arc and sites on the free arc. We first explain how the previous estimates on harmonic measures can be used to derive estimates on connection probabilities. The following lemma is instrumental in this approach.

**Lemma 9.9.** *Let  $(\Omega, a, b)$  be a Dobrushin domain. For any site  $x$  on the free arc of  $\Omega$ , we have*

$$\sqrt{\mathbf{HM}_\circ(W_x)} \leq \phi_\Omega^{a,b}(x \leftrightarrow \text{wired arc}) \leq \sqrt{\mathbf{HM}_\bullet(B_x)}, \quad (9.11)$$

where  $B_x$  is the black face corresponding to  $x$ , and  $W_x$  is any closest white face that is not on the free arc.

**Proof** Since  $x$  is on the free boundary of  $\Omega$ , there exists a white face on the free arc of  $\Omega_\circ$  which is adjacent to  $B_x$ : denote by  $e$  the edge between these faces. As noted before, since the edge  $e$  is along the free arc, the winding  $W_\gamma(e_a, e)$  of the exploration path  $\gamma$  at

$e$  is constant, and depends only on the direction of  $e$ . This implies that

$$\phi_{\Omega}^{a,b}(e \in \gamma) = |F(e)|.$$

In addition,  $e$  belongs to  $\gamma$  if and only if  $x$  is connected to the wired arc, which implies that  $|F(e)|$  is exactly equal to  $\phi_{\Omega}^{a,b}(x \leftrightarrow \text{wired arc})$ . Proposition 9.5 thus implies the claim.  $\square$

With this lemma at our disposal, the different estimates can be proved. Throughout the proof, the notation  $c_i(\beta)$  will be used for constants that depend neither on  $n$  nor on sites  $x, y$  or on boundary conditions. When they do not depend on  $\beta$ , they are denoted by  $c_i$  (it is the case for the upper bounds). Recall the definition of  $R_n^\beta$ :

$$R_n^\beta = [-\beta n, \beta n] \times [0, 2n]. \quad (9.12)$$

Let  $\partial_+ R_n^\beta$  (resp.  $\partial_- R_n^\beta$ ) be the *top side*  $[-\beta n, \beta n] \times \{2n\}$  (resp. *bottom side*  $[-\beta n, \beta n] \times \{0\}$ ) of the rectangle  $R_n^\beta$ . We begin with a lower bound on connection probabilities.

**Proposition 9.10** (connection probability for one point on the bottom side). *Let  $\beta > 0$ , there exists a constant  $c(\beta) > 0$  such that for any  $n \geq 1$ ,*

$$\phi_{R_n^\beta}^0(x \leftrightarrow u) \geq \frac{c(\beta)}{n} \quad (9.13)$$

for all  $x = (x_1, 0) \in \partial_- R_n^\beta$ ,  $u = (u_1, 2n) \in \partial_+ R_n^\beta$ , satisfying  $|x_1|, |u_1| \leq \beta n/2$ .

**Proof** The probability that  $x$  and  $u$  are connected in the rectangle with free boundary conditions can be written as the probability that  $x$  is connected to the wired arc in  $(R_n^\beta, u, u)$  (where the wired arc consists of a single vertex). The previous lemma, together with the estimate of Lemma 9.6, concludes the proof.  $\square$

We now study the probability that two boundary points on the bottom edge of  $R_n^\beta$  are connected to the top edge, with boundary conditions wired on the top side and free on the other sides.

**Proposition 9.11** (connection probability for two points on the bottom side). *There exists a constant  $c > 0$  (uniform in  $\beta, n$ ) such that for any rectangle  $R_n^\beta$  and any two points  $x, y$  on the bottom side  $\partial_- R_n^\beta$ ,*

$$\phi_{R_n^\beta}^{a_n, b_n}(x, y \leftrightarrow \text{wired arc}) \leq \frac{c}{\sqrt{|x - y|n}}, \quad (9.14)$$

where  $a_n$  and  $b_n$  denote respectively the top-left and top-right corners of the rectangle  $R_n^\beta$ .

The proof is based on the following lemma, which is a strong form of the so-called half-plane one-arm probability estimate (see Subsection 3.1 for a further discussion of this result). For  $x$  on the bottom side of  $R_n^\beta$  and  $k \geq 1$ , denote by  $\mathcal{B}_k(x)$  the box centered at  $x$  with diameter  $k$  for the *graph distance*. The required lemma can be stated:

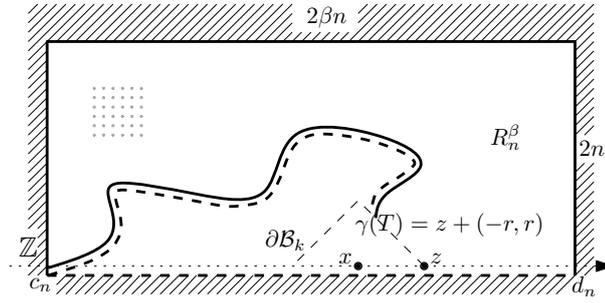


Figure 9.5: The Dobrushin domain  $(R_n^\beta, c_n, d_n)$ , together with the exploration path up to time  $T$ .

**Lemma 9.12.** *There exists a constant  $c_5 > 0$  (uniform in  $n, \beta$  and the choice of  $x$ ) such that for all  $k \geq 0$ ,*

$$\phi_{R_n^\beta}^{a_n, b_n}(\mathcal{B}_k(x) \leftrightarrow \text{wired arc}) \leq c_5 \sqrt{\frac{k}{n}}. \quad (9.15)$$

**Proof** Consider  $n, k, \beta > 0$ , and the box  $R_n^\beta$  with one point  $x \in \partial_+ R_n^\beta$ . (9.15) becomes trivial if  $k \geq n$ , so we can assume that  $k \leq n$ . For any choice of  $\beta' \geq \beta$ , the monotonicity between boundary conditions implies that the probability that  $\mathcal{B}_k(x)$  is connected to the wired arc  $\partial_+ R_n^\beta$  in  $(R_n^\beta, a_n, b_n)$  is smaller than the probability that  $\mathcal{B}_k(x)$  is connected to the wired arc in the Dobrushin domain  $(R_n^{\beta'}, c_n, d_n)$ , where  $c_n$  and  $d_n$  are the bottom-left and bottom-right corners of  $R_n^{\beta'}$ . From now on, replace  $\beta$  by  $\beta + 1$ , and consider the new domain  $(R_n^\beta, c_n, d_n)$ . Notice that  $\mathcal{B}_k$  is then included in  $R_n^\beta$  and that the right-most site of  $\mathcal{B}_k$  is at a distance at least  $n$  from the wired arc.

Let  $T$  denote the hitting time – for the exploration path naturally parametrized by the number of steps – of the set of medial edges bordering (the black faces corresponding to) the sites of  $\mathcal{B}_k(x)$ ; set  $T = \infty$  if the exploration path never reaches this set, so that  $\mathcal{B}_k$  is connected to the wired arc if and only if  $T < \infty$ .

Let  $z$  be the right-most site of the box  $\mathcal{B}_k(x)$ . Consider now the event  $\{z \leftrightarrow \text{wired arc}\}$ . By conditioning on the curve up to time  $T$  (and on the event  $\{\mathcal{B}_k(x) \leftrightarrow \text{wired arc}\}$ ), we obtain

$$\begin{aligned} \phi_{R_n^\beta}^{c_n, d_n}(z \leftrightarrow \text{wired arc}) &= \phi_{R_n^\beta}^{c_n, d_n}[\mathbb{I}_{T < \infty} \cdot \phi_{R_n^\beta}^{c_n, d_n}(z \leftrightarrow \text{wired arc} \mid \gamma[0, T])] \\ &= \phi_{R_n^\beta}^{c_n, d_n}[\mathbb{I}_{T < \infty} \cdot \phi_{R_n^\beta \setminus \gamma[0, T]}^{\gamma(T), d_n}(z \leftrightarrow \text{wired arc})], \end{aligned}$$

where the second inequality used the domain Markov property and the fact that it is sufficient for  $z$  to be connected to the wired arc in the new domain (since it is then automatically connected to the wired arc of the original domain).

On the one hand, since  $z$  is at a distance at least  $n$  from the wired arc (thanks to the

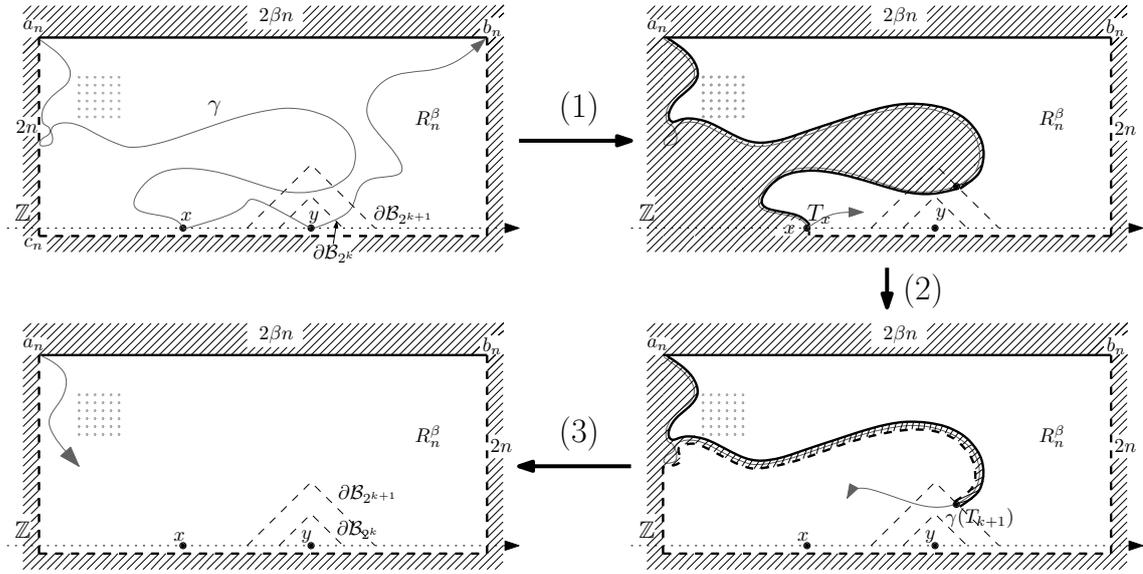


Figure 9.6: This picture presents the different steps in the proof of Proposition 9.11: we first (1) condition on  $\gamma[0, T_x]$  and use the uniform estimate (9.9) of Lemma 9.8, then (2) condition on  $\gamma[0, T_{k+1}]$  and use the estimate (9.10) of Lemma 9.8, in order to (3) conclude with Lemma 9.12.

new choice of  $\beta$ ), Lemma 9.9 can be combined with Item (9.9) of Lemma 9.8 to obtain

$$\phi_{R_n^\beta}^{c_n, d_n}(z \leftrightarrow \text{wired arc}) \leq \frac{c_3}{\sqrt{n}}.$$

On the other hand, if  $\gamma(T)$  can be written as  $\gamma(T) = z + (-r, r)$ , with  $0 \leq r \leq k$ , then the arc  $z + l_r(-r)$  disconnects the free arc from  $z$  in the domain  $R_n^\beta \setminus \gamma[0, T]$ , while if  $\gamma(T) = z + (-r, 2k - r)$ , with  $k + 1 \leq r \leq 2k$ , then the arc  $z + l_r(-r)$  still disconnects the free arc from  $z$ . Using once again Lemma 9.9, this time with Lemma 9.7, we obtain that a.s.

$$\phi_{R_n^\beta \setminus \gamma[0, T]}^{\gamma(T), d_n}(z \leftrightarrow \text{wired arc}) \geq \frac{c_4}{\sqrt{r}} \geq \frac{c_4}{\sqrt{2k}}.$$

This estimate being uniform in the realization of  $\gamma[0, T]$ , we obtain

$$\frac{c_4}{\sqrt{2k}} \phi_{R_n^\beta}^{c_n, d_n}(T < \infty) \leq \phi_{R_n^\beta}^{c_n, d_n}(z \leftrightarrow \text{wired arc}) \leq \frac{c_3}{\sqrt{n}},$$

which implies the desired claim (9.15).  $\square$

**Proof of Proposition 9.11** Let us take two sites  $x$  and  $y$  on  $\partial_- R_n^\beta$ . As in the previous proof, the larger the  $\beta$ , the larger the corresponding probability,  $\beta$  can thus be chosen in such a way that there are no boundary effects. In order to prove the estimate, we express

the event considered in terms of the exploration path  $\gamma$ . If  $x$  and  $y$  are connected to the wired arc,  $\gamma$  must go through two boundary edges which are adjacent to  $x$  and  $y$ , which are denoted by  $e_x$  and  $e_y$ . Notice that  $e_x$  has to be discovered by  $\gamma$  before  $e_y$  is.

Now, define  $T_x$  to be the hitting time of  $e_x$ , and  $T_k$  to be the hitting time of the set of medial edges bordering (the black faces associated with) the sites of  $\mathcal{B}_{2^k}(y)$ , for  $k \leq k_0 = \lfloor \log_2 |x - y| \rfloor$  – where  $\lfloor \cdot \rfloor$  is the integer part of a real number. If the exploration path does not cross this ball before hitting  $e_x$ , set  $T_k = \infty$ . With these definitions, the probability that  $e_x$  and  $e_y$  are both on  $\gamma$  can be expressed as

$$\phi_{R_n^\beta}^{a_n, b_n}(x, y \leftrightarrow \text{wired arc}) = \phi_{R_n^\beta}^{a_n, b_n}(e_x, e_y \in \gamma) \quad (9.16)$$

$$= \sum_{k=0}^{k_0} \phi_{R_n^\beta}^{a_n, b_n}(e_y \in \gamma, T_x < \infty, T_{k+1} < T_k = \infty) \quad (9.17)$$

$$= \sum_{k=0}^{k_0} \phi_{R_n^\beta}^{a_n, b_n}[\mathbb{I}_{T_{k+1} < T_k = \infty} \cdot \mathbb{I}_{T_x < \infty} \cdot \phi_{R_n^\beta}^{a_n, b_n}(e_y \in \gamma | \gamma[0, T_x])], \quad (9.18)$$

where the third equality is obtained by conditioning on the exploration path up to time  $T_x$ . Recall that  $e_y$  belongs to  $\gamma$  if and only if  $y$  is connected to the wired arc. Moreover, if  $\{T_k = \infty\}$ ,  $y$  is at a distance at least  $2^k$  from the wired arc in  $R_n^\beta \setminus \gamma[0, T_x]$ . Hence, the domain Markov property, item (9.9) of Lemma 9.8 and Lemma 9.9 give that, on  $\{T_k = \infty\}$ ,

$$\phi_{R_n^\beta}^{a_n, b_n}(e_y \in \gamma | \gamma[0, T_x]) = \phi_{R_n^\beta \setminus \gamma[0, T_x]}^{x, b_n}(y \leftrightarrow \text{wired arc}) \leq \frac{c_3}{\sqrt{2^k}} \quad a.s.$$

By plugging this uniform estimate into (9.18), and removing the condition on  $T_k = \infty$ , we obtain

$$\phi_{R_n^\beta}^{a_n, b_n}(e_x, e_y \in \gamma) \leq \sum_{k=0}^{k_0} \frac{c_3}{\sqrt{2^k}} \phi_{R_n^\beta}^{a_n, b_n}[\mathbb{I}_{T_{k+1} < \infty} \cdot \phi_{R_n^\beta}^{a_n, b_n}(T_x < \infty | \gamma[0, T_{k+1}])],$$

where we conditioned on the path up to time  $T_{k+1}$ . Now,  $e_x$  belongs to  $\gamma$  if and only if  $x$  is connected to the wired arc. Assuming  $\{T_{k+1} < \infty\}$ , the vertical segment connecting  $\gamma(T_{k+1})$  to  $\mathbb{Z}$  – of length at most  $2^{k+1}$  – disconnects the wired arc from  $x$  in the domain  $R_n^\beta \setminus \gamma[0, T_{k+1}]$ . For  $k+1 < k_0$ , this vertical segment is at distance at least  $\frac{1}{2}|x - y|$  from  $x$ . Applying the domain Markov property and item (9.10) of Lemma 9.8, we deduce that, for  $k+1 < k_0$ , on  $\{T_{k+1} < \infty\}$ ,

$$\phi_{R_n^\beta}^{a_n, b_n}(e_x \in \gamma | \gamma[0, T_{k+1}]) = \phi_{R_n^\beta \setminus \gamma[0, T_{k+1}]}^{\gamma(T_{k+1}), b_n}(x \leftrightarrow \text{wired arc}) \leq 2c_4 \frac{\sqrt{2^{k+1}}}{|x - y|} \quad a.s..$$

Making use of this uniform bound, we obtain

$$\begin{aligned}
 & \phi_{R_n^\beta}^{a_n, b_n}(x, y \leftrightarrow \text{wired arc}) \\
 & \leq 2c_3c_4 \sum_{k=0}^{k_0-2} \frac{\sqrt{2^{k+1}}}{\sqrt{2^k}|x-y|} \phi_{R_n^\beta}^{a_n, b_n}(T_{k+1} < \infty) + 2c_3 \frac{\phi_{R_n^\beta}^{a_n, b_n}(T_x < \infty)}{\sqrt{2^{k_0-1}}} \\
 & \leq \frac{\sqrt{2}c_3c_4c_5}{|x-y|\sqrt{n}} \sum_{k=0}^{k_0-2} \sqrt{2^k} + \frac{2c_3c_5}{\sqrt{n}2^{k_0-1}} \\
 & \leq \frac{c}{\sqrt{n}|x-y|},
 \end{aligned}$$

using also Lemma 9.12 (twice) for the second inequality.  $\square$

We are now in a position to prove our result.

**Proof of Theorem 9.1** Let  $\beta > 0$ ,  $n > 0$ , and also  $R_n^\beta$  defined as previously.

**Step 1: lower bound for free boundary conditions.** Let  $N_n$  be the number of connected pairs  $(x, u)$ , with  $x \in \partial_- R_n^\beta$ , and  $u \in \partial_+ R_n^\beta$ . The expected value of this quantity is equal to

$$\phi_{R_n^\beta}^0[N_n] = \sum_{\substack{u \in \partial_+ R_n^\beta \\ x \in \partial_- R_n^\beta}} \phi_{R_n^\beta}^0(x \leftrightarrow u).$$

Proposition 9.10 directly provides the following lower bound on the expectation by summing over the  $(\beta n)^2$  pairs of points  $(x, u)$  far enough from the corners, *i.e.* satisfying the condition of the proposition:

$$\phi_{R_n^\beta}^0[N_n] \geq c_6(\beta)n$$

for some  $c_6(\beta) > 0$ .

On the other hand, if  $x$  and  $u$  (resp.  $y$  and  $v$ ) are pair-wise connected, then they are also connected to the horizontal line  $\mathbb{Z} \times \{n\}$  which is (vertically) at the middle of  $R_n^\beta$ . Moreover, the domain Markov property implies that the probability – in  $R_n^\beta$  with free boundary conditions – that  $x$  and  $y$  are connected to this line is smaller than the probability of this event in the rectangle of half height with wired boundary conditions on the top side. In the following, assume without loss of generality that  $n$  is even and set  $m = n/2$ , so that the previous rectangle is  $R_m^{2\beta}$ , and define  $a_m$  and  $b_m$  as before. Using the FKG inequality, and also the symmetry of the lattice, we get

$$\phi_{R_n^\beta}^0(x \leftrightarrow u, y \leftrightarrow v) \leq \phi_{R_m^{2\beta}}^{a_m, b_m}(x, y \leftrightarrow \text{wired arc}) \phi_{R_m^{2\beta}}^{a_m, b_m}(\bar{u}, \bar{v} \leftrightarrow \text{wired arc}),$$

where  $\bar{u}$  and  $\bar{v}$  are the projections on the real axis of  $u$  and  $v$ . Summing the bound provided by Proposition 9.11 on all sites  $x, y \in \partial_- R_n^\beta$  and  $u, v \in \partial_+ R_n^\beta$ , we obtain

$$\phi_{R_n^\beta}^0[N_n^2] \leq c_7 m^2 \leq c_7 n^2$$

for some constant  $c_7 > 0$ . Now, by the Cauchy-Schwarz inequality,

$$\phi_{R_n^\beta}^0(\mathcal{C}_v(R_n^\beta)) = \phi_{R_n^\beta}^0(N_n > 0) = \phi_{R_n^\beta}^0[(\mathbb{I}_{N_n > 0})^2] \geq \frac{\phi_{R_n^\beta}^0[N_n]^2}{\phi_{R_n^\beta}^0[N_n^2]} \geq c_6(\beta)^2/c_7,$$

since  $\phi_{R_n^\beta}^0[N_n] = \phi_{R_n^\beta}^0[N_n \mathbb{I}_{N_n > 0}]$ . We have thus reached the claim.

**Step 2: lower and upper bounds for general boundary conditions.** Using the ordering between boundary conditions, the lower bound that was previously proved for free boundary conditions actually implies the lower bound for any boundary conditions  $\xi$ .

For the upper bound, consider a rectangle  $R$  with dimensions  $n \times m$  with  $m \in [\beta_1 n, \beta_2 n]$  and with boundary conditions  $\xi$ . Using once again (3.11), it is sufficient to address the case of wired boundary conditions, and in this case, the probability that there exists a dual crossing from the left side to the right side is at least  $c_- = c_-(1/\beta_2, 1/\beta_1)$ , since the dual model has free boundary conditions. We deduce, using the self-duality property, that

$$\phi_R^\xi(\mathcal{C}_v(R)) \leq 1 - \phi_R^1(\mathcal{C}_h^*(R)) = 1 - \phi_{R^*}^0(\mathcal{C}_h(R^*)) \leq 1 - c_-, \quad (9.19)$$

where the notation  $\mathcal{C}_h^*$  is used for the existence of a horizontal dual crossing, and  $R^*$  is as usual the dual graph of  $R$  (note that the invariance by  $\pi/2$ -rotations was implicitly used). This concludes the proof of Theorem 9.1. □

### 3 Consequences for the FK-Ising and the (spin) Ising models

In this section gathers several implications of the previous theorem. We would like to emphasize the fact that uniformity on boundary conditions is crucial for all these applications, and Theorem 4.4 would not suffice in these case.

#### 3.1 Critical exponents for the FK-Ising and the Ising models

##### Power-law decay of the magnetization at criticality

Let us start with stating an easy consequence of Theorem 9.1. Consider the box  $\Lambda_n = [-n, n]^2$ , its boundary being denoted as usual by  $\partial\Lambda_n$ . Let us also introduce the annulus  $A_{m,n} = \Lambda_n \setminus \Lambda_m$  of radii  $m < n$  centered on the origin, and denote the event that there exists an open circuit surrounding  $\Lambda_m$  in this annulus by  $\mathcal{C}(A_{m,n})$ .

**Corollary 9.13** (circuits in annuli). *For every  $\beta < 1$ , there exists a constant  $c_\beta > 0$  such that for all  $n$  and  $m$ , with  $m \leq \beta n$ ,*

$$\phi_{A_{m,n}}^0(\mathcal{C}(A_{m,n})) \geq c_\beta.$$

**Proof** This follows from Theorem 9.1 applied in the four rectangles  $R_B = [-n, n] \times [-n, -m]$ ,  $R_L = [-n, -m] \times [-n, n]$ ,  $R_T = [-n, n] \times [m, n]$  and  $R_R = [m, n] \times [-n, n]$ . Indeed, if there exists a crossing in each of these rectangles in the “hard” direction, one can construct from them a circuit in  $S_{m,n}$ .

Now, consider any of these rectangles,  $R_B$  for instance. Its aspect ratio is bounded by  $2/(1-\beta)$ , so that Theorem 9.1 implies that there is a horizontal crossing with probability at least

$$\phi_{R_B}^0(\mathcal{C}_H(R_B)) \geq c > 0.$$

Combined with the FKG inequality, this allows us to conclude: the desired probability is at least  $c_\beta = c^4 > 0$ .  $\square$

**Proposition 9.14** (power-law decay of the magnetization). *For  $p = p_{\text{sd}}$ , there exists a unique infinite-volume FK-Ising measure. For this measure, there is almost surely no infinite open cluster. Moreover, there exist constants  $\alpha, c > 0$  such that for all  $n \geq 0$ ,*

$$\phi_{p_{\text{sd}},2}(0 \leftrightarrow \partial\Lambda_n) \leq \frac{c}{n^\alpha}. \quad (9.20)$$

This result also applies to the Ising model: the magnetization at the origin decays at least as a power law.

**Remark 9.15.** *It is known from Onsager’s work that the connection probability follows a power law as  $n \rightarrow \infty$ , described by the one-arm plane exponent  $\alpha_1 = 1/8$ . It should be possible to prove the existence and the value of this exponent using conformal invariance, as well as the arm exponents for a larger number of arms. More precisely, one would need to consider the probability of crossing an annulus a certain (fixed) number of times in the scaling limit, and analyze the asymptotic behavior of this probability as the modulus tends to  $\infty$ . Theorem 9.1 then implies the so-called quasi-multiplicativity property, which allows one to deduce, using concentric annuli, the existence and the value of the arm exponents for the discrete model.*

**Proof** First note that it is classical that the non-existence of infinite clusters implies the uniqueness of the infinite-volume measure: it is thus sufficient to prove (9.20)). Consider the annuli  $A_n = A_{2^n, 2^{n+1}}$  for  $n \geq 1$ , and  $\mathcal{C}^*(A_n)$  the event that there is a dual circuit in  $A_n^*$ . Corollary 9.13 implies the existence of a constant  $c > 0$  such that

$$\phi_{A_n}^1(\mathcal{C}^*(A_n)) \geq c$$

for all  $n \geq 1$ . By successive conditionings, we then obtain

$$\phi(0 \leftrightarrow \partial\Lambda_{2^N}) \leq \prod_{n=0}^{N-1} \phi_{A_n}^1((\mathcal{C}^*(A_n))^c) \leq (1-c)^N,$$

and the desired result follows.  $\square$

**$n$ -point functions for the FK-Ising and the Ising models**

Since the work of Onsager [Ons44], it is known that for the Ising model at criticality, the magnetization at the middle of a square of side length  $2m$  with (+) boundary conditions decays like  $m^{-1/8}$ . It is then tempting to say that the correlation of two spins at distance  $m$  in the plane (in the infinite-volume limit, say) decays like  $m^{-1/4}$ , and this is indeed what happens. To the knowledge of the authors, there is no straightforward generalization of Onsager's work that allows us to derive this without difficult computations. However, this result can be made rigorous very easily with the help of Theorem 9.1. Here, the result is given for two-point correlation functions only, but exponents for  $n$ -spin correlations, for instance, can be obtained using exactly the same method.

Let us first use Theorem 9.1 to interpret Onsager's result in terms of the random-cluster representation.

**Lemma 9.16.** *Let  $\Lambda_m$  be the square  $[-m, m]^2$  with arbitrary boundary conditions  $\xi$ . Then there exist two constants  $c_1$  and  $c_2$  (independent of  $m$  and  $\xi$ ) such that*

$$c_1 m^{-1/8} \leq \phi_{\Lambda_m}^{\xi}(0 \leftrightarrow \partial\Lambda_m) \leq c_2 m^{-1/8}.$$

**Proof** This is a consequence of Onsager's result for wired boundary conditions (since it is derived in terms of the Ising model with (+) boundary conditions), which provides the upper bound by monotonicity. Using Theorem 9.1, a lower bound independent of the boundary conditions can be obtained by enforcing the existence of a circuit in the annulus  $A_{m/2, m}$ , and using the FKG inequality. For that, we just need to lower the constant, using monotonicity: the connection probability conditionally on the fact that there is a wired annulus around the origin is indeed larger than the connection probability with wired boundary conditions on  $\partial\Lambda_m$ .  $\square$

The result for two-point correlation functions in the infinite-volume Ising model can now be stated.

**Proposition 9.17.** *Consider the Ising model  $\mu_{\beta_c}$  on  $\mathbb{Z}^2$  at critical temperature. There exist two positive constants  $C_1$  and  $C_2$  such that*

$$C_1 |x - y|^{-1/4} \leq \mu_{\beta_c}[\sigma_x \sigma_y] \leq C_2 |x - y|^{-1/4},$$

where for any  $x, y \in \mathbb{Z}^2$ ,  $\sigma_x$  and  $\sigma_y$  denote the spins at  $x$  and  $y$ .

**Proof** The 2-spin correlation  $\mu_{\beta_c}[\sigma_x \sigma_y]$  can be expressed, in the corresponding random-cluster representation, as the probability of the event  $\{x \leftrightarrow y\}$ . Let now  $m$  be the integer part of  $|x - y|/4$ . The upper bound is easy and does not rely on Theorem 9.1: the event that  $x$  is connected to  $y$  implies that  $x$  is connected to  $x + \partial\Lambda_m$  and that  $y$  is connected to  $y + \partial\Lambda_m$ . Using the domain Markov property, these two events are independent conditionally on the boundaries of the boxes being open: together with the previous lemma, this provides the upper bound.

Let us turn now to the lower bound. We can enforce the existence of a connected “8” in

$$[(x + \Lambda_{2m+2}) \cup (y + \Lambda_{2m+2})] \setminus [(x + \Lambda_m) \cup (y + \Lambda_m)]$$

that surrounds both  $x$  and  $y$  and separates them: this costs only a positive constant  $\alpha$ , independent of  $m$ , using Theorem 9.1 in well-chosen rectangles and the FKG inequality. Using once again the FKG inequality, we get that

$$\phi_{p_{sd},2}(x \leftrightarrow y) \geq \alpha \phi_{p_{sd},2}(x \leftrightarrow x + \partial\Lambda_{2m+2}) \cdot \phi_{p_{sd},2}(y \leftrightarrow y + \partial\Lambda_{2m+2}),$$

and combined with the previous lemma, this yields the desired result.  $\square$

Recently, Chelkak and Izyurov [CI11] introduced a modification of the fermionic observable which permits an explicit computation of the scaling limit of two-point functions in a finite domain. It appears that they are indeed conformally invariant. Chelkak, Hongler and Izyurov also announced a computation of the  $n$ -point functions using the same observable.

### Half-plane one-arm exponent for the FK-Ising model and boundary magnetization for the Ising model

As a by-product of our proofs, in particular of the estimates of Section 1, one can also obtain the value of the critical exponent for the boundary magnetization in the Ising model, near a free boundary arc (assuming it is smooth), and the corresponding one-arm half-plane exponent for the FK-Ising model.

Let us first consider the one-point magnetization  $\mu_{\Omega}^{a,b}[\sigma_x]$  for the Ising model at criticality in a discrete domain  $(\Omega, a, b)$  with free boundary conditions on the counterclockwise arc  $(ab)$ , and  $(+)$  boundary conditions on the other arc  $(ba)$ .

**Proposition 9.18.** *There exist positive constants  $c_1$  and  $c_2$  such that for any discrete domain  $(\Omega, a, b)$  with  $a = (-n, 0)$  and  $b = (n, 0)$  ( $n \geq 0$ ), containing the rectangle  $R_n = [-n, n] \times [0, n]$  and such that its boundary contains the lower arc  $[-n, n] \times \{0\}$ , we have*

$$c_1 n^{-1/2} \leq \mu_{\Omega}^{a,b}[\sigma_0] \leq c_2 n^{-1/2},$$

*uniformly in  $n$ .*

**Proof** The magnetization at the origin can be expressed, in the corresponding random-cluster representation, as the probability that the origin is connected to the wired counterclockwise arc  $(ba)$ . By Lemma 9.9, this probability can be compared to the harmonic measures  $\mathbf{HM}_{\circ}$  and  $\mathbf{HM}_{\bullet}$ , for which estimates similar to the estimates in Lemmas 9.7 and 9.8 hold.  $\square$

This result can be equivalently stated for the one-arm half-plane probability for random-cluster models:

**Proposition 9.19.** *Consider the rectangle  $R_n = [-n, n] \times [0, n]$ . There exist positive constants  $c_1$  and  $c_2$  such that for any boundary conditions  $\xi$  such that the bottom side  $\partial^- R_n$  is free, one has*

$$c_1 n^{-1/2} \leq \phi_{R_n}^\xi(0 \leftrightarrow \partial^+ R_n) \leq c_2 n^{-1/2},$$

*uniformly over all  $n$ .*

**Proof** The upper bound can be obtained using monotonicity and the previous proposition, since (+) boundary conditions in the Ising model correspond to wired boundary conditions in the corresponding random-cluster representation. For the lower bound, by Theorem 9.1 and the FKG inequality, we can enforce the existence of a crossing in the half-annulus  $R_n \setminus R_{n/2}$  that disconnects 0 from  $\partial R_n \setminus \partial^- R_n$  to the price of a constant independent of  $\xi$ . Using monotonicity and FKG, the probability that 0 is connected by an open path to this crossing (conditionally on its existence) is larger than the probability that 0 is connected to the boundary with wired boundary conditions on  $\partial R_n \setminus \partial^- R_n$ , without conditioning. Hence, the lower bound of the previous proposition gives the desired result.  $\square$

**Remark 9.20.** *Note that contrary to the power laws established using the SLE technology, there are no potential logarithmic corrections here – as is the case with the “universal” arm exponents for percolation (corresponding to 2 and 3 arms in the half-plane, and 5 arms in the plane). Furthermore, one can follow the same standard reasoning as for percolation, based on the RSW lower bound, to prove that the two- and three-arm half-plane exponents, with alternating “types” (primal or dual), have values 1 and 2 respectively, see Chapter 10.*

## 3.2 Spatial mixing at criticality

Theorem 9.1 also provides estimates on spatial mixing for both the FK-Ising and the Ising models. In the following proposition, an example of decorrelation between events for the FK-Ising model is given.

**Proposition 9.21.** *There exist  $c, \alpha > 0$  such that for any  $k \leq n$ ,*

$$|\phi_{p_{sd},2}(A \cap B) - \phi_{p_{sd},2}(A)\phi_{p_{sd},2}(B)| \leq c \left(\frac{k}{n}\right)^\alpha \phi_{p_{sd},2}(A)\phi_{p_{sd},2}(B) \quad (9.21)$$

*for any event  $A$  (resp.  $B$ ) depending only on the edges in the box  $S_k$  (resp. outside  $B_n$ ).*

**Proof** First, it is sufficient to prove

$$|\phi_{p_{sd},2,\Lambda_n}^\xi(A) - \phi_{p_{sd},2,\Lambda_n}^1(A)| \leq c \left(\frac{k}{n}\right)^\alpha \phi_{p_{sd},2,\Lambda_n}^\xi(A)$$

for any boundary conditions  $\xi$  and any event  $A$  depending on edges in  $\Lambda_k$ .

CLAIM: There exists a coupling  $P$  on configurations  $(\omega_\xi, \omega_1)$  with the following properties:

- $\omega_\xi$  (resp.  $\omega_1$ ) has law  $\phi_{p_{sd},2,\Lambda_n}^\xi$  (resp.  $\phi_{p_{sd},2,\Lambda_n}^1$ ).
- if  $\omega_1$  contains a closed circuit in  $\Lambda_n \setminus \Lambda_k$ , let  $\Gamma$  be the exterior most such circuit. Then  $\Gamma$  is also closed in  $\omega_\xi$  and  $\omega_1$  and  $\omega_\xi$  coincide inside  $\Gamma$ .

**Proof of the claim** Consider uniform random variables  $U_e$  for every edge  $e$  and index the edges in an arbitrary way. Sample both configurations based on the same random variables  $U_e$  from the exterior, meaning that after  $k$  steps, consider the edge with one end-point connected to the boundary of  $\Lambda_n$  by an open path which has the smallest index. If there are no such edges, pick the edge with smallest index (this will happen only if you discover a closed circuit). This edge is declared open if  $U_e$  is smaller than the conditional probability to be open knowing the boundary conditions and the already determined edges. Note that  $\omega_1$  is larger than  $\omega_\xi$  by comparison between boundary conditions. Therefore, any closed circuit in  $\omega_1$  will also be closed in  $\omega_\xi$ . The configurations inside a closed circuit of  $\omega_1$  coincide since they have been constructed from the same uniform random variables, with the same free boundary conditions in the restricted domain.  $\square$

Now, since  $A$  depends only on the edges in  $\Lambda_k$ , we can prove that conditionally on  $A$ , there exists a dual circuit in  $\phi_{p_{sd},2,\Lambda_n}^1$  with probability  $1 - c(k/n)^\alpha$ . Let  $E$  be this event. We deduce

$$\begin{aligned} \phi_{p_{sd},2,\Lambda_n}^\xi(A) &\geq \phi_{p_{sd},2,\Lambda_n}^\xi(A \cap E) \\ &= P(\omega_\xi \in A \cap E) \\ &\geq P(\omega_1 \in A \cap E) \\ &= \phi_{p_{sd},2,\Lambda_n}^1(A \cap E) \\ &\geq (1 - c(k/n)^\alpha) \phi_{p_{sd},2,\Lambda_n}^1(A) \end{aligned}$$

where in the third line, we used the fact that if  $\omega_1$  belongs to  $E$ , then  $\omega_\xi$  belongs to  $E$  and both configurations coincide in  $\Lambda_k$ . In particular, if  $\omega_1 \in A$  then  $\omega_\xi \in A$ .

Similarly, there exists a coupling  $\tilde{P}$  on configurations  $(\omega_\xi, \omega_1)$  with the following properties:

- $\omega_\xi$  (resp.  $\omega_1$ ) has law  $\phi_{p_{sd},2,\Lambda_n}^\xi$  (resp.  $\phi_{p_{sd},2,\Lambda_n}^1$ ).
- if  $\omega_\xi$  contains an open circuit in  $\Lambda_n \setminus \Lambda_k$ , let  $\tilde{\Gamma}$  be the exterior most such circuit. Then  $\tilde{\Gamma}$  is also open in  $\omega_1$  and  $\omega_1$  and  $\omega_\xi$  coincide inside  $\tilde{\Gamma}$ .

If  $F$  denotes the event that there is an open circuit in  $\Lambda_n \setminus \Lambda_k$ , we find

$$\begin{aligned} \phi_{p_{sd},2,\Lambda_n}^1(A) &\geq \phi_{p_{sd},2,\Lambda_n}^1(A \cap F) \\ &= \tilde{P}(\omega_1 \in A \cap F) \\ &\geq \tilde{P}(\omega_\xi \in A \cap F) \\ &= \phi_{p_{sd},2,\Lambda_n}^\xi(A \cap F) \\ &\geq (1 - c(k/n)^\alpha) \phi_{p_{sd},2,\Lambda_n}^\xi(A) \end{aligned}$$

where once again, we used in the third line that if  $\omega_\xi \in F$ , then  $\omega_1 \in F$ , and both configurations coincide on  $\Lambda_k$  so that  $\omega_\xi \in A$  implies that  $\omega_1 \in A$ . We also used the fact that conditionally on  $A$ , there is an open circuit in  $\Lambda_n \setminus \Lambda_k$  with probability  $1 - c(k/n)^\alpha$ .

□

More generally, Theorem 9.1 would lead to ratio mixing properties, with an explicit polynomial estimate. Away from criticality, estimates of this type can be established by using the rate of spatial decay for the influence of a single site. At criticality, the correlation between distant events does not boil down to correlations between points and a finer argument must be found. Crossing-probability estimates which are *uniform in boundary conditions* are perfectly suited for these problems.

### 3.3 Polynomial bounds on mixing-time

Recently, Lubetzky and Sly [LS10] used spatial mixing properties of the Ising model in order to derive an important conjecture on the mixing time of the Glauber dynamics of the Ising model at criticality:

**Theorem 9.22** (Lubetzky and Sly [LS10]). *There exists  $\alpha > 0$  such that the mixing time of the Glauber dynamics on a  $n \times n$  box is bounded by  $n^\alpha$  for every  $n > 0$  and every boundary conditions.*

As a key step, they harness Theorem 9.1 in order to prove a suitable analogue of Proposition 9.21. Together with tools from the analysis of Markov chains, the spatial mixing property provides polynomial upper bounds on the inverse spectral gap of the Glauber dynamics (and also on the total variation mixing time).

## 4 Russo-Seymour-Welsh for the Ising model

**Proof of Theorem 9.2** We consider the Edward-Sokal coupling. The boundary conditions related to the  $-$  boundary conditions are wired.

Events  $A_{n,2n}$ ,  $A_{n/2,n}^*$ , and  $A_{2n,4n}^*$  occur simultaneously with probability larger than  $c_2^3$  using Corollary 9.13. Now, the occurrence of these three events guarantee the existence of at least one circuit in  $\tilde{A}_{n,2n}$  not connected to the boundary. Therefore, the random coloring of the clusters gives pluses to this circuit with probability  $1/2$ . All together, we find that

$$\mu_{\beta_c, A_{n/2, 4n}}^-(\mathcal{C}(\tilde{A}_{n, 2n})) \geq c_2^3/2.$$

□

**Proof of Theorem 9.3** This proof follows the same lines as the one of Theorem 4.4, therefore we only sketch it now. Let us first fix  $\alpha = 1$ . Exponential decay of correlation for subcritical random-cluster model implies (via the Edward-Sokal coupling)

$$\left| \mu_{\beta, R_{n,n}^K}^{\xi}(A) - \mu_{\beta, R_{n,n}^K}^{\phi}(A) \right| \leq c n e^{-\varepsilon K \log n} \mu_{\beta, R_{n,n}^K}(A) \quad (9.22)$$

for some  $\varepsilon$  small enough and for any event  $A$  depending only on sites inside  $R_n$ . Therefore, using the usual symmetry arguments, we obtain

$$\mu_{\beta, R_n^K}^{\xi}(C_h(R_n)) \geq c$$

for every boundary condition  $\xi$ , where  $c = c(1)$  is small enough and  $K$  large enough.

Now, fix  $\alpha = 3/2$ . Running along the lines of the proof of Proposition 4.8, the event  $A$  has also a probability bounded away from 0 uniformly in  $n$ . Now, the construction of the domain  $G_0(\Gamma_1, \Gamma_2)$  can be adapted (one must be careful about the geometry specific to the triangular lattice, but a simple modification yields the result). Observing the boundary conditions on  $G_0(\Gamma_1, \Gamma_2)$ , two arcs are already  $+$ , and the others are in the worse case  $-$ . Therefore, there is a crossing of pluses between the two  $+$  arcs with probability larger than  $1/2$ , using the comparison between boundary conditions and the fact that pluses and minuses have the same law (this replaces duality). In conclusion, the proof of Proposition 4.8 works *mutatis mutandis*, and Theorem 9.3 follows.  $\square$

# Chapter 10

## Crossing probabilities in topological rectangles

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**Abstract:** We consider the FK-Ising model in two dimension at criticality. We obtain RSW-type crossing probabilities bounds in arbitrary topological rectangles, uniform with respect to the boundary conditions, generalizing results of Chapter 9 and [CS09]. Our result relies on new discrete complex analysis techniques, introduced in [Che11].

We detail some applications, in particular the computation of so-called universal exponents and crossing bounds for the classical Ising model. It is based on the article *Crossing probabilities in topological rectangles for the planar FK-Ising model*, written with Dmitri Chelkak and Clément Hongler [CDCH11a].

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Given a topological rectangle  $(\Omega, a, b, c, d)$  (i.e. a bounded simply connected subdomain of  $\mathbb{Z}^2$  with four marked boundary points) and boundary conditions  $\xi$ , denote by  $\phi_\Omega^\xi$  the critical FK-Ising probability measure on  $\Omega$  with boundary conditions  $\xi$  and by  $(ab) \leftrightarrow (cd)$  the event that there is a *crossing* between the arcs  $(ab)$  and  $(cd)$ , i.e. that  $(ab)$  and  $(cd)$  are connected in the FK configuration.

Let us denote by  $\ell_\Omega[(ab), (cd)]$  the discrete extremal length between  $(ab)$  and  $(cd)$  in  $\Omega$  with unit conductances (see Section 1 for a precise definition). Informally speaking,  $\ell_\Omega[(ab), (cd)]$  measures the distance between  $(ab)$  and  $(cd)$  from a random walk or electrical resistance point of view.

Our main result is a bound for FK-Ising crossing probabilities in terms of discrete extremal length only:

**Theorem 10.1.** *Let  $M > 0$ . There exists  $\delta \in (0, \frac{1}{2})$  such that*

$$\delta \leq \phi_\Omega^\xi [(ab) \leftrightarrow (cd)] \leq 1 - \delta$$

*for any boundary conditions  $\xi$  and for any topological rectangle  $(\Omega, a, b, c, d)$  with*

$$\ell_\Omega [(ab), (cd)] \in \left[ \frac{1}{M}, M \right].$$

Such crossing probabilities bounds, uniform with respect to the boundary conditions, have been obtained in a (straight) rectangle in Theorem 9.1; asymptotic exact computations of crossing probability in arbitrary domains with specific boundary conditions have been derived in [CS09, Theorem 6.1]. In this paper, the crossing bounds hold in *general topological rectangles with general boundary conditions*, and are independent of the local geometry of the boundary. Roughly speaking, the result is a generalization of Theorem 9.1 to possibly “rough” discrete domains; this is for instance needed in order to deal with domains generated by random interfaces (which usually have fractal scaling limits).

As in [DCHN10], our result relies on discrete complex analysis: to connect the FK-Ising model with discrete complex analysis objects, we use the discrete analytic observable for the FK-Ising model introduced by Smirnov [Smi10a] and crossing probability representation (in terms of harmonic measure) introduced by Chelkak and Smirnov [CS09]. To obtain the desired estimate, we adapt these results and use the new harmonic measure techniques developed by Chelkak in [Che11].

Crossing probabilities estimates play a very important role in rigorous statistical mechanics, in particular for percolation models. They constitute the key argument enabling the use of the following techniques:

- Spatial decorrelation: probabilities of certain events in disjoint ‘well separated’ sets can be factorized at the expense of uniformly controlled constants. The main ingredients to do so are the spatial Markov property of the model and the crossing probabilities.
- Regularity estimates and precompactness: the crossing probabilities are instrumental to pass to the scaling limit, by obtaining a priori regularity estimates on the discrete random curves arising in the model.
- Discretization of continuous results: thanks to uniform estimates, one can connect the discrete models (at finite scales) to their continuous limits, and transfer results from the latter to the former.

While the RSW bounds given by Theorem 9.1 already allow for a number of interesting applications of these techniques (see for instance [CN09, LS10, CGN10, GP]), the stronger version of the RSW-type estimates provided by Theorem 10.1 increases the scope of applications. In particular, we get the following consequences:

- Arm exponents: thanks to crossing probabilities, the (microscopic) arm exponents for the FK-Ising model can be related to the (macroscopic) SLE arms exponents, which in turn can be computed using stochastic calculus techniques. The microscopic arm exponents are crucial to understand the fine structure of the phase transition of percolation [Kes87, Nol08], as well as as for interface regularity [AB99] and noise sensitivity [GP] questions. In Section 3 below, a number of results concerning the microscopic arm exponents are obtained (notably the universal arm exponents).

- Crossing probabilities for the spin Ising model: their conformal invariance was investigated numerically in [LPSA94]. Theorem 10.1 allows us to get RSW bounds for the critical Ising model with certain boundary conditions (that imply non-triviality of those in [LPSA94]). Such crossing probabilities can be used to understand the spin-Ising interfaces, in particular in presence of free boundary conditions. See Corollary 10.22 below.
- Coupling of discrete and continuous interfaces: it is useful to couple the critical FK-Ising interfaces and their scaling limit SLE(16/3), in such a way that they are close to each other and that whenever the SLE(16/3) interface hits the boundary of the domain, so does the discrete interface with high probability. Such couplings are in particular useful in order to obtain the full scaling limit of discrete interfaces [CN06, KS10].

## 1 Discrete complex analysis

In the section, we introduce the discrete harmonic measures and random walk partition functions that will be used in this chapter. A number of their properties are provided, namely factorization properties and uniform comparability results, obtained in [Che11]. Finally, we relate certain elementary critical FK-Ising model probabilities to discrete harmonic measure, notably using the fermionic observable (see [CS09] for details on how to use fermionic observables to obtain bound on crossing probabilities with free/wired/free/wired boundary conditions). These results will be brought together in the next section to prove Theorem 10.1.

In the rest of this paper, for two real-valued functions  $f, g$  (generally defined on discrete domains), we will use the notation  $f \lesssim g$  if there exists a constant  $c > 0$  such that  $f \leq cg$  and  $f \asymp g$  if there exists two constants  $c_1, c_2 > 0$  such that  $c_1 f \leq g \leq c_2 f$ .

### 1.1 Graph

For a planar graph  $G$ , we denote by  $\mathcal{E}(G)$  the set of its edges. Most of the time  $G$  will be identified with the set of its vertices, which we will also call *sites*. For any two vertices  $x, y \in G$ , we write  $x \sim y$  if they are adjacent and we denote by  $xy \in \mathcal{E}(G)$  the edge between them.

In this paper, we will consider finite connected and simply connected graphs that are made of the union of faces of the square grid  $\mathbb{Z}^2$  (vertices are points of  $\mathbb{Z}^2$  and vertices at distance 1 are linked by an edge). We will call these *discrete domains*.

For a discrete domain  $\Omega$ , we denote by  $\partial\Omega \subset \Omega$  its boundary (when we view  $\Omega$  a domain consisting of the union of its faces); most of the time, we will identify  $\partial\Omega$  with the set of its vertices, called *the boundary vertices*. We denote by  $\text{Int}(\Omega)$  the interior of the graph, defined as  $\Omega \setminus \partial\Omega$ . We denote by  $\partial_{\text{ext}}\mathcal{E}(\Omega)$  the set of *external edges* of  $\Omega$ , defined as the set of edges of  $\mathcal{E}(\mathbb{Z}^2) \setminus \mathcal{E}(\Omega)$  incident to a vertex of  $\Omega$ , *counted with multiplicity*: if an edge of  $\mathcal{E}(\mathbb{Z}^2) \setminus \mathcal{E}(\Omega)$  is incident to two vertices of  $\Omega$ , it appears as two distinct elements

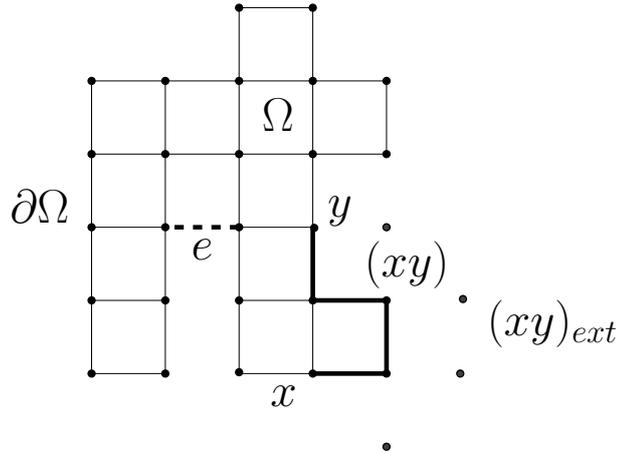


Figure 10.1: A domain  $\Omega$  with two points  $x$  and  $y$  on its boundary. The set  $(xy)$  and  $(xy)_{\text{ext}}$  are depicted. The edge  $e$  appears twice in  $\mathcal{E}_{\text{ext}}$ .

of  $\partial_{\text{ext}}\mathcal{E}(\Omega)$ . We identify the edges of  $\partial_{\text{ext}}\mathcal{E}(\Omega)$  with the set  $\partial_{\text{ext}}\Omega$  of *external boundary vertices*, they are the formal endpoints in  $\mathbb{Z}^2 \setminus \Omega$  of the edges of  $\partial\mathcal{E}(\Omega)$ .

For two points  $x, y \in \partial\Omega$ , we denote by  $(xy) \subset \partial\Omega$  the counterclockwise arc of  $\partial\Omega$  from  $x$  to  $y$  (including  $x$  and  $y$ ); as usual we identify  $(xy)$  with the set of the vertices located on it; we will frequently identify  $x \in \partial\Omega$  with the arc  $(xx)$ ; we denote by  $(xy)_{\text{ext}}$  the set of vertices of  $\partial_{\text{ext}}\Omega$  adjacent to  $(xy)$ . We call a discrete domain  $\Omega$  with four marked vertices  $a, b, c, d \in \partial\Omega$  in counterclockwise order a *topological rectangle*.

We denote by  $(\mathbb{Z}^2)^*$  the dual of  $\mathbb{Z}^2$ : the vertices of  $(\mathbb{Z}^2)^*$  are the (centers) of the faces of  $\mathbb{Z}^2$  and vertices at distance 1 are linked by an edge. Given a discrete domain  $\Omega$ , the *dual domain*  $\Omega^*$  is the induced subgraph of  $(\mathbb{Z}^2)^*$  whose vertices are the faces  $\Omega$ . We denote by  $\partial\Omega^*$  the set of vertices of  $\Omega^*$  corresponding to faces of  $\Omega$  sharing an edge with  $\partial\Omega$ . We denote by  $\partial_{\text{ext}}\Omega^*$  the set of *external dual vertices*, corresponding to the faces of  $\mathbb{Z}^2 \setminus \Omega$  adjacent to  $\partial\Omega$ , with multiplicity: it is in bijection with the edges of  $\partial\Omega$ .

## 1.2 Laplacians, harmonic measures and random walks

Let  $\Omega$  be a discrete domain, with boundary vertices  $\partial\Omega$  and external boundary vertices  $\partial_{\text{ext}}\Omega$ . Consider a collection of nonnegative conductances  $\mathcal{C} = (\mathbf{c}_e)_e$  defined on the set of the edges  $\mathcal{E}$  and the set of external boundary edges  $\partial_{\text{ext}}\mathcal{E}(\Omega)$ ; we call the conductances on  $\mathcal{E}(\Omega)$  the *bulk conductances* and the conductances on  $\partial_{\text{ext}}\mathcal{E}(\Omega)$  the *boundary conductances*. In this paper, the bulk conductances are always assumed to be 1.

With this set of conductances is associated a Laplacian  $\Delta_{\mathcal{C}}$  defined (for a function

$f : \Omega \cup \partial_{\text{ext}}\Omega \rightarrow \mathbb{R}$ ) by:

$$\begin{aligned} \Delta_{\mathcal{C}} f(x) &:= \frac{1}{\lambda_x} \sum_{y \sim x} \mathbf{c}_{xy} (f(y) - f(x)) \quad \forall x \in \Omega \\ \lambda_x &:= \sum_{y \sim x} \mathbf{c}_{xy} \end{aligned}$$

In this paper, the collection of conductances that we will consider are equal to 1 on the edges of  $\Omega$ , and less or equal than 1 on the boundary edges.

For  $x, y \in \Omega$ , we denote by  $Z_{\Omega; \mathcal{C}}[x, y]$  the partition function of the random walks (RW)  $\omega$  in  $\Omega$  with conductances  $\mathcal{C}$  from  $x$  to  $y$ , absorbed by  $\partial_{\text{ext}}\Omega$ . The possible realizations are the sequences  $\omega_1, \dots, \omega_n$  of vertices such that  $\omega_i$  is adjacent to  $\omega_{i+1}$  for each  $i$ ,  $\omega_1 = x$ ,  $\omega_2, \dots, \omega_{n-1} \in \Omega \setminus \{y\}$  and  $\omega_n = y$ . The partition function is defined by

$$\begin{aligned} Z_{\Omega; \mathcal{C}}[x, y] &:= \sum_{\omega: x \rightarrow y} \prod_{k=1}^{\text{length}(\omega)-1} \frac{\mathbf{c}_{\omega_k \omega_{k+1}}}{\lambda_{\omega_k}} \\ &= \mathbb{P}\{\text{RW with generator } \Delta_{\mathcal{C}} \text{ starting from } x \text{ hits } y \text{ before } \partial_{\text{ext}}\Omega\} \end{aligned}$$

When the context is clear, we will omit the set of conductances  $\mathcal{C}$  in the subscripts.

Let  $(cd) \subset \partial\Omega$  be a boundary arc. We define for  $x \in \Omega$

$$\begin{aligned} Z_{\Omega}[x, (cd)] &:= \sum_{y \in (cd)} Z_{\Omega}[x, y] \\ &= \mathbb{P}\{\text{RW with generator } \Delta_{\mathcal{C}} \text{ starting from } x \text{ hits } (cd) \text{ before } \partial_{\text{ext}}\Omega\}. \end{aligned}$$

It is easy to check that  $x \mapsto Z_{\Omega}[x, (cd)]$  is a  $\Delta_{\mathcal{C}}$ -harmonic function on  $\Omega \setminus (cd)$  which has boundary conditions 1 on  $(cd)$  and boundary conditions 0 on  $\partial_{\text{ext}}\Omega$ .

If  $(ab), (cd) \subset \partial\Omega$  are boundary arcs, we define

$$Z_{\Omega}[(ab), (cd)] := \sum_{x \in (ab)} Z_{\Omega}[x, (cd)].$$

Given a discrete domain  $\Omega$ , we define in the same manner partition functions of random walks on  $\Omega^*$ , taking  $\partial\Omega^*$  and  $\partial_{\text{ext}}\Omega^*$  instead of  $\partial\Omega$  and  $\partial_{\text{ext}}\Omega$  (again, we assume that the bulk conductances are all 1).

### 1.3 Discrete extremal length

A very useful tool when dealing with discrete harmonic measures in topological rectangle is a discrete version of the extremal length. It measures the distance, from the discrete harmonic measures point of view, between two arcs of a domain, in a particularly robust manner. In this paper, we will mostly use it to compare partition functions of random walks on  $\Omega$  and on the dual graph  $\Omega^*$ .

Consider a topological rectangle  $(\Omega, a, b, c, d)$  and a collection of conductances  $\mathcal{C}$  (bulk conductances are always 1). Denote by  $\mathcal{C}_{(\Omega, a, b, c, d)}^{\text{DN}}$  the set of conductances  $\mathcal{C}$ , except that

the conductances to the edges incident to a vertex of  $(bc) \cup (da)$  are set to 0: in other words, the Laplacian  $\Delta_{\mathcal{C}_{(\Omega,a,b,c,d)}^{\text{DN}}}$  is the generator of the random walk generated by  $\Delta_{\mathcal{C}}$  reflected by the arcs  $(bc)$  and  $(da)$  (more precisely: reflected by the edges of  $\partial\mathcal{E}(\Omega)$  incident to  $(bc) \cup (da)$ ).

Following [Che11], we define the extremal length  $\ell_{\Omega;\mathcal{C}}[(ab), (cd)]$  by

$$\ell_{\Omega;\mathcal{C}}[(ab), (cd)] := \left( \mathcal{Z}_{\Omega;\mathcal{C}_{(\Omega,a,b,c,d)}^{\text{DN}}}[(ab), (cd)] \right)^{-1}.$$

When no set of conductances is specified, like in Theorem 10.1, all conductances are set to 1.

The discrete extremal length is particularly powerful because of its robustness: the discrete extremal lengths on a discrete domain with different boundary conductances are uniformly comparable. Also, the discrete extremal length on a rectangle and its dual are comparable (note that such a general result would not be true for partition functions of random walks with purely Dirichlet boundary conditions):

**Theorem 10.2** ([Che11]). *Let  $\mu > 1$ . Let  $(\Omega, a, b, c, d)$  be a topological rectangle and consider a set of conductances  $\mathcal{C}$  on  $\Omega$  with boundary conductances in  $\left[\frac{1}{\mu}, \mu\right]$ . Let  $\Omega^*$  be the dual to  $\Omega$  and let  $\mathcal{C}^*$  be a set of conductances on  $\Omega^*$  with boundary conductances in  $\left[\frac{1}{\mu}, \mu\right]$ . Then we have*

$$\ell_{\Omega;\mathcal{C}}[(ab), (cd)] \asymp \ell_{\Omega^*;\mathcal{C}^*}[(ab)^*, (bc)^*],$$

where the constants in  $\asymp$  depend on  $\mu$  only.

When the extremal length is of order 1 (like in the statement of Theorem 10.1), then so are the partition functions of random walks with Dirichlet boundary conditions:

**Theorem 10.3** ([Che11]). *Let  $M > 1$  and  $\mu > 1$ . For any topological rectangle  $(\Omega, a, b, c, d)$  and any set of conductances  $\mathcal{C}$  with boundary conductances in  $\left[\frac{1}{\mu}, \mu\right]$ , if*

$$\frac{1}{M} \leq \ell_{\Omega;\mathcal{C}}[(ab), (cd)] \leq M$$

then

$$\ell_{\Omega;\mathcal{C}}[(bc), (da)] \asymp 1, \quad \mathcal{Z}_{\Omega;\mathcal{C}}[(ab), (cd)] \asymp 1, \quad \mathcal{Z}_{\Omega;\mathcal{C}}[(bc), (da)] \asymp 1,$$

where the constants in  $\asymp$  depend on  $M$  and  $\mu$  only.

**Remark 10.4.** *It is actually proven in [Che11] that we have*

$$\ell_{\Omega;\mathcal{C}}[(ab)(bc)] \ell_{\Omega;\mathcal{C}}[(bc), (da)] \asymp 1,$$

uniformly over all topological rectangles.

### 1.4 Factorization results

In this section, we review the main results of [Che11] concerning factorization properties of discrete harmonic measure. While in the continuum such results are rather easy to derive (for instance using explicit expressions and conformal invariance), it requires a much more delicate analysis to obtain them (up to uniform constants) on the discrete level.

In this subsection, we assume that all the bulk conductances are 1 and that all the boundary conductances are in  $\left[\frac{1}{\mu}, \mu\right]$  for some  $\mu \geq 1$ .

**Theorem 10.5** ([Che11]). *For any discrete domain  $\Omega$  and any boundary points  $a, b, c \in \partial\Omega$ , we have*

$$Z_{\Omega}[a, (bc)] \asymp \left( \frac{Z_{\Omega}[a, b] Z_{\Omega}[a, c]}{Z_{\Omega}[b, c]} \right)^{\frac{1}{2}},$$

where the constants in  $\asymp$  depend on  $\mu$  only.

The following estimate will also be needed. It involves a discrete version of the cross-ratio (the left-hand side of 10.1):

**Theorem 10.6** ([Che11]). *Let  $M > 0$ . Then for any  $(\Omega, a, b, c, d)$  with  $\ell_{\Omega}[(ab), (cd)] \leq M$ , we have*

$$\sqrt{\frac{Z_{\Omega}[a, d] Z_{\Omega}[b, c]}{Z_{\Omega}[a, b] Z_{\Omega}[c, d]}} \asymp Z_{\Omega}[(ab), (cd)], \tag{10.1}$$

where the constants in  $\asymp$  depend on  $M$  and  $\mu$  only.

### 1.5 Separators

A crucial concept in the following study is the notion of separators: they will indeed allow us to perform some efficient surgery of the discrete domains.

Informally speaking, separators are discrete curves that separate domain in two pieces, in a “good” manner from harmonic measure point of view: the product of partition functions of random walks in the two pieces is of the same order as the partition function of random walks in the original domain.

In this subsection, we assume again that all the bulk conductances on the discrete domains are 1 and that all the boundary conductances are in  $\left[\frac{1}{\mu}, \mu\right]$  for some  $\mu \geq 1$ . If  $(\Omega, a, b, c, d)$  is a topological rectangle, a *separating curve* between  $(ab)$  and  $(cd)$  is a simple discrete curve  $\Gamma$  in  $\Omega$  separating  $(ab)$  from  $(cd)$ ; we denote by  $\Omega_{\Gamma, (ab)}$  and  $\Omega_{\Gamma, (cd)}$  the connected components of  $\Omega \setminus \Gamma$  containing  $(ab)$  and  $(cd)$  respectively.

**Theorem 10.7** ([Che11]). *Let  $M > 1$ . Take a topological rectangle  $(\Omega, a, b, c, d)$  such that  $Z_{\Omega}[(ab), (cd)] \leq M$ . For any  $k \in \left[\frac{Z}{M}, \frac{M}{Z}\right]$ , there exists a separating curve  $\Gamma \subset \Omega$  between  $(ab)$  and  $(cd)$  such that we have*

$$\begin{aligned} Z_{\Omega_{\Gamma, (ab)}}[(ab), \Gamma] \cdot Z_{\Omega_{\Gamma, (cd)}}[\Gamma, (cd)] &\asymp Z_{\Omega}[(ab), (cd)], \\ Z_{\Omega_{\Gamma, (cd)}}[\Gamma, (cd)] &\asymp k \cdot Z_{\Omega_{\Gamma, (ab)}}[(ab), \Gamma], \end{aligned} \tag{10.2}$$

where the constants in  $\asymp$  depend on  $M$  and  $\mu$  only.

We will call *separator* a separating curve satisfying 10.2. Let us give a corollary that will be particularly useful for us:

**Corollary 10.8.** *Let  $M > 1$ . Then there exists  $\epsilon \in (0, 1)$  (depending on  $M$  only) such that for any topological rectangle  $(\Omega, a, b, c, d)$  with  $Z := Z_\Omega[(ab), (cd)] \leq M$  and any  $\kappa \in [\frac{Z}{\epsilon}, \epsilon]$  there exists a separating curve  $\Gamma \subset \Omega$  between  $(ab)$  and  $(cd)$  with*

$$\begin{aligned} Z_{\Omega_{\Gamma, (ab)}}[(ab), \Gamma] \cdot Z_{\Omega_{\Gamma, (cd)}}[\Gamma, (cd)] &\asymp Z_\Omega[(ab), (cd)], \\ Z_\Omega[\Gamma, (cd)] &\in [\epsilon\kappa, \kappa], \end{aligned}$$

where the constant in  $\asymp$  depends on  $M$  and  $\mu$  only.

**Proof** By Theorem 10.7, there exists  $C_1, C_2, C_3, C_4 > 0$  such that for any  $k \in [\frac{Z}{M}, \frac{M}{Z}]$  we have

$$C_1 Z \leq Z_\Omega[(ab), \Gamma] \cdot Z_\Omega[\Gamma, (cd)] \leq C_2 Z \quad \text{and} \quad C_3 k \leq \frac{Z_\Omega[\Gamma, (cd)]}{Z_\Omega[(ab), \Gamma]} \leq C_4 k.$$

Hence, we obtain

$$\sqrt{C_1 C_3 k Z} \leq Z_\Omega[\Gamma, (cd)] \leq \sqrt{C_2 C_4 k Z}.$$

Take  $\epsilon := \min\{\sqrt{C_1 C_3 / (C_2 C_4)}, \sqrt{C_2 C_4 / M}\}$ . If  $\kappa \in [\frac{Z}{\epsilon}, \epsilon]$ , we can choose  $k := \frac{\kappa^2}{C_2 C_4 Z} \in [\frac{Z}{M}, \frac{M}{Z}]$  in Theorem 10.7 to get the result.  $\square$

We will also need the following corollary, which says that we can split a topological rectangle in “fair” shares:

**Corollary 10.9.** *Let  $M > 1$ . For any topological rectangle  $(\Omega, a, b, c, d)$  with  $M^{-1} \leq \ell_\Omega[(ab), (cd)] \leq M$ , there exists separating curve  $\Gamma \subset \Omega$  between  $(ab)$  and  $(cd)$  such that we have*

$$\ell_{\Omega_{(ab)}}[(ab), \Gamma] \asymp \ell_{\Omega_{(cd)}}[(cd), \Gamma] \asymp \ell_\Omega[(ab), (cd)],$$

where the constants in  $\asymp$  depend on  $M$  only.

**Proof** By Theorem 10.3, we have that  $Z_\Omega[(ab), (cd)] \asymp 1$  (where the constant depends on  $M$  only). Applying Theorem 10.7 with  $k = 1$ , we obtain a simple curve  $\Gamma$  separating  $(ab)$  from  $(cd)$  with

$$Z_{\Omega_{(ab)}}[(ab), (xy)] \asymp Z_{\Omega_{(cd)}}[(xy), (cd)] \asymp Z_\Omega[(ab), (cd)],$$

where the constants in  $\asymp$  depend on  $M$  only. Applying once more Theorem 10.3, we get the result.  $\square$

## 1.6 From FK-Ising model to discrete harmonic measure

In this section, we relate critical FK-Ising crossing probabilities with free/wired/free/wired boundary conditions to discrete harmonic measures. The main tool consists of the fermionic observable. It has been used in [CS09] in order to obtain the scaling limit of FK-Ising crossing probabilities under free/wired/free/wired boundary conditions.

The probability that two arcs wired arcs are connected (with free boundary conditions elsewhere) can be bounded by above in terms of discrete harmonic measure.

Let  $\mathcal{C}_\circ$  denote the set of unit conductances on the edges of  $\Omega_\delta^*$  and let  $Z_\circ$  be the corresponding random walk partition function. Let  $\mathcal{C}_\bullet$  be the set of conductances on  $\Omega_\delta$ , where each bulk edge has conductance 1, the boundary edges incident to  $(bc) \cup (da)$  have conductance 1 and the boundary edges incident to  $(ab) \cup (cd)$  have conductance  $\frac{2}{\sqrt{2+1}}$ .

**Proposition 10.10.** *For any  $M > 0$ , for any  $(\Omega, a, b, c, d)$  topological rectangle with  $Z_\Omega[(ab), (cd)] \leq M$ , we have*

$$\phi_\Omega^{(ab), (cd)} [(ab) \leftrightarrow (cd)] \lesssim \sqrt{Z_\Omega[(ab), (cd)]},$$

where the constant in  $\lesssim$  depends on  $M$  only.

The proof is given below. It follows the ideas of the proof of [CS09, Theorem 6.1], where the above crossing probability is computed in the scaling limit.

Let us recall that when we degenerate the arc  $(ab)$  to a singleton, we find the upper bound of Proposition 9.5

**Corollary 10.11.** *With the notation of Proposition 10.10, we have*

$$\phi_\Omega^{(cd)} [a \leftrightarrow (cd)] \lesssim \sqrt{Z_\Omega[a, (cd)]},$$

where the constant in  $\lesssim$  is universal.

If we also degenerate the arc  $(cd)$  to a singleton, we have the following double-sided harmonic measure estimate for the probability that two boundary vertices are connected with free boundary conditions.

Consider a discrete domain  $\Omega$  and its dual  $\Omega^*$ . Let  $\mathcal{C}^*$  be the set of unit conductances on  $\Omega^*$ . Let  $\mathcal{C}$  be the set of conductances on  $\Omega$ , where each bulk edge has conductance 1 and each boundary edge has conductance  $\frac{2}{\sqrt{2+1}}$ .

**Proposition 10.12.** *Let  $\Omega$  be a discrete domain. For any two sites  $a, b \in \partial\Omega$ , we have*

$$\sqrt{Z_{\Omega^*; \mathcal{C}^*} [a^*, b^*]} \lesssim \phi_\Omega^0 (a \leftrightarrow b) \lesssim \sqrt{Z_{\Omega; \mathcal{C}} [a, b]},$$

for any  $a^* \in \partial\Omega^*$  at distance  $\frac{\sqrt{2}}{2}$  from  $a$  and  $b^*$  at distance  $\frac{\sqrt{2}}{2}$  from  $b$ . The constants in  $\lesssim$  are universal.

This proposition is a restatement in new notations of Proposition 9.5 when the wired arc is degenerated to a singleton.

**Proof of Proposition 10.10** Fix a domain  $(\Omega, a, b, c, d)$  and consider the critical FK-Ising model with boundary conditions  $(ab), (cd)$  (i.e wired/free/wired/free) on it. In [CS09, Proof of Theorem 6.1], two discrete holomorphic observables  $F_1$  and  $F_2$  for this model are introduced, and it is shown that there exists a unique linear combination of  $F$  of  $F_1, F_2$  and a unique  $\kappa \in \mathbb{R}$  such that a discrete version  $H$  of  $\text{Imm}(\int F^2)$  satisfies the following boundary conditions:

$$H = 0 \text{ on } (da), H = 1 \text{ on } (cd) \text{ and } H = \kappa \text{ on } (ab)_{\text{ext}} \cup (bc)_{\text{ext}}.$$

This discrete function  $H$  is  $\Delta_c$ -subharmonic on  $\Omega \setminus ((ab) \cup (cd))$ . The constant  $\kappa$  is shown to be in one-to-one correspondence with  $\phi_{\Omega}^{(ab),(cd)}[(ab) \leftrightarrow (cd)]$ ; from [CS09, Formula 6.6], we get in particular that

$$\sqrt{\kappa} \asymp \phi_{\Omega}^{(ab),(cd)}[(ab) \leftrightarrow (cd)], \quad (10.3)$$

where the constants are universal.

Let  $c_{\text{in}}$  be a vertex of  $\text{Int}(\Omega)$  adjacent to  $c$ . By the construction of  $H$  (see [CS09, Proof of Theorem 6.6]), we have that  $H(c_{\text{in}}) \geq H(c)$ . If we now consider the function  $H - \kappa$ , we obtain the following estimate in terms of the discrete harmonic measure

$$0 \leq H(c_{\text{in}}) - \kappa \leq (1 - \kappa) Z_{\Omega}[a, (cd)] - \kappa Z_{\Omega}[a, (bc)],$$

which leads to

$$\kappa \leq \frac{Z_{\Omega}[a, (cd)]}{Z_{\Omega}[a, (bc)]}.$$

Using the factorization result for the harmonic measure (Proposition 10.5), we get

$$\kappa \lesssim \frac{Z_{\Omega}[a, (cd)]}{Z_{\Omega}[a, (bc)]} \asymp \sqrt{\frac{Z_{\Omega}[a, c] Z_{\Omega}[a, d] Z_{\Omega}[b, c]}{Z_{\Omega}[a, b] Z_{\Omega}[a, c] Z_{\Omega}[c, d]}} = \sqrt{\frac{Z_{\Omega}[a, d] Z_{\Omega}[b, c]}{Z_{\Omega}[a, b] Z_{\Omega}[c, d]}}.$$

Using the assumption  $Z_{\Omega}[(ab), (cd)] \leq M$ , we get by Theorem 10.6 that

$$\kappa \lesssim \sqrt{\frac{Z_{\Omega}[a, d] Z_{\Omega}[b, c]}{Z_{\Omega}[a, b] Z_{\Omega}[c, d]}} \asymp Z_{\Omega}[(ab), (cd)].$$

Hence, (10.3) implies

$$\phi_{\Omega}^{(ab),(cd)}[(ab) \leftrightarrow (cd)] \asymp \sqrt{\kappa} \lesssim \sqrt{Z_{\Omega}[(ab), (cd)]}.$$

□

## 2 Proof of Theorem 10.1

In this section, we will be considering partition functions of random walks on a topological rectangles, and will omit the dependence on the domain in the notation when the context is clear. Given two boundary arcs  $\Gamma_1, \Gamma_2 \subset \partial\Omega$ , we will denote  $Z_\bullet[\Gamma_1, \Gamma_2]$  the partition function function of random walks on  $\Omega$  as previously defined, with unit conductances everywhere, except on the external edges incident to  $\Gamma_1 \cup \Gamma_2$ , where the conductances are set to  $\frac{2}{\sqrt{2+1}}$ .

**Lemma 10.13.** *Let  $M > 1$ . For any  $(\Omega, a, b, c, d)$  with  $Z_\bullet[(ab), (cd)] \leq M$ , we have*

$$\phi^{(cd)}(a \leftrightarrow (cd), b \leftrightarrow (cd)) \lesssim \sqrt{\frac{Z_\bullet[a, (cd)]Z_\bullet[b, (cd)]}{Z_\bullet[(ab), (cd)]}},$$

where the constant in  $\lesssim$  depends only on  $M$ .

**Proof** Constants in  $\asymp$  and  $\lesssim$  are depending only on  $M$ . Note that  $Z_\bullet[a, (cd)] \leq Z_\bullet[(ab), (cd)] \leq M$ . Fix  $\epsilon = \epsilon(M) \in (0, 1)$  as given by Corollary 10.8. Then we have two cases:

**Case 1:**  $Z_\bullet[a, (cd)] > \frac{\epsilon}{3}Z_\bullet[(ab), (cd)]$  or  $Z_\bullet[b, (cd)] > \frac{\epsilon}{3}Z_\bullet[(ab), (cd)]$ .

Suppose we are in the first case (the other case is symmetric). Then Corollary 10.11 implies

$$\begin{aligned} \phi_\Omega^{(cd)}(a, b \leftrightarrow (cd)) &\leq \phi_\Omega^{(cd)}(b \leftrightarrow (cd)) \leq \sqrt{Z_\bullet[b, (cd)]} \\ &\lesssim \sqrt{\frac{Z_\bullet[a, (cd)]Z_\bullet[b, (cd)]}{Z_\bullet[(ab), (cd)]}}. \end{aligned}$$

**Case 2:**  $Z_\bullet[a, (cd)] \leq \frac{\epsilon}{3}Z_\bullet[(ab), (cd)]$  and  $Z_\bullet[b, (cd)] \leq \frac{\epsilon}{3}Z_\bullet[(ab), (cd)]$ .

By Corollary 10.8 (setting  $\kappa := \frac{1}{3}Z_\bullet[(ab), (cd)]$ ), there exists a separator  $\Gamma_a$  between  $a$  and  $(cd)$  such that

$$Z_\bullet[(ab), (cd)] \leq Z_\bullet[\Gamma_a, (cd)] \leq \frac{1}{3}Z_\bullet[(ab), (cd)]. \quad (10.4)$$

Denote by  $\Omega_a$  the connected component of  $\Omega \setminus \Gamma_a$  containing  $a$ .

Similarly, there exists a separator  $\Gamma_b$  of  $b$  and  $(cd)$  such that

$$Z_\bullet[(ab), (cd)] \lesssim Z_\bullet[\Gamma_b, (cd)] \leq \frac{1}{3}Z_\bullet[(ab), (cd)]. \quad (10.5)$$

Denote by  $\Omega_b$  the connected component of  $\Omega \setminus \Gamma_b$  containing  $b$ .

Note that the two separators do not intersect:  $\Omega_a \cap \Omega_b = \emptyset$ . Otherwise, their union would separate the whole arc  $(ab)$  from  $(cd)$ , which is contradictory since

$$Z_\bullet[\Gamma_a \cup \Gamma_b, (cd)] \leq Z_\bullet[\Gamma_a, (cd)] + Z_\bullet[\Gamma_b, (cd)] \leq 2/3 \cdot Z_\bullet[(ab), (cd)].$$

We are thus facing the following topological picture: the two arcs  $\Gamma_a$  and  $\Gamma_b$  are not intersecting and are separating  $a, b$  and  $(cd)$ . Wiring the arc  $\Gamma_a$  and  $\Gamma_b$ , we find:

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \leq \phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a] \phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b] \phi_{\Omega \setminus (\Omega_a \cup \Omega_b)}^{(cd), \Gamma_a \cup \Gamma_b}[\Gamma_a \cup \Gamma_b \leftrightarrow (cd)].$$

Let us deal with the first term on the right-side. Using Corollary 10.11 and the fact that  $\Gamma_a$  is a separator between  $a$  and  $(cd)$ , we obtain

$$\phi_{\Omega_a}^{\Gamma_a}[a \leftrightarrow \Gamma_a] \leq \sqrt{Z_\bullet[a, \Gamma_a]} \asymp \sqrt{\frac{Z_\bullet[a, (cd)]}{Z_\bullet[\Gamma_a, (cd)]}} \lesssim \sqrt{\frac{Z_\bullet[a, (cd)]}{Z_\bullet[(ab), (cd)]}},$$

where in the last inequality we used (10.4). Similarly:

$$\phi_{\Omega_b}^{\Gamma_b}[b \leftrightarrow \Gamma_b] \lesssim \sqrt{\frac{Z_\bullet[b, (cd)]}{Z_\bullet[(ab), (cd)]}}$$

For the last term, we get

$$\begin{aligned} \phi_{\Omega \setminus (\Omega_a \cup \Omega_b)}^{(cd), \Gamma_a \cup \Gamma_b}[\Gamma_a \cup \Gamma_b \leftrightarrow (cd)] &\leq \phi_{\Omega \setminus (\Omega_a \cup \Omega_b)}^{(cd), \Gamma_a \cup \Gamma_b \cup (ab)}[\Gamma_a \cup \Gamma_b \cup (ab) \leftrightarrow (cd)] \\ &\leq \sqrt{Z_\bullet[\Gamma_a \cup \Gamma_b \cup (ab), (cd)]} \\ &\leq \sqrt{2Z_\bullet[(ab), (cd)]} \end{aligned}$$

where in the second inequality we used Proposition 10.10 and in the third, (10.4) and (10.5). Putting everything together we find

$$\phi_\Omega^{(cd)}[a, b \leftrightarrow (cd)] \lesssim \sqrt{\frac{Z_\bullet[a, (cd)] Z_\bullet[b, (cd)]}{Z_\bullet[(ab), (cd)]}}.$$

□

Thanks to the two-point function estimate given by Lemma 10.13, we can now prove Theorem 10.1, which relies mostly on a second-moment estimate.

**Proof of Theorem 10.1** Let  $M > 1$ . Once again, constants in  $\asymp$ ,  $\lesssim$  and  $\gtrsim$  depend only on  $M > 0$ . Fix a domain  $(\Omega, a, b, c, d)$  with  $Z = Z_\bullet[(ab), (cd)] \in [M^{-1}, M]$  ( $Z_\bullet$  is as defined before Lemma 10.13).

Using the monotonicity with respect to the boundary conditions, in order to get a lower bound for the crossing probabilities that is uniform with respect to the boundary

conditions, it is enough to get such a bound for free boundary conditions. Similarly, it is sufficient to get it in the case of fully wired boundary conditions in order to get an upper bound on crossing probabilities.

Using the self-duality of the model, we see that obtaining an upper bound for the probability of a crossing  $(ab) \leftrightarrow (cd)$  on  $\Omega$  (with wired boundary conditions) is equivalent to obtaining a lower bound for the probability of a crossing  $(bc)^* \leftrightarrow (da)^*$  for the critical FK-Ising model on  $\Omega^*$  (with free boundary condition). It is hence enough to bound from below the probability  $\phi_{\Omega^*}^0[(bc)^* \leftrightarrow (da)^*]$  of a dual crossing from  $(bc)^*$  to  $(da)^*$  (by a constant depending on  $M$  only). The extremal length  $\ell_{\Omega;1}[(ab)^*, (cd)^*]$  is of the same order as  $\ell_{\Omega;1}[(ab), (cd)]$  by Theorems 10.2 and 10.3, so it is enough to prove the lower bound of Theorem 10.1.

So, we only need to prove a lower bound for crossing probabilities with free boundary conditions. As mentioned earlier, the proof consists of a second-moment estimate on the random variable

$$N := \sum_{u \in (ab), v \in (cd)} \phi_{\Omega}^0[u \leftrightarrow v] \mathbb{I}_{u \leftrightarrow v}. \quad (10.6)$$

**Step 1:** First moment of  $N$ .

Let us start with estimating the first moment:

$$\begin{aligned} \phi_{\Omega}^0[N] &= \sum_{u \in (ab), v \in (cd)} \phi_{\Omega}^0[u \leftrightarrow v]^2 \gtrsim \sum_{w \in (ab)^*, t \in (cd)^*} Z_{\Omega^*;1}(w \leftrightarrow t) \\ &= Z_{\Omega^*;1}[(ab)^*, (cd)^*] \asymp Z_{\bullet}[(ab), (cd)], \end{aligned}$$

Note that in order to obtain the first inequality, we used Proposition 10.12. For the last one, we used the comparability of harmonic measures for neighboring dual vertices.

**Step 2:** Second moment of  $N$ .

Corollary 10.9 applied in  $(\Omega, a, b, c, d)$  gives a separator  $\Gamma \subset \Omega$  between  $(ab)$  and  $(cd)$  splitting  $\Omega$  in two parts of comparable sizes (in terms of harmonic measure):

$$Z_{\bullet}[(ab), \Gamma] \asymp Z_{\bullet}[\Gamma, (cd)] \asymp Z_{\bullet}[(ab), (cd)] \asymp 1. \quad (10.7)$$

We find:

$$\begin{aligned} \phi_{\Omega}^0[N^2] &= \sum_{u, v \in (ab), u', v' \in (cd)} \phi_{\Omega}^0[u \leftrightarrow v] \phi_{\Omega}^0[u' \leftrightarrow v'] \phi_{\Omega}^0[u \leftrightarrow v, u' \leftrightarrow v'] \\ &\leq \sum_{u, v \in (ab), u', v' \in (cd)} \phi_{\Omega}^0[u \leftrightarrow \Gamma] \phi_{\Omega}^0[u' \leftrightarrow \Gamma] \phi_{\Omega}^0[v \leftrightarrow \Gamma] \phi_{\Omega}^0[v' \leftrightarrow \Gamma] \phi_{\Omega}^0[u, u' \leftrightarrow \Gamma, v, v' \leftrightarrow \Gamma]. \end{aligned}$$

Let  $\Omega_1$  and  $\Omega_2$  be the connected components of  $\Omega \setminus \Gamma$  containing  $(ab)$ , and  $(cd)$  respectively. Wiring the arc  $\Gamma$ , the right-hand side factorizes into the product of two terms

$$\begin{aligned} S_{\Omega_1} &= \sum_{u, v \in (ab)} \phi_{\Omega_1}^{\Gamma}[u \leftrightarrow \Gamma] \phi_{\Omega_1}^{\Gamma}[v \leftrightarrow \Gamma] \phi_{\Omega_1}^{\Gamma}[u, v \leftrightarrow \Gamma], \\ S_{\Omega_2} &= \sum_{u', v' \in (cd)} \phi_{\Omega_2}^{\Gamma}[u' \leftrightarrow \Gamma] \phi_{\Omega_2}^{\Gamma}[v' \leftrightarrow \Gamma] \phi_{\Omega_2}^{\Gamma}[u', v' \leftrightarrow \Gamma]. \end{aligned}$$

Assume for a moment that we possess the bounds

$$S_{\Omega_1} \lesssim Z_{\bullet}[(ab), \Gamma]^{3/2} \quad \text{and} \quad S_{\Omega_2} \lesssim Z_{\bullet}[\Gamma, (cd)]^{3/2}. \quad (10.8)$$

They imply, thanks to the definition of separators,

$$\phi_{\Omega}^0[N^2] \leq (Z_{\bullet}[(ab), \Gamma] \cdot Z_{\bullet}[\Gamma, (cd)])^{3/2} \lesssim Z_{\bullet}[(ab), (cd)]^{3/2}. \quad (10.9)$$

**Step 3:** Proof of the two estimates in (10.8).

We only show the first one, since the second one is the same. Using Lemma 10.13 and Corollary 10.11, we find

$$\begin{aligned} S_{\Omega_1} &= \sum_{u, v \in (ab)} \phi_{\Omega_1}^{\Gamma}[u \leftrightarrow \Gamma] \phi_{\Omega_1}^{\Gamma}[v \leftrightarrow \Gamma] \phi_{\Omega_1}^{\Gamma}[u, v \leftrightarrow \Gamma] \\ &\lesssim \sum_{u, v \in (ab)} \frac{Z_{\bullet}(u, \Gamma) Z_{\bullet}(v, \Gamma)}{\sqrt{Z_{\bullet}[(uv), \Gamma]}} \end{aligned}$$

Note that for any sequence of positive real numbers  $(u_n)_{n \geq 0}$ , and  $\alpha > 0$ , a comparison between series and integral implies

$$\sum_{k=1}^n u_k \left( \sum_{j=1}^k u_j \right)^{\alpha-1} \leq \frac{1}{\alpha} \left( \sum_{k=1}^n u_k \right)^{\alpha}. \quad (10.10)$$

Say that  $u < v$  if  $u$  and  $v$  are found in this order when going along the arc  $(ab)$  in the counterclockwise order. In our case, (10.10) implies that,

$$\begin{aligned} \sum_{u, v \in (ab)} \frac{Z_{\bullet}[u, \Gamma] Z_{\bullet}[v, \Gamma]}{\sqrt{Z_{\bullet}[(uv), \Gamma]}} &\leq 2 \sum_{u < v \in (ab)} \frac{Z_{\bullet}[u, \Gamma] Z_{\bullet}[v, \Gamma]}{\sqrt{Z_{\bullet}[(uv), \Gamma]}} \\ &= 2 \sum_{v \in (ab)} Z_{\bullet}[v, \Gamma] \sum_{u \in (av)} \frac{Z_{\bullet}[u, \Gamma]}{\sqrt{Z_{\bullet}[(uv), \Gamma]}} \\ &\leq \sum_{v \in (ab)} Z_{\bullet}[v, \Gamma] \sqrt{Z_{\bullet}[(av), \Gamma]} \\ &\leq \sum_{v \in (ab)} Z_{\bullet}[u, \Gamma] \sqrt{Z_{\bullet}[(ab), \Gamma]} \\ &\leq Z_{\bullet}[(ab), \Gamma]^{\frac{3}{2}}, \end{aligned}$$

thus giving (10.8).

**Step 4:** Lower bound for crossing probabilities.

By the Cauchy-Schwarz inequality,

$$\phi_{\Omega}^0((ab) \leftrightarrow (cd)) = \phi_{\Omega}^0(N > 0) = \phi_{\Omega}^0[(\mathbb{I}_{N>0})^2] \geq \frac{\phi_{\Omega}^0[N^2]}{\phi_{\Omega}^0[N^2]} \gtrsim \frac{Z_{\bullet}[(ab), (cd)]^2}{Z_{\bullet}[(ab), (cd)]^{3/2}},$$

where we used the two first steps. Now, by Theorem 10.3, we get that both the numerator and the denominator are of order 1. Our bound depends on  $M$  only.  $\square$

### 3 Arm exponents

In this section  $\phi$  denotes the unique infinite-volume measure at  $q = 2$ ,  $p = p_c(2)$ . Define  $\Lambda_n(x) := x + [-n, n]^2$  and  $\Lambda_n = \Lambda_n(0)$ . Also set  $S_{n,N}(x) = \Lambda_N(x) \setminus \Lambda_n(x)$  and  $S_{n,N} = S_{n,N}(0)$ .

Fix a sequence  $\sigma$  of "open"  $o$  or "closed"  $c$ . We say that a path is  $o$ -connected if it is connected and  $c$ -connected if it is dual connected. Fix  $\sigma = \sigma_1.. \sigma_j$ . For  $n < N$ , define  $A_\sigma(n, N)$  to be the event that there are  $j$  disjoint paths from  $\partial\Lambda_n$  to  $\partial\Lambda_N$  with are  $\sigma_i$ -connected, for  $i \leq j$  where the paths are indexed in counter-clockwise order. We set  $A_\sigma(N)$  to be  $A_\sigma(k, N)$  where  $k$  is the smallest possible integer such that the event is non-empty. For instance,  $A_o(n, N)$  is the one-arm event corresponding to the existence of a crossing from the inner to the outer boundary of  $\Lambda_N \setminus \Lambda_n$ .

A classical use of Theorem 10.1 implies that there exists  $\beta_\sigma$  and  $\beta'_\sigma$  such that

$$(n/N)^{\beta_\sigma} \leq \phi[A_\sigma(n, N)] \leq (n/N)^{\beta'_\sigma}.$$

It is therefore natural to predict that there exists a *critical exponent*  $\alpha_\sigma \in (0, \infty)$  such that

$$\phi[A_\sigma(n, N)] = (n/N)^{-\alpha_\sigma + o(1)},$$

where  $o(1)$  is a quantity converging to 0 as  $n/N$  goes to 0. The quantity  $\alpha_\sigma$  is called an *arm-exponent*.

Before starting, note that an important consequence of Proposition 9.21 is the following: the probability of arms does not really depend on the boundary conditions. In particular,

$$\phi(A_\sigma(n, N) \mid \mathcal{F}_{\mathbb{Z}^2 \setminus \Lambda_{2N}}) \simeq \phi(A_\sigma(n, N)) \quad \text{a.s.} \tag{10.11}$$

uniformly in  $n, N$ , where  $\mathcal{F}_\Omega$  is the  $\sigma$ -algebra generated by edges in  $\Omega$ .

#### 3.1 Quasi-multiplicativity

The following proposition is crucial in the understanding of arm-exponents:

**Theorem 10.14** (Quasi-multiplicativity). *Fix a sequence  $\sigma$ . For every  $n_1 < n_2 < n_3$ , we have*

$$\phi[A_\sigma(n_1, n_3)] \simeq \phi[A_\sigma(n_1, n_2)] \cdot \phi[A_\sigma(n_2, n_3)].$$

Define  $\Lambda_n(x) := x + [-n, n]^2$  and  $\Lambda_n = \Lambda_n(0)$ . Also set  $S_{n,N}(x) = \Lambda_N(x) \setminus \Lambda_n(x)$  and  $S_{n,N} = S_{n,N}(0)$ .

Let us define the notion of well-separated arms. In words, well-separated arms extend slightly outside the boxes and their ends are at macroscopic distance of each others, see Fig. 10.2. More precisely, for  $\delta > 0$ ,  $j$  arms  $\gamma_1, \dots, \gamma_j$  paths with end-points  $x_k = \gamma_k \cap \partial\Lambda_n$ ,  $y_k = \gamma_k \cap \partial\Lambda_N$  are said to be *well-separated* if

- points  $y_k$  are at distance larger than  $2\delta N$  from each others.

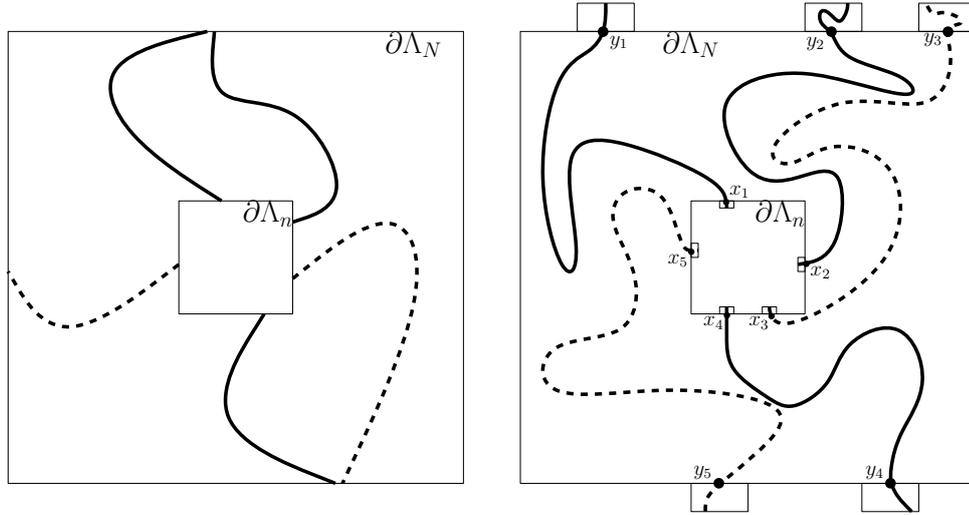


Figure 10.2: On the left, the five-arm event  $A_{ocococ}(n, N)$ . On the right, the event  $A_{ocococ}^{sep}(n, N)$  with well-separated arms. Note that these arms are not at macroscopic distance of each others inside the domain, but only at their end-points.

- points  $x_k$  are at distance larger than  $2\delta n$  from each others.
- For every  $k$ ,  $y_k$  is  $\sigma_k$ -connected to distance  $\delta N$  of  $S_{n, N}$  in  $\Lambda_{\delta N}(y_k)$ ,
- For every  $k$ ,  $x_k$  is  $\sigma_k$ -connected to distance  $\delta n$  of  $S_{n, N}$  in  $\Lambda_{\delta n}(x_k)$ .

Let  $A_\sigma^{sep; \delta}(n, N) = A_\sigma^{sep}(n, N)$  be the event that  $A_\sigma(n, N)$  holds true and there exist arms realizing  $A_\sigma(n, N)$  which are  $\delta$  well-separated. The previous definition has several convenient properties.

**Proposition 10.15.** *Fix  $\delta < 1$  small enough. For every  $n_1 \leq n_2 \leq \frac{n_3}{2}$ ,*

$$\phi[A_\sigma^{sep}(n_1, n_3)] \geq \phi(A_\sigma^{sep}(n_1, n_2)) \cdot \phi[A_\sigma^{sep}(2n_2, n_3)].$$

This proposition has the following easy consequence. Fix  $p \in (0, 1)$  and  $\delta < 1$  small enough. There exists  $\alpha = \alpha(\delta) > 0$  such that for every  $n_1 \leq n_2 \leq n_3$ ,

$$\phi[A_\sigma^{sep}(n_1, n_2)] \leq \left(\frac{n_3}{n_2}\right)^\alpha \cdot \phi[A_\sigma^{sep}(n_1, n_3)] \quad (10.12)$$

$$\phi[A_\sigma^{sep}(n_2, n_3)] \leq \left(\frac{n_2}{n_1}\right)^\alpha \cdot \phi[A_\sigma^{sep}(n_1, n_3)]. \quad (10.13)$$

To prove this inequalities, it suffices to see that  $\phi[A_\sigma^{sep}(2n, N)]$  is also bounded from below by a power of  $(n/N)$ . This is an easy consequence of Theorem 10.1.

**Proof** We have

$$\begin{aligned}\phi[A_\sigma^{sep}(n_1, n_2) \cap A_\sigma^{sep}(2n_2, n_3)] &= \phi[A_\sigma^{sep}(n_1, n_2) | A_\sigma^{sep}(2n_2, n_3)] \cdot \phi[A_\sigma^{sep}(2n_2, n_3)] \\ &\geq \phi[A_\sigma^{sep}(n_1, n_2)] \cdot \phi[A_\sigma^{sep}(2n_2, n_3)]\end{aligned}$$

thanks to (10.11) and it suffices to prove that  $\phi[A_\sigma^{sep}(n_1, n_2) \cap A_\sigma^{sep}(2n_2, n_3)]$  and  $\phi[A_\sigma^{sep}(n_1, n_3)]$  are comparable. To do so, condition on  $A_\sigma^{sep}(n_1, n_2) \cap A_\sigma^{sep}(2n_2, n_3)$  and construct  $j$  disjoint tubes of width  $\varepsilon = \varepsilon(\delta)$  connecting  $(y_k + \Lambda_{\delta n_2}) \setminus \Lambda_{n_2}$  to  $(y_k + \Lambda_{2\delta n_2}) \cap \Lambda_{2n_2}$  for every  $k \leq j$ . It is simple to show that this is topologically possible when  $\delta$  is small enough. Via Theorem 10.1, the  $\sigma_k$ -paths connecting  $x_k$  to  $\partial\Lambda_{2\delta n_2}(x_k) \cap \Lambda_{n_2}$ , and  $y_k$  to  $\partial\Lambda_{\delta n_2}(y_k) \setminus \Lambda_{n_2}$  can be connected by a  $\sigma_k$ -path with positive probability  $c = c(\delta, p_0)$ . Therefore,

$$\phi(A_\sigma^{sep}(n_1, n_3)) \geq c\phi[A_\sigma^{sep}(n_1, n_2) \cap A_\sigma^{sep}(2n_2, n_3)],$$

thus concluding the proof.  $\square$

Our main objective is now the following result:

**Proposition 10.16.** *Fix  $\sigma$ . For every  $n < N$ ,*

$$\phi[A_\sigma^{sep}(n, N)] \asymp \phi[A_\sigma(n, N)].$$

Indeed, if  $A_\sigma^{sep}(n, N)$  and  $A_\sigma(n, N)$  have uniformly comparable probabilities, Theorem 10.14 follows from the previous statement, as we can see in the following proof:

**Proof of Theorem 10.14** We have for  $n_1 \leq n_2 \leq n_3$ :

$$\begin{aligned}\phi[A_\sigma(n_1, n_3)] &\leq \phi[A_\sigma(n_1, n_2) | A_\sigma(2n_2, n_3)] \cdot \phi[A_\sigma(2n_2, n_3)] \\ &\asymp \phi[A_\sigma(n_1, n_2)] \cdot \phi[A_\sigma(2n_2, n_3)] \\ &\asymp \phi[A_\sigma^{sep}(n_1, n_2)] \cdot \phi[A_\sigma^{sep}(2n_2, n_3)] \\ &\lesssim \phi[A_\sigma^{sep}(n_1, n_2)] \cdot \phi[A_\sigma^{sep}(n_2, n_3)] \\ &\leq \phi[A_\sigma(n_1, n_2)] \cdot \phi[A_\sigma(n_2, n_3)],\end{aligned}$$

where in the second line we used (10.11), in the third, Proposition 10.16, and in the fourth, (10.13). Now,

$$\begin{aligned}\phi[A_\sigma(n_1, n_3)] &\asymp \phi[A_\sigma^{sep}(n_1, n_3)] \\ &\geq \phi[A_\sigma^{sep}(n_1, n_2)] \cdot \phi[A_\sigma^{sep}(2n_2, n_3)] \\ &\asymp \phi[A_\sigma(n_1, n_2)] \cdot \phi[A_\sigma(2n_2, n_3)] \\ &\geq \phi[A_\sigma(n_1, n_2)] \cdot \phi[A_\sigma(n_2, n_3)],\end{aligned}$$

where in the first and third lines, we used Proposition 10.16, in the second Proposition 10.15, and in the last,  $A_\sigma(n_2, n_3) \subset A_\sigma(2n_2, n_3)$ .  $\square$

Therefore, we only need to prove Proposition 10.16. Let us start with the following two lemmas:

**Lemma 10.17.** *For any  $\varepsilon > 0$ , there exists  $T > 0$  such that for every  $n > 0$*

$$\phi_{S_{n,2n}}^\xi(\exists T \text{ disjoint crossings of } S_{n,2n}) \leq \varepsilon$$

*uniformly in boundary conditions  $\xi$ .*

**Proof** It is sufficient to show that for  $\varepsilon > 0$ , there exists  $T > 0$  such that the probability of  $T$  disjoint vertical crossings of  $[0, 4n] \times [0, n]$  is bounded by  $\varepsilon$  uniformly in  $n$  and the boundary conditions. In fact, we only need to prove that conditionally on the existence of  $k$  crossings, the existence of another crossing is bounded from above by some constant  $c < 1$ .

In order to prove this statement, condition on the  $k$ -th left-most crossing  $\gamma_k$ . Assume without loss of generality that  $\gamma_k$  is a dual crossing. Construct a subdomain of  $[0, 4n] \times [0, n]$  by considering the connected component of  $[0, 4n] \times [0, n] \setminus \gamma_k$  containing  $\{4n\} \times [0, n]$ . The configuration in  $\Omega$  is a random-cluster configuration with boundary conditions  $\xi$  on the outside and free elsewhere (*i.e.* on the arc bordering the dual arc  $\gamma_k$ ). Now, Theorem 10.1 implies that  $\Omega$  is crossed from left to right by a primal *and* a dual crossing with probability bounded from below by a universal constant. Indeed, cut the domain  $\Omega$  into two domains  $\Omega_1 = \Omega \cap [0, 4n] \times [0, n/2]$  and  $\Omega_2 = \Omega \cap [0, 4n] \times [n/2, n]$  and assume  $\Omega_1$  is horizontally crossed and  $\Omega_2$  is horizontally dual crossed). This prevents the existence of an additional vertical crossing or dual crossing, therefore implying the claim.  $\square$

The previous proof harnesses Theorem 10.1 in a crucial way, the left boundary of  $\Omega$  being possibly very rough, previous results on crossing estimates would not have been strong enough.

**Lemma 10.18.** *For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every  $2n \leq N$ ,*

$$\phi_{S_{n/2,2N}}^\xi(\text{any set of crossings of } S_{n,N} \text{ can be made well separated}) \geq 1 - \varepsilon$$

*uniformly in boundary conditions  $\xi$ .*

**Proof** Fix  $n$  and the boundary conditions  $\xi$ .

Consider  $T$  large enough so that there exist more than  $T$  disjoint crossings of  $S_{n,2n}$  with probability less than  $\varepsilon$ .

Fix  $\delta > 0$  such that in *any subdomain* of the annulus  $S_{\delta r,r}$ ,  $\partial\Lambda_{\delta r}$  is not connected or dual connected to  $\partial\Lambda_r$  with probability  $1 - \varepsilon/T$ , uniformly in the domain and the boundary conditions on  $S_{\delta r,r}$ . This fact can be proved easily using Theorem 10.1.

We can assume with probability  $1 - 8\varepsilon$  that no crossing ends at distance less than  $\delta N$  of a corner of  $S_{n,N}$ . It is thus sufficient to work with vertical crossings in the rectangle  $[-N, N] \times [n, N]$ .

Now, condition on the left-crossing  $\gamma_1$  of  $[-N, N] \times [n, N]$  and set  $y$  to be the ending point of  $\gamma_1$  on the top. As before, construct the domain  $\Omega$  to be the connected component of  $\{N\} \times [n, N]$  in  $[-N, N] \times [n, N] \setminus \gamma_1$ . We can assume with probability  $1 - \varepsilon/T$  that no

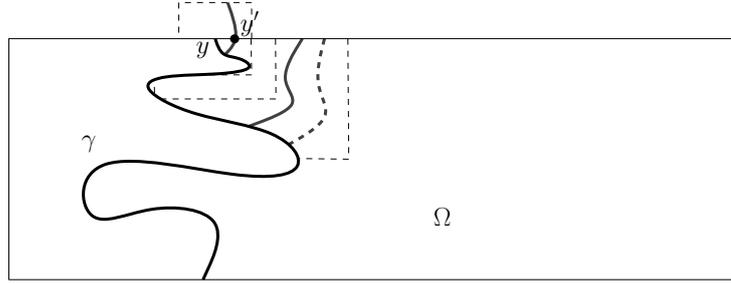


Figure 10.3: The construction of open and closed paths extending the crossing and preventing other crossings of finishing close to the path.

vertical crossing will land at distance  $\delta N$  of  $y$  by ensuring that  $\Omega \cap S_{\delta^2 N, \delta N}(y)$  contains open and dual-open circuits. Moreover, Theorem 10.1 allows us to construct a path  $P$  in  $\Lambda_{\delta^2 N}(y) \setminus ([-N, N] \times [n, N] \setminus \Omega)$  connecting  $\gamma_1$  to the top of  $\Lambda_{\delta^2 N}(y)$  with probability  $c > 0$ . This construction costed  $c\varepsilon/T$  and  $\gamma_1$  is guaranteed to be isolated from other crossings. Iterating the construction  $T$  times, we find the result.

The same reasoning applies to the interior side and we obtain the result.  $\square$

**Proof of Proposition 10.16** The lower bound  $\phi[A_\sigma^{sep}(n, N)] \leq \phi[A_\sigma(n, N)]$  is straightforward. Let us prove the upper bound for  $S_{2^n, 2^N}$ , first with only the separation on the exterior. Define  $A_\sigma^{sep/ext}(2^n, 2^k)$  to be the event  $A_\sigma(2^n, 2^k)$  with separation on the exterior only. Let  $B_k$  be the event that crossings in  $S_{2^{k-1}, 2^k}$  can be made separated. Lemma 10.18 ensures that  $\phi(B_k^c) \leq \varepsilon$ . Note that  $A_\sigma(2^n, 2^k) \cap B_k \subset A_\sigma^{sep/ext}(2^n, 2^k)$ . We thus have

$$\begin{aligned} \phi[A_\sigma(2^n, 2^N)] &\leq \sum_{k=n}^{N-1} \phi[A_\sigma(2^n, 2^k), B_k, B_{k+1}^c, \dots, B_{N-1}^c] \\ &\leq \sum_{k=n}^{N-1} \phi[A_\sigma(2^n, 2^k), B_k, B_{k+2}^c, B_{k+4}^c, \dots] \end{aligned}$$

Since annuli are separated by macroscopic areas, we can use (10.11) repeatedly to find

$$\begin{aligned} \phi(A_\sigma(2^n, 2^N)) &\leq \sum_{k=n}^{N-1} \phi[A_\sigma(2^n, 2^k), B_k] C \phi(B_{k+2}) C \phi(B_{k+4}) \dots \\ &\leq \sum_{k=n}^{N-1} \phi[A_\sigma^{sep/ext}(2^n, 2^k)] (C\varepsilon)^{(N-n)/2} \\ &\leq \left( \sum_{k=n}^{N-1} (2^{N-n})^\alpha (C\varepsilon)^{(N-n)/2} \right) \phi[A_\sigma^{sep/ext}(2^n, 2^N)] \end{aligned}$$

where we used (10.12) in the third line. Choosing  $\varepsilon$  small enough, we obtain  $\delta$  such that

$$\phi[A_\sigma(2^n, 2^N)] \leq \phi[A_\sigma^{sep/ext}(2^n, 2^N)]$$

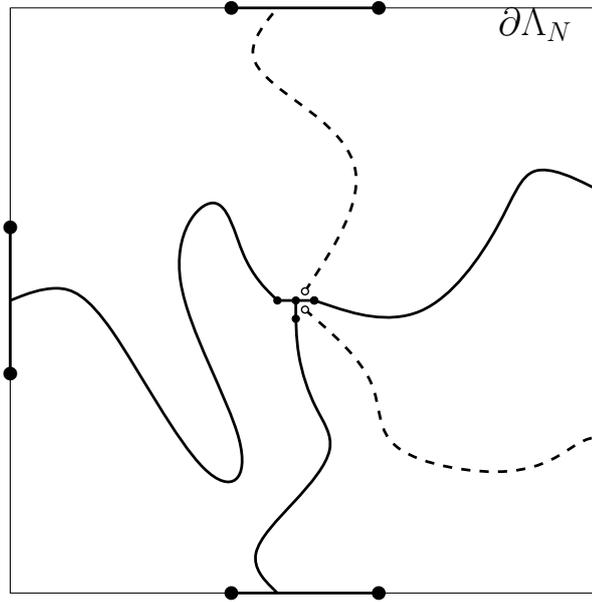


Figure 10.4: Only one site per rectangle can satisfy the following topological picture.

One can then obtain the separation on the interior in the same way. Now, fix  $n < N$  arbitrary. define  $s, r$  by the formulæ  $2^{s-1} < n \leq 2^s$  and  $2^r \leq N < 2^{r+1}$ . We have

$$\phi[A_\sigma(n, N)] \leq \phi[A_\sigma(2^s, 2^r)] \asymp \phi[A_\sigma^{sep}(2^s, 2^r)] \asymp \phi[A_\sigma^{sep}(n, N)]$$

using (10.12) and (10.13) a last time. □

We mention a classical corollary of the comparison between well-separated arms and usual arms: one can choose a landing sequence  $I = (I_k)_{k \leq j}$  of disjoint areas of size  $\delta$  on the boundary of the square  $Q = [-1, 1]^2$ , found in counter-clockwise order following  $\partial Q$ .

Let  $A_\sigma^I(n, N)$  be the event that there exist arms from the interior to the exterior of  $S_{n, N}$ , and such that  $\gamma_k$  ends on  $NI_k$ .

**Corollary 10.19.** *Fix  $j > 0$ . For any choice of  $I$ ,  $\sigma$ ,  $n < N$ , we have*

$$\phi[A_\sigma^I(n, N)] \asymp \phi[A_\sigma(n, N)].$$

### 3.2 Universal exponents

**Theorem 10.20.** *For every  $0 < k < n \leq L_p$ ,*

$$\phi[A_{ococ}(k, n)] \asymp (k/n)^2, \quad \phi[A_{oc}^{HP}(k, n)] \asymp k/n, \quad \phi[A_{oco}^{HP}(k, n)] \asymp (k/n)^2.$$

where  $A_\sigma^{HP}(n, N)$  is the existence of  $j$  paths in  $[-N, N] \times [0, N] \setminus [-n, n] \times [0, n]$  from  $[-n, n] \times [0, n]$  to  $([-N, N] \times [0, N])^c$ .

**Proof** We treat the first case only, since the others are similar and actually technically easier. We only need to look at the case  $k = 1$  via quasi-multiplicativity.

Let us first prove the lower bound. Fix  $n < L_p$ . Consider the following construction: assume there exist a horizontal crossing of  $[-n, n] \times [-n/4, 0]$  and a dual horizontal crossing of  $[-n, n] \times [0, n/4]$ . This happens with probability bounded from below by  $c > 0$  not depending on  $n$ . By conditioning on the lowest interface  $\Gamma$  between an open and a closed crossing of  $[-n, n] \times [-n/4, n/4]$ , the configuration above it is a random-cluster configuration with free boundary conditions. Let  $\Omega$  be the connected component of  $\Lambda_n \setminus \Gamma$  containing  $[-n, n] \times \{n\}$ . Assume that  $[-n/4, 0] \times [-n, n] \cap \Omega$  is dual crossed horizontally, and that  $[0, n/4] \times [-n, n] \cap \Omega$  is crossed horizontally. The probability of this event is once again bounded from below uniformly in  $n$ , thanks to Theorem 10.1. Note that we need a strong form of crossing probabilities in order to guarantee the existence of the last crossing since the boundary of  $\Omega$  can be very rough.

Summarizing, all these events occur with probability larger than  $c' > 0$ . Moreover, the existence of all these crossings implies the existence of a site in  $\Lambda_{n/4}$  with five arms emanating from it. The union bound implies

$$(n/4)^2 \phi[A_{ocococ}(n/4)] \geq c'.$$

In order to prove an upper bound for  $\phi[A_{ocococ}(n)]$ , recall that it suffices to show it for well-separated arms for which we choose landing sequences. Consider the event described in Fig. 10.4. Topologically, no two sites in  $\Lambda_n$  can satisfy this event simultaneously, which implies the claim.  $\square$

This result has an interesting corollary:

**Corollary 10.21.** *Fix  $p \in (0, 1)$ . There exists  $\alpha > 0$  such that for every  $0 < k < n \leq L_p$ ,*

$$\begin{aligned} \phi[A_{ocococ}(k, n)] &\lesssim (k/n)^{2+\alpha} \\ \phi[A_{ococ}(k, n)] &\gtrsim (k/n)^{2-\alpha}. \end{aligned}$$

The 'six-arm' event will be important for convergence to SLE. The 'four-arm event' is important for the existence of pivotal sites (see Chapter 12).

**Proof** Fix  $n < N$ , we have

$$\phi(A_{ocococ}(n, N)) \asymp \phi(A_{ocococ}(n, N), \text{no arm finishing at the bottom}).$$

Conditioning on five arms (starting the exploration from the bottom for instance), it can be shown that

$$\phi(A_{ocococ}(n, N), \text{no arm finishing at the bottom}) \leq \phi^0(A_c(n, N))\phi(A_{ococ}(n, N)).$$

The result follows from Theorem 10.20 and the fact that Theorem 10.1 implies

$$\phi^0(A_c(n, N)) \leq (n/N)^\alpha$$

for some  $\alpha > 0$ . The same proof works with *ococ* replacing *ocococ*.  $\square$

## 4 Other applications

### 4.1 Spin-Ising crossing probabilities

Thanks to the Edwards-Sokal coupling, we can couple the FK-Ising and the spin-Ising model, and derive from Theorem 10.1 crossing probabilities bounds for the spin Ising model.

While it is impossible to obtain crossing probabilities for the critical spin-Ising that would be uniform with respect to the boundary conditions (the probability of crossing of + spins with – boundary conditions everywhere tends to 0 in the scaling limit, as can be seen using SLE techniques), it is possible to get nontrivial bounds that allow to deal with spin-Ising interfaces, notably in presence of free boundary conditions (which is the setup considered in [LPSA94]).

**Corollary 10.22.** *Let  $M > 1$ . Then there exists  $\delta \in (0, \frac{1}{2})$  such that the following holds: Let  $(\Omega, a, b, c, d)$  be a topological rectangle with  $\frac{1}{M} \leq \ell_\Omega[(ab), (cd)] \leq M$ . Consider the critical Ising model on  $(\Omega, a, b, c, d)$  with free boundary conditions on  $(ab) \cup (cd)$  and + boundary conditions on  $(bc) \cup (da)$ . Then we have*

$$\delta \leq \mathbb{P}[\text{There is a crossing of } - \text{ spins } (ab) \leftrightarrow (cd)] \leq 1 - \delta.$$

**Remark 10.23.** *By monotonicity of the spin-Ising model with respect to the boundary conditions, this result implies that the probabilities of – crossings in topological rectangles with free boundary conditions (the setup considered in [LPSA94]) are also bounded away from below. By self-duality (for topological reason there cannot be both a – crossing between  $(ab)$  and  $(cd)$  and a + crossing between  $(bc)$  and  $(da)$ ) and symmetry between – and + spins, such crossing probabilities are also bounded from above.*

**Proof of Corollary 10.22** Let us show a lower bound only (the upper bound can be obtained by self-duality arguments).

The Edwards-Sokal coupling enables us to couple this Ising model with an FK-Ising model with boundary conditions  $(bc) \cup (da)$  (the sites on  $(bc) \cup (da)$  are wired, and the sites on  $(ab) \cup (cd)$  are free). Use Corollary 10.9 to split  $\Omega$  into three “fair shares”  $(\Omega_1, a, x_a, x_b, b)$ ,  $(\Omega_2, x_b, x_a, x_c, x_d)$  and  $(\Omega_3, c, d, x_d, x_c)$ , with

$$\ell_{\Omega_1}[(ax_a), (x_b b)] \asymp \ell_{\Omega_2}[(x_b x_a), (x_c x_d)] \asymp \ell_{\Omega_3}[(cd), (x_d x_c)] \asymp 1$$

(the constants depend on  $M$  only). By Theorem 10.1 there exists  $\alpha > 0$  such that with probability at least  $\alpha$ , there is no FK crossing  $(ax_a) \leftrightarrow (x_b b)$  in  $\Omega_1$ , with probability at least  $\alpha$  there is no FK crossing  $(cd) \leftrightarrow (x_d x_c)$ , with probability at least  $\alpha$  there is an FK-Ising crossing  $(x_b x_a) \leftrightarrow (x_d x_c)$ . So, with probability at least  $\alpha^3$ , we can ensure that there is an FK-Ising crossing  $(ab) \leftrightarrow (cd)$  in  $\Omega$ , that does not touch  $(bc) \cup (da)$ . Sampling a spin-Ising configuration from the FK-Ising model, we get that with probability at least  $\frac{1}{2}\alpha^3$ , there is an FK-Ising crossing will take sign – (since it is not connected to  $(bc) \cup (da)$ ), hence a – spin crossing  $(ab) \leftrightarrow (cd)$ .  $\square$

# Chapter 11

## Convergence to chordal SLE(3) and chordal SLE(16/3)

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**Abstract:** This chapter presents a proof of convergence of interfaces for FK-Ising and Ising to the chordal Schramm-Loewner Evolutions of parameters  $\kappa = 16/3$  and 3 respectively. It is inspired of the article *Convergence of Ising interfaces to Schramm's SLEs*, written with D. Chelkak, C. Hongler, A. Kemppainen and S. Smirnov [CDCH<sup>+</sup>11b]. Let us mention that the proof of convergence to  $SLE_{16/3}$  was first published in [Kem09, Smi10b]. Section 4 sketches an alternative proof of the main technical step in [Kem09, Smi10b] based on the previous chapter and Section 7 is a new result.

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There are many different ways of defining conformal invariance. In Chapter 7, a model was said to be conformally invariant if there exists a family of conformally covariant 'relevant observables' in the scaling limit. Following Aizenman's suggestion to look at interfaces, we show that FK-Ising and Ising interfaces are conformally invariant. For both models, the Dobrushin boundary conditions allow us to isolate a single interface (in the Ising case, between +1 and -1, and in the FK-Ising, between open and dual-open clusters) and we thus restrict ourselves to this context. Our aim is to prove that these interfaces, in the scaling limit, form a family of conformally invariant curves in the following sense:

**Definition 11.1.** *A family of random continuous curve  $\gamma_{(\Omega,a,b)}$  defined on simply connected domains  $\Omega$  with two marked points  $a$  and  $b$  on the boundary is conformally invariant if for any  $(\Omega, a, b)$  and any conformal map<sup>1</sup>  $\psi : \Omega \rightarrow \mathbb{C}$ ,*

$$\psi \circ \gamma_{(\Omega,a,b)} \quad \text{has the same law as} \quad \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

In 1999, Schramm proposed a natural candidate for the possible conformally invariant families of non-intersecting curves. He noticed that interfaces of models further satisfy the *domain Markov property* (see Definition 11.6) which, together with the assumption of

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<sup>1</sup>conformal means holomorphic and one-to-one.

conformal invariance, determine the possible family of curves. In [Sch00], he introduced the *Schramm-Loewner Evolution* – SLE in short. The SLE( $\kappa$ ), for  $\kappa > 0$  is a (random) Loewner chain with driving process  $\sqrt{\kappa}B_t$ , where  $B_t$  is a standard Brownian motion. Such a definition is not completely straightforward.

The fact that SLEs can be ‘encoded’ via Brownian motions paves the way to the use of standard techniques such as stochastic calculus in order to study the properties of the model. Consequently, SLEs are now fairly well understood: path properties have been derived in [RS05], their Hausdorff dimension can be computed [Bef04, Bef08a], etc... In addition to this, several critical exponents can be related to properties of the interfaces, and thus be computed using SLE. Therefore, proving convergence of interfaces of a model of statistical physics towards an SLE leads to a deep understanding of the phase transition. We refer to [Law05, Wer40, Wer05] for complete expositions on Schramm-Loewner Evolutions and related conformally invariant processes.

One of the first and most fundamental model for which convergence to SLE is known is site percolation on the triangular lattice [Smi01, Smi05, CN07] (it converges to SLE(6)). The convergence result enables us to compute of several exponents such as polychromatic arm-exponents [LSW01b, LSW01a], the monochromatic one-arm exponent [LSW02], the exponent  $\beta$  of the infinite-cluster density  $\theta(p)$  (the polychromatic four-arm exponent and the one-arm exponent can be related, via Kesten scaling relations [Kes87] to the exponent for  $\theta(p)$ , see Chapter 12 for further details), etc... In [LSW04a], loop-erased random walks were shown to converge to SLE(2). In [SS05], an *ad-hoc* model, called the harmonic explorer, was shown to converge to SLE(4).

The FK-Ising and Ising models are conformally invariant in the sense that they possess conformally covariant families of observables. As mentioned earlier, this *a priori* weaker result should in fact be sufficient to prove conformal invariance of interfaces. The goal of this section is to explain this step.

Convergence of random parametrized curves (say with time-parameter in  $[0, 1]$ ) is in the sense of the **weak topology** inherited from the following distance on curves:

$$d(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{u \in [0,1]} |\gamma_1(u) - \gamma_2(\phi(u))|, \quad (11.1)$$

where the infimum is taken over all reparametrizations (*i.e.* strictly increasing continuous functions  $\phi: [0, 1] \rightarrow [0, 1]$  with  $\phi(0) = 0$  and  $\phi(1) = 1$ ).

Let us begin with the FK-Ising model.

**Theorem 11.2** (Smirnov-Kemppainen [Smi10b, Kem09]). *Let  $\Omega$  be a simply connected domain with two marked points  $a, b$  on the boundary. Let  $\gamma_\delta$  be the interface of the critical FK-Ising with Dobrushin boundary conditions on  $(\Omega_\delta, a_\delta, b_\delta)$ . Then the law of  $\gamma_\delta$  converges weakly, as  $\delta \rightarrow 0$ , to the chordal Schramm-Loewner Evolution with  $\kappa = 16/3$ , for the topology associated to the curve distance.*

A similar statement holds for the spin-Ising model, with a different value of  $\kappa$ :

**Theorem 11.3.** *Let  $(\Omega, a, b)$  be a simply connected domain with two marked points on the boundary. Let  $\gamma_\delta$  be the interface of the critical Ising model with Dobrushin boundary*

conditions on the Dobrushin domain  $(\Omega_\delta, a_\delta, b_\delta)$ . Then  $(\gamma_\delta)_{\delta>0}$  converges weakly, as  $\delta \rightarrow 0$ , to the (chordal) Schramm-Loewner Evolution with parameter  $\kappa = 3$  for the topology associated to the curve distance.

The strategy to prove that a family of parametrized curves converges to SLE( $\kappa$ ) follows three steps:

- First, prove that the family of curves is tight.
- Then, show that any sub-sequential limit is a time-changed Loewner chain with a continuous driving process.
- Finally, show that the only possible driving process for the sub-sequential limits is  $\sqrt{\kappa}B_t$  where  $B_t$  is a standard Brownian motion.

The main step is the third one. In order to identify the Brownian motion as being the only possible driving process for the curve, we find computable martingales expressed in terms of the limiting curve. In our case, these martingales will be the limits of fermionic observables. The fact that these (explicit) functions are martingales allows us to deduce martingale properties of the driving process. More precisely, we aim to use Lévy's theorem: a continuous real-valued process  $X$  such that  $X_t$  and  $X_t^2 - at$  are martingales is necessarily  $\sqrt{a}B_t$ .

The chapter is organized as follows. The first section is a crash course on SLE. The second one deals with precompactness of FK-Ising interfaces. The third one presents a criterion to prove that these sub-sequential limits are Loewner chains. The fourth one contains the proof that FK-Ising interfaces converge to SLE(16/3). The fifth one contains the convergence result of Ising interfaces. The sixth section explains the first step of the program for general random-cluster models with cluster-weight  $q \geq 1$ .

## 1 Crash course on Schramm-Loewner Evolution

We do not aim for completeness (see [Law05, Wer40, Wer05] for details). We simply introduce notions needed in the next sections. Recall that a domain is a simply connected open set not equal to  $\mathbb{C}$ .

Set  $\mathbb{H}$  to be the upper half-plane. Fix a compact set  $K \subset \overline{\mathbb{H}}$  such that  $H = \mathbb{H} \setminus K$  is still simply connected. For such a domain  $H$ , Riemann's mapping theorem guarantees the existence of a conformal map from  $H$  onto  $\mathbb{H}$ . Moreover, there is *a priori* three real degrees of freedom in the choice of the conformal map, so that it is possible to fix its asymptotic when  $z$  goes to  $\infty$ . Let  $g$  be the unique conformal map from  $H$  onto  $\mathbb{H}$  such that

$$g(z) := z + \frac{C}{z} + O\left(\frac{1}{z^2}\right).$$

The proof of the existence of this map is not completely obvious and requires the reflexion principle. The constant  $C$  is called the *h-capacity* of  $H$ . It acts like a capacity: it is increasing in the domain and the *h-capacity* of  $\lambda K$  is  $\lambda^2$  times the *h-capacity* of  $K$ .

There is a natural way to parametrize continuous curves  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{H}$  with  $\gamma(0) = 0$  and with  $\gamma$  going to  $\infty$  when  $t \rightarrow \infty$ . For every  $s$ , let  $H_s$  be the connected component of  $\mathbb{H} \setminus \gamma[0, s]$  containing  $\infty$ . We denote by  $K_s$  the *hull* created by  $\gamma[0, s]$ , i.e. the compact set  $\overline{\mathbb{H}} \setminus H_s$ . From the previous paragraph,  $K_s$  has a certain  $h$ -capacity  $C_s$ . The continuity of the curve guarantees that  $C_s$  grows continuously, so that it is possible to parametrize the curve in such a way that  $C_s = 2t$  at time  $t$ . This parametrization is called the  *$h$ -capacity parametrization*. Note that in general, the previous operation is not a proper reparametrization, since any part of the curve hidden from  $\infty$  will not make the  $h$ -capacity grow, and thus will be mapped to the same point for the new curve.

From now on, assume the curve is parametrized via  $h$ -capacity. In particular, the curve can be encoded<sup>2</sup> via the family of conformal maps  $g_t$  from  $H_t$  to  $\mathbb{H}$ , such that

$$g_t(z) := z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

Under mild conditions, the infinitesimal evolution of the family  $(g_t)$  can be studied and it implies the existence of a continuous real valued process  $W_t$  such that for every  $t$  and  $z \in H_t$ ,

$$\partial_t g_t(z) := \frac{2}{g_t(z) - W_t}.$$

The process  $W_t$  is called the *driving process*. This equation can be derived for general growing hulls, the typical required hypothesis in order to do so is the following 'local growth' condition:

*Local Growth Condition:* for any  $t \geq 0$  and for any  $\varepsilon$ , there exists  $\delta > 0$  such that for any  $s \leq t$ , the diameter of  $g_s(K_{s+\delta} \setminus K_s)$  is smaller than  $\varepsilon$ , where  $K_s = \mathbb{H} \setminus H_s$  is the hull created by  $\gamma_s$ .

It is important to notice that the procedure is revertible. If a continuous function  $W_t$  is given, it is possible to reconstruct the hull  $K_t$  as the set of points  $z$  for which the previous differential equation already blew up.

We are now in a position to define Schramm-Loewner Evolutions:

**Definition 11.4** (SLE in the upper half-plane). *The chordal Schramm-Loewner Evolution in  $\mathbb{H}$  with parameter  $\kappa > 0$  is the (random) Loewner chain with (random) driving process  $W_t := \sqrt{\kappa}B_t$ , where  $B_t$  is a standard Brownian motion.*

Loewner chains in other domains are easy to define via conformal mapping.

**Definition 11.5** (SLE in general domains). *Fix a domain  $\Omega$  with two points on the boundary  $a$  and  $b$  and assume it has a nice boundary (for instance a Jordan curve). The chordal Schramm-Loewner Evolution with parameter  $\kappa > 0$  in  $(\Omega, a, b)$  is the image of the Schramm-Loewner Evolution in the upper half-plane by a conformal map from  $(\mathbb{H}, 0, \infty)$  onto  $(\Omega, a, b)$ .*

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<sup>2</sup>In fact only the hull associated to the curve can be encoded via conformal maps.

To conclude this paragraph, let us mention the fact that these curves are natural scaling limits for interfaces of conformally invariant models. In order to explain this fact, let us introduce the notion of *domain Markov property* for a family of random growing curves.

**Definition 11.6.** *A family of random continuous curves  $\gamma_{(\Omega,a,b)}$  (parametrized via  $h$ -capacity) in simply connected domains is said to satisfy the domain Markov property if for every  $(\Omega, a, b)$ , and every  $t > 0$ , the law of the curve  $\gamma[t, \infty)$  conditionally on  $\gamma[0, t]$  is the same as the law of  $\gamma_{(\Omega_t, \gamma_t, b)}$ , where  $\Omega_t$  is the connected component of  $\Omega \setminus \gamma_t$  containing  $b$ .*

Discrete interfaces in models of statistical physics naturally satisfy this property, and therefore their limit also do. Schramm proved the following result in [Sch00], which in some way justifies SLEs as natural candidates for limits of interfaces.

**Theorem 11.7** (Schramm, [Sch00]). *Every family of Loewner chains  $\gamma_{(\Omega,a,b)}$  which*

- *is conformally invariant,*
- *satisfies the domain Markov property,*
- *satisfies that  $\gamma_{(\mathbb{H},0,\infty)}$  is scale invariant,*

*is a chordal Schramm-Loewner Evolution with parameter  $\kappa \in [0, \infty)$ .*

## 2 Tightness of interfaces for FK-Ising

In this section, the following theorem is proved:

**Theorem 11.8.** *Fix a domain  $(\Omega, a, b)$ , the family  $(\gamma_\delta)_{\delta>0}$  of random interfaces for critical FK-Ising in  $(\Omega, a, b)$  is tight for the topology associated to the curve distance.*

The question of tightness for curves in the plane has been studied in the groundbreaking paper [AB99]. In that paper, it is proved that a sufficient condition for tightness is the absence, at every scale, of annuli crossed back and forth an unbounded number of times.

More precisely, for  $x \in \Omega$  and  $r < R$ , let  $S_{r,R}(x) = (x + [-R, R]^2) \setminus (x + [-r, r]^2)$  and define  $\mathcal{A}_k(x; r, R)$  to be the event that there exist  $k$  crossing of the curve  $\gamma_\delta$  between outer and inner boundaries of  $S_{r,R}(x)$ .

**Theorem 11.9** (Aizenman-Burchard [AB99]). *Let  $\Omega$  be a simply connected domain and let  $a$  and  $b$  be two marked points on its boundary. Denote by  $\mathbb{P}_\delta$  the law of a random curve  $\tilde{\gamma}_\delta$  on  $\Omega_\delta$  from  $a_\delta$  to  $b_\delta$ . If there exist  $k \in \mathbb{N}$ ,  $C_k < \infty$  and  $\Delta_k > 2$  such that for all  $\delta < r < R$  and  $x \in \Omega$ ,*

$$\mathbb{P}_\delta(\mathcal{A}_k(x; r, R)) \leq C_k \left(\frac{r}{R}\right)^{\Delta_k},$$

*then the family of curves  $(\tilde{\gamma}_\delta)$  is tight.*

We now show how to exploit this theorem in order to prove Theorem 11.8. The main tool is Corollary 9.13 (which follows from Theorem 9.1).

**Proof of Theorem 11.8** Fix  $x \in \Omega$ ,  $\delta < r < R$  and recall that the lattice has mesh size  $\delta$ . Let  $k$  to be fixed later. We first prove that

$$\phi_{\Omega_\delta, p_{sd}, 2}^{a_\delta, b_\delta}(\mathcal{A}_{2k}(x; r, 2r)) \leq c^k \quad (11.2)$$

for some constant  $c < 1$  uniform in  $x, k, r, \delta$  and the configuration outside of  $S_{r, 2r}(x)$ .

If  $\mathcal{A}_{2k}(x; r, 2r)$  holds, then there are (at least)  $k$  open paths, alternating with  $k$  dual paths, connecting the inner boundary of the annulus to its outer boundary. Since the paths are alternating, one can deduce that there are  $k$  open crossings, each one being surrounded by closed crossings. Hence, using successive conditionings and the comparison between boundary conditions, the probability for each crossing is smaller than the probability that there is a crossing in the annulus with wired boundary conditions (since these boundary conditions maximize the probability of the event). We obtain

$$\phi_{\Omega_\delta, p_{sd}}^{a_\delta, b_\delta}(\mathcal{A}_{2k}(x; r, 2r)) \leq \left[ \phi_{S_{r, 2r}(x), p_{sd}, 2}^1(S_{r, 2r} \text{ is crossed}) \right]^k.$$

Using Corollary 9.13,  $\phi_{S_{r, 2r}(x), p_{sd}, 2}^1(S_{r, 2r} \text{ is crossed}) \leq 1 - c_2 < 1$ , and (11.2) follows.

One can further fix  $k$  large enough so that  $c^k < \frac{1}{8}$ . Now, one can decompose the annulus  $S_{r, R}(x)$  into roughly  $\ln_2(R/r)$  annuli of the form  $S_{r, 2r}(x)$ , so that for the previous  $k$ ,

$$\phi_{\Omega_\delta, p_{sd}, 2}^{a_\delta, b_\delta}(\mathcal{A}_{2k}(x; r, R)) \leq \left( \frac{r}{R} \right)^3. \quad (11.3)$$

Hence, Theorem 11.9 implies that the family  $(\gamma_\delta)$  is tight.  $\square$

### 3 sub-sequential limits of FK-Ising interfaces are Loewner chains

In the previous paragraph, traces of interfaces in Dobrushin domains were shown to be tight. We would now like to parametrize any sub-sequential limit curve as a Loewner chain, *i.e.* via its  $h$ -capacity. In this case, we say that the curve is a *time-changed Loewner chain*.

**Theorem 11.10.** *Any sub-sequential limit of the family  $(\gamma_\delta)_{\delta > 0}$  of FK-Ising interfaces is a time-changed Loewner chain.*

As emphasized in the first section of this chapter, not every continuous curve is a time-changed Loewner chain, therefore an additional argument is needed, especially since the limiting curve of FK interfaces is fractal-like and has many double points. A general characterization for a parametrized non-selfcrossing curve in  $(\Omega, a, b)$  to be a time-changed Loewner chain is the following:

- its  $h$ -capacity must be continuous,

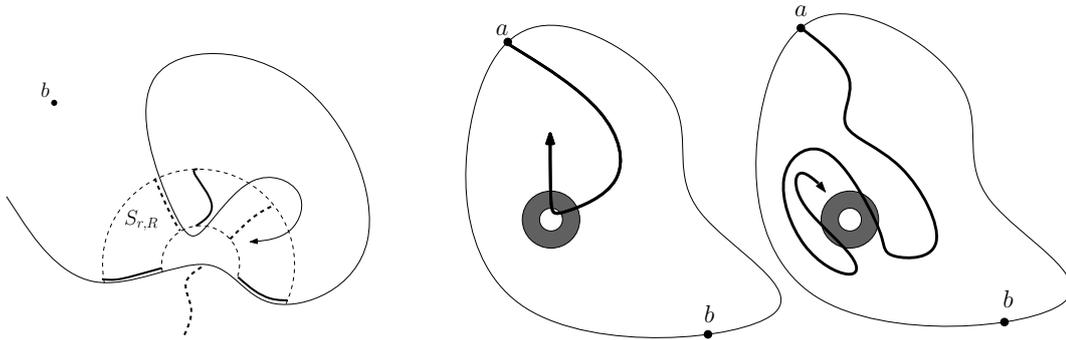


Figure 11.1: **Left:** An example of a fjord. Seen from  $b$ , the  $h$ -capacity (roughly speaking, the size) of the hull does not grow much while the curve is in the fjord. The event involves six alternating open and closed crossings of the annulus. **Right:** Conditionally on the beginning of the curve, the crossing of the annulus is unforced on the left, while it is forced on the right.

- its  $h$ -capacity must be strictly increasing
- the curve grows locally seen from infinity in the following sense: for any  $t \geq 0$  and for any  $\varepsilon$ , there exists  $\delta > 0$  such that for any  $s \leq t$ , the diameter of  $g_s(K_{s+\delta} \setminus K_s)$  is smaller than  $\varepsilon$ , where  $K_s = \mathbb{H} \setminus H_s$  is the hull created by  $\gamma[0, s]$ .

The first condition is automatically satisfied by continuous curves. The third one follows from the other two when the curve is continuous, so that the only condition to check is the second one. This condition can be understood as being the fact that the tip of the curve is visible from  $b$  at every time. In other words, the family of hulls created by the curve (*i.e.* the complement of the connected component of  $\Omega \setminus \gamma_t$  containing  $b$ ) is strictly increasing. This is the case if the curve does not enter long fjords created by its past at every scale, see Fig. 11.1.

In the case of FK interfaces, this corresponds to so-called six-arm events, and it boils down to proving that  $\Delta_6 > 2$ . We already proved this result in 10, and we show at the end of this subsection how it indeed implies that scaling limits are Loewner chains. Before that, we present a more general criterion characterizing Loewner chains.

Recently, Kemppainen and Smirnov [KS10] proved a 'structural theorem' characterizing random continuous curves that can be parametrized as Loewner chains. We describe it now.

For a family of parametrized curves  $(\gamma_\delta)_{\delta>0}$ , define Condition  $(\star)$ :

*Condition  $(\star)$ : There exist  $C > 1$  and  $\Delta > 0$  such that for any  $0 < \delta < r < R/C$ , for any stopping time  $\tau$  and for any annulus  $S_{r,R}(x)$  not containing  $\gamma_\tau$ , the probability that  $\gamma_\delta$  crosses the annulus  $S_{r,R}(x)$  (from the outside to the inside) after time  $\tau$  **while it is not forced to enter**  $S_{r,R}(x)$  **again** is smaller than  $C(r/R)^\Delta$ , see Fig. 11.1.*

Roughly speaking, the previous condition is a uniform bound on unforced crossings. Note that it is necessary to assume that the crossing is unforced.

**Theorem 11.11.** *If a family of curves  $(\gamma_\delta)$  satisfies Condition  $(\star)$ , then it is tight for the topology associated to the curve distance. Moreover, any sub-sequential limit  $\gamma$  is a time-changed Loewner chain and  $\gamma$  is the trace of the family of hulls generated by  $\gamma$ .*

Tightness is almost obvious, since Condition  $(\star)$  implies the hypothesis in Aizenman-Burchard's theorem. The hard part is the proof that Condition  $(\star)$  guarantees that the  $h$ -capacity of sub-sequential limits is strictly increasing and that they create Loewner chains. The reader is referred to [KS10] for a proof of this statement.

**Proof of Theorem 11.10** Corollary 9.13 implies Condition  $(\star)$  without difficulty.  $\square$

## 4 sub-sequential limits of FK-Ising interfaces are Loewner chains (alternative proof)

Let us now sketch another way of proving Theorem 11.10. It does not require Theorem 11.11 and it harnesses Theorem 10.1 only. More precisely, we will be using Theorem 10.1 and two of its corollaries: the 6-arm exponent in the plane is greater than 2 and the three arm exponent on the boundary is equal to 2. We refer to [Wer07] for additional details on this method.

We need to prove that the  $h$ -capacity is strictly increasing. Let us consider the discrete explorations directly in the upper half-plane, and already parametrized by their  $h$ -capacity. The idea is to proceed in three steps. Let  $\sigma_\delta(z)$  (resp.  $\sigma(z)$ ) be the time at which  $z$  is disconnected from infinity by the discrete curve  $\gamma_\delta$  (resp. the continuous curve  $\gamma$ ).

STEP 1: SIMULTANEOUSLY FOR EVERY  $z$ ,  $\sigma_\delta(z)$  CONVERGES TO  $\sigma(z)$  ALMOST SURELY. This is due to the fact that if one point  $z$  does not satisfy this property, the discrete model has to possess six arms of alternative colors (or three arms on the boundary of alternative colors). Yet, the six arm event has exponent larger than 2 and does not happen anywhere in the domain with probability going to 1.

STEP 2: FOR ANY  $u < u'$ , THERE EXISTS  $v \in (u, u')$  SUCH THAT  $\gamma(v) \notin \gamma[0, u] \cup \partial\mathbb{H}$ . Fix a dense family of points on  $\gamma[0, u] \cup \partial\mathbb{H}$ . Each of these points does not belong to the curve  $\gamma[0, \infty]$  almost surely, thanks to Theorem 10.1. Therefore, none of these points belongs to  $\gamma[0, \infty]$  almost surely. This implies that  $\gamma[u, u']$  cannot be included in  $\gamma[0, u] \cup \partial\mathbb{H}$ .

STEP 3: FOR EVERY RATIONAL  $u < u'$ ,  $K_u \neq K_{u'}$ . Recall that  $K_u$  is the hull created by  $\gamma[0, u]$ . It is thus sufficient to prove that there exists  $v \in (u, u')$  such that  $\gamma(v) \notin K_u \cup \partial\mathbb{H}$ . We already know from the second step that there exists  $\gamma(v) \notin \gamma[0, u] \cup \partial\mathbb{H}$ . Thus  $\gamma(v)$  is in one of the connected components of  $\mathbb{H} \setminus \gamma[0, u]$ . Assume it is not in the unbounded one. The first step implies that

$$\sigma_\delta[\gamma(v)] \leq \frac{v + \sigma[\gamma(v)]}{2}$$

with probability going to 1. It immediately implies that  $\sigma_\delta[\gamma_\delta(v)] < v$  for  $\delta$  small enough, which is impossible since discrete curves  $\gamma_\delta$  do not have triple points.

## 5 Convergence of FK-Ising interfaces to SLE(16/3)

The FK fermionic observable is now proved to be a martingale for the discrete curves and to identify the driving process of any sub-sequential limit of FK-Ising interfaces.

**Lemma 11.12.** *Let  $\delta > 0$ . The FK fermionic observable  $M_n^\delta(z) = F_{\Omega_\delta \setminus \gamma[0,n], \gamma_n, b_\delta}(z)$  is a martingale with respect to  $(\mathcal{F}_n)$  where  $\mathcal{F}_n$  is the  $\sigma$ -algebra generated by the FK interface  $\gamma[0, n]$ .*

**Proof** For a Dobrushin domain  $(\Omega_\delta^\circ, a_\delta, b_\delta)$ , the slit domain created by “removing” the first  $n$  steps of the exploration path is again a Dobrushin domain. Conditionally on  $\gamma[0, n]$ , the law of the FK-Ising model in this new domain is exactly  $\phi_{\Omega_\delta^\circ \setminus \gamma[0,n]}^{\gamma_n, b_\delta}$ . This observation implies that  $M_n^\delta(z)$  is the random variable  $1_{z \in \gamma_\delta} e^{\frac{1}{2}iW_{\gamma_\delta}(z,b)}$  conditionally on  $\mathcal{F}_n$ , therefore it is automatically a martingale.  $\square$

**Proposition 11.13.** *Any sub-sequential limit of  $(\gamma_\delta)_{\delta>0}$  which is a Loewner chain is the (chordal) Schramm-Loewner Evolution with parameter  $\kappa = 16/3$ .*

**Proof** Consider a sub-sequential limit  $\gamma$  in the domain  $(\Omega, a, b)$  which is a Loewner chain. Let  $\phi$  be a map from  $(\Omega, a, b)$  to  $(\mathbb{H}, 0, \infty)$ . Our goal is to prove that  $\tilde{\gamma} = \phi(\gamma)$  is a chordal SLE(16/3) in the upper half-plane.

Since  $\gamma$  is assumed to be a Loewner chain,  $\tilde{\gamma}$  is a growing hull from 0 to  $\infty$  parametrized by its  $h$ -capacity. Let  $W_t$  be its continuous driving process. Also define  $g_t$  to be the conformal map from  $\mathbb{H} \setminus \tilde{\gamma}[0, t]$  to  $\mathbb{H}$  such that  $g_t(z) = z + 2t/z + O(1/z^2)$  when  $z$  goes to  $\infty$ .

Fix  $z' \in \Omega$ . For  $\delta > 0$ , recall that  $M_n^\delta(z')$  is a martingale for  $\gamma_\delta$ . Since the martingale is bounded,  $M_{\tau_t}^\delta(z')$  is a martingale with respect to  $\mathcal{F}_{\tau_t}$ , where  $\tau_t$  is the first time at which  $\phi(\gamma_\delta)$  has an  $h$ -capacity larger than  $t$ . Since the convergence is uniform,  $M_t(z') := \lim_{\delta \rightarrow 0} M_{\tau_t}^\delta(z')$  is a martingale with respect to  $\mathcal{G}_t$ , where  $\mathcal{G}_t$  is the  $\sigma$ -algebra generated by the curve  $\tilde{\gamma}$  up to the first time its  $h$ -capacity exceeds  $t$ . By definition, this time is  $t$ , and  $\mathcal{G}_t$  is the  $\sigma$ -algebra generated by  $\tilde{\gamma}[0, t]$ .

Recall that  $M_t(z')$  is related to  $\phi(z')$  via the conformal map from  $\mathbb{H} \setminus \tilde{\gamma}[0, t]$  to  $\mathbb{R} \times (0, 1)$ , normalized to send  $\tilde{\gamma}_t$  to  $-\infty$  and  $\infty$  to  $\infty$ . This last map is exactly  $\frac{1}{\pi} \ln(g_t - W_t)$ . Setting  $z = \phi(z')$ , we obtain that

$$\sqrt{\pi}M_t^z := \sqrt{\pi}M_t(z') = \sqrt{[\ln(g_t(z) - W_t)]'} = \sqrt{\frac{g_t'(z)}{g_t(z) - W_t}} \quad (11.4)$$

is a martingale. Recall that, when  $z$  goes to infinity,

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{and} \quad g'_t(z) = 1 - \frac{2t}{z^2} + O\left(\frac{1}{z^3}\right) \quad (11.5)$$

For  $s \leq t$ ,

$$\begin{aligned} \sqrt{\pi} \cdot \mathbb{E}[M_t^z | \mathcal{G}_s] &= \mathbb{E}\left[\sqrt{\frac{1 - 2t/z^2 + O(1/z^3)}{z - W_t + 2t/z + O(1/z^2)}} \mid \mathcal{G}_s\right] \\ &= \frac{1}{\sqrt{z}} \mathbb{E}\left[1 + \frac{1}{2}W_t/z + \frac{1}{8}(3W_t^2 - 16t)/z^2 + O(1/z^3) \mid \mathcal{G}_s\right] \\ &= \frac{1}{\sqrt{z}} \left(1 + \frac{1}{2}\mathbb{E}[W_t | \mathcal{G}_s]/z + \frac{1}{8}\mathbb{E}[3W_t^2 - 16t | \mathcal{G}_s]/z^2 + O(1/z^3)\right). \end{aligned}$$

Taking  $s = t$  yields

$$\sqrt{\pi} \cdot M_s^z = \frac{1}{\sqrt{z}} \left(1 + \frac{1}{2}W_s/z + \frac{1}{8}(3W_s^2 - 16s)/z^2 + O(1/z^3)\right).$$

Since  $\mathbb{E}[M_t^z | \mathcal{G}_s] = M_s^z$ , terms in the previous asymptotic development can be matched together so that  $\mathbb{E}[W_t | \mathcal{G}_s] = W_s$  and  $\mathbb{E}[W_t^2 - \frac{16}{3}t | \mathcal{G}_s] = W_s^2 - \frac{16}{3}s$ . Since  $W_t$  is continuous, Lévy's theorem implies that  $W_t = \sqrt{\frac{16}{3}}B_t$  where  $B_t$  is a standard Brownian motion.

In conclusion,  $\gamma$  is the image by  $\phi^{-1}$  of the chordal Schramm-Loewner Evolution with parameter  $\kappa = 16/3$  in the upper half-plane. This is exactly the definition of the chordal Schramm-Loewner Evolution with parameter  $\kappa = 16/3$  in the domain  $(\Omega, a, b)$ .  $\square$

**Proof of Theorem 11.2** By Theorem 11.8, the family of curves is tight. Using Theorem 11.10, any sub-sequential limit is a time-changed Loewner chain. Consider such a sub-sequential limit and parametrize it by its  $h$ -capacity. Proposition 11.13 then implies that it is the Schramm-Loewner Evolution with parameter  $\kappa = 16/3$ . The possible limit being unique, the claim is proved.  $\square$

## 6 Convergence to SLE(3) for spin Ising interfaces

The proof of Theorem 11.3 is very similar to the proof of Theorem 11.2, except that we work with the spin Ising fermionic observable instead of the FK-Ising one.

First, let us mention a slight simplification compared to the FK-Ising case. Theorem 9.2 implies tightness. Interestingly, it is not necessary to prove that possible scaling limits are Loewner chains. Indeed, the only interest of Theorem 11.10 is to prove that the  $h$ -capacity increases strictly. If one forgets about this condition, it is still possible to describe the hull created by the interface. If one identifies it to be the same as SLE(3), it

immediately implies that the interface hull is a simple curve, since SLE(3) is simple. As a corollary, the interface itself converges to SLE(3). It is therefore sufficient to prove that the driving process is the same as  $\sqrt{3}B_t$ .

The only point differing from the identification of the driving process in the FK-Ising is the fact that the spin fermionic observable is a martingale for the curve. We prove this fact now and leave the remainder of the proof as an exercise. Let  $\gamma$  be the interface in the critical Ising model with Dobrushin boundary conditions.

**Lemma 11.14.** *Let  $\delta > 0$ , the spin fermionic observable  $M_n^\delta(z) = F_{\Omega_\delta^\circ \setminus \gamma[0,n], \gamma(n), b_\delta}(z)$  is a martingale with respect to  $(\mathcal{F}_n)$  where  $\mathcal{F}_n$  is the  $\sigma$ -algebra generated by the exploration process  $\gamma[0, n]$ .*

**Proof** It is sufficient to check that  $F_\delta(z)$  has the martingale property when  $\gamma = \gamma(\omega)$  makes one step  $\gamma_1$ . In this case  $\mathcal{F}_0$  is the trivial  $\sigma$ -algebra, so that we wish to prove

$$\mu_{\beta_c, \Omega}^{a,b} [F_{\Omega_\delta^\circ \setminus [a_\delta \gamma_1], \gamma_1, b_\delta}(z)] = F_{\Omega_\delta^\circ, a_\delta, b_\delta}(z). \quad (11.6)$$

Write  $Z_{\Omega_\delta^\circ, a_\delta, b_\delta}$  (resp.  $Z_{\Omega^\circ \setminus [a_\delta x], x, b_\delta}$ ) for the partition function of the Ising model with Dobrushin boundary conditions on  $(\Omega_\delta^\circ, a_\delta, b_\delta)$  (resp.  $(\Omega^\circ \setminus [a_\delta x], x, b_\delta)$ ), i.e.  $Z_{\Omega^\circ \setminus [a_\delta x], x, b_\delta} = \sum_\omega (\sqrt{2} - 1)^{|\omega|}$ . Note that  $Z_{\Omega^\circ \setminus [a_\delta x], x, b_\delta}$  is almost the denominator of  $F_{\Omega_\delta^\circ \setminus [a_\delta x], x, b_\delta}(z_\delta)$ . By definition,

$$\begin{aligned} Z_{\Omega_\delta^\circ, a_\delta, b_\delta} \mu_{\beta_c, \Omega}^{a,b}(\gamma_1 = x) &= (\sqrt{2} - 1) Z_{\Omega^\circ \setminus [a_\delta x], x, b_\delta} \\ &= (\sqrt{2} - 1) e^{i\frac{1}{2}W_\gamma(x, b_\delta)} \frac{\sum_{\omega \in \mathcal{E}_{\Omega^\circ \setminus [a_\delta x]}(x, z_\delta)} e^{-i\frac{1}{2}W_\gamma(x, z_\delta)} (\sqrt{2} - 1)^{|\omega|}}{F_{\Omega_\delta^\circ \setminus [a_\delta x], x, b_\delta}(z_\delta)} \\ &= e^{i\frac{1}{2}W_\gamma(a_\delta, b_\delta)} \frac{\sum_{\omega \in \mathcal{E}_{\Omega_\delta^\circ}(a_\delta, z_\delta)} e^{-i\frac{1}{2}W_\gamma(a_\delta, z_\delta)} (\sqrt{2} - 1)^{|\omega|} \mathbf{1}_{\{\gamma_1 = x\}}}{F_{\Omega_\delta^\circ \setminus [a_\delta x], x, b_\delta}(z_\delta)} \end{aligned}$$

In the second equality, we used the fact that  $\mathcal{E}_{\Omega_\delta^\circ \setminus [a_\delta x]}(x, z_\delta)$  is in bijection with configurations of  $\mathcal{E}_{\Omega_\delta^\circ}(a_\delta, z_\delta)$  such that  $\gamma_1 = x$  (there is still a difference of weight of  $\sqrt{2} - 1$  between two associated configurations). Thus giving:

$$\mu_{\beta_c, \Omega}^{a,b}(\gamma_1 = x) F_{\Omega_\delta^\circ \setminus [a_\delta x], x, b_\delta}(z_\delta) = \frac{\sum_{\omega \in \mathcal{E}(a_\delta, z_\delta)} e^{-i\frac{1}{2}W_\gamma(a_\delta, z_\delta)} (\sqrt{2} - 1)^{|\omega|} \mathbf{1}_{\{\gamma_1 = x\}}}{e^{-i\frac{1}{2}W_\gamma(a_\delta, b_\delta)} Z_{\Omega_\delta^\circ, a_\delta, b_\delta}}.$$

The same holds for all possible first steps. Summing over all possibilities, we obtain the expectation on one side of the equality and  $F_{\Omega_\delta^\circ, a_\delta, b_\delta}(z_\delta)$  on the other side, thus proving (11.6).  $\square$

## 7 Precompactness of interfaces in random-cluster models with $q \geq 1$

This section is devoted to the first step of the program in the case of general random-cluster models with  $q \geq 1$ . Condition  $(\star)$  seems difficult to prove for  $q \neq 1, 2$  since we do not possess crossing estimates which are valid uniformly in the boundary conditions. Nevertheless, it is still possible, using arguments similar to those in chapter 4, to prove the criterion of Theorem 11.9. We deduce the following result:

**Theorem 11.15.** *Fix a domain  $(\Omega, a, b)$ , the family  $(\gamma_\delta)_{\delta>0}$  of random interfaces for the critical random-cluster model in  $(\Omega, a, b)$  with cluster weight  $q \geq 1$  is tight for the topology associated to the curve distance.*

**Proof** We fix  $q \geq 1$  and  $p = p_{sd}(q)$  and we drop them from the notations. We must prove that there exists  $k > 0$  such that, uniformly in  $x \in \Omega$ ,  $0 < \delta < r < R$  and boundary conditions  $\xi$ ,

$$\phi_{\Omega_\delta}^\xi(\mathcal{A}_k(x; r, R)) \leq C_k \left(\frac{r}{R}\right)^{\Delta_k},$$

for some universal  $C_k$  and  $\Delta_k > 2$ . In order to do so, it is sufficient to show that for any  $\varepsilon > 0$ , there exists  $k > 0$  such that

$$\phi_{S_{n,2n}}^\xi(\mathcal{A}_k(0; n, 2n)) \leq \varepsilon$$

uniformly in  $n$  and  $\xi$ . Simplifying one more time, it is in fact sufficient (make a picture) to show that for any  $\varepsilon > 0$ , there exists  $k > 0$  such that

$$\phi_{[0,3n] \times [0,n]}^\xi(\exists k \text{ alternating closed/open vertical crossings}) \leq \varepsilon$$

uniformly in  $n$  and  $\xi$ . The final simplification is slightly more complicated. When conditioning on the existence of  $l-1$  alternating closed/open vertical paths, the event that there exists an additional (say open) crossing takes place in a random domain  $D_l \subset [0, 3n] \times [0, n]$  with boundary conditions  $\xi$  on  $\partial D_l \cap \partial[0, 3n] \times [0, n]$  and free boundary conditions elsewhere<sup>3</sup>. In particular, the boundary conditions are dominated by wired boundary conditions on  $\partial D_l \cap \partial[0, 3n] \times [0, n]$  and free elsewhere. Then, if  $k$  is large enough, we must find some  $l < k$  for which  $\partial D_l \cap \partial[0, 3n] \times [0, n]$  is very narrow (say smaller than  $2n/k$  on each side). Therefore, it is sufficient to prove that there exist  $c, \varepsilon > 0$ , such that the following holds true for every  $n$  and every  $a, b$  two sites on the bottom side at distance  $\varepsilon n$  of each others, and  $c, d$  two points on the top side at distance  $\varepsilon n$  of each other

$$\phi_{[0,3n] \times [0,n]}^{(ab),(cd)}[(ab) \leftrightarrow (cd)] \leq c < 1,$$

where boundary conditions  $(ab), (cd)$  are wired on  $(ab)$  and  $(cd)$  and free elsewhere.

<sup>3</sup>Since it boils down to conditioning on the right-most crossings on the left, and the left-most on the right

Note first that if  $(ab)$  and  $(cd)$  can be separated by a vertical line, the result is then easy, since duality and symmetry imply that the previous probability is smaller than  $1/2$  (we leave this as an easy exercise). Therefore, one can assume that  $(ab)$  and  $(cd)$  cannot be separated. Making the two intervals slightly bigger, we can even assume that they are on top of each others.

At the end, the following result would be sufficient to imply the theorem: there exists  $c < 1$  such that uniformly in  $n$ ,

$$\phi_{\mathbb{Z} \times [0, n]}^{[0, n/4], [in, (i+1/4)n]}([0, n/4] \leftrightarrow [in, (i + 1/4)n]) \leq c < 1.$$

Let  $c(n)$  be the probability that  $[0, n/4] \times [0, n]$  is dual-crossed horizontally.

CASE 1:  $c(n)$  IS SMALLER OR EQUAL TO  $1/2$ : The probability that  $[n/4, n/2] \times [0, n]$  and  $[-n/4, 0] \times [0, n]$  are dual crossed vertically is larger than  $1 - c(n) \geq 1/2$ , using duality and the comparison between boundary conditions. We deduce that with probability  $1/2^2$ ,  $[n/4, n/2] \times [0, n]$  and  $[-n/4, 0] \times [0, n]$  are dual-crossed vertically simultaneously. Conditionally on this event, boundary conditions in the area between the left-most dual crossing of  $[-n/4, 0] \times [0, n]$  and the right-most crossing of  $[n/4, n/2] \times [0, n]$  are dominated by the free/wired/free/wired boundary conditions on  $[-n/2, n/2] \times [0, n]$ . Therefore, the area between the two vertical dual-crossings is dual-crossed horizontally with probability larger than  $1/2$  using duality. Overall, we find that  $[0, n/4]$  and  $[in, (i + 1/4)n]$  are disconnected with probability  $1/8$  and the claim is proved.

CASE 2:  $c(n)$  IS LARGER THAN  $1/2$ : Define  $u_n$  in such a way that the probability to dual-cross  $[0, u_n] \times [0, n]$  horizontally equals  $1/2$ . Note that  $u_n \geq n/4$  by definition.

Consider the event  $B$  that  $[0, u_n] \times [0, n]$  is dual-crossed horizontally, and that  $[-u_n, 0] \times [0, n]$  and  $[u_n, 2u_n] \times [0, n]$  are dual-crossed vertically. This event has probability larger than  $1/8$  thanks to the FKG inequality and the definition of  $u_n$  (here again we used duality and the comparison between boundary conditions).

Condition on the left most closed path crossing  $[-u_n, 0] \times [0, n]$  and the top most closed path crossing  $[0, u_n] \times [0, n]$ . Following the proof of Proposition 4.8, the construction of a symmetric domain is then straightforward and we can say that with probability bounded away from 0, see Fig. 11.2, these two closed paths are dual-connected. Now, condition on the top/left-most connection between these two closed paths, and on the right-most closed vertical crossing of  $[u_n, 2u_n] \times [0, n]$ , one can once again construct a symmetric domain and prove that these closed paths are connected with positive probability. At the end, we constructed with positive probability a closed path from  $[-u_n, 0]$  to  $[u_n, 2u_n]$ . This path prevents the existence of a path between  $[0, n/4]$  and  $[in, (i + 1/4)n]$ , which concludes the proof.  $\square$

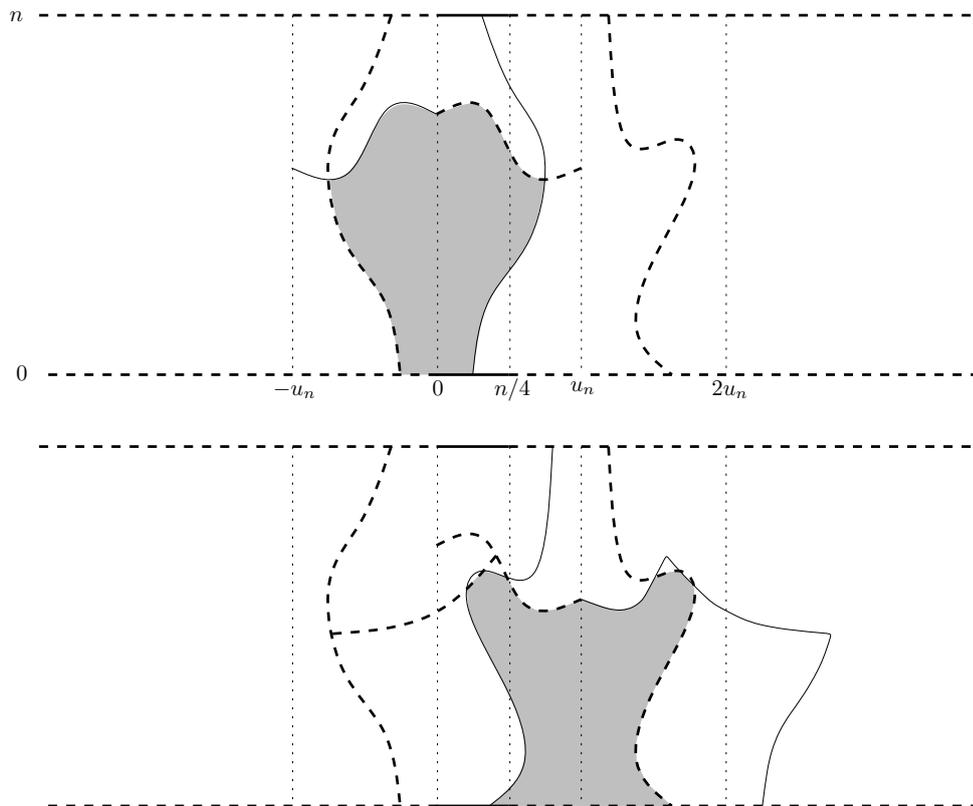


Figure 11.2: The two symmetric domains considered in the proof.

# Chapter 12

## Near-critical planar FK-Ising model

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**Abstract:** We study the near-critical FK-Ising model. First, a determination of the correlation length defined via crossing probabilities is provided. Second, a striking phenomenon about the near-critical behavior of FK-Ising is highlighted, which is completely missing from the case of standard percolation: in any monotone coupling of FK configurations  $\omega_p$  (e.g., in the one introduced in [Gri95]), as one raises  $p$  near  $p_c$ , the new edges arrive in a fascinating self-organized way, so that the correlation length is not governed anymore by the amount of pivotal edges at criticality. In particular, it is smaller than the heat-bath dynamical correlation length determined in the forthcoming [GP].

We also include a discussion of near-critical and dynamical regimes for general random-cluster models. For the heat-bath dynamics in critical random-cluster models, we conjecture that there is a regime of  $q$  values where there exist macroscopic pivots yet there are no exceptional times. These are the first natural models that are expected to be noise sensitive but not dynamically sensitive. This chapter is inspired by the article *The near-critical planar FK-Ising model*, written with Christophe Garban and Gábor Pete.

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**Near-critical regime and correlation length.** Beyond the understanding of the critical and non-critical phases (which was the subject of previous chapters), the principal goal of statistical physics is to study the phase transition itself, and in particular the behavior of macroscopic properties (for instance, the density of the infinite-cluster for  $p > p_c(q)$ ) near the critical point. It is possible to relate the critical regime to these thermodynamical properties via the study of the so-called *near-critical regime*. This regime was investigated in [Kes87] in the case of percolation. Many works followed afterward, culminating in a rather good understanding of dynamical and near-critical phenomena in standard percolation [SS10, GPS10a, NW09, GPS10b, GPS]. The goal of this chapter is to discuss the near-critical regime in the random-cluster case, and more precisely in the FK-Ising case.

The near-critical regime is the study of the random-cluster model of edge-parameter  $p$  in the box of size  $L$  when  $(p, L)$  goes to  $(p_c, \infty)$ . Note that, on the one hand, if  $p$  goes to

$p_c$  very quickly the configuration in the box of size  $L$  will look critical. On the other hand, if  $p$  goes to  $p_c$  (from above) too slowly, the random-cluster model will look supercritical. The typical scale  $L = L(p)$  separating these two regimes is called the **correlation length** (or **characteristic length**). In rough terms, if  $p$  is slightly above  $p_c(2) = \sqrt{2}/(1 + \sqrt{2})$ , the correlation length  $L(p)$  is the scale below which things still look somewhat *critical* and above which the infinite cluster starts being *visible*. In the subcritical regime, it corresponds to the scale above which the fact that  $p$  is subcritical becomes apparent.

**Definition of correlation length in the case of percolation ( $q = 1$ ).** The critical regime is often characterized by the fact that crossing probabilities remain strictly between 0 and 1. Formally, consider *rectangles*  $R$  of the form  $[0, n] \times [0, m]$  for  $n, m > 0$ , and translations of them. We denote by  $\mathcal{C}_v(R)$  the event that there exists a *vertical crossing* in  $R$ , a path from the bottom side  $[0, n] \times \{0\}$  to the top side  $[0, n] \times \{m\}$  that consists only of open edges. The celebrated Russo-Seymour-Welsh theorem shows that in the case of critical percolation, crossing probabilities of rectangles of bounded aspect ratio remain bounded away from 0 and 1. A natural way of describing the picture as being critical is to check that crossing probabilities are neither near 0 nor near 1. Mathematically, it is thus natural to define the correlation length for every  $p < p_c = 1/2$  and  $\varepsilon > 0$  as

$$L_\varepsilon(p) := \inf \{n > 0 : \mathbb{P}_p(\mathcal{C}_v([0, n]^2)) \leq \varepsilon\},$$

and, when  $p > p_c = 1/2$ , as  $L_\varepsilon(p) := L_\varepsilon(1 - p)$ , where  $1 - p$  is the dual edge-weight. The dependence on  $\varepsilon$  is not relevant, since it can be proved ([Kes87, Nol08]) that for any  $\varepsilon > 0$ ,

$$L_{1/4}(p) \asymp L_\varepsilon(p),$$

where  $\asymp$  means that there exist constants  $0 < A_\varepsilon, B_\varepsilon < \infty$  such that

$$A_\varepsilon L_{1/4}(p) \leq L_\varepsilon(p) \leq B_\varepsilon L_{1/4}(p).$$

The correlation length was shown to behave like  $|p - p_c(1)|^{-4/3+o(1)}$  in the case of percolation [SW01].

**Definition of the correlation length for FK-Ising ( $q = 2$ ).** The first result of this chapter is the determination of the behavior of  $L(p)$  when  $p$  goes to  $p_c$  for  $q = 2$ . Before stating the main result, let us give a proper definition of *correlation length* in this setting. Since the Russo-Seymour-Welsh theorem has been generalized to the FK-Ising case in Chapter 9, it is natural to characterize the *critical regime* once again by the fact that crossing probabilities remain strictly between 0 and 1. An important difference from the  $q = 1$  case is that one has to take into account the effect of *boundary conditions*:

**Definition 12.1** (Correlation length). *Fix  $q = 2$  and  $\rho > 0$ . For any  $n \geq 1$ , let  $R_n$  be the rectangle  $[0, n] \times [0, \rho n]$ .*

*If  $p < p_c(2)$ , for every  $\varepsilon > 0$  and boundary condition  $\xi$ , define*

$$L_{\rho,\varepsilon}^\xi(p) := \inf \{n > 0 : \phi_{p,2,R_n}^\xi(\mathcal{C}_v(R_n)) \leq \varepsilon\} .$$

If  $p > p_c(2)$ , define similarly

$$L_{\rho,\varepsilon}^\xi(p) := \inf \{n > 0 : \phi_{p,2,R_n}^\xi(\mathcal{C}_v(R_n)) \geq 1 - \varepsilon\} .$$

Our main result on the correlation length can be stated as follows.

**Theorem 12.2.** *Fix  $q = 2$ . For every  $\varepsilon, \rho > 0$ , there is a constant  $c = c(\varepsilon, \rho)$  such that*

$$c \frac{1}{|p - p_c|} \leq L_{\rho,\varepsilon}^\xi(p) \leq c^{-1} \frac{1}{|p - p_c|} \log \frac{1}{|p - p_c|}$$

for all  $p \neq p_c$  whatever the choice of the boundary condition  $\xi$  is.

Note that the left-hand side of the previous theorem has the following reformulation, which we state as a theorem of its own (this result is interesting on its own since it provides estimates on crossing probabilities which are uniform in boundary conditions away from the critical point):

**Theorem 12.3** (RSW-type crossing bounds). *For  $\lambda > 0$  and  $\rho > 0$ , there exist two constants  $0 < c_- \leq c_+ < 1$  such that for any rectangle  $R$  with side lengths  $n$  and  $m \in [\frac{1}{\rho}n, \rho n]$ , any  $p \in [p_c - \frac{\lambda}{n}, p_c + \frac{\lambda}{n}]$  and any boundary condition  $\xi$ , one has*

$$c_- \leq \phi_{R,p,2}^\xi(\mathcal{C}_v(R)) \leq c_+ .$$

The main ingredient of the proof of the latter theorem (and the most interesting one) is Smirnov’s fermionic observable. This observable is defined in Dobrushin domains (with a free and a wired boundary arc), and is a key ingredient in the proof of conformal invariance at criticality. Nevertheless, its importance goes much beyond that proof, in particular because it can be related to connectivity properties of the FK-Ising model. We study its properties away from the critical point, and estimate its behavior near the free arc of Dobrushin domains. It implies estimates on the probability for sites of the free arc to be connected to the wired arc. This in turn allows us to perform a second-moment estimate on the number of connections between sites of the free arc and the wired arc, therefore implying crossing probabilities in Dobrushin domains. All that remains is to get rid of Dobrushin boundary conditions (which is not as simple as one might hope) in order to obtain crossing probabilities with free boundary conditions.

In [CHI11], Chelkak, Hongler and Izyurov show that

$$\phi_{p_c,q=2}(0 \leftrightarrow \partial[-n, n]^2) \asymp n^{-1/8} \tag{12.1}$$

using conformal invariance techniques. Together with Theorem 12.2, this implies:

**Theorem 12.4.** *Assuming (12.1), there exists a constant  $c > 0$  such that if  $p > p_c(2)$ ,*

$$\phi_{p,2}(0 \leftrightarrow \infty) \geq c \left( \frac{|p - p_c|}{\log 1/|p - p_c|} \right)^{1/8} .$$

The result

$$\langle \sigma_0 \rangle_\beta^+ = \phi_{p,2}(0 \leftrightarrow \infty) \asymp |\beta - \beta_c|^{1/8},$$

as  $\beta > \beta_c$  tends to  $\beta_c$ , goes back to Onsager [Ons44]. Nevertheless, the proof of Theorem 12.4 is of some value, since the result of [CHI11] and the techniques in this chapter extend to isoradial graphs (with additional work) while Onsager's technology is restricted to the square lattice.

**The random-cluster model through its phase transition** The previous way to look at the near-critical regime may seem slightly artificial. It is more natural to study the random-cluster model through its phase transition by constructing a monotone coupling of random-cluster models with fixed cluster-weight  $q \geq 1$ . Then, properties of the monotone coupling (which can be thought of as a dynamics following the evolution of  $p$  between 0 and 1) near  $p_c$  will describe the near-critical regime.

In the case of standard bond percolation ( $q = 1$ ), such a monotone coupling simply consists of i.i.d. Uniform $[0, 1]$  labels on the edges, and a percolation configuration  $\omega_p$  of density  $p$  is the set of bonds with labels at most  $p$ . It is straightforward to interpret this coupling as an asymmetric dynamical percolation: starting from critical percolation at time zero, as time goes on, whenever the clock of a bond rings, we open that bond; we can also run time backwards and close the bonds that ring. Now, the question is: in this monotone coupling, how fast does the system enter the supercritical and subcritical regimes as  $p$  changes near  $p_c$ ?

This **near-critical window** in percolation was studied by Kesten in [Kes87], then by [BCKS01, Nol08, NW09, GPS10b, GPS]. It turns out that its size is governed by the expected number of macroscopically pivotal edges, *i.e.* edges having four alternative (dual-primal) open paths starting from them and going to macroscopic distance. In Subsection 2.1, we will describe in more detail the mechanism governing this near-critical window, but let us introduce roughly the main reasons for macroscopic pivots to govern the near-critical behavior. Let us set  $\alpha_4(n)$  to be the probability at criticality that an edge has four alternative dual-primal open paths going to a distance  $n$ . Getting from  $\omega_{p_c}$  to  $\omega_{p_c + \Delta p}$  in the box of size  $L$ , the system is moving out of stationarity, and roughly  $L^2 |\Delta p|$  edges are switched from closed to open. If one assumes that the configuration still looks critical at  $p_c + \Delta p$ , then the probability of an edge being pivotal is roughly  $\alpha_4(L)$  during the whole process and the number of pivotal edges which are switched between  $p_c$  and  $p_c + \Delta p$  is roughly  $L^2 \alpha_4(L) |\Delta p|$ . Now, Kesten proved the following **stability** result: as long as  $L^2 \alpha_4(L) |\Delta p| = O(1)$ , there are not much more pivotal points in  $\omega_p$  than at criticality, hence, despite the monotonicity of the dynamics, changes do not speed up significantly (compared to symmetric dynamical percolation), and the macroscopic geometry starts changing significantly only at  $L^2 \alpha_4(L) |\Delta p| \asymp 1$ . On the opposite,  $L^2 \alpha_4(L) |\Delta p| \gg 1$  means that we have really left the near-critical regime, *i.e.*, the window of size  $L$  has become well-connected since many closed pivotal points (they were preventing macroscopic open paths) have switched to open.

Summarizing, the scale at which the critical regime becomes the supercritical regime is given by  $L^2 \alpha_4(L) |\Delta p| \asymp 1$ . The same reasoning can be applied for the subcritical regime.

In particular, the correlation length is given (up to constants) by the relation

$$(L_\varepsilon(p))^2 \alpha_4(L_\varepsilon(p)) |p - p_c| \asymp 1. \quad (12.2)$$

In the previous equality, we did not specify  $\rho$  since it is irrelevant, see the discussion in the previous paragraphs. The main principle we shall extract from this discussion can be stated as follows:

**Phenomenon 12.5.** *In percolation ( $q = 1$ ), the near-critical behavior is governed by the amount of pivotal points at criticality.*

To our knowledge, it has been widely believed in the community that basically the same mechanism should hold in the case of random-cluster models. Namely, once we understand the geometry of the set of pivotal points, we may readily deduce information on its near-critical behavior. In fact, this is not the case.

Let us consider the case of the FK-Ising. It has been shown in [Gar11] that the critical FK-Ising probability  $\alpha_4^{FK}(n)$  for a site to be pivotal behaves like  $n^{-35/24+o(1)}$  when  $n$  goes to infinity. If pivotal points were governing the near-critical regime, the correlation length should satisfy

$$(L_\varepsilon^{FK}(p))^2 \alpha_4(L_\varepsilon^{FK}(p)) |p - p_c| \asymp 1$$

which would give

$$L_\varepsilon^{FK}(p) = |p - p_c(2)|^{-\frac{24}{13}+o(1)} \gg |p - p_c(2)|^{-1}. \quad (12.3)$$

This brings us to the following observation.

**Phenomenon 12.6.** *The correlation length in FK-Ising is much smaller than what the intuition coming from standard percolation ( $q = 1$ ) would predict. In other words, as one raises the parameter  $p$ , the supercritical regime appears “faster” than what would be dictated simply by the amount of pivotal edges at criticality: new edges arrive in a very non-uniform, self-organized manner. Pivotal edges are still an important aspect of the mechanism that governs the near-critical behavior, yet, as we shall discuss more in Sections 2 to 2.2, a striking **self-organized near-criticality** appears.*

Let us mention possible explanations for this phenomenon. First, there is a basic phenomenon in the FK( $p, q$ ) models for  $q \geq 2$  that is very relevant to the above discussion: the difference between the average densities of edges for  $p = p_c(q) + \Delta p$  and  $p = p_c(q)$  is not proportional to  $\Delta p$ , but larger than that, with an exponent given by the so-called *specific heat* of the model. (We will discuss this in more detail in Section 2.3.) A first guess could be that the discrepancy in (12.3) is a result of the fact that  $\Delta p$  is not the density of the new edges arriving, and this should have been taken into account in the computation using the pivotal exponent. Nevertheless, this is only partially right: the specific heat exponent itself is not large enough to account for this discrepancy (in fact, for  $q = 2$  it equals 0). In fact, a self-organizational mechanism kicks in. In standard percolation, when  $\Delta p \ll 1$ , on the way from  $\omega_{p_c}$  to  $\omega_{p_c+\Delta p}$  new points arrive in a “Poissonian” way. This is no longer the case with FK-Ising: the arriving edges tend to prefer “strategic” locations,

*i.e.*, edges which are pivotal at large scales. In other words, near  $p_c$ , the arriving edges depend in a very sensitive way on the current configuration. This subtle balance between the current configuration and the conditional law of the arriving edges is representative of a self-organized mechanism. In Sections 2 to 2.2, we will discuss the reasons for this self-organized mechanism and the consequences of this observation. Besides some facts that can be rigorously proved, most of the underlying self-organization scheme remains to be understood.

Another point of view on the problem, which might explain the discrepancy, is the fact that for  $q > 1$  Russo's formula has to be modified. The probability to be pivotal must be replaced by the influence of an edge on the event that a box of size  $n$  is crossed. This influence should then behave like  $n^{-\iota(q)}$  at criticality. Then, Kesten's scaling relation  $(2 - \xi_4(q))\nu(q) = 1$ , where  $\xi_4(q)$  and  $\nu(q)$  are the critical exponents of the pivotal events and the correlation length respectively, coming from (12.2) is still valid with the critical exponent  $\xi_4(q)$  replaced by the exponent  $\iota(q)$  governing the behavior of the influence. This subtlety and the fact that  $\xi_4(q) \neq \iota(q)$  seem to be new.

**On the dynamical and near-critical behavior for other values of  $q$ .** Finally, we investigate what happens for other values of  $q$ . Since the mathematical understanding of critical FK( $q$ ) models is very limited when  $q \notin \{0, 1, 2\}$ , the study relies on predictions from physics. We will mostly focus on the case  $q \in [1, 4]$ , where the FKG inequality holds and the phase transition is conjectured to be continuous (*i.e.* there is a unique infinite-volume measure at criticality, having no infinite cluster). It is therefore natural to consider the near-critical regime. In this case yet again, pivotal points do not seem to control the behavior of the near-critical regime. Critical exponents are indeed violating Kesten's relations.

We will also discuss briefly noise-sensitivity and dynamical sensitivity of random-cluster models with  $q \in [1, 4]$ . Indeed, the study of the near-critical regime of percolation (especially in [GPS10b, GPS]) was conducted in parallel to the study of the dynamical percolation. It is also possible to define dynamical critical random-cluster models and to study the influence of pivotal edges on the existence of exceptional times (times for which an infinite-cluster exists). As for the near-critical regime, the situation seems much more complicated than the percolation one. In particular, the existence of pivotal edges is not equivalent to the dynamical sensitivity of the model (it is expected to be equivalent to the noise sensitivity though). We refer to this section for further details on these phenomena.

Random-cluster models with  $q > 4$  are not as interesting as those with  $q \leq 4$ . Indeed, the phase transition is of first order and no near-critical regime exists. In addition, the models are not expected to be noise-sensitive or dynamically sensitive. The last subsection of this chapter is devoted to their study.

**Organization of the chapter.** The chapter is organized as follows. In the first section, we prove Theorems 12.3 and 12.2.

In the second section, we study the self-organized phenomenon in detail. We start with explaining why pivotal points are crucial in the understanding of the near-critical regime

of percolation. Then, we present Grimmett’s monotone coupling, allowing to follow the evolution of the random-cluster model through its phase transition. Finally we explain how the self-organized phenomenon acts concretely.

The third section contains a discussion of other values of  $q$ .

The last one mentions the interesting case of dynamical random-cluster models.

## 1 Proofs of the main results on the correlation length (Theorems 12.3, 12.2 and 12.4)

In this section, a point will be identified with its complex coordinate. We will be working with the fermionic observable  $F$ . Recall from Chapter 8 that the observable satisfies the following relations inside the domain:

**Proposition 12.7.** *Let  $p \in (0, 1)$  and  $X$  with four neighbors in  $G \setminus \partial G$ , we have*

$$\Delta_p F(e_X) = 0, \tag{12.4}$$

where the operator  $\Delta_p$  is defined by

$$\Delta_p g(e_X) := \frac{\cos[2\alpha]}{4} \left( \sum_{Y \sim X} g(e_Y) \right) - g(e_X). \tag{12.5}$$

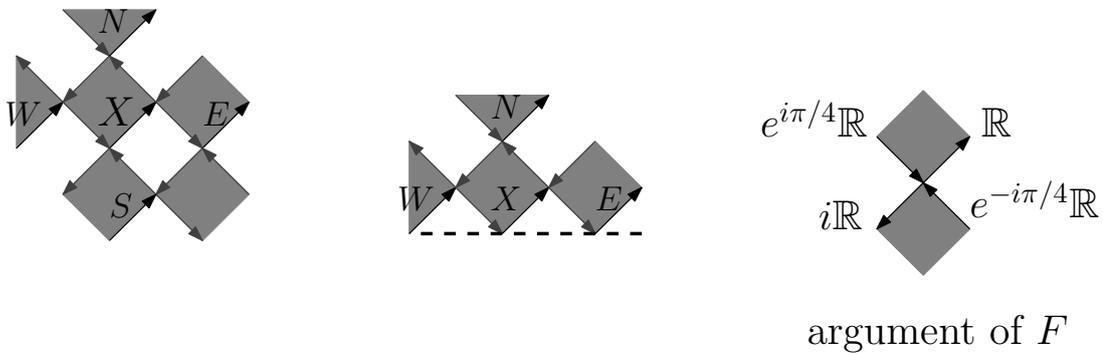


Figure 12.1: **Left:** An edge inside the domain: it has four edges oriented the same way at distance two. **Right:** An edge on the free arc with the associated indexation.

Observe that  $\alpha(p) = 0$  if and only if  $p = p_c$ . In this case, the observable is discrete harmonic inside the domain. As mentioned before, this is one of the main ingredients of Smirnov’s proof of conformal invariance: when properly rescaled, the observable converges to an harmonic map. Boundary conditions for  $F$  correspond to discretizations of the Riemann-Hilbert problem. These boundary conditions are quite complicated to study at a discrete level, and Smirnov used a discrete primitive  $H$  of the (imaginary part of)  $F^2$  to handle them. The function  $H$  was then solving an approximated Dirichlet problem.

In particular, the use of  $H$  made the estimation of  $F$  on the free arc possible. More precisely,  $F$  was related to the square root of modified harmonic measures (see Proposition 9.5). This fact was crucial in the proof of Theorem 9.1. Omitting the details, let us say that in 'Dobrushin domains'  $(G, a, b)$ , the probability at criticality for a site  $x$  on the free arc to be connected to the wired arc (which is the modulus of the observable, thanks to Lemma 5.9) is of the order of the square-root of the harmonic measure of the wired arc seen from  $x$ . Equivalently, for a dual site  $u$  on the wired arc, the probability of being dual-connected to the free arc is of order of the square-root of the harmonic measure of the free arc seen from  $u$ .

We will be using this fact for two nice infinite Dobrushin domains:

- The infinite strip  $S_n = \mathbb{Z} \times [0, n]$  of height  $n$ . Denote by  $\phi_{S_n, p}^{-\infty, \infty}$  the random-cluster measure with parameter  $p$ , free boundary conditions on the bottom and wired boundary conditions on the top. The probability at criticality for a dual-site on the top to be dual-connected to the free arc is of order  $1/\sqrt{n}$  (since the harmonic measure of the free arc is  $1/n$  via the Gambler's ruin).
- The upper half-plane  $\mathbb{H}$ . Denote by  $\phi_{\mathbb{H}, p}^{0, \infty}$  the random-cluster measure with parameter  $p$ , free boundary conditions on  $\mathbb{Z}_+ = \{0, 1, \dots\}$  and wired boundary conditions on  $\mathbb{Z}_- = \{\dots, -2, -1, 0\}$ . The probability at criticality for the dual site adjacent to  $-n$  to be dual-connected to the free arc is of order  $1/\sqrt{n}$  for the same reason as for the strip.

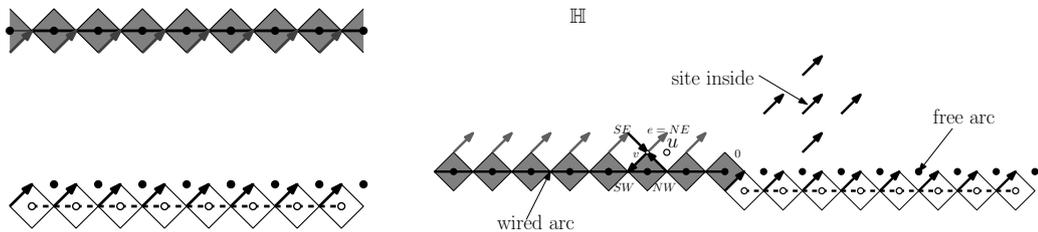


Figure 12.2: **Left:** The strip. **Right:** The upper half-plane. In this case,  $\tau$  is the hitting time of grey edges.

### 1.1 Integrability relations of the fermionic observable away from criticality

Away from the critical point, the primitive  $H$  is not available anymore. Nevertheless,  $F_p$  is massive harmonic inside the domain. In fact,  $F_p$  satisfies very explicit relations on the free arc of the domain as well. Precisely, Lemma 8.6 shows that

$$\Delta_p F_p(e_X) = \frac{\cos 2\alpha}{2(1 + \cos(\pi/4 - \alpha))} [F_p(e_W) + F_p(e_N)] + \frac{\cos(\pi/4 + \alpha)}{1 + \cos(\pi/4 - \alpha)} F_p(e_E) - F_p(e_X) = 0$$

for  $X$  on the free arc. When  $p = p_c$ , the sum of the coefficients on the right equals 0, which means that the observable has an interpretation in terms of reflected random-walks. This relates to the discretization of the Riemann-Hilbert boundary problem. It provides an alternative strategy to handle the scaling limit of the observable.

Away from criticality, we can also interpret these relations in terms of a random process. Define the Markov process with generator  $\Delta_p$ , which one can interpret as the random walk of a massive particle. We write this process  $(X_n^{(p)}, m_n^{(p)})$  where  $X_n^{(p)}$  is a random walk with jump probabilities defined in terms of  $\Delta_p$  — the proportionality between jump probabilities is the same as the proportionality between coefficients — and  $m_n^{(p)}$  is the mass associated to this random walk. The law of the random walk starting at an edge  $x$  is denoted  $\mathbb{P}_p^x$ . In order to simplify the notation, we drop the dependency in  $p$  in  $(X_n^{(p)}, m_n^{(p)})$  and simply write  $(X_n, m_n)$ . Note that the mass of the walk decays by a factor  $\cos 2\alpha$  at each step inside the domain (on the free arc, it decays by some constant that we do not explicit here).

Define  $\tau$  to be the hitting time of the wired arc, more precisely, of set  $\partial$  of medial edges pointing north-east and having one end-point on the wired arc (the grey edges in Fig. 12.2). The fact that  $\Delta_p F_p = 0$  for every edge  $x \notin \partial$  implies for any  $t \geq 0$

$$F_p(x) = \mathbb{E}_p^x[F_p(X_{t \wedge \tau})m_{t \wedge \tau}]. \quad (12.6)$$

Since  $m_\tau^p \leq 1$ ,  $F_p(X_{t \wedge \tau})m_{t \wedge \tau}$  is uniformly integrable and (12.6) can be improved into

$$F_p(x) = \mathbb{E}_p^x[F_p(X_\tau)m_\tau]. \quad (12.7)$$

This will be the principal tool in our study.

**Proposition 12.8.** *Let  $\lambda > 0$ . There exists  $C_1 = C_1(\lambda)$  such that for every  $n > 0$  and  $p_c \geq p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{S_n, p}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) \geq \frac{C_1}{\sqrt{n}}. \quad (12.8)$$

*There exists  $C_2 = C_2 > 0$  such that for every  $n > 0$  and  $p < p_c - \frac{C_2 \log n}{n}$ ,*

$$\phi_{S_n, p}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) \leq \frac{C_2}{n^4}. \quad (12.9)$$

**Proof** In both cases, we study the probability for a point on the free arc to be connected to the wired arc. In particular, Lemma 5.9 implies that quantities on the left of (12.8) and (12.9) are equal to  $|F(e_0)|$  (or  $F(e_0)$  in this case, since the winding is fixed on the boundary). Moreover, (12.7) allows us to write

$$\phi_{S_n, p}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) = F(e_0) = \mathbb{E}_p^0[F_p(X_\tau)m_\tau]$$

Yet, recall that  $F_p(X_\tau) = \phi_{S_n, p}^{-\infty, \infty}(X_\tau \overset{*}{\leftrightarrow} \mathbb{Z})$  by duality and Lemma 5.9 again. We deduce

$$\begin{aligned} \phi_{S_n, p}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) &= \mathbb{E}_p^0[\phi_{S_n, p}^{-\infty, \infty}(X_\tau \overset{*}{\leftrightarrow} \mathbb{Z})m_\tau] \\ &= \phi_{S_n, p}^{-\infty, \infty}(in \overset{*}{\leftrightarrow} \mathbb{Z})\mathbb{E}_p^0[m_\tau] \end{aligned}$$

Let us first deal with (12.8). Since  $p > p_c - \frac{\lambda}{n}$ ,  $m_p$  is larger than  $1 - c(\lambda/n)^2$  for some  $c > 0$ . We deduce that

$$\mathbb{E}^x[m_\tau] \geq C$$

for some  $C = C(\lambda)$ . In addition to this,

$$\phi_{S_n,p}^{-\infty,\infty}(in \overset{*}{\leftrightarrow} \mathbb{Z}) \geq \frac{C}{\sqrt{n}},$$

where we used the estimate of the probability at criticality for a dual site of the wired arc to be connected to the free arc, together with the fact that  $p < p_c$  implies that the dual model is supercritical. Plugging both inequalities together, we obtain (12.8).

Let us now turn to (12.9). When  $p < p_c - C(\log n)/n$ , we use the expansion of  $\alpha$  near  $p_c$  and  $\cos 2\alpha \leq 1 - c(\log n)/n$  (for some constant  $c = c(C)$ ) to deduce

$$\phi_{S_n,p}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) \leq \mathbb{E}_p^0[m_\tau] \leq \mathbb{E}_p^0[(1 - c_2(\log n)/n)^\tau] \approx n^{c'}$$

for  $c' = c'(C)$ . In order to conclude,  $c'$  can be chosen larger than 4 by tuning  $C$ .  $\square$

The previous proof of (12.8) was based on a comparison with the estimates at criticality: when  $p > p_c - \lambda/n$  the connection probabilities are of the same order as the critical ones. We push further this reasoning in the following proposition.

**Proposition 12.9.** *For any  $\lambda > 0$ , there exists  $C_3 = C_3(\lambda) > 0$  such that for every  $n > 0$  and  $p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{\mathbb{H},p}^{0,\infty}(n \leftrightarrow \mathbb{Z}_-) \geq \frac{C_3}{\sqrt{n}}. \tag{12.10}$$

Let us first prove a straightforward yet technical result. It should be compared to Lemma 5.9.

**Lemma 12.10.** *Let  $u$  be a dual vertex adjacent to the wired arc of  $\mathbb{H}$ ,*

$$F_p(e_u) \asymp \phi_{\mathbb{H},p}^{0,\infty}(u \overset{*}{\leftrightarrow} \mathbb{Z}_+),$$

where  $e_u$  is the edge pointing north-east and adjacent to  $u$ , and  $\asymp$  means that the ratio is uniformly bounded away from 0 and  $\infty$ .

**Proof** If  $v$  is the vertex of the medial lattice on the left of  $u$ , the relation around  $v$  (5.6) gives  $F(NW) + F(SE) = e^{i\alpha}(F(NE) + F(SW))$ , where edges are indexed with respect to **the direction they are pointing to** (see Fig. 12.2). Since we know the complex argument modulo  $\pi$  of the observable, we can project the relation on  $e^{-i\pi/4}\mathbb{R}$ . Then, the argument modulo  $2\pi$  of the observable at  $NW$  and  $SW$  is in fact determined, since the winding on the boundary is deterministic (it equals  $-\pi/2$  for  $NW$ , and  $-\pi$  for  $SW$ ). Therefore, we find

$$e^{i\pi/4}F(NW) - \cos(\pi/4 - \alpha)iF(SW) = \cos(\pi/4 + \alpha)F(NE).$$

Lemma 5.9 then implies

$$\begin{aligned} e^{i\pi/4}F(NW) &= |F(NW)| = \phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+) \\ iF(SW) &= |F(SW)| = \phi_{\mathbb{H},p}^{0,\infty}(u-1 \overset{\star}{\leftrightarrow} \mathbb{Z}_+). \end{aligned}$$

Using the fact that  $NE = e_u$ , we deduce

$$\cos(\pi/4 + \alpha)F(e_u) = \phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+) - \cos(\pi/4 - \alpha)\phi_{\mathbb{H},p}^{0,\infty}(u-1 \overset{\star}{\leftrightarrow} \mathbb{Z}_+).$$

Now,  $\phi_{\mathbb{H},p}^{0,\infty}(u-1 \overset{\star}{\leftrightarrow} \mathbb{Z}_+) \leq \phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+)$  thanks to the comparison between boundary conditions. We deduce

$$\frac{1 - \cos(\pi/4 - \alpha)}{\cos(\pi/4 + \alpha)} \phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+) \leq F(e_u) \leq \frac{1}{\cos(\pi/4 + \alpha)} \phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+)$$

which is the claim.  $\square$

We are now in a position to prove the proposition.

**Proof of Proposition 12.9** Fix  $n > 0$  and  $p \geq p_c - \frac{\lambda}{n}$  and denote the fermionic observable in  $(\mathbb{H}, 0, \infty)$  by  $F_p$ . Lemma 12.10 implies

$$F_p(n) = \mathbb{E}_p^n[F_p(X_\tau)m_\tau] \asymp \mathbb{E}_p^n[\phi_{\mathbb{H},p}^{0,\infty}(X_\tau \overset{\star}{\leftrightarrow} \mathbb{Z}_+)m_\tau].$$

We know that

$$\phi_{\mathbb{H},p}^{0,\infty}(u \overset{\star}{\leftrightarrow} \mathbb{Z}_+) \geq C_3/\sqrt{|u|},$$

thus implying

$$F_p(n) \geq C_4 \mathbb{E}_p^n[\phi_{\mathbb{H},p}^{0,\infty}(X_\tau \overset{\star}{\leftrightarrow} \mathbb{Z}_+) m_\tau] \geq C_4 C_3 \mathbb{E}_p^n[m_\tau^p/\sqrt{|X_\tau|}] \quad (12.11)$$

for two universal constants  $C_3, C_4 > 0$ . Therefore, it is sufficient to prove that  $|X_\tau|$  is not larger than  $n$  and that  $m_\tau$  is larger than some constant  $\varepsilon > 0$  with probability bounded away from 0 uniformly in  $n$ . The second condition can be replaced by the fact that  $\tau \leq n^2$  for instance.

First, note that it is sufficient to prove that  $X_t$  exits  $[0, 2n] \times [0, n]$  through  $[0, 2n] \times \{n\}$  in less than  $n^2/2$  steps with probability larger than some constant  $c > 0$  not depending on  $n$ . Indeed, the walk has then a uniformly positive probability to exit the domain in less than  $\frac{1}{2}n^2$  steps, and that  $|X_\tau| \leq n$ .

Consider  $(X_t)_{t \leq n^2/2} = (A_t, B_t)_{t \leq n^2/2}$  conditioned on the event that  $(X_t)_{t \leq n^2/2}$  visits the free arc less than  $n$  times. The probability that the first coordinate is less than  $n$  for every  $t \leq n^2/2$  is bounded away from 0 uniformly in  $n$  (since the number of visits of  $(X_t)$  to 0 is less than  $n$ ,  $A_t$  can be compared to a symmetric random walk with a deterministic drift of order  $rn$  for  $r < 1$ ). Now, conditioned on the visits of  $(X_t)$  to the free arc,  $(A_t)$  and  $(B_t)$  are independent. Thus,  $(B_t)$  is a reflected random walk at the origin conditioned on the fact that it does not visit 0 more than  $n$  times. In time  $n^2/2$ , it reaches height  $n$  with probability bounded away from 0 uniformly in  $n$ . The claim follows.  $\square$

### 1.2 Proof of Theorem 12.3

We first prove crossing probabilities in rectangles with specific boundary conditions. Then, we use these crossings to construct crossings in arbitrary rectangles with free boundary conditions.

**Crossing in rectangles with Dobrushin boundary conditions.** Let us first use the estimates obtained in the previous section to prove crossing probabilities in the strip and the half-plane. The proof follows a second moment argument.

**Proposition 12.11.** *Fix  $\lambda > 0$ . There exists  $C_6 = C_6(\lambda) > 0$  such that for every  $n > 0$  and every  $p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{S_n, p}^{-\infty, \infty}([-n, n] \leftrightarrow in + \mathbb{Z}) \geq C_6$$

and

$$\phi_{\mathbb{H}, p}^{-\infty, \infty}([3n, 4n] \leftrightarrow \mathbb{Z}_-) \geq C_6.$$

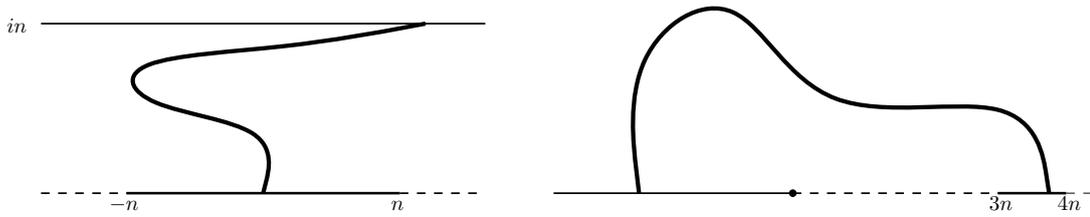


Figure 12.3: The two crossing events of Proposition 12.11.

**Proof** We present the proof for  $S_n$  (a similar argument works for  $\mathbb{H}$ ). Let  $N$  be the (random) number of sites on  $[-n, n]$  which are connected by an open path to  $in + \mathbb{Z}$ . Proposition 12.8 implies that

$$\phi_{S_n, p}^{-\infty, \infty}(N) = \sum_{x \in [-n, n]} \phi_{S_n, p}^{-\infty, \infty}(x \leftrightarrow in + \mathbb{Z}) \geq (2n + 1) \frac{C_1}{\sqrt{n}} \geq 2C_1 \sqrt{n}. \quad (12.12)$$

Moreover,

$$\phi_{S_n, p}^{-\infty, \infty}(N^2) \leq \phi_{S_n, p_c}^{-\infty, \infty}(N^2).$$

The right hand side is a quantity at the critical point and was already studied in Chapter 9 (in fact, only very related quantities were studied, but the generalization is straightforward). In particular, it was proved in this chapter that

$$\phi_{S_n, p_c}^{-\infty, \infty}(N^2) \leq C_6 n.$$

Cauchy-Schwarz thus implies that

$$\phi_{S_n, p}^{-\infty, \infty}([-n, n] \leftrightarrow in + \mathbb{Z}) \geq \phi_{S_n, p}^{-\infty, \infty}(N > 0) \geq 2C_1^2 / C_6$$

uniformly in  $n$ . □

It is now easy to reduce crossing probabilities in the strip and the half-plane to crossing probabilities in (possibly very large) rectangles. The idea is that a crossing cannot explore too much of the strip or the half-plane, since there exist slightly supercritical dual crossings preventing it.

**Proposition 12.12.** *Fix  $\lambda > 0$ , there exist  $C_7 > 0$  and  $M > 0$  such that for every  $n > 0$  and  $p_c > p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}([-n, n] \leftrightarrow in + \mathbb{Z}) \geq C_7$$

and

$$\phi_{[-Mn, Mn] \times [0, Mn], p}^{-Mn, 0}([3n, 4n] \leftrightarrow \mathbb{Z}_-) \geq C_7.$$

**Proof** As before, we do it in the case of the strip. Fix  $M$  large enough so that at criticality, the probability that there exists a vertical dual crossing with free boundary conditions of  $[n, Mn] \times [0, n]$  exceeds  $1 - C_6/3$  (use Theorem 9.1 to prove this fact). Then, with probability  $C_6/3$ , there will exist a crossing of  $[-n, n]$  to  $in + \mathbb{Z}$  and two dual vertical crossings in  $[n, Mn] \times [0, n]$  and  $[-Mn, -n] \times [0, n]$ . The domain Markov property and the comparison between boundary conditions imply the result.  $\square$

**Crossing in rectangles with free boundary conditions.** A consequence of Proposition 12.12 is the existence of crossings inside a box with free boundary conditions everywhere. Indeed, the previous result only deals *a priori* with domains where a part of the boundary is already wired but this condition can be removed.

**Proposition 12.13.** *Fix  $\lambda > 0$ . There exist  $C_8, M > 0$  such that for every  $n > 0$  and  $p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{[-Mn, Mn] \times [0, n], p}^0([-Mn, Mn] \times [0, n/2] \text{ is cross. vert.}) \geq C_8.$$

**Proof** Fix  $M$  so that the Proposition 12.12 holds true. Let  $A_n$  be the event that  $[-Mn, Mn] \times [0, n/2]$  is crossed vertically. We have for every  $n > 0$ ,

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(A_n) \geq C_7.$$

Let  $B_n$  be the event that  $[-Mn, Mn] \times [n/2, n]$  is dual crossed horizontally. Theorem 9.1 implies that

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(B_n | A_n) \geq c$$

for some constant  $c > 0$  uniform in  $n$  and  $p < p_c$ . Now,

$$\begin{aligned} \phi_{[-Mn, Mn] \times [0, n], p}^0(A_n) &\geq \phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(A_n | B_n) \\ &\geq \phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(A_n \cap B_n) \\ &= \phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(B_n | A_n) \cdot \phi_{[-Mn, Mn] \times [0, n], p}^{(i+M)n, (i-M)n}(A_n) \\ &\geq c \cdot C_7. \end{aligned}$$

□

We now prove that crossings of rectangles of any aspect ratio do exist.

**Lemma 12.14.** *Fix  $\lambda > 0$  and  $\kappa > 0$ , there exists  $C_9 = C_9(\kappa) > 0$  such that for every  $n$  and  $p > p_c - \frac{\lambda}{n}$ ,*

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([0, \kappa n] \times [0, n] \text{ is crossed horizontally}) \geq C_9.$$

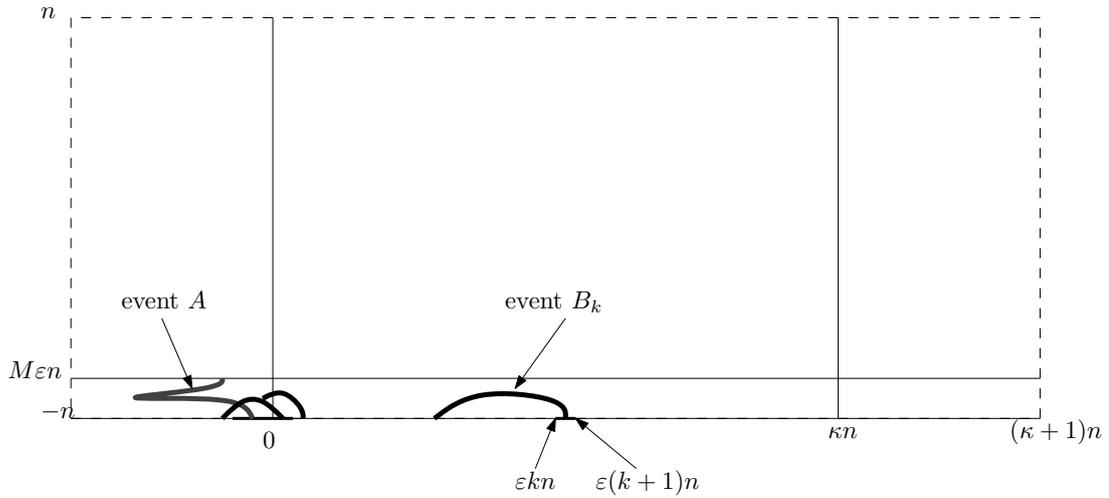


Figure 12.4: The intersections of events  $A$  and  $B_k$  create a crossing of the rectangle  $[-n, n] \times [0, \kappa n]$ .

**Proof** Fix  $M = M(\lambda)$  as in Propositions 12.12 and 12.13. Let  $\epsilon = 1/(2M)^2$ . Let  $A$  be the event that there exists a crossing from  $[-\epsilon n, \epsilon n]$  to  $iM\epsilon n + \mathbb{Z}$ . Now, let  $B_k$  be the event that there exists a path in  $\mathbb{Z} \times [0, M\epsilon n]$  from  $[(k+1)\epsilon n, (k+2)\epsilon n]$  to  $[(k-1)\epsilon n, k\epsilon n]$ . We have

$$\begin{aligned} & \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([0, \kappa n] \times [0, n] \text{ is crossed horizontally}) \\ & \geq \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0 \left( A \cap \bigcap_{k=0}^{\kappa/\epsilon-1} B_k \right) \\ & = \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(A) \prod_{k=0}^{\kappa/\epsilon-1} \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(B_k | A, B_r, r < k). \end{aligned}$$

Yet,

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(A) \geq \phi_{[-n, n] \times [0, n/(2M)], p}^0(A).$$

Now, the event  $A$  in  $[-n, n] \times [0, n/(2M)]$  correspond to the existence of a crossing from the bottom to the middle, but with the additional constraint that it starts between  $[-\varepsilon n, \varepsilon n]$ . The union bound and the comparison between boundary conditions implies that

$$\phi_{[-n, n] \times [0, n/(2M)], p}^0(A) \geq \varepsilon \phi_{[-n, n] \times [0, n/(2M)], p}^0([-n, n] \times [0, n/(4M)]) \geq \varepsilon C_8.$$

Moreover, since  $1/(4M) = M\varepsilon$ , we find

$$\phi_{[-n, n] \times [-n, (\kappa+1)n], p}^0(B_k | A, B_r, r < k) \geq \phi_{[(k-M)\varepsilon n, (k+M)\varepsilon n] \times [0, M\varepsilon n], p}^{k\varepsilon n, \infty}(B_k) \geq C_7$$

using the comparison between boundary conditions and Proposition 12.12. Altogether, we obtain that

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([-n, n] \times [0, \kappa n] \text{ is crossed horizontally}) \geq C_8 C_7^{\kappa/\varepsilon}.$$

□

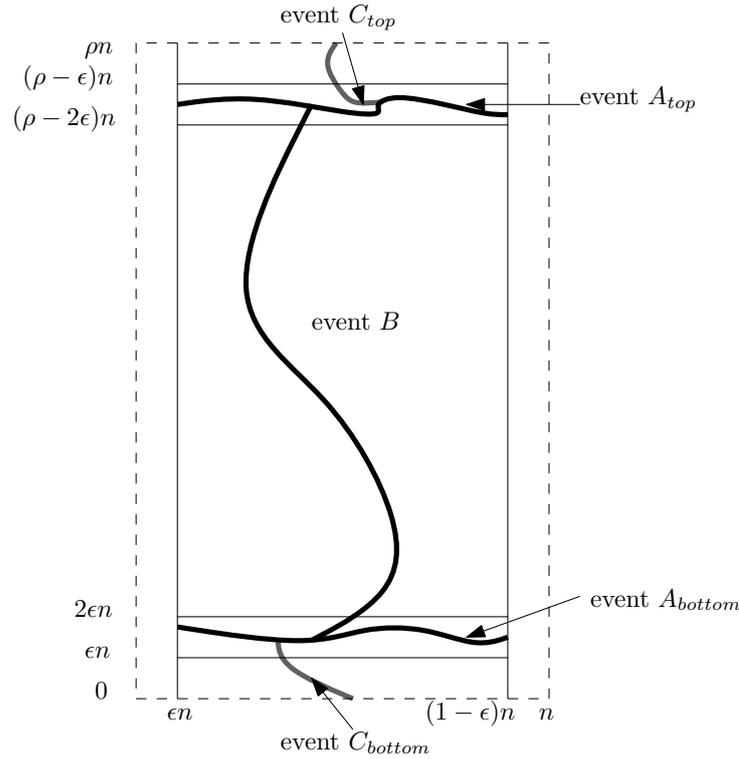


Figure 12.5: The five events involved in the proof of Theorem 12.3.

**Proof of Theorem 12.3** Fix  $\varepsilon < 1/(4M)$ . Let  $A_{bottom}$  and  $A_{top}$  be the events that  $[\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon, 2\varepsilon n]$  and  $[\varepsilon n, (1 - \varepsilon)n] \times [(\rho - 2\varepsilon)n, (\rho - \varepsilon)n]$  are crossed horizontally.

Let  $B$  be the event that  $[\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon n, (\rho - \varepsilon)n]$  is crossed vertically. Let  $C_{bottom}$  and  $C_{top}$  be the events that  $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$  and  $[\varepsilon n, (1 - \varepsilon)n] \times [(\rho - 2\varepsilon)n, \rho n]$  are crossed vertically. By Lemma 12.14, the events  $A_{bottom}$ ,  $A_{top}$  and  $B$  have probability bounded away from 0 uniformly in  $n$ . The FKG inequality implies that their intersection also has this property. Now, conditionally on  $A_{bottom}$ ,  $C_{bottom}$  has probability larger than the probability that there exists a crossing in  $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$  with wired boundary condition on the top and free boundary condition on the bottom. Proposition 12.12 implies that this probability is larger than  $C_7$  since  $(1 - 2\varepsilon)/(2\varepsilon) > 2M$  (the important thing is that the rectangle is  $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$  is wide enough). The same reasoning can be applied to  $C_{top}$  ergo the claim follows.  $\square$

### 1.3 Proofs of Theorems 12.2 and 12.4

Let us start with the following lemma:

**Lemma 12.15.** *There exists  $C_{11} > 0$  such that*

$$\phi_p(0 \leftrightarrow \partial[-n, n]^2) \leq C_{11}n^{-3} \quad (12.13)$$

for every  $n$  large enough and every  $p \leq p_c - C_{11} \frac{\log n}{n}$ .

**Proof** Equation 12.9 implies the existence of  $C_{10} > 0$  such that

$$\phi_{S_{n,p}}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) \leq \frac{C_{10}}{n^4}$$

for  $p \leq p_c - C_{10} \frac{\log n}{n}$ . The reasoning described in the proof of Theorem 8.1 applies here and gives

$$\phi_p(0 \leftrightarrow \partial[-n, n]^2) \leq \frac{C_{11}}{n^3}.$$

which implies readily the claim.  $\square$

**Proof of Theorem 12.2** Fix  $C_{11} > 0$  as defined in Lemma 12.15. Theorem 12.3 implies the lower bound trivially. For the upper bound, it suffices to show that for any  $\kappa > 0$ ,

$$\phi_{[-n, n] \times [-\kappa n, \kappa n], p}^1([-n, n] \times [-\kappa n, \kappa n] \text{ is crossed horizontally}) \rightarrow 0$$

whenever  $(n, p) \rightarrow (\infty, 0)$  with  $p \leq p_c - C_{11} \frac{\log n}{n}$ . Fix  $\varepsilon > 0$  and  $\kappa > 0$ .

Theorem 9.1 implies the existence of  $\delta > 0$  such that the probability that there exists a crossing of  $[-n, -(1 - 2\delta)n] \times [-\kappa n, \kappa n]$  with wired boundary conditions is smaller than  $\varepsilon/3$  for any  $p < p_{sd}$  and  $n > 0$ .

Define  $A_n$  to be the event that the annulus

$$S_n := [-n, n] \times [-\kappa n, \kappa n] \setminus [-(1 - \delta)n, (1 - \delta)n] \times [-(\kappa - \delta)n, (\kappa - \delta)n]$$

contains a close circuit surrounding the inner box. Note that there exists  $\eta > 0$  such that

$$\phi_{p,S_n}^1(A_n) \geq \eta$$

thanks to Theorem 9.1 again. Now, let  $B_n$  be the event that  $[-(1-\delta)n, (1-\delta)n] \times [-(\kappa-\delta)n, (\kappa-\delta)n]$  contains a cluster of diameter  $\delta n$ . Since  $A_n$  is decreasing and  $B_n$  depends only on edges inside  $[-(1-\delta)n, (1-\delta)n] \times [-(\kappa-\delta)n, (\kappa-\delta)n]$ , we obtain

$$\phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^1(A_n | B_n) \geq \phi_{p,S_n}^1(A_n) \geq \eta.$$

In particular,

$$\begin{aligned} \eta \phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^1(B_n) &\leq \phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^1(B_n \cap A_n) \\ &\leq \phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^1(B_n | A_n) \\ &\leq \phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^0(B_n). \end{aligned}$$

Lemma 12.15 and the definition of  $C_{11}$  implies that

$$\phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^0(B_n) \longrightarrow 0 \quad \text{when } n \rightarrow 0. \quad (12.14)$$

Therefore,  $\phi_{p,[-n,n] \times [-\kappa n, \kappa n]}^1(B_n) \rightarrow 0$ .

In order to conclude, notice that if the rectangle is crossed horizontally, then  $[-n, -(1-2\delta)n] \times [-\kappa n, \kappa n]$  or  $[(1-2\delta)n, n] \times [-\kappa n, \kappa n]$  are crossed horizontally, or  $B_n$  occurs. Since the first two events have probability less than  $\varepsilon/3$ , and the last one has probability less than  $\varepsilon$  for  $n$  large enough: it implies the claim readily.  $\square$

Let us now turn to the proof of Theorem 12.4. We have just proved that, for  $\rho > 0$  and  $\varepsilon > 0$ , there exists  $c = c(\varepsilon, \rho)$  such that for any  $n \geq \frac{c}{p_c - p} \log \frac{1}{p_c - p}$ ,

$$\phi_{p,[0,n] \times [0,\rho n]}^1(\mathcal{C}_h([0, n] \times [0, \rho n])) \leq \varepsilon.$$

The next lemma asserts that crossing probabilities in fact converge to 0 very quickly as soon as  $n$  is larger than the correlation length.

**Lemma 12.16.** *For any  $p < p_c$ , there exists  $L(p)$  such that*

$$\frac{c}{p_c - p} \leq L(p) \leq \frac{1}{c(p_c - p)} \log \frac{1}{p_c - p}$$

and

$$\phi_{p,[0,2^k L(p)] \times [0,2^{k+1} L(p)]}^1(\mathcal{C}_h([0, 2^k L(p)] \times [0, 2^{k+1} L(p)])) \leq e^{-2^k}$$

for any  $k \geq 0$ .

**Proof** For  $n > 0$ , let

$$u_n := \max \left\{ \phi_{p, [0, n] \times [0, 2n]}^1 \left( \mathcal{C}_h([0, n] \times [0, 2n]) \right), \phi_{p, [0, n]^2}^1 \left( \mathcal{C}_h([0, n]^2) \right) \right\}.$$

We are going to show that

$$u_{2n} \leq 25 u_n^2. \quad (12.15)$$

First, cutting vertically the domain  $[0, 2n]^2$  into two rectangles, together with comparison between boundary conditions, imply that

$$\phi_{p, [0, 2n]^2}^1 \left( \mathcal{C}_h([0, 2n]^2) \right) \leq \phi_{p, [0, n] \times [0, 2n]}^1 \left( \mathcal{C}_h([0, n] \times [0, 2n]) \right)^2 \leq u_n^2. \quad (12.16)$$

Second, cutting vertically the domain  $[0, 2n] \times [0, 4n]$  into two, together with comparison between boundary conditions again, imply that

$$\phi_{p, [0, 2n] \times [0, 4n]}^1 \left( \mathcal{C}_h([0, 2n] \times [0, 4n]) \right) \leq \phi_{p, [0, n] \times [0, 4n]}^1 \left( \mathcal{C}_h([0, n] \times [0, 4n]) \right)^2.$$

Now, consider the rectangles

$$\begin{aligned} R_1 &:= [0, n] \times [0, 2n] \\ R_2 &:= [0, n] \times [n, 3n] \\ R_3 &:= [0, n] \times [2n, 4n] \\ R_4 &:= [0, n] \times [n, 2n] \\ R_5 &:= [0, n] \times [2n, 3n] \end{aligned}$$

These rectangles have the property that whenever  $[0, n] \times [0, 4n]$  is crossed horizontally, at least one of the rectangles  $R_i$  is crossed (in the horizontal direction for  $R_1$ ,  $R_2$  and  $R_3$ , and vertically otherwise). We deduce, using the comparison between boundary conditions, that

$$\phi_{p, [0, n] \times [0, 4n]}^1 \left( \mathcal{C}_h([0, n] \times [0, 4n]) \right) \leq 5 u_n,$$

and hence

$$\phi_{p, [0, 2n] \times [0, 4n]}^1 \left( \mathcal{C}_h([0, 2n] \times [0, 4n]) \right) \leq (5 u_n)^2. \quad (12.17)$$

Combining (12.16) and (12.17), we obtain (12.15). Iterating that, we easily obtain that, for every  $k \geq 0$ ,

$$25 u_{2^k n} \leq (25 u_n)^{2^k}.$$

By Theorem 12.2, if  $p < p_c$  and  $n \geq \frac{c}{p_c - p} \log \frac{1}{p_c - p}$ , where  $c = \max\{c(1/100, 2), c(1/100, 1)\}$ , then  $u_n$  satisfies

$$25 u_n \leq 1/\varepsilon.$$

Therefore, the lemma follows for  $L(p) = \frac{c}{p_c - p} \log \frac{1}{p_c - p}$ .  $\square$

**Proof of Theorem 12.4** Fix  $p > p_c$ . Let

$$R_k := [0, L(p)2^k] \times [-L(p), L(p)(2^{k+1} - 1)] \quad \text{if } k \text{ is even,}$$

and

$$R_k := [0, L(p)2^{k+1}] \times [-L(p), L(p)(2^k - 1)] \quad \text{if it is odd.}$$

Define  $E_k$  to be the event that  $R_k$  is crossed in the 'long' direction. The FKG inequality implies that

$$\begin{aligned} \phi_p^0(0 \leftrightarrow \infty) &\geq \phi_p^0(0 \leftrightarrow \{L(p)\} \times [-L(p), L(p)]) \cdot \prod_{k \geq 0} \phi_p^0(E_k) \\ &\geq \frac{1}{4} \phi_p^0(0 \leftrightarrow \partial[-L(p), L(p)]^2) \cdot \prod_{k \geq 0} (1 - e^{-2^k}) \\ &\geq c(L(p))^{-1/8}, \end{aligned}$$

where  $c > 0$ . We used Lemma 12.16 to get the second line, and the lower bound of Theorem 12.2 and (12.1) to get the third inequality.  $\square$

## 2 Near-critical behavior: a fascinating self-organized near-criticality emerges

As promised, we start the discussion with the *near-critical regime* in standard percolation. Our goal here is to provide a self-contained explanation of the fact that the near-critical correlation length and the behavior of the percolation probability  $\theta(p)$  as  $p \searrow p_c$  are governed by the amount of pivotals at criticality. Then we explain why this picture must be flawed in the case of random-cluster models.

### 2.1 Pivotal points govern the near-critical regime of percolation

Recall that the correlation length is defined as follows: fix some small  $\varepsilon > 0$ . For any  $n \geq 1$ , let  $R_n$  be the  $[0, n] \times [0, n]$  square for bond percolation on  $\mathbb{Z}^2$ , and for any  $p = p_c + \Delta p > p_c$ , define

$$L_\varepsilon(p) = L(p) := \inf \{n \geq 1 \text{ s.t. } \mathbb{P}_p(\mathcal{C}_v(R_n)) \geq 1 - \varepsilon\},$$

where  $\mathbb{P}_p$  is the probability measure of bond-percolation with parameter  $p$ . Let us start with explaining the fact that things look supercritical above  $L(p)$ . For  $n \geq L(p)$ , the probability to have a left-to-right crossing in the box  $[0, n]^2$  is larger than  $1 - \varepsilon$  by definition. Russo-Seymour-Welsh theory implies that the probability to have a left-to-right crossing in the rectangle  $[0, 3n] \times [0, n]$  is greater than  $1 - g(\varepsilon)$ , where  $g(\varepsilon)$  goes to 0 as  $\varepsilon \rightarrow 0$ . Then, using well-known arguments, one can show that the ‘‘geometry’’ of  $\omega_p$  above  $L(p)$  stochastically dominates a certain **supercritical** percolation model of parameter  $1 - \phi(g(\varepsilon))$ , where  $\phi(x)$  goes to 0 as  $x \rightarrow 0$ . In this sense, things indeed look supercritical

above  $L(p)$ . This step would have been simpler if one had worked directly with long rectangles  $[0, 3n] \times [0, n]$  in the definition of  $L(p)$  instead of the symmetric  $R_n$ . However, the symmetry of  $R_n$  will be relevant to the explanation of the following "opposite" fact: things look critical below  $L(p)$ . We need to obtain RSW estimates for the dual percolation. By planar duality, if  $n < L(p)$ , then the probability to have a top-to-bottom **dual** crossing in  $R_n$  is greater than  $\varepsilon$ . Russo-Seymour-Welsh theory then implies that the probability to have a dual left-to-right crossing in  $[0, 3n] \times [0, n]$  is greater than  $\psi(\varepsilon)$  (where  $\psi(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ ). This means that at scales smaller than  $L(p)$  we do have the uniform crossing probabilities that make the configuration look critical.

**Goal:** How can one estimate  $L(p)$  as a function of  $|p - p_c|$ ?

As explained in the introduction, it is quite simple to guess what  $L(p)$  should be. Indeed, the quantity is naturally related to thermodynamical quantities of the model via *Kesten's scaling relations*:

**Theorem 12.17** ([Kes87], see also [Wer07, Nol08] for modern expositions). *For  $L(p) = L_\varepsilon(p)$ , one always has*

$$L(p)^2 \alpha_4(L(p)) \asymp \frac{1}{|p - p_c|}, \quad (12.18)$$

$$\theta(p) \asymp \alpha_1(L(p)), \quad (12.19)$$

where the constants in  $\asymp$  depend on  $\varepsilon > 0$ .

Let us sketch the proof now:

**Proof** Let us start with the first relation. In the following, let  $f_n$  denote the indicator function of the left-to-right crossing event in the domain  $R_n$ .

The intuition was already given in the introduction: for crossing events of scale  $n$  (i.e., for rectangles of diameter  $\asymp n$ ), there are  $\Theta(n^2 \alpha_4(n))$  points on average which are pivotal for the crossing event. Now, in the standard monotone coupling, from  $\omega_{p_c}$  to  $\omega_p$ , each of these pivotal points flips with probability of order  $|p - p_c|$ . Therefore, it is tempting to believe that as far as  $n^2 \alpha_4(n) |p - p_c| \ll 1$ , it is unlikely for the crossing event to change, while once  $n^2 \alpha_4(n) |p - p_c| \gg 1$ , many pivotal points are flipped and things should start being highly connected.

A few things are a bit "fishy" in this intuition: one of them is that one might have  $f_n(\omega_p) = 1$  and  $f_n(\omega_{p_c}) = 0$  together with the fact that from  $\omega_{p_c}$  to  $\omega_p$ , none of the initial pivotal points for  $f_n$  had been switched: the pivotal switch might happen at a point that was not pivotal originally. On the way from  $\omega_{p_c}$  to  $\omega_p$ , if one stayed at equilibrium as it is the case for example in dynamical percolation, one would still be able to conclude something based on such considerations, but one difficulty here is that as we follow the monotone coupling, we leave the "critical regime".

The nice idea from [Kes87] to overcome this near-critical bias is to apply Russo's formula simultaneously to the crossing event as well as to the **four-arm** event. Indeed,

for the crossing event, one can check that as long as  $n \leq L(p) = L_\varepsilon(p)$ ,

$$\begin{aligned} \frac{d}{dp} \mathbb{E}_p[f_n] &= \sum_{\text{sites } x} \mathbb{P}_p[x \text{ is pivotal}] \\ &\asymp n^2 \alpha_4^p(n). \end{aligned} \tag{12.20}$$

To go from the first line to the second one is in fact non-trivial: one needs to prove that below the correlation length, the main contribution in Russo’s formula comes from bulk points rather than boundary points. The technology involved here is quasi-multiplicativity and separation of arms. One can prove these even if we are not at the critical point, since as far as  $n \leq L(p)$ , one still has RSW estimates both for the primal and the dual model.

Now the key observation is the following one:

$$\begin{aligned} \left| \frac{d}{dp} \alpha_4^p(n) \right| &\leq \sum_x \mathbb{P}_p[x \text{ is pivotal for the 4-arm event}] \begin{cases} \text{using Russo's} \\ \text{formula again} \end{cases} \\ &\leq O(1) \alpha_4^p(n/3) \sum_{|x| \geq 2n/3} \mathbb{P}_p[x \text{ has the 4-arm event to distance } n/3] \\ &\leq O(1) \alpha_4^p(n) n^2 \alpha_4^p(n). \end{aligned} \tag{12.21}$$

The second inequality uses quasi-multiplicativity along with a dyadic summation to show that the main contribution arises from large-scale pivotal points. The third inequality uses quasi-multiplicativity again.

The combination of (12.20) and (12.21) implies that for all  $n < L(p)$ , the variation of  $p \mapsto \log(\alpha_4^p(n))$  is controlled (up to constants which depend only on  $\varepsilon$ ) by the variation of  $p \mapsto \mathbb{E}_p[f_n]$  which is of course bounded. This implies that

$$\alpha_4^p(n) \asymp \alpha_4(n), \tag{12.22}$$

where the constants involved in  $\asymp$  depend only on  $\varepsilon > 0$ .

Now integrating (12.20) and using (12.22), one can conclude about the correlation length: for  $n \leq L(p)$ ,

$$\begin{aligned} \mathbb{E}_p[f_n] - \mathbb{E}_{p_c}[f_n] &\asymp \int_{p_c}^p n^2 \alpha_4^u(n) du \\ &\asymp |p - p_c| n^2 \alpha_4(n). \end{aligned}$$

In particular, for  $n = L(p)$ , since  $\mathbb{E}_p[f_n] - \mathbb{E}_{p_c}[f_n] \asymp 1$ , we obtain our desired estimate (12.18).

The **near-critical stability** of the four-arm probability (12.22) also follows from [GPS]. The philosophy behind the previous argument is that, as far as few pivotal points are touched, percolation remains “critical”. In particular, critical exponents remain unchanged below  $L(p)$ . We have seen this in (12.22) for the case of  $\alpha_4(n)$ , but the same argument works for the one-arm event  $\alpha_1(n)$ : namely, for  $n < L(p)$ ,

$$\alpha_1^p(n) \asymp \alpha_1(n).$$

This immediately implies the second scaling relation:

$$\begin{aligned} \theta(p) &:= \mathbb{P}_p[0 \leftrightarrow \infty] \\ &\asymp \mathbb{P}_p[0 \leftrightarrow \partial B(0, L(p))] \left\{ \begin{array}{l} \text{since above } L(p), \text{ there} \\ \text{is a “dense” infinite cluster} \end{array} \right. \\ &= \alpha_1^p(L(p)) \\ &\asymp \alpha_1(L(p)). \end{aligned}$$

□

The knowledge of critical arm-exponents  $\xi_1(1)$  and  $\xi_4(1)$  for site-percolation on the triangular lattice  $\mathbb{T}$  allows for an estimation of  $L(p)$  and  $\theta(p)$ :

$$\begin{aligned} L_{\mathbb{T}}(p) &= (p - p_c)^{-4/3+o(1)} \\ \theta_{\mathbb{T}}(p) &= (p - p_c)^{5/36+o(1)} \end{aligned}$$

as  $p \searrow p_c = 1/2$ .

As we discussed at length in the introduction, the first scaling relation (12.18) does not hold in the FK-Ising case (see Phenomenon 12.6 and (12.3)). Nevertheless, note that the heuristic of the second scaling relation should be very general. Contrarily to the first scaling relation, it gives the right prediction for the correlation length. Indeed, the thermodynamical quantities  $L(p)$  and  $\theta(p)$  have pendants in the FK-Ising case. Onsager’s determination of the magnetization, together with the Edwards-Sokal coupling implies that

$$\theta_{\mathbb{Z}^2}^{\text{FK}}(p) := \phi_{p,2}(0 \leftrightarrow \infty) \asymp |p - p_c|^{1/8} \tag{12.23}$$

and

$$\phi_{p_c,2}(0 \leftrightarrow \partial[-n, n]^2) \asymp n^{-1/8}. \tag{12.24}$$

From these two relations, the second scaling relation (which does not harness any pivotal event) implies that the correlation length should behave like  $1/|p - p_c|$  for FK-Ising, which is the right prediction. Also note that (12.24) has been proved using conformal invariance techniques in [CHI11]. It would be interesting to make sense of the second scaling relation in the FK-Ising case in order to provide a derivation of the exponent 1/8 for the magnetization which would be independent of Onsager’s computation.

Let us conclude this paragraph by mentioning another result proved in [Kes87]. The correlation length is sometimes defined as the inverse rate of exponential decay of the connectivity probabilities in subcritical:

$$\frac{1}{L(p)} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_p((0, 0) \leftrightarrow (n, 0)).$$

Kesten proved that this definition is coherent with the definition in terms of crossing probabilities. Nonetheless, this correspondence is known only in the percolation case and

is not fully understood in the FK-Ising case. Furthermore, this definition is less tractable when studying near-critical regimes. It is therefore useless to consider existing estimates on the inverse rate of exponential for the FK-Ising model (see Chapter 8 *e.g.*) to prove that the correlation length defined as before behaves like  $|p - p_c|^{-1}$ .

## 2.2 The monotone increasing Markov process on cluster configurations

We would now like to understand the behavior of random-cluster models when  $p$  varies from 0 to 1. Let us first present a monotone coupling  $\mu$  between random-cluster models with fixed  $q \geq 1$ , which was first considered by Grimmett in [Gri95]. This section is not restricted to  $q = 2$  and applies for any  $q \geq 1$ .

It turns out that it is non-trivial to construct explicitly such a measure  $\mu$  (note that on the contrary the existence of abstract monotone couplings follows easily from a generalized Strassen's theorem). Instead of constructing explicitly a joint coupling  $\mu$ , Grimmett obtains this monotone coupling for a graph  $G = (V, E)$  as the invariant measure  $\mu$  of a natural Markov Process on the space  $\Omega := [0, 1]^E$ . This technique is usually employed to simulate a single Gibbs measure and we propose to provide this example first.

**Heat-bath dynamics** Assume we wish to simulate the random-cluster measure  $\phi_{G,p,q}^0$  on the graph  $G = (V, E)$ . First note that the random-cluster model with parameters  $(p, q)$  has the following property:

**Property 12.18.** *For any edge  $e = [xy]$ ,*

$$\phi_{G,p,q}^0(e \text{ is open} \mid \omega \text{ on } G \setminus \{e\}) = \begin{cases} p & , \text{ if } x \overset{\omega}{\longleftrightarrow} y \text{ in } G \setminus \{e\} \\ \frac{p}{p+(1-p)q} & , \text{ otherwise.} \end{cases}$$

This ‘‘almost local’’ rule for the conditional law of an edge  $e$  knowing the external environment enables us to consider the **heat-bath dynamics**:

**Definition 12.19 (Heat-bath dynamics or Sweeny algorithm).** *Let  $G = (V, E)$  be any finite graph. The random-cluster heat-bath dynamics on  $G$  is defined as follows: each edge  $e \in E$  is updated at rate 1 (the exponential clocks being independent) and when a clock rings at  $e$ , its status  $\omega(e)$  is resampled according to the conditional law given in Property 12.18.*

It is straightforward to check that this dynamics has the following properties: it is a reversible Markov chain with state space  $\{0, 1\}^E$  and its invariant measure is  $\phi_{G,p,q}^0$ . This dynamics has been studied both for theoretical reasons (see [Gri95, Gri06]) and practical ones (see [DGS07] for a good account of recent works). For instance, via the Edwards-Sokal coupling, it turns out to provide a faster way than classical Glauber dynamics to sample Ising models (at least in dimension  $d = 2$ ). More sophisticated algorithms are known for integer values of  $q$  (for example, the Swendsen-Wang algorithm), however the above dynamics has the advantage that it works for all real values of  $q$  and is probably more tractable for rigorous analysis.

**Grimmett’s dynamics and monotone coupling** Let us go back to Grimmett’s monotone coupling and briefly describe it (see [Gri95, Gri06] for a detailed exposition). Let  $G = (V, E)$  be a finite subgraph of  $\mathbb{Z}^2$  and  $\Omega$  be the space  $[0, 1]^E$ . Each  $Z \in \Omega$  decomposes into a monotone family of edge configurations  $(\omega_p)_{0 \leq p \leq 1} = (\omega_p(Z))_{0 \leq p \leq 1}$ , where for each  $p \in [0, 1]$  and any  $e \in E$ :

$$\omega_p(Z)(e) := 1_{Z(e) \leq p}.$$

The goal is to find a measure  $\mu = \mu_G$  on  $\Omega$  in a such a way that all the “projections”  $\omega_p(Z)$  with  $Z \sim \mu$  follow the random-cluster probability measure of parameters  $(p, q)$  on  $\{0, 1\}^E$  with free boundary conditions. It is not hard to see what this Markov process  $(Z_t)_{t \geq 0}$  should be. Assume that for all  $p \in [0, 1]$ , the projection  $\omega_p(Z_t)$  is also a Markov process and that its invariant measure is  $\phi_{G,p,q}^0$ , then it is natural to expect  $\omega_p(Z_t)$  to be the heat-bath dynamics that we defined previously. Indeed, if  $(Z_t)_{t \geq 0}$  is such that each edge is updated at rate one (the exponential clocks on each edge being independent), this means that **simultaneously** for all  $p \in [0, 1]$ , the law of the update at  $e$  needs to be compatible with Property 12.18. For any  $e = \langle x, y \rangle \in E$ , let  $\mathcal{D}_e \subset \{0, 1\}^E$  be the event that there is a path of open edges in  $E \setminus \{e\}$  connecting  $x$  and  $y$ . For any  $e \in E$  and any  $Z \in \Omega$ , define

$$T_e(Z) := \inf\{p \in [0, 1] \text{ s.t. } \omega_p(Z) \in \mathcal{D}_e\}.$$

Assume one is running the dynamics and that at a time  $t$  the edge  $e$  rings. Let  $Z_{t-}$  be the current configuration (before the update). Let  $\mathcal{U}_e$  be the random variable corresponding to the new label at  $e$  knowing  $Z_{t-}$ . In particular, we know the value of  $T = T_e(Z_{t-})$ . Since  $p \geq T$  is equivalent to  $\omega_p(Z_{t-}) \in \mathcal{D}_e$ , in order for  $Z_t$  to match the heat-bath dynamics on the projection  $\omega_p(Z_t)$  for all  $p$  simultaneously, conditionally on the value  $T = T_e(Z_{t-})$ , the update variable  $\mathcal{U}_e$  must satisfy

$$\mathbb{P}[\mathcal{U}_e \leq p] := \begin{cases} p & \text{if } p \geq T \\ \frac{p}{p+(1-p)q} & \text{if } p < T. \end{cases} \quad (12.25)$$

Fortunately,  $q \geq 1$  implies that this is a valid distribution function, hence we can simply define  $\mathcal{U}_e$  to be a sample from this distribution. Note that  $\mathcal{U}_e$  has an absolutely continuous part plus a **Dirac point mass** (for  $q > 1$ ) on  $T$ , namely  $[T - \frac{T}{T+(1-T)q}]\delta_T$ .

This discussion motivates the introduction of the Markov chain  $Z_t$  on the state space  $X_\Lambda = [0, 1]^{E(\Lambda)}$  where labels on the edges are updated at rate one according to the above conditional law (given by  $\mathcal{U}_e$ ,  $e \in E(\Lambda)$ ). This is precisely the dynamics that was considered by Grimmett in [Gri95] (see also [HJL02] where this dynamics was revisited).

Constructing an infinite-volume version of the previous dynamics is not straightforward. Nevertheless, one has the following asymptotic statement from [Gri95, Gri06] to light the way.

**Proposition 12.20** (Infinite Volume Limit [Gri95]). *For each  $n \geq 1$ , let  $\Lambda_n := [-n, n]^d$ . Let  $\xi$  be some initial configuration in  $X := [0, 1]^{E(\mathbb{Z}^2)}$ . Consider the random-cluster heat-bath dynamics  $Z_t^{\Lambda_n}$  on  $\Lambda_n$  with **free** boundary conditions and which starts from the initial*

state  $Z_0^{\Lambda_n} \equiv \xi_{|\Lambda_n}$ . Then, as  $n \rightarrow \infty$ , the process  $(Z_t^{\Lambda_n})$  weakly converges to a **Markov** process  $(Z_t^{\text{free}})_{t \geq 0}$  which starts from the initial configuration  $Z_0^{\text{free}} = \xi$ .

Furthermore, as  $t \rightarrow \infty$ ,  $Z_t^{\text{free}}$  weakly converges to an invariant measure  $\mu$  on  $X$ .

If, in the limiting procedure, one uses **wired** boundary conditions instead, one obtains at the limit a Markov process  $(Z_t^{\text{wired}})_{t \geq 0}$ . The processes  $Z_t^{\text{wired}}$  and  $Z_t^{\text{free}}$  might possibly have different transition kernels but they both have the same  $\mu$  as the **unique** invariant measure.

The underlying dynamics here is **non-Fellerian**, and the limiting Markov process in the above theorem is derived from the monotonicity properties inherent to the dynamics. In particular, the relationship between this Markov process and its formal generator (we will not write it down explicitly here) would need to be investigated. This seems to be a non-trivial task for the present dynamics. Therefore, we will not assume any explicit **transition rule** for the infinite-volume dynamics  $Z_t^{\text{free}}$  (or  $Z_t^{\text{wired}}$ ) and will restrict ourselves to the “compact case”.

**The Markov property of the monotone coupling** Let us prove that Grimmett’s coupling leads to a monotone increasing **Markov** process (as  $p$  varies) on the cluster configurations, *i.e.*, on the space  $\{0, 1\}^E$ . We will not rely on this Markovian property later on, nevertheless, it provides a nice picture of the self-organization scheme near  $p_c(q)$ . Namely, as one raises  $p$  near  $p_c$ , new edges arrive in a complicated fashion, yet depending only on the current configuration  $\omega_p$ .

**Proposition 12.21.** *Let  $G = (V, E)$  be a finite subgraph of  $\mathbb{Z}^2$ . Let  $Z$  be sampled according to  $\mu$ . Then the monotone family of projections  $(\omega_p(Z))_{0 \leq p \leq 1}$ , seen as a random process in the “time” variable  $p$ , is a non-decreasing **inhomogeneous Markov process** on the space  $\{0, 1\}^E$ .*

**Proof** We wish to prove that conditioned on the projections  $(\omega_u(Z))_{0 \leq u \leq p}$ , the conditional law of the higher configurations  $(\omega_u(Z))_{p \leq u \leq 1}$  depends only on  $\omega_p(Z)$ . To achieve this, it is enough to prove the Lemma 12.22 below. Before stating the lemma, we introduce some notations. For  $p \in [0, 1]$ , decompose the configuration  $Z$  into the triple  $(\omega_p, Z^{\leq p}, Z^{>p})$  defined as

$$\omega_p = \omega_p(Z_\Lambda); \quad Z^{\leq p} = \begin{cases} Z & \text{if } Z \leq p \\ 1 & \text{otherwise} \end{cases}; \quad Z^{>p} = \begin{cases} Z & \text{if } Z > p \\ 0 & \text{otherwise} \end{cases}.$$

Note that

$$\omega_p = \omega_p(Z^{\leq p}) = \omega_p(Z^{>p}), \tag{12.26}$$

and that  $Z$  can be recovered from the triple  $(\omega_p, Z^{\leq p}, Z^{>p})$ .

**Lemma 12.22.** *Conditioned on the value of the first component  $\omega_p$ , the other two components  $Z^{\leq p}$  and  $Z^{>p}$  are conditionally independent.*

**Proof of the lemma** Fix  $p \in [0, 1]$  and omit it from the notation  $\omega = \omega_p$  to make space for a time variable  $t$ .

We basically follow the construction of the measure  $\mu$  as the limiting measure of the Markov process  $Z_t$ , except that we divide the randomness used along the Markov chain into three components, the second and third being independent conditionally on the first one. Namely, define a Markov process

$$(\omega_t, Z_t^{\leq p}, Z_t^{> p})_{t \geq 0} \in \{0, 1\}^{E(\Lambda)} \times [0, 1]^{E(\Lambda)} \times [0, 1]^{E(\Lambda)},$$

where edges are updated at rate one, in such a way that the relations (12.26) between the three coordinates hold for all  $t \geq 0$ . To be consistent at  $t = 0$ , the process either from the empty state  $(\omega_0, Z_0^{\leq p}, Z_0^{> p}) \equiv (\mathbf{0}, \mathbf{1}, \mathbf{1})$  or the full state  $(\mathbf{1}, \mathbf{0}, \mathbf{0})$ , where  $\mathbf{0}$  and  $\mathbf{1}$  denote the vectors all 0 and all 1 respectively. Then, instead of sampling  $\mathcal{U}_e$  directly, let us proceed stepwise: first look whether  $\omega_{t-}$  satisfies  $\mathcal{D}_e$  or not. If it does, then let  $\omega_t(e) := 1$  with probability  $p$ . If  $\omega_{t-} \notin \mathcal{D}_e$ , then let  $\omega_t(e) := 1$  with probability  $p/(p + (1-p)q)$ . This is exactly the heat-bath dynamics for  $\phi_{G,p,q}^0$ . Note that this part of the dynamics does not use the two components  $(Z^{\leq p}, Z^{> p})$ .

Let us describe how to update the component  $Z_t^{\leq p}$ . If, after the update,  $\omega_t(e)$  equals 0, then we fix  $Z_t^{\leq p}(e) := 1$ . otherwise (if  $\omega_t(e) = 1$ ), we use the following variable:

$$T_e^{\leq p}(Z^{\leq p}) := \inf\{u \in [0, p] : \omega^u(Z^{\leq p}) \in \mathcal{D}_e\}.$$

Note that  $T_e^{\leq p}(Z^{\leq p}) = T_e(Z)$  on the event  $T_e(Z) \leq p$ . Otherwise (*i.e.*  $\omega_p(Z) \notin \mathcal{D}_e$ ), we set  $T_e^{\leq p} = p$ . In either case, it is important here that no information about the third component  $Z^{> p}$  has been used.

Next, recall the update random variable  $\mathcal{U}_e$  from the previous subsection (see (12.25)). It needed as an input the value of  $T_e(Z_{t-})$ . Let  $\mathcal{U}_e^{\leq p}$  be the same random variable here, with input the value of  $T_e^{\leq p}(Z_{t-}^{\leq p})$ . Remembering that we are in the case  $\omega_t(e) = 1$ , update the value of  $Z_t^{\leq p}$  as follows, independently of everything:

$$Z_t^{\leq p}(e) \sim \mathcal{L}[\mathcal{U}_e^{\leq p} | \mathcal{U}_e^{\leq p} \leq p],$$

where  $\mathcal{L}$  stands for the law of the variable. We define  $(Z_t^{> p})$  in the same fashion, using  $\mathcal{U}_e^{> p}$ . In particular, the evolutions of  $(Z_t^{\leq p})$  and  $(Z_t^{> p})$  are sampled out of the evolution of  $(\omega_t)$  plus some randomness in each case that are independent from each other, hence the conditional independence of  $Z^{\leq p}$  and  $Z^{> p}$  is satisfied.

To conclude the proof, one just has to notice that if one defines

$$Z_t := \begin{cases} Z_t^{\leq p} & \text{if } \omega_t(e) = 1 \\ Z_t^{> p} & \text{else,} \end{cases}$$

then  $(Z_t)_{t \geq 0}$  is exactly the Markov chain which was considered by Grimmett in [Gri95]. (This is not hard to check; an important feature here is that if  $T_e > p$ , then the conditional law  $\mathcal{L}[\mathcal{U}_e | \mathcal{U}_e \leq p]$  does not depend on the exact value of  $T_e$ , and a similar thing holds for  $\mathcal{U}_e^{> p}$  when  $T_e \leq p$ ). In particular, from [Gri95], it converges to the unique invariant

measure  $\mu_\Lambda$ , which inherits its conditional independence property. This finishes the proof of Lemma 12.22 and hence of Proposition 12.21.  $\square$

**Proposition 12.23.** *This Markovian property extends to the infinite volume limit  $\mu$  on  $X = [0, 1]^{E(\mathbb{Z}^2)}$ .*

Indeed, the same procedure works, but one has to be a bit careful with the initial state of our Markov chain: following [Gri95], with the slightly asymmetric projection convention we have chosen, we need to start from the full state  $(\omega_0, Z_0^{\leq p}, Z_0^{> p}) \equiv (\mathbf{1}, \mathbf{0}, \mathbf{0})$ . In the case  $q = 2$ , this is not very important since there is a unique infinite volume limit for all values of  $p$ . However for larger values of  $q$ , these considerations do matter. We will not enter in more detail here; see [Gri95, Gri06] for a detailed exposition on the infinite volume limit of  $Z$  together with its projection  $\omega_p(Z)$ .

### 2.3 Specific Heat of random-cluster model

The first non-trivial effect which occurs in the near-critical random-cluster model is the fact that the derivative of the edge-intensity blows up around  $p_c$ . This implies that edges appear much faster in the monotone coupling near  $p_c$  than it does for percolation. Let us study the FK-Ising ( $q = 2$ ) case. Define the edge-intensity function as follows: for all  $p \in [0, 1]$ , let

$$\mathcal{I}(p) := \phi_{p,2}(e \text{ is open}),$$

where  $e$  is any edge of  $\mathbb{Z}^2$ . It is not hard to check that at the critical (and self-dual) point  $p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}$ , one has

$$\mathcal{I}(p_c) = \frac{1}{2}.$$

A relevant quantity to us is the derivative in  $p$  of the edge-density  $d\mathcal{I}(p)/dp$ . It corresponds to the average rate at which new edges appear in any possible monotone coupling  $(\omega_{p,2})_{p \in [0,1]}$ . This quantity is linked to the so-called **specific heat** of the Ising model which measures the variance of the total energy  $H(\sigma)$  in the Ising model<sup>1</sup>. The relationship between these quantities is detailed in [GH]. From the results on the specific heat of the Ising model known since [Ons44, FF69], one obtains in the infinite volume case that, at  $p = p_c$ ,

$$\left. \frac{d}{dp} \right|_{p=p_c} \mathcal{I}(p) = \infty.$$

More precisely, one can extract the following logarithmic behavior around  $p_c$ :

$$\frac{d}{dp} \mathcal{I}(p) \sim a \log \frac{1}{|p - p_c|},$$

<sup>1</sup>Notably, the variance of the random variable  $H$  at a given inverse temperature  $\beta$  is also given by the derivative of  $\mathbb{E}_\beta[H]$  in  $\beta$ .

as  $p \rightarrow p_c$ . The finite-volume study of the specific heat ([Ons44, FF69]) leads to the following estimate: let  $\mathbb{T}_n$  be the torus  $\mathbb{Z}^2/n\mathbb{Z}^2$  and let  $p \mapsto \mathcal{I}_n(p)$  denote the edge-intensity for random-cluster model on  $\mathbb{T}_n$ , then

$$\frac{d}{dp} \Big|_{p=p_c} \mathcal{I}_n(p) \asymp \log n.$$

The extension to planar domains  $\Omega_n := \frac{1}{n}\mathbb{Z}^2 \cap \Omega$  will be carried out in [GH], based on the recent results from [Hon10] and [BdT10, BdT11].

In conclusion, these estimates show that in a window of size  $n$ , as one raises  $p$  near  $p_c$ , more edges will suddenly arrive. Nevertheless, the discrepancy is only logarithmic. It is not sufficient to explain the error in the power law of the correlation length. The reason for this matter is the existence of emerging clouds, which we discuss now.

## 2.4 Existence of emerging clouds

We are about to describe the main feature of the self-organized behavior that appears in the monotone coupling of random-cluster models. We restrict ourselves to the finite case, since the transition rule for the infinite volume Markov process  $(Z_t)_{t \geq 0}$  has not been established. Let then  $\Lambda$  be a finite box in  $\mathbb{Z}^2$  (or a torus  $\mathbb{Z}^2/n\mathbb{Z}^2$ ). **In this subsection, we fix  $q = 2$ .** The following proposition gives the first hint of some “non-linear” behavior:

**Proposition 12.24.** *For any  $N \geq 1$ , let  $(\omega_p(Z_\Lambda))_{p \in [0,1]}$  be a monotone coupling in the box  $\Lambda$ . The probability that clouds of at least  $N$  edges appear simultaneously in  $\omega_p(Z_\Lambda)$  at some  $p \in (0,1)$  converges to 1 when the size of the box  $\Lambda \nearrow \mathbb{Z}^2$ .*

This proposition is very easy to prove, yet one already sees here that the monotone Markovian coupling  $(\omega_{\Lambda,p})_{0 \leq p \leq 1}$  has a nature that is very different from the  $q = 1$  case.

**Proof** Let us consider the sets  $E_1, E_2$  and  $E_3$  in  $\Lambda$  (which is assumed to be large enough) as defined in Fig. 12.6.

Now let us sample  $Z_0 = Z^\Lambda t = 0$  according to the invariant measure  $\mu_\Lambda$ , and let us run the dynamics for a unit time. With positive probability, all edges in  $E$  are updated and their labels at time 1 satisfy the following: all labels in  $E_2$  are smaller than  $1/4$ , the edge  $e_0 = \langle (0,0), (1,0) \rangle$  gets a label in  $(1/4, 1/2)$ , and all other labels in  $E_1 \cup E_3$  are larger than  $3/4$ . Under such circumstances, all edges  $e \in E_1 \setminus \{e_0\}$  are such that  $T_e(Z_{t=1}) = Z_1(e_0)$ . It could be that this situation evolves later on, but we have that, with positive probability, none of the edges in  $E_2 \cup E_3 \cup \{e_0\}$  are updated from time 1 to time 2. Knowing this, again with positive probability, all edges in  $E_1$  are updated from time 1 to time 2 and all of them take exactly the value  $u := Z_1(e_0)$  (this is due to the Dirac mass  $\delta_u$  in the law  $\mathcal{U}_e$ ). Since we started at equilibrium,  $Z_{t=2}$  has the equilibrium law, and edges in  $E_2$  are all open or all closed in the projections of  $Z_{t=2}$ . This shows that with positive probability, at least  $N$  edges appear simultaneously as one raises  $p$ .

Now, if  $\Lambda$  is getting very large, we can divide the box into a lattice of  $2N \times 2N$  squares. Starting from  $Z_{\Lambda,0} \sim \mu_\Lambda$ , the above strategy works in each box independently

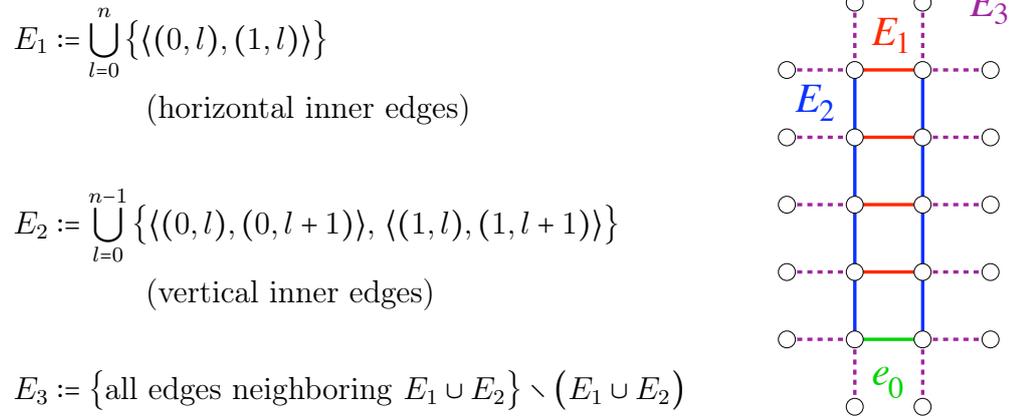


Figure 12.6: The definition of the sets  $E_1$ ,  $E_2$ ,  $E_3$  and the edge  $e_0 \in E_1$ .

of what happens in other boxes. Stated like that, it looks wrong, since obviously the dynamics itself is not independent from one square to another, but all that is needed in the above procedure is a positive lower bound on the probability that this “scenario” happens. Using the structure of the dynamics, it is not hard to see that if  $y_1, \dots, y_K$  denote the indicator functions of the events that the scenario happened in the squares  $i \in \{1, \dots, K\}$ , then there is an independent product of Bernoulli  $\varepsilon > 0$  variables which is stochastically dominated by our vector  $(y_1, \dots, y_K)$ . In particular, the emergence of clouds is somewhat ergodic in the plane.

By changing slightly the argument, one can show that there are such clouds for any open interval of the variable  $p \in [0, 1]$ . □

The above “naive” proof is not quantitative at all. Therefore, many natural questions on these **emerging clouds** remain: how do they look, how large are they, how does their law depend on the level  $p$  at which they appear? In particular, recovering the correlation length from such geometric considerations appears to be quite a challenging program. We shall discuss some of these questions in the next subsection.

**An intuitive explanation for the clouds** We end this subsection by a hand-waving argument why these clouds of simultaneously opening edges appear and may play an important role in the dynamics of any monotone coupling. Consider a monotone coupling  $(\omega_p, \omega_{p+\Delta p})$ . Due to the factor  $q^{\# \text{ clusters}}$  in the partition function, FK configurations  $\omega_p$  tend to have as many clusters as possible. Without this factor, one would be in the case of  $q = 1$ , *i.e.*, standard percolation, and the edge intensity would be exactly  $p$ . With  $q = 2$ , say, the random-cluster configuration tries to maximize the number of clusters, hence the edge-intensity drops to a smaller value  $\mathcal{I}(p) < p$ . In some sense, there is a fight between entropy (under the product measure  $p^{\# \text{ open edges}} (1-p)^{\# \text{ closed edges}}$ , most configurations have edge-intensity  $p$ ) and energy (which would correspond here to  $-\log(q^{\# \text{ clusters}})$ ). When one goes from  $p$  to  $p + \Delta p$ , new edges are added due to the entropy effect, but in such a way

that not so many clusters will merge into a single one. A good strategy for adding many edges without a significant increase in energy is the following **storing mechanism**. Say we have two “neighboring” large clusters in  $\omega_p$  with closed edges going from one to the other (these closed edges are then large-scale pivotal edges). Once we decide to open one of them, it does not cost more energy to open a few others.

Now, we have just seen that the monotone coupling is Markovian in  $p$ : in particular, the only way for this storing mechanism to actually happen is to have some values of  $p$  where the system can simultaneously open several edges. This indeed can happen, due to the atom in the update distribution, as shown in Lemma 12.24, and the construction there was indeed a simple example of edges arriving simultaneously between two neighboring large clusters (the two components of  $E_2$ ).

It is worth noticing that this heuristic explanation (based on entropy/energy considerations plus the Markov property) hints that this “non-linear phenomenon” should be much stronger near the critical point. Indeed, near  $p_c(q = 2)$ , there are many neighboring large clusters (i.e., many large scale pivotal points), which makes the storing mechanism more efficient. Away from criticality, this is not the case anymore. This intuition explains, for example, why we observe a blow-up of the derivative of the edge-intensity near  $p_c$ , and why the emerging clouds are more important there.

## 2.5 Questions on the structure of emerging clouds

**Finite volume case** To start with, let us define properly the notion of **cloud** for a finite graph  $G = (V, E)$ . Given a sample  $Z = Z_G \in [0, 1]^E$  from Grimmett’s monotone coupling  $\mu_\Lambda$ , for an edge  $e \in E$ , let  $\text{Cloud}(e)$  be the set of edges which appear simultaneously with  $e$ :

$$\text{Cloud}(e) := \{f \in E(\Lambda) \text{ s.t. } Z(f) = Z(e)\}.$$

The previous subsection shows that there are non-trivial clouds with positive probability. Let us consider the case of  $G = \Lambda_n := [-n, n]^2$  with free boundary conditions. (Another natural choice would be to consider discrete tori  $\Lambda_n := \mathbb{Z}^2/n\mathbb{Z}^2$ ). We strongly suspect the following behavior:

**Question 12.25** (Macroscopic clouds near  $p_c$ ). *For all  $n \geq 1$ , with  $\mu_{\Lambda_n}$ -probability at least a universal constant  $c > 0$ , there is at least one **macroscopic cloud** in  $\Lambda_n$ , i.e., whose diameter is larger than  $cn$ . Furthermore, with probability going to 1 as  $n \rightarrow \infty$ , the labels of such macroscopic clouds concentrate around the critical value  $p_c(q = 2)$ .*

To answer such a question, it is natural to run the dynamics at equilibrium (i.e.,  $Z_0^n \sim \mu_{\Lambda_n}$ ) for a short amount of time that is given precisely by the rescaling

$$\tau_n := \frac{1}{n^2 \xi_4(2)} = n^{-13/24+o(1)}.$$

Doing so, only finitely many macroscopic pivotal edges will be resampled, and it is easy to convince ourselves that with positive probability at least two of them will pick the same

label thus creating a macroscopic cloud. This intuition is close to being rigorous, since we have at our disposal a 'stability property' from the forthcoming [GP] which suggests that the "geometry" of  $Z_{t=\tau_n}^n$  could be recovered with high precision from  $Z_0^n$  plus the updates of the initially macroscopically important edges (neglecting the "smaller" updates). However, the stability result holds only for  $\omega_{p_c}$ , not the entire coupling  $Z$ . Thus, a certain control on the *concentration* of the labels around  $p_c$  would be helpful not only for the second part of Question 12.25.

The intuition that big clouds should appear only around the critical point can be translated into the following conjecture:

**Question 12.26** (Local clouds away from  $p_c$ ). *For any  $\delta > 0$ , emerging clouds with labels outside of  $(p_c - \delta, p_c + \delta)$  are **local** in the sense that the largest such cloud in  $\Lambda_n$  should be of logarithmic size.*

A natural way to attack this question would be via a **coupling** argument. Namely, prove that one has a coupling  $(Z_{\Lambda_n}^{\geq p_c + \delta}, \tilde{Z}_{\Lambda_n}^{\geq p_c + \delta})$  (see the notation in Subsection 2.2) whose marginals are  $\mu_{\Lambda_n}^{\geq p_c + \delta}$ , and whose coordinates are identical on a small neighborhood of the origin, but with probability at least  $\lambda^k$  (with  $\lambda \in (0, 1)$ ) they are independent of each other outside a box of size  $k$  (an exponential decay of correlations). Such a statement is doable for the supercritical (or subcritical) random-cluster model measure  $\phi_{\mathbb{Z}^2, p, q}$ , yet what makes it harder here is the lack of a "DLR (spatial Markov) property" for our monotone coupling  $\mu_{\Lambda_n}$ .

Finally, it would be interesting to prove some quantitative results on the size of the emerging clouds in the finite volume case ( $\Lambda_n$ ). See [GH], where this question is further discussed.

**Infinite volume case** We now move to the infinite-volume coupling where we basically know nothing about the emerging clouds. At least, the clouds are well-defined objects, since there is a unique limiting measure  $\mu_{\mathbb{Z}^2}$  of Grimmett's monotone coupling and for any  $e \in E(\mathbb{Z}^2)$ ,  $\text{Cloud}(e)$  can still be defined relatively to a sample  $Z$  from  $\mu_{\mathbb{Z}^2}$ . The first embarrassing question is the following one:

**Question 12.27.** *Prove that a.s. there exist non-trivial emerging clouds.*

This does not follow directly from the existence of non-trivial clouds in the finite volume case. Assuming the above question, the next natural question would be

**Question 12.28.** *Is it the case that emerging clouds are a.s. finite ?*

See [GH], where this question is discussed in more detail.

## 2.6 What about the influence of an edge?

Before moving on to the study of random-cluster models with other values of  $q$ , let us mention a natural approach to a "geometric" understanding of near-critical random-cluster

model. As a continuation of the work by Kesten on near-critical percolation [Kes87], Russo’s formula should be replaced by a slightly different formula. We already presented this fact in Chapter 4. Let us recall it now.

Fix an increasing event  $A$ . As in the case of percolation, the intuition suggests that the derivative of  $\phi_{G,p,q}^\xi(A)$  with respect to  $p$  is mostly governed by the influence of one single edge, switching from closed to open. The following definition is therefore natural in this setting. The (*conditional*) *influence* on  $A$  of the edge  $e \in E$ , denoted by  $I_A^p(e)$ , is defined as

$$I_A^p(e) := \phi_{G,p,q}^\xi(A|e \text{ is open}) - \phi_{G,p,q}^\xi(A|e \text{ is closed}).$$

With this notations, we have the following formula:

**Proposition 12.29** (See [Gri06]). *Let  $q \geq 1$  and  $\varepsilon > 0$ ; for any random-cluster measure  $\phi_{G,p,q}^\xi$  with  $p \in [\varepsilon, 1 - \varepsilon]$  and any increasing event  $A$ ,*

$$\frac{d}{dp} \phi_{G,p,q}^\xi(A) \asymp \sum_{e \in E} I_A^p(e),$$

where the constants in  $\asymp$  depend on  $q$  and  $\varepsilon$  only.

It is tempting to use this extension of Russo’s formula to see what our results on the correlation length (Theorem 12.2) may imply on the *influences*  $I_A^p(e)$ . To avoid boundary issues, let us consider the case of the torus  $\mathbb{T}_n := \mathbb{Z}^2/n\mathbb{Z}^2$ , and let  $A_n$  be the event that there is an open circuit with non-trivial homotopy in  $\mathbb{T}_n$ . It is easy to check (by self-duality) that  $\phi_{p_c,2}(A_n) \leq 1/2$ . The results from Section 1 can easily be generalized to the torus. In particular, there exists a constant  $\lambda > 0$  such that if  $p_n := p_c(2) + \lambda \frac{\log n}{n}$ , then

$$\phi_{p_n,2}(A_n) \geq 3/4.$$

Using the above Proposition 12.29, this says that

$$\int_{p_c}^{p_c + \lambda \frac{\log n}{n}} (I_{A_n}^p(e_{hor}) + I_{A_n}^p(e_{ver})) dp \geq \Omega(1) \frac{1}{n^2},$$

where  $e_{hor}$  and  $e_{ver}$  are any horizontal and vertical edges in  $\mathbb{T}_n$ . Since it is natural to expect that on the interval  $[p_c, p_c + \lambda \frac{\log n}{n}]$ , *influences* behave reasonably smoothly, the following conjecture should hold.

**Conjecture 12.30.** *For any  $n \geq 1, \lambda > 0, p \in [p_c - \frac{\lambda}{n}, p_c + \frac{\lambda}{n}]$  and any  $e \in \mathbb{T}_n$ ,*

$$I_{A_n}^p(e) \geq c \frac{1}{n \log n},$$

where  $c = c(\lambda)$  is some positive constant.

In fact since it is reasonable to conjecture that in Theorem 12.2, one has actually  $L_{\rho,\varepsilon}^\xi(p) \asymp |p - p_c|^{-1}$ , one may strengthen the previous conjecture into the following one:

**Conjecture 12.31.** For any  $n \geq 1, \lambda > 0, p \in [p_c - \frac{\lambda}{n}, p_c + \frac{\lambda}{n}]$  and any  $e \in \mathbb{T}_n$ ,

$$c \frac{1}{n} < I_{A_n}^p(e) < c^{-1} \frac{1}{n},$$

where  $c = c(\lambda)$  is some positive constant.

These conjectures are beyond reach with the techniques of the present paper.

### 3 Other values of $q$

As promised in the introduction, most of this section will rely on predictions from physics to investigate what happens for general  $q \in [1, 4]$ .

#### 3.1 Some useful critical exponents

Let us start with collecting several useful exponents.

- $\xi_1(q)$  denotes the *one-arm* exponent
- $\xi_4(q)$  denotes the *four-arm* exponent
- $\alpha = \alpha(q)$  describes the behavior of the **specific heat** near  $p_c(q)$ . That is,  $C^{\text{FK}_q}(p) \approx |p - p_c(q)|^{\alpha(q)}$ . We will come back to its interpretation in Subsection 3.2.
- $\beta = \beta(q)$  describes the behavior of the “magnetization”: this can be interpreted as

$$\phi_{p,q}(0 \leftrightarrow \infty) \approx (p - p_c(q))^{\beta(q)},$$

as  $p \rightarrow p_c(q)^+$ .

- $\nu = \nu(q)$  corresponds to the correlation length:  $L^{\text{FK}_q}(p) \approx |p - p_c(q)|^{-\nu(q)}$ .
- $\eta = \eta(q)$  corresponds to the correlation function  $\mathbb{P}_{p_c(q)}[x \leftrightarrow y] \approx |x - y|^{-\eta(q)}$ . In particular, assuming RSW, this exponent is twice the *one-arm* exponent  $\xi_1(q)$ .

Let us summarize the physics predictions on these exponents in the following table. The expressions are simplified using the term  $u = u(q) = \frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2}) = 2 - \frac{8}{\kappa(q)}$ .

| Exponents            | predictions                                  |
|----------------------|--|
| $u = u(q)$           | $\frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2})$  |
| $\alpha = \alpha(q)$ | $\frac{2(1-2u)}{3(1-u)}$                     |
| $\beta = \beta(q)$   | $\frac{1+u}{12}$                             |
| $\nu = \nu(q)$       | $\frac{2-u}{3(1-u)}$                         |
| $\eta = \eta(q)$     | $\frac{1-u^2}{2(2-u)}$                       |
| $\xi_1 = \xi_1(q)$   | $\frac{1-u^2}{4(2-u)}$                       |
| $\xi_4 = \xi_4(q)$   | $\frac{5}{2} - \frac{3}{4}u - \frac{1}{2-u}$ |

Most of the previous critical exponents can be found, for example, in [Wu82], except the *four-arm* or *pivotal* exponent which is more of a geometric nature. This latter exponent is computed in [Gar11] using  $SLE_\kappa$  calculations and assuming the (conjectured) correspondence

$$\kappa = \kappa(q) := \frac{4\pi}{\arccos(-\frac{\sqrt{q}}{2})}.$$

This SLE exponent was also derived by Wendelin Werner [Wer09a]. Assuming a proof of **conformal invariance** for the critical random-cluster model with parameter  $q$  as well as a proof of **quasi-multiplicativity** on the discrete level, this would say that

$$\alpha_4^{\text{FK}_q}(n) = n^{-\xi_4(q)+o(1)}.$$

Such an estimate is of course far from reach at the moment, except for  $q = 1$  (see [SW01]) and  $q = 2$  (see [Gar11]), but in this section we will assume that it holds.

### 3.2 Near-critical behavior for $q \in (1, 4]$ and self-organized monotone coupling

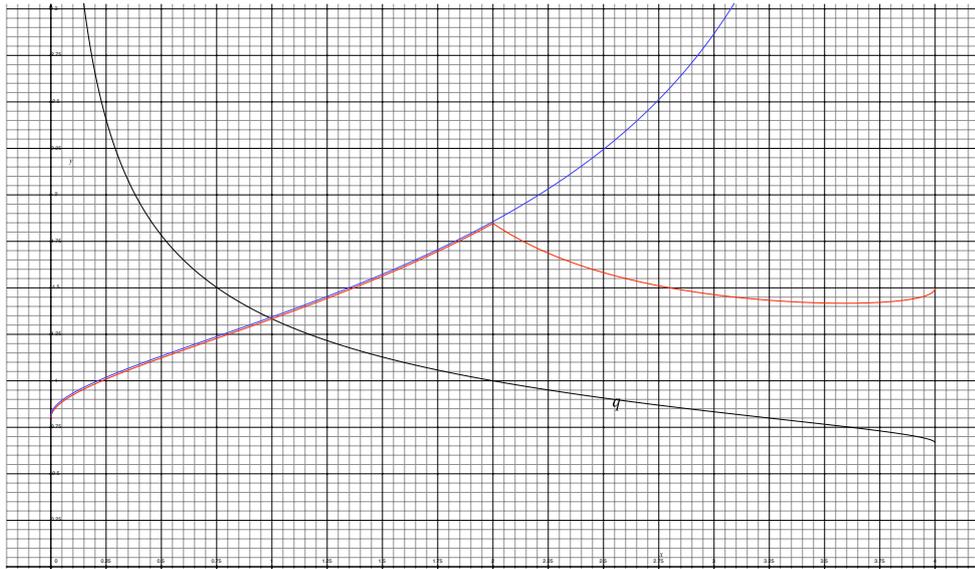


Figure 12.7: The blue curve corresponds to what the correlation length exponent in the critical random-cluster model with parameter  $q$  would be if the monotone coupling happened to be “Poissonian”. The red curve is a refinement of the blue one where the Specific Heat is taken into account. Finally, the black curve represents the actual correlation length exponent.

Let us consider the random-cluster model on  $\mathbb{Z}^2$  with fixed cluster-weight  $q \in (1, 4]$  (we drop it for several notations). In this subsection, our goal is to illustrate that there is a

strong self-organized mechanism within this monotone Markov process which goes beyond the Specific Heat effect. To show this, we will take for granted the Specific Heat exponent (which describes the critical blow up of the edge-intensity), and based on this, we will estimate what would be the correlation length exponent if there was no self-organized mechanism (i.e., if new edges simply arrived in a Poissonian way).

Let us first describe our setup: we will restrict ourselves to a finite but very large window  $\Lambda_n := [-n, n]^2$  with, say, wired boundary conditions. Now, starting from a critical random-cluster configuration  $\omega_{p_c}$  in  $\Lambda_n$ , we raise  $p = p_c + \Delta p$  until macroscopic effects start being non-negligible. If  $p_0$  is the value where we stop, we should thus obtain the relation  $L(p_0) \asymp n$ .

Now, the specific heat exponent  $\alpha = \alpha(q)$  has the following interpretation when  $q \geq 2$ :

$$\frac{d}{dp} \mathcal{I}^{\text{FK}_q}(p) \asymp \left( \frac{1}{|p - p_c(q)|} \right)^{\alpha(q)},$$

where  $\mathcal{I}^{\text{FK}_q}(p) := \phi_{\mathbb{Z}^2, p, q}(e \text{ is open})$  is the random-cluster edge-intensity. From this result (which is at the level of a prediction in the physics literature), it is reasonable to expect that in the finite volume range, one has

$$\frac{d}{dp} \mathcal{I}_n^{\text{FK}_q}(p) \asymp n^{\alpha(q)},$$

as far as  $n \lesssim L(p)$  (where  $\mathcal{I}_n^{\text{FK}_q}(p)$  denotes the edge-intensity in  $\Lambda_n$  with, say, wired boundary conditions). This is known in the case  $q = 2$  (where  $\alpha(2) = 0$  with logarithmic blow-ups), where the finite-size behavior matches with the infinite volume one as we have seen in Section 2.3 (see [FF69] for more precise results).

On the other hand, when  $q \in [1, 2)$ , the specific heat exponent  $\alpha(q) \in [-2/3, 0)$  in some sense measures the second order variation of the partition function of random-cluster model around  $(p, q) = (p_c(q), q)$ . In this case, there is no blow-up of the derivative of the edge-intensity, and one has

$$\frac{d}{dp} \mathcal{I}_n^{\text{FK}_q}(p) \asymp 1,$$

as far as  $n \lesssim L(p)$ . In particular, the specific heat exponent in our analysis will play a role only for  $q \in [2, 4]$ .

Let us then start from a critical configuration  $\omega_{p_c}$  in  $\Lambda_n$  (with wired conditions) and let us raise  $p$  to the level  $p = p_c + \Delta p$  in such a way that one still has  $n \lesssim L(p)$ . From the above discussion, one expects that about  $n^2 \Delta p n^{\alpha(q) \wedge 1}$  new edges will arrive. If we assume the absence of self-organization, *i.e.*, if edges arrive more or less independently of the current configuration (except possibly a local rate which would depend on whether the endpoints of the edge are connected or not), then each of these arrivals should be macroscopic pivotal flips with probability about  $n^{-\xi_4(q)}$  (we implicitly harnessed the fact that the pivotal exponent does not vary below the critical length). Therefore, at  $n \approx L(p)$ , we expect

$$n^2 \Delta p n^{\alpha(q) \wedge 1} n^{-\xi_4(q)} \approx 1.$$

Let  $L^{\text{Poiss}}(p)$  denote the correlation length obtained via the above analysis. We find

$$L^{\text{Poiss}}(p) \approx \left( \frac{1}{|p - p_c|} \right)^{\frac{1}{2 - \xi_4(q) + \alpha(q)\wedge 1}},$$

which is represented as a function of  $q$  by the red curve in Figure 12.7 (the blue curve represents the result of the same analysis when specific heat blow-ups are not taken into account).

As one can see from Figure 12.7, the actual correlation length  $L(p)$  is much smaller than  $L^{\text{Poiss}}(p)$  when  $q \in (1, 4]$ , which reveals for all these random-cluster models non-trivial self-organized schemes as  $p$  increases near  $p_c(q)$ .

## 4 Noise versus dynamical sensitivity in the heat-bath dynamics

We end this paper with a slightly tangential discussion on the heat-bath dynamics for the random-cluster models with  $q > 1$ . A rigorous treatment of the special case  $q = 2$  will be presented in [GP]. The purpose of the first subsection is to highlight the following interesting phenomenon:

**Phenomenon 12.32** (conjectural). *There seem to exist “natural” critical two-dimensional systems (i.e., scale-invariant and so on), which have the property that they have pivotal points at all scales (hence are expected to be noise sensitive), but for which there are no exceptional times of infinite clusters along the natural heat-bath dynamics (hence are not dynamically sensitive).*

The second subsection is a rigorous treatment of the case  $q > 25.72$  (which should then also hold for all  $q > 4$  but is yet conjectural).

### 4.1 On the dynamical sensitivity of random-cluster models when $q \in (1, 4]$

The (conjectural) Phenomenon 12.32 is rather surprising since so far it was believed that noise sensitivity and dynamical sensitivity (i.e. the existence of exceptional times with an infinite cluster) were more or less equivalent. One can see this phenomenon as another illustration of the fact that a good understanding of the structure of the set of pivotal points is by far not enough to answer questions on noise sensitivity or dynamical sensitivity. Recall that all the current proofs of noise sensitivity and dynamical sensitivity for critical percolation ( $q = 1$ ) rely on the Fourier Spectrum of percolation which is a very different object compared to the pivotal points (see [GPS10a]). In fancier words, the Fourier Spectrum of random-cluster models should still be concentrated on “high frequencies” (once projected on a well-chosen orthogonal basis which would depend on

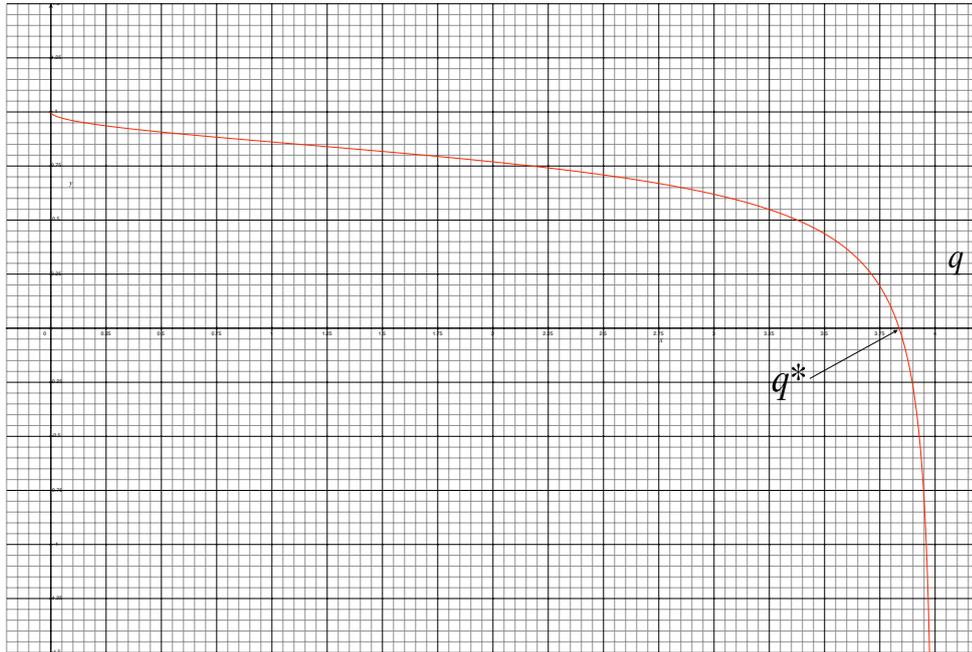


Figure 12.8: The red curve represents the upper bound on the Hausdorff dimension of exceptional times for random-cluster models,  $q \in [1, 4]$  which is obtained heuristically in this subsection. It highlights an interesting transition as the variable  $q$  increases. Note that the red curve gives the correct bound of  $31/36$  for  $q = 1$  (proved in [GPS10a]).

$q$ ). Nevertheless, it seems that when  $q$  gets closer to 4, the *lower tail* of these spectrums would not be thin enough to allow exceptional times.

Let us now discuss the reasons for this phenomenon in greater detail. As for the increasing coupling, an infinite-volume heat-bath dynamics on  $\mathbb{Z}^2$  can be constructed for every  $(p, q)$  with  $q \geq 1$ . These dynamics are constructed as monotone limits of the *wired* or *free* finite volume heat-bath dynamics and are unique as soon as the infinite-volume measure is unique at  $(p, q)$ . Fix  $q \in [1, 4]$  and  $p = p_c(q)$ . In this case, the infinite-volume measure is expected to be unique. Set  $(\omega_t^{\text{FK}_q})_{t \geq 0}$  to be the unique critical dynamics obtained either from the free or the wired limit. The natural question raised by the discussion of the percolation case is the following:

**Question 12.33.** *For  $q \in (1, 4]$ , are there exceptional times  $t$  almost surely for which there is an infinite cluster in  $\omega_t^{\text{FK}_q}$ ? If “yes” and if  $\mathcal{E}_q$  denotes the random set of these exceptional times, what is the (almost sure) Hausdorff dimension of  $\mathcal{E}_q$ ?*

We conjecture the following property for which we will then give a heuristic proof based on the above exponents from the physics literature.

**Conjecture 12.34.** *Let*

$$q^* := 4 \cos^2\left(\frac{\pi}{4}\sqrt{14}\right) \approx 3.83$$

- For  $q \in (q^*, 4]$ ,  $\text{FK}_q$  percolation is NOT dynamically sensitive.
- If  $q \in [1, q^*]$ , one has a.s.

$$\dim(\mathcal{E}_q) \leq \frac{1 - 8u(q) + 2u(q)^2}{3u(q)^2 - 8u(q)}, \tag{12.27}$$

where, as above  $u(q) := \frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2}) = 2 - \frac{8}{\kappa(q)}$ .

This conjecture hints that there is a critical  $q_c \in [1, q^*]$ , above which there are no exceptional times. It is natural to expect that  $q \mapsto \mathbb{E}[\dim(\mathcal{E}_q)]$  is continuous and thus that  $q_c > 1$ . It is quite possible that  $q_c = q^*$ .

**Heuristic explanation of the conjecture.** Due to the complicated structure of the  $\text{FK}_q$  near-critical percolation highlighted in the previous section, one cannot easily dominate the union of critical configurations  $\omega_t^{\text{FK}_q}$ , where  $t$  spans an interval  $I$  of length  $\varepsilon > 0$ , by a slightly supercritical  $\text{FK}_q$  configuration. Instead, one can rely on a more artificial near-critical version, where new edges are added to a critical random-cluster configuration in a Poissonian way. The correlation length  $\tilde{L}(p)$  of this near-critical model can be derived from the *four-arm* critical exponent as in the case  $q = 1$  (this assumes RSW and the analog of near-critical stability describe in Subsection 2.1). This gives

$$\tilde{L}(p) \approx \left( \frac{1}{|p - p_c(q)|} \right)^{\frac{1}{2 - \xi_4(q)}}.$$

Below this correlation length, the artificial near-critical configuration has the same connectivity properties as a critical random-cluster configuration. In particular, if our configurations  $\omega_t^{\text{FK}_q}$ ,  $t \in I$  are dominated by a configuration  $\tilde{\omega}_{p_c + \varepsilon}$ , one obtains

$$\begin{aligned} \mathbb{P}[I \cap \mathcal{E}_q \neq \emptyset] &\leq \mathbb{P}[0 \overset{\omega_{p_c + \varepsilon}}{\longleftrightarrow} \tilde{L}(p_c + \varepsilon)] \\ &\lesssim \mathbb{P}[0 \overset{\omega_{p_c}}{\longleftrightarrow} \tilde{L}(p_c + \varepsilon)] \\ &\approx \tilde{L}(p_c + \varepsilon)^{-\xi_1(q)} \approx \varepsilon^{\frac{\xi_1(q)}{2 - \xi_4(q)}}. \end{aligned}$$

Since one needs  $O(\varepsilon^{-1})$  intervals  $I$  to cover  $[0, 1]$ , a first moment argument implies that a.s.

$$\dim_H(\mathcal{E}_q) \leq 1 - \frac{\xi_1(q)}{2 - \xi_4(q)}.$$

Plugging in the expressions from Subsection 3.1 explains the second part of Conjecture 12.34. For the first part, it is enough to solve the equation  $\frac{\xi_1(q)}{2 - \xi_4(q)} = 1$ , so that above its solution, the dimension would be “negative”, i.e., we expect the set  $\mathcal{E}_q$  to be almost surely empty. The solution of this equation is given by  $q^*$ .

Finally, let us mention that (12.27) will be made rigorous in the special case  $q = 2$  in the forthcoming paper [GP].

## 4.2 Random-cluster models with $q > 4$

The purpose of this subsection is to briefly explain what occurs when  $q > 4$ . In this case, the phase transition is expected to be first order. In particular, the correlation length does not go to infinity when  $p$  goes to  $p_c$ . Therefore, the previous discussion of the near-critical regime does not make sense. We can still discuss a possible noise sensitivity. Yet, exponential decay of correlation suggest that the model is not noise sensitive. As an illustration, we exploit the known exponential decay estimates for  $\mathbb{P}_{p_c(q)}^{\text{FK}_q, \text{free}}$  when  $q$  is large enough (*i.e.*,  $q > 25.72$ ) in order to obtain the following result. Of course, the theorem is expected to hold for all  $q > 4$ .

**Theorem 12.35.** *When  $q$  is large enough, there are no exceptional times for the infinite free boundary heat-bath dynamics on critical  $\text{FK}_q$  configurations.*

**Proof** The proof is based on [Gri06, Theorem 6.35] which states that in dimension  $d = 2$ , and if  $q > 25.72$ , then at the critical point  $p_c(q) = p_{\text{sd}}(d)$ , one has for the **free** infinite volume limit:

$$\phi_{p_c(q), q}^0(0 \longleftrightarrow \partial[-n, n]^2) \leq C \exp(-c(q)n),$$

where  $c(q) > 0$  is a positive constant which depends only on  $q > 25.72$ . By monotonicity, this result implies that for the random-cluster measure on the finite box  $\Lambda_N := [-N, N]^2$  endowed with free boundary conditions, then for all radius  $n$  such that  $2n \leq N$ , one has

$$\phi_{\Lambda_N, p_c(q), q}^0(x \longleftrightarrow \partial(x + \Lambda_n)) \leq C \exp(-c(q)n),$$

for all points  $x \in \Lambda_n$ . This in turn implies that for all  $x \in \Lambda_n$ , the probability to have a *four-arm* event around  $x$  of radius  $n$  is bounded above by the same exponential bound  $\exp(-c(q)n)$ .

Now, let us fix a large radius  $n \gg 1$ . Our goal is to find a small upper bound on

$$g(n) := \phi_{p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \overset{\omega_t}{\longleftrightarrow} \partial\Lambda_n)$$

for the free boundary infinite-volume heat-bath dynamics on  $\mathbb{Z}^2$ . Since, as we discussed earlier, this infinite volume limit is obtained as an increasing limit of finite volume heat-bath dynamics and since the event under consideration is a cylinder event, one has

$$g(n) = \lim_{N \rightarrow \infty} \phi_{\Lambda_N, p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \overset{\omega_t}{\longleftrightarrow} \partial\Lambda_n).$$

Using the random variable  $X_n = X_n^{(N)}$  to denote the number of flips over the time interval  $[0, 1]$  for the event  $\{0 \longleftrightarrow \partial\Lambda_n\}$ , one easily obtains

$$\begin{aligned} \mathbb{E}[X_n] &\leq O(1) n^2 \sup_{x \in \Lambda_n} \phi_{\Lambda_N, p_c(q), q}^0(x \longleftrightarrow x + \partial(\Lambda_{n/2})) \\ &\leq O(1) n^2 \exp(-c(q)n/2). \end{aligned}$$

This in turn implies the bound

$$\phi_{\Lambda_N, p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \xleftrightarrow{\omega_t} \partial\Lambda_n) \leq O(1) n^2 \exp(-c(q)n/2),$$

which gives a uniform upper bound in  $N$ . In particular,  $g(n) \leq O(1) n^2 \exp(-c(q)n/2)$  and thus, taking  $n \rightarrow \infty$ , one concludes that a.s. there are no exceptional times for the critical free random-cluster measure on  $\mathbb{Z}^2$  with  $q$  large enough.  $\square$

## Part III

# The $O(n)$ -models and the self-avoiding walk



# Chapter 13

## The planar $O(n)$ -model

---

**Abstract:** This chapter describes the classes of planar spin and loop  $O(n)$ -models. These models were introduced in order to provide a unifying family for spin models<sup>1</sup>.

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This short chapter is organized as follows. First, spin and loop  $O(n)$ -models are defined, and the (conjectured) phase transition occurring in this models, called the Berezinsky-Kosterlitz-Thouless phase transition, is described briefly. The second section deals with the self-avoiding walk model. This model is introduced formally and a few useful facts are reminded.

### 1 The family of $O(n)$ -models

#### 1.1 Spin $O(n)$ -models

After the introduction of the Ising model by Lenz [Len20], and the conjecture by Ising that no phase transition was occurring, many physicists tried to find natural generalizations of the model exhibiting a phase transition. In [HK34], Heller and Kramers describe the classical version of the celebrated quantum Heisenberg model where spins are vectors of the three-dimensional sphere  $\mathbb{S}_3$ . Later, Stanley generalized this model by allowing spins to be on the sphere  $\mathbb{S}_n$  in dimension  $n$  [Sta68] (the model was studied in the case  $n = 2$  in [VL66]). We refer to [DG76] for a historic of the subject.

The spin  $O(n)$ -model can be defined on any graph. However, we restrict ourselves to the hexagonal lattice  $\mathbb{H}$ . Let  $G$  be a finite subgraph of  $\mathbb{H}$ . The spin  $O(n)$ -model with *free boundary conditions* is a random assignment  $\sigma \in \mathbb{S}_n^G$  of spins  $\sigma_x \in \mathbb{S}_n$  to vertices of  $G$ , where  $\sigma_x$  denotes the spin at site  $x$ . The *Hamiltonian* of the model is defined by

$$H_G^f(\sigma) := - \sum_{x \sim y} \langle \sigma_x | \sigma_y \rangle.$$

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<sup>1</sup>In opposition to random-cluster models which are providing a unifying family of models for percolation models.

where the summation is over all pairs of neighboring sites  $x, y$  in  $G$ , and  $\langle \cdot | \cdot \rangle$  is the scalar product in dimension  $n$ . The *partition function* of the model is

$$Z_{\beta, G}^f := \int_{\sigma \in \mathbb{S}_n^G} d\sigma \exp[-\beta H_G^f(\sigma)], \quad (13.1)$$

where  $\beta$  is the inverse temperature of the model and  $d\sigma$  the tensor product of  $|G|$  measures  $k_n dx$  ( $dx$  is the Lebesgue measure on  $\mathbb{S}_n$ ), where  $k_n$  is chosen in such a way that  $\int d\sigma \pi_i(\sigma_x)^2 = 1$  ( $\pi_i$  is the projection on the  $i$ -th coordinate). When  $n = 1$ , we obtain the Ising model. The case  $n = 2$  is called the  $XY$ -model and the  $n = 3$  is the (classical) Heisenberg model.

## 1.2 Loop $O(n)$ -models

This model, introduced in [DMNS81] on the hexagonal lattice, is a lattice gas of non-intersecting loops. More precisely, consider configurations of non-intersecting simple loops on a finite subgraph of the hexagonal lattice and introduce two parameters: a loop-weight  $n \geq 0$  (in fact  $n \geq -2$ ) and an edge-weight  $x > 0$ , and ask the probability of a configuration to be proportional to  $n^{\#\text{ loops}} x^{\#\text{ edges}}$ .

Alternatively, an interface between two boundary points could be added: in this case configurations are composed of non-intersecting simple loops and one self-avoiding interface (avoiding all the loops) from  $a$  to  $b$ .

The  $O(0)$ -model with an interface is the self-avoiding walk from  $a$  to  $b$ , since no loop is allowed. It is worth mentioning that a connection between the self-avoiding walk and spin  $O(0)$ -model (which does not really make sense) was mentioned in [DG72]. In the next paragraph, the loop  $O(1)$ -model will be related to the high-temperature expansion of the Ising model on the hexagonal lattice.

## 1.3 Connection between spin and loop $O(n)$ -models

In fact, loop  $O(n)$ -models were introduced as approximations of the high-temperature expansion of the spin  $O(n)$ -models. Instead of the partition function in (13.1), consider the simplified partition function

$$\tilde{Z}_{x, G}^f := \int_{\sigma \in \mathbb{S}_n^G} d\sigma \prod_{[ab] \in E[G]} (1 + x \langle \sigma_a | \sigma_b \rangle). \quad (13.2)$$

Strictly speaking, the partition functions  $Z_{\beta, G}^f$  and  $\tilde{Z}_{\beta, G}^f$  coincide only in the limit  $\beta$  approaching 0 yet the two models are expected to belong to the same universality class. In the Ising case,  $Z_{\beta, G}^b$  is the integral of

$$\prod_{[ab] \in E[G]} \left( \frac{e^\beta + e^{-\beta}}{2} + \frac{e^\beta - e^{-\beta}}{2} \langle \sigma_a | \sigma_b \rangle \right).$$

Thus, up to a universal multiplicative constant, it is equal to  $\tilde{Z}_{x, G}^b$  for  $x := \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}}$ . In other words, when  $n = 1$ , the previous replacement is not an approximation.

As in the high temperature expansion of the Ising model,  $\tilde{Z}_{x,G}$  can be expanded in powers of  $\beta$ .

$$\begin{aligned}\tilde{Z}_{x,G}^f &= \int_{\sigma \in \mathbb{S}_n^G} d\sigma \prod_{[ab] \in E[G]} (1 + x \langle \sigma_a | \sigma_b \rangle) \\ &= \sum_{\sigma} \sum_{\gamma \subset E} \prod_{e \in [ab] \in \gamma} \langle \sigma_a | \sigma_b \rangle \\ &= \int_{\sigma \in \mathbb{S}_n^G} d\sigma \sum_{\gamma \subset E[G]} x^{|\gamma|} \prod_{[ab] \in \gamma} \langle \sigma_a | \sigma_b \rangle \\ &= \sum_{\gamma \subset E[G]} x^{|\gamma|} \int_{\sigma \in \mathbb{S}_n^G} d\sigma \prod_{[ab] \in \gamma} \langle \sigma_a | \sigma_b \rangle.\end{aligned}$$

Now, the integral equals 0 except if edges in  $\gamma$  form a collection of non-intersecting loops. In the latter case, the weight equals  $n^{\#\text{ loops}}$ . We thus obtain the loop  $O(n)$ -model.

#### 1.4 Phase transition in planar $O(n)$ -models

The planar spin  $O(1)$ -model being the Ising model, we already discussed its phase transition extensively. The phase transition in the spin  $O(n)$ -model when  $n \geq 2$  is very different from the phase transition in random-cluster models. The case of the  $O(2)$ -model is already interesting: the planar  $XY$ -model is never ordered at any temperature. Nevertheless, there is a qualitative change of behavior in the model:

- At very low inverse-temperature, spin correlations decay exponentially fast in the distance between the spins [MS77].
- At very high inverse-temperature, spin correlations decay as a power in the distance between the spins [FS81].

Moreover, physics considerations suggest that there exists a critical inverse-temperature  $\beta_c$  separating the two phases: for  $\beta > \beta_c$ , correlations decay as power laws while for  $\beta < \beta_c$ , they decay exponentially fast. A phase transition of the previous type is called a *Berezinsky-Kosterlitz-Thouless* phase transition. This type of phase transition is named after Berezinsky and Kosterlitz-Thouless who introduced it nonrigorously for the planar  $XY$ -model in two independent papers [Ber72] and [KT73]. The main differences with phase transitions previously described in this document are the following: there is no ordered phase, no global symmetry is broken through the phase transition. Moreover, the order of the phase transition is infinite (the free energy is infinitely differentiable but not analytic at the transition).

Other values of  $n$  are very different: Polyakov conjectured in 1975 that no phase transition occurs whenever  $n \geq 3$  [Pol75]. Polyakov's conjecture is generally accepted, even so it is not completely unanimous. We mention that the existence or absence of phase transitions are still open mathematical questions of great interest.

The loop  $O(n)$ -model exhibits a greater variety of critical behavior than the spin  $O(n)$ -model. Similarly to the spin  $O(n)$ -models, some loop  $O(n)$ -models are expected to have a Berezinsky-Kosterlitz-Thouless phase transition. In this case, the definition of the phase transition corresponds to the existence of  $x_c \in (0, \infty)$  such that

- For  $x < x_c$ , the probability of  $a$  and  $b$  being on the same loop decays exponentially fast in the distance between  $a$  and  $b$ .
- For  $x > x_c$ , the probability of  $a$  and  $b$  being on the same loop decays as a power in the distance between  $a$  and  $b$ .

Unsurprisingly, the loop  $O(n)$ -models are expected to exhibit a Berezinsky-Kosterlitz-Thouless phase transition if and only if  $n \in [-2, 2]$  (no phase transition is predicted to occur when  $n > 2$ ). In this range, Bernard Nienhuis [Nie82, Nie84] proposed the following conjecture, supported by physics arguments:

**Conjecture 13.1.** *The critical value is given by  $x_c(n) = 1/\sqrt{2 + \sqrt{2 - n}}$ .*

The conjecture was rigorously established for two cases only. When  $n = 1$ , the critical value is related to the critical temperature of the Ising model, since the  $O(1)$ -model is the high-temperature expansion of the spin Ising model. In particular, an adaptation of the argument in Chapter 8 implies the result. When  $n = 0$ , it will be proved in Chapter 14 that  $\sqrt{2 + \sqrt{2}}$  is the connective constant of the hexagonal lattice, so that  $x_c(0) = 1/\sqrt{2 + \sqrt{2}}$ .

In this context, the model exhibits one critical behavior at  $x_c(n)$  and another on the interval  $(x_c(n), +\infty)$ , both being conformally invariant: the interface should converge to an SLE, see Chapter 17. Both regimes are critical yet different since the parameter  $\kappa$  in the scaling limit is not the same. Another way to put it is to say that it corresponds to 'dilute' and 'dense' phases (when in the limit the loops are simple and non-simple correspondingly), see Fig. 17.2 in Chapter 17.

It would be of great interest to show that a phase transition indeed occurs at  $x_c(n)$ . Unfortunately, no obvious monotonicity exists in the model, and the existence of a phase transition itself remains a mystery. Even for large values of  $n$  (which should be easier), a mathematical proof of the absence of phase transition is still missing (this corresponds to the fact that spin  $O(n)$ -models are conjectured not to have a phase transition for  $n \geq 3$ ).

## 2 The $O(0)$ -model: the self-avoiding walk

When taking the limit of  $n$  goes to 0 of the loop  $O(n)$  model with one interface, the model contains one polymer and no loops. This is the self-avoiding walk on the hexagonal lattice. As was mentioned in the introduction, self-avoiding walks were defined by Flory in 1953 [Flo53], roughly thirty years before the loop  $O(n)$ -model. Forgetting about the  $O(0)$ -model, it is possible to define self-avoiding walks on any transitive lattice.

First, let  $c_n = c_n(\mathcal{L})$  be the number of  $n$ -step self-avoiding walks starting at the origin on a transitive lattice  $\mathcal{L}$ . Denote by  $\gamma_n$  the set of such walks. It is in general elementary

to provide exponential bounds on  $c_n$ . Moreover, by looking at the first  $n$  steps of a  $n+m$ -steps self-avoiding walk and the  $m$  last ones, every elements of  $\gamma_{n+m}$  can be mapped to one element of  $(\gamma_n, \gamma_m)$  in a one-to-one fashion. Thus,  $c_{n+m} \leq c_n c_m$  and Fekete's subadditive lemma implies that  $c_n^{1/n}$  converges to a constant called the connective constant  $\mu$  of the lattice  $\mathcal{L}$ .

Two natural classes of questions can be asked on self-avoiding walks: the *combinatorial* and the *geometrical* ones. Honesty forces us to admit that neither of the two kinds of questions is very well understood, and the number of results on self-avoiding walks is quite restricted (at least in low dimensions). We refer to the last chapter for open questions concerning self-avoiding walks.

One of the first questions coming to mind is can the bounds on  $c_n$  be improved? Namely, subadditivity implies  $c_n \geq \mu^n$ , but what about the other bound? The best currently available result is forty years old:

**Theorem 13.2** (Hammersley-Welsh,[HW62]). *For any transitive lattice  $\mathcal{L}$ , there exists  $c > 0$  such that for every  $n$ ,*

$$\mu^n \leq c_n \leq e^{c\sqrt{n}} \mu^n.$$

In fact, this result has been improved by Kesten in dimension three and higher [Kes63b, Kes64b].

Since it will be used in the following chapters, the argument is presented now (it is extracted from the lecture notes [BDCGS11]) in the case of the hypercubic lattices.

For a self-avoiding walk  $\gamma$ , denote by  $\gamma_1(i)$  the first spatial coordinate of  $\gamma(i)$ .

**Definition 13.3.** *An  $n$ -step bridge is an  $n$ -step self-avoiding walk  $\gamma$  such that for every  $1 \leq i \leq n$ ,*

$$\gamma_1(0) < \gamma_1(i) \leq \gamma_1(n) \tag{13.3}$$

*An  $n$ -step half-space walk is an  $n$ -step self-avoiding walk  $\gamma$  such that for every  $1 \leq i \leq n$ ,*

$$\gamma_1(0) < \gamma_1(i) \tag{13.4}$$

*Let  $b_n$  (reps.  $h_n$ ) be the number of  $n$ -step bridges (resp. half-plane walks) with  $\gamma(0) = 0$  for  $n > 1$ , and  $b_0 = 1$ .*

While  $(c_n)$  is a submultiplicative sequence, the sequence  $(b_n)$  is obviously supermultiplicative so that the connective constant  $\mu_{\text{Bridge}}$  for bridges can be defined by

$$\mu_{\text{Bridge}} = \lim_{n \rightarrow \infty} b_n^{1/n} = \sup_{n \geq 1} b_n^{1/n}.$$

Furthermore,

$$b_n \leq \mu_{\text{Bridge}}^n \leq \mu^n. \tag{13.5}$$

We will use the following result on integer partitions which dates back to 1917, due to Hardy and Ramanujan [HR17].

**Theorem 13.4.** For an integer  $A \geq 1$ , let  $P_D(A)$  denote the number of ways of writing  $A = A_1 + \dots + A_k$  with  $A_1 > \dots > A_k \geq 1$ , for any  $k \geq 1$ . Then

$$\log P_D(A) \sim \pi \left( \frac{A}{3} \right)^{1/2}$$

as  $A \rightarrow \infty$ .

**Lemma 13.5.**  $h_n \leq P_D(n)b_n$  for all  $n \geq 1$ .

**Proof** Set  $n_0 = 0$  and inductively define

$$A_{i+1} = \max_{j > n_i} (-1)^i (\gamma_1(j) - \gamma_1(n_i))$$

and

$$n_{i+1} = \max\{j > n_i : (-1)^i (\gamma_1(j) - \gamma_1(n_i)) = A_{i+1}\}.$$

In words,  $j = n_1$  maximises  $\gamma_1(j)$ ,  $j = n_2$  minimises  $\gamma_1(j)$  for  $j > n_1$ ,  $n_3$  maximises  $\gamma_1(j)$  for  $j > n_2$ , and so on in an alternating pattern. In addition  $A_1 = \gamma_1(n_1) - \gamma_1(n_0)$ ,  $A_2 = \gamma_1(n_1) - \gamma_1(n_2)$  and so on. Moreover, the  $n_i$  are chosen to be the last times these extrema are attained.

This procedure stops at some step  $K \geq 1$  when  $n_K = n$ . Since the  $n_i$  are chosen maximal, it follows that  $A_{i+1} < A_i$ . Note that  $K = 1$  if and only if  $\gamma$  is a bridge, and in that case  $A_1$  is the span of  $\gamma$ . Let  $h_n[a_1, \dots, a_k]$  denote the number of  $n$ -step half-space walks with  $K = k$ ,  $A_i = a_i$  for  $i = 1, \dots, k$ . We observe that

$$h_n[a_1, a_2, a_3, \dots, a_k] \leq h_n[a_1 + a_2, a_3, \dots, a_n]. \quad (13.6)$$

To obtain this, reflect the part of the walk  $(\gamma(j))_{j \geq n_1}$  across the line  $\gamma_1 = A_1$ . Repeating this inequality gives

$$h_n[a_1, \dots, a_k] \leq h_n[a_1 + \dots + a_k] = b_{n, a_1 + \dots + a_k}.$$

where  $b_{n,A}$  be the number of  $n$ -step bridges with  $\gamma_1(n) - \gamma_1(0) = A$ . So we can bound

$$h_n = \sum_{k \geq 1} \sum_{a_1 > \dots > a_k > 0} h_n[a_1, \dots, a_k] \leq \sum_{k \geq 1} \sum_{a_1 > \dots > a_k > 0} b_{n, a_1 + \dots + a_k} = \sum_{A=1}^n P_D(A) b_{n,A}.$$

Bounding  $P_D(A)$  by  $P_D(n)$ , we obtain  $h_n \leq P_D(n) \sum_{A=1}^n b_{n,A} = P_D(n)b_n$  as claimed.  $\square$

We can now prove an upper bound on the number of self-avoiding walks:

**Proof of Theorem 13.2** We first prove

$$c_n \leq \sum_{m=0}^n h_{n-m} h_{m+1}, \tag{13.7}$$

using the decomposition defined as follows: given an  $n$ -step self-avoiding walk  $\gamma$ , let

$$x_1 = \min_{0 \leq i \leq n} \gamma_1(i), \quad m = \max\{i : \gamma_1(i) = x_1\}.$$

Write  $e_1$  for the unit vector in the first coordinate direction of  $\mathbb{Z}^d$ . Then (after translating by  $\gamma(m)$ ) the walk  $(\gamma(m), \gamma(m+1), \dots, \gamma(n))$  is an  $(n-m)$ -step half-space walk, and (after translating by  $\gamma(m) - e_1$ ) the walk  $(\gamma(m) - e_1, \gamma(m), \gamma(m-1), \dots, \gamma(1), \gamma(0))$  is an  $(m+1)$ -step half-space walk. This proves (13.7).

Next we apply Lemma 13.5 in (13.7) and the supermultiplicativity of self-avoiding bridges to get

$$\begin{aligned} c_n &\leq \sum_{m=0}^n P_D(n-m) P_D(m+1) b_{n-m} b_{m+1} \\ &\leq b_{n+1} \sum_{m=0}^n P_D(n-m) P_D(m+1). \end{aligned}$$

Fix  $B > B' > \pi(\frac{2}{3})^{1/2}$ . By Theorem 13.4, there exists  $K > 0$  such that  $P_D(A) \leq K \exp(B'(A/2)^{1/2})$  and consequently

$$P_D(n-m) P_D(m+1) \leq K^2 \exp \left[ B' \left( \sqrt{\frac{n-m}{2}} + \sqrt{\frac{m+1}{2}} \right) \right].$$

The bound  $x^{1/2} + y^{1/2} \leq (2x + 2y)^{1/2}$  now gives

$$c_n \leq (n+1) K^2 e^{B' \sqrt{n+1}} b_{n+1} \leq e^{B \sqrt{n}} b_{n+1}$$

if  $n \geq n_0(B)$ . By (13.5), the result follows. □

We mention the following corollary:

**Corollary 13.6.** *For  $n \geq n_0(B)$ ,*

$$b_n \geq c_{n-1} e^{-B \sqrt{n-1}} \geq \mu^{n-1} e^{-B \sqrt{n-1}}.$$

*In particular,  $b_n^{1/n} \rightarrow \mu$  and so  $\mu_{\text{Bridge}} = \mu$ .*



# Chapter 14

## The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$

---

**Abstract:** We prove that the connective constant of the hexagonal lattice equals  $\sqrt{2 + \sqrt{2}}$  using a parafermionic observable in the  $O(0)$ -model. This chapter is inspired of the paper *The connective constant of the honeycomb lattice equals  $\sqrt{2 + \sqrt{2}}$*  with Stanislav Smirnov to appear in *Annals of Mathematics* [DCS10]. We also compute the connective constant of other two classical lattices.

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Using Coulomb gas formalism, B. Nienhuis [Nie82, Nie84] proposed physical arguments for  $\mu$  to have the value  $\sqrt{2 + \sqrt{2}}$ . We rigorously prove this statement. While our methods are different from those harnessed by Nienhuis, they are similarly motivated by considerations of vertex operators in the  $O(n)$  model.

**Theorem 14.1.** *For the hexagonal lattice,*

$$\mu = \sqrt{2 + \sqrt{2}}.$$

It will be convenient to consider walks between *mid-edges* of  $\mathbb{H}$ , *i.e.* centers of edges of  $\mathbb{H}$  (the set of mid-edges will be called  $H$ ). We will write  $\gamma : a \rightarrow E$  if a walk  $\gamma$  starts at  $a$  and ends at some mid-edge of  $E \subset H$ . In the case  $E = \{b\}$ , we simply write  $\gamma : a \rightarrow b$ . The *length*  $\ell(\gamma)$  of the walk is the number of vertices belonging to  $\gamma$ .

We will work with the (increasing in  $x$ ) sum

$$Z(x) = \sum_{\gamma : a \rightarrow H} x^{\ell(\gamma)} \in (0, +\infty].$$

This sum does not depend on the choice of  $a$ . Establishing  $\mu = \sqrt{2 + \sqrt{2}}$  is equivalent to showing that  $Z(x) = +\infty$  for  $x > 1/\sqrt{2 + \sqrt{2}}$  and  $Z(x) < +\infty$  for  $x < 1/\sqrt{2 + \sqrt{2}}$ . To this effect, we first restrict walks to bounded domains and weight them counting their winding.

The vertex operator obtained leads to a *parafermionic observable* which is a generalization of the spin fermionic observable. To simplify formulæ, below we set  $x_c := 1/\sqrt{2 + \sqrt{2}}$  and  $j = e^{i2\pi/3}$ .

The chapter is organized as follows. In Section 1, the parafermionic observable is introduced and its principal property is derived. Section 2 contains the proof of Theorem 14.1. Section 3 presents conjectures on self-avoiding walks related to the parafermionic observable. Section 4 computes the connective constant of other two notable planar lattices.

## 1 Parafermionic observable

A (hexagonal lattice) *domain*  $\Omega \subset H$  is a union of all mid-edges emanating from a given collection of vertices  $V(\Omega)$  (see Fig. 14.1): a mid-edge  $z$  belongs to  $\Omega$  if at least one end-point of its associated edge is in  $\Omega$ , it belongs to  $\partial\Omega$  if only one of them is in  $\Omega$ . We further assume  $\Omega$  to be simply connected, *i.e.* having a connected complement.

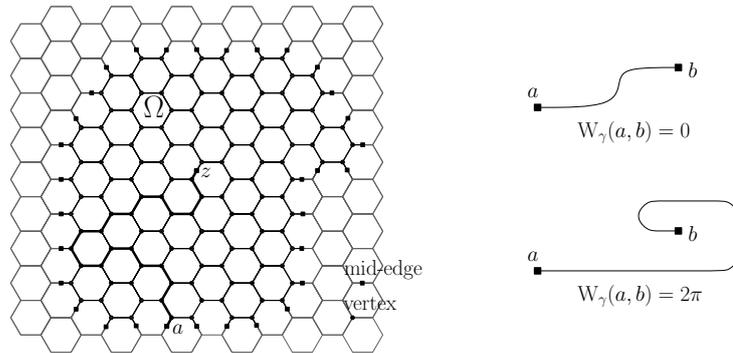


Figure 14.1: **Left.** A domain  $\Omega$  whose mid-edges are pictured by small black squares. Vertices of  $V(\Omega)$  correspond to circles. **Right.** Winding of a curve  $\gamma$ .

**Definition 14.2.** The winding  $W_\gamma(a, b)$  of a self-avoiding walk  $\gamma$  between mid-edges  $a$  and  $b$  (not necessarily the start and the end) is the total rotation of the direction in radians when  $\gamma$  is traversed from  $a$  to  $b$ , see Fig. 14.1.

The *parafermionic observable* is defined as follows: for  $a \in \partial\Omega$ ,  $z \in \Omega$ , set

$$F(z) = F(a, z, x, \sigma) = \sum_{\gamma \subset \Omega: a \rightarrow z} e^{-i\sigma W_\gamma(a, z)} x^{\ell(\gamma)}.$$

**Lemma 14.3.** If  $x = x_c$  and  $\sigma = \frac{5}{8}$ , then  $F$  satisfies the following relation for every vertex  $v \in V(\Omega)$ :

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0, \tag{14.1}$$

where  $p, q, r$  are the mid-edges of the three edges adjacent to  $v$ .

Note that with  $\sigma = 5/8$ , the term  $e^{-i\sigma W_\gamma(a,z)}$  gives a weight  $\lambda$  or  $\bar{\lambda}$  per left or right turn of  $\gamma$ , where

$$\lambda = \exp\left(-i\frac{5}{8} \cdot \frac{\pi}{3}\right) = \exp\left(-i\frac{5\pi}{24}\right).$$

**Proof** In this proof, we further assume that  $p, q$  and  $r$  are oriented counter-clockwise around  $v$ . Note that  $(p-v)F(p) + (q-v)F(q) + (r-v)F(r)$  is a sum of contributions  $c(\gamma)$  over all possible walks  $\gamma$  finishing at  $p, q$  or  $r$ . For instance, if the walk ends at the mid-edge  $p$ , the contribution will be given by

$$c(\gamma) = (p-v)e^{-i\sigma W_\gamma(a,p)} x_c^{\ell(\gamma)}.$$

One can partition the set of walks  $\gamma$  finishing at  $p, q$  or  $r$  into pairs and triplets of walks in the following way, see Fig 14.2:

- If a walk  $\gamma_1$  visits all three mid-edges  $p, q, r$ , it means that the edges belonging to  $\gamma_1$  form a self-avoiding path plus (up to a half-edge) a self-avoiding loop from  $v$  to  $v$ . One can associate to  $\gamma_1$  the walk passing through the same edges, but exploring the loop from  $v$  to  $v$  in the other direction. Hence, walks visiting the three mid-edges can be grouped in pairs.
- If a walk  $\gamma_1$  visits only one mid-edge, it can be associated to two walks  $\gamma_2$  and  $\gamma_3$  that visit exactly two mid-edges by prolonging the walk one step further (there are two possible choices). The reverse is true: a walk visiting exactly two mid-edges is naturally associated to a walk visiting only one mid-edge by erasing the last step. Hence, walks visiting one or two mid-edges can be grouped in triplets.

If one can prove that the sum of contributions for each pair and each triplet vanishes, then the total sum is zero.

Let  $\gamma_1$  and  $\gamma_2$  be two walks that are grouped as in the first case. Without loss of generality, we assume that  $\gamma_1$  ends at  $q$  and  $\gamma_2$  ends at  $r$ . Note that  $\gamma_1$  and  $\gamma_2$  coincide up to the mid-edge  $p$  since  $(\gamma_1, \gamma_2)$  are matched together. We deduce

$$\ell(\gamma_1) = \ell(\gamma_2) \quad \text{and} \quad \begin{cases} W_{\gamma_1}(a,q) = W_{\gamma_1}(a,p) + W_{\gamma_1}(p,q) = W_{\gamma_1}(a,p) - \frac{4\pi}{3} \\ W_{\gamma_2}(a,r) = W_{\gamma_2}(a,p) + W_{\gamma_2}(p,r) = W_{\gamma_1}(a,p) + \frac{4\pi}{3} \end{cases}.$$

In order to evaluate the winding of  $\gamma_1$  between  $p$  and  $q$ , we used the fact that  $a$  is on the boundary and  $\Omega$  is simply connected. Therefore,

$$\begin{aligned} c(\gamma_1) + c(\gamma_2) &= (q-v)e^{-i\sigma W_{\gamma_1}(a,q)} x_c^{\ell(\gamma_1)} + (r-v)e^{-i\sigma W_{\gamma_2}(a,r)} x_c^{\ell(\gamma_2)} \\ &= (p-v)e^{-i\sigma W_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} (j\bar{\lambda}^4 + \bar{j}\lambda^4) = 0 \end{aligned}$$

where the last equality is due to the chosen value  $\lambda = \exp(-i5\pi/24)$ .

Let  $\gamma_1, \gamma_2, \gamma_3$  be three walks matched as in the second case. Without loss of generality, we assume that  $\gamma_1$  ends at  $p$  and that  $\gamma_2$  and  $\gamma_3$  extend  $\gamma_1$  to  $q$  and  $r$  respectively. As before, we easily find that

$$\ell(\gamma_2) = \ell(\gamma_3) = \ell(\gamma_1) + 1 \quad \text{and} \quad \begin{cases} W_{\gamma_2}(a,r) = W_{\gamma_2}(a,p) + W_{\gamma_2}(p,q) = W_{\gamma_1}(a,p) - \frac{\pi}{3} \\ W_{\gamma_3}(a,r) = W_{\gamma_3}(a,p) + W_{\gamma_3}(p,r) = W_{\gamma_1}(a,p) + \frac{\pi}{3} \end{cases}.$$

Following the same steps as above, we obtain

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3) = (p-v)\varepsilon^{-i\sigma W_{\gamma_1}(a,p)} x_c^{\ell(\gamma_1)} (1 + x_c j \bar{\lambda} + x_c \bar{j} \lambda) = 0.$$

Here is the *only* place where we use the crucial fact that  $x_c^{-1} = \sqrt{2 + \sqrt{2}} = (2 \cos \frac{\pi}{8})$ .

The claim follows readily by summing over all pairs and triplets.  $\square$

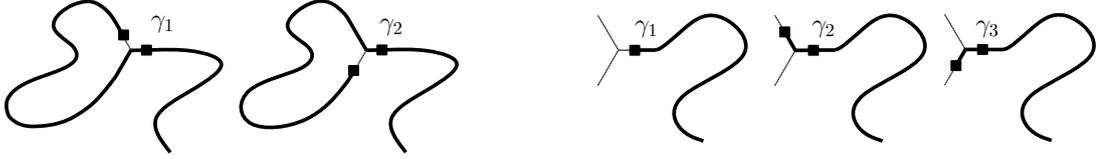


Figure 14.2: **Left:** a pair of walks visiting the three mid-edges and matched together. **Right:** a triplet of walks, one visiting one mid-edge, the other two visiting two mid-edges, which are matched together.

**Remark 14.4.** *Coefficients above are three cube roots of unity multiplied by  $p-v$ , so that the left-hand side can be seen as a discrete integral along an elementary contour on the dual lattice. The fact that the integral of the parafermionic observable along discrete contours vanishes is a glimpse of conformal invariance of the model, see Section 3.*

## 2 Proof of Theorem 14.1

**Counting argument in a strip domain.** We consider a vertical strip domain  $S_T$  composed of  $T$  strips of hexagons, and its finite version  $S_{T,L}$  cut at height  $L$  at an angle of  $\pi/3$ , see Fig. 14.3. Namely, position a hexagonal lattice  $\mathbb{H}$  of meshsize 1 in  $\mathbb{C}$  so that there exists a horizontal edge  $e$  with mid-edge  $a$  being 0. Then

$$V(S_T) = \{z \in V(\mathbb{H}) : 0 \leq \operatorname{Re}(z) \leq \frac{3T+1}{2}\},$$

$$V(S_{T,L}) = \{z \in V(S_T) : |\sqrt{3}\operatorname{Im}(z) - \operatorname{Re}(z)| \leq 3L\}.$$

Denote by  $\alpha$  the left boundary of  $S_T$ , by  $\beta$  the right one. Symbols  $\varepsilon$  and  $\bar{\varepsilon}$  denote the top and bottom boundaries of  $S_{T,L}$ . Introduce the following positive quantities:

$$A_{T,L}^x := \sum_{\gamma \subset S_{T,L}: a \rightarrow \alpha \setminus \{a\}} x^{\ell(\gamma)},$$

$$B_{T,L}^x := \sum_{\gamma \subset S_{T,L}: a \rightarrow \beta} x^{\ell(\gamma)},$$

$$E_{T,L}^x := \sum_{\gamma \subset S_{T,L}: a \rightarrow \varepsilon \cup \bar{\varepsilon}} x^{\ell(\gamma)}.$$

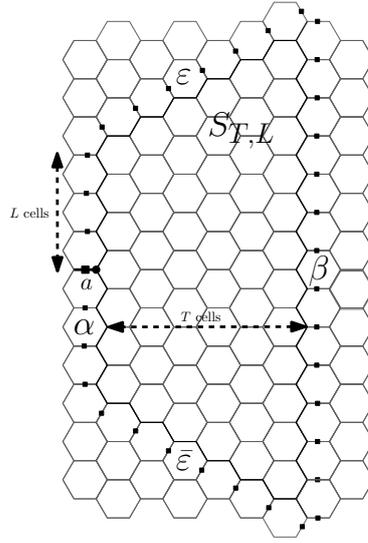


Figure 14.3: Domain  $S_{T,L}$  and boundary parts  $\alpha$ ,  $\beta$ ,  $\varepsilon$  and  $\bar{\varepsilon}$ .

**Lemma 14.5.** *When  $x = x_c$ , we have*

$$1 = c_\alpha A_{T,L}^{x_c} + B_{T,L}^{x_c} + c_\varepsilon E_{T,L}^{x_c}, \quad (14.2)$$

where  $c_\alpha = \cos\left(\frac{3\pi}{8}\right)$  and  $c_\varepsilon = \cos\left(\frac{\pi}{4}\right)$ .

**Proof** Sum the relation (14.1) over all vertices in  $V(S_{T,L})$ . Values at interior mid-edges disappear and we arrive at

$$0 = - \sum_{z \in \alpha} F(z) + \sum_{z \in \beta} F(z) + j \sum_{z \in \varepsilon} F(z) + \bar{j} \sum_{z \in \bar{\varepsilon}} F(z). \quad (14.3)$$

Using the symmetry of the domain, we deduce  $F(\bar{z}) = \bar{F}(z)$ , where  $\bar{x}$  is the symmetric of  $x$  with respect to the real axis. Observe that the winding of any self-avoiding walk from  $a$  to the bottom part of  $\alpha$  is  $-\pi$  while the winding to the top part is  $\pi$ . We conclude

$$\sum_{z \in \alpha} F(z) = F(a) + \sum_{z \in \alpha \setminus \{a\}} F(z) = 1 + \frac{e^{-i\sigma\pi} + e^{i\sigma\pi}}{2} A_{T,L}^x = 1 - \cos\left(\frac{3\pi}{8}\right) A_{T,L}^x = 1 - c_\alpha A_{T,L}^x.$$

Above, we have used the fact that the only walk from  $a$  to  $a$  is of length 0. Similarly, the winding from  $a$  to any half-edge in  $\beta$  (resp.  $\varepsilon$  and  $\bar{\varepsilon}$ ) is 0 (resp.  $\frac{2\pi}{3}$  and  $-\frac{2\pi}{3}$ ), therefore

$$\sum_{z \in \beta} F(z) = B_{T,L}^x \quad \text{and} \quad j \sum_{z \in \varepsilon} F(z) + \bar{j} \sum_{z \in \bar{\varepsilon}} F(z) = \cos\left(\frac{\pi}{4}\right) E_{T,L}^x = c_\varepsilon E_{T,L}^x.$$

The lemma follows readily by plugging these three formulæ in (14.3).  $\square$

Observe that sequences  $(A_{T,L}^x)_{L>0}$  and  $(B_{T,L}^x)_{L>0}$  are increasing in  $L$  and are bounded for  $x \leq x_c$  thanks to (14.2) and the monotonicity in  $x$ . Thus they have limits

$$\begin{aligned} A_T^x &= \lim_{L \rightarrow \infty} A_{T,L}^x = \sum_{\gamma \subset S_T: a \rightarrow \alpha \setminus \{a\}} x^{\ell(\gamma)}, \\ B_T^x &= \lim_{L \rightarrow \infty} B_{T,L}^x = \sum_{\gamma \subset S_T: a \rightarrow \beta} x^{\ell(\gamma)}. \end{aligned}$$

When  $x = x_c$ , via (14.2) again, we conclude that  $(E_{T,L}^{x_c})_{L>0}$  decreases and converges to a limit  $E_T^{x_c} = \lim_{L \rightarrow \infty} E_{T,L}^{x_c}$ . Then, (14.2) implies

$$1 = c_\alpha A_T^{x_c} + B_T^{x_c} + c_\varepsilon E_T^{x_c}. \quad (14.4)$$

**Proof of Theorem 14.1** Let us first prove that  $Z(x_c) = +\infty$ , which implies  $\mu \geq \sqrt{2 + \sqrt{2}}$ . Suppose that for some  $T$ ,  $E_T^{x_c} > 0$ . As noted before,  $E_{T,L}^{x_c}$  decreases in  $L$  and

$$Z(x_c) \geq \sum_{L>0} E_{T,L}^{x_c} \geq \sum_{L>0} E_T^{x_c} = +\infty,$$

which completes the proof. Assume on the contrary that  $E_T^{x_c} = 0$ , then (14.4) simplifies to

$$1 = c_\alpha A_T^{x_c} + B_T^{x_c}. \quad (14.5)$$

Observe that walks entering into the count of  $A_{T+1}^{x_c}$  and not in  $A_T^{x_c}$  have to visit some vertex adjacent to the right edge of  $S_{T+1}$ . Cutting such a walk at the first such point (and adding half-edges to the two halves), we obtain two walks crossing  $S_{T+1}$  (these walks are usually called bridges). We conclude that

$$A_{T+1}^{x_c} - A_T^{x_c} \leq x_c (B_{T+1}^{x_c})^2. \quad (14.6)$$

Combining (14.5) for  $T$  and  $T + 1$  with (14.6), we can write

$$\begin{aligned} 0 &= 1 - 1 = (c_\alpha A_{T+1}^{x_c} + B_{T+1}^{x_c}) - (c_\alpha A_T^{x_c} + B_T^{x_c}) \\ &= c_\alpha (A_{T+1}^{x_c} - A_T^{x_c}) + B_{T+1}^{x_c} - B_T^{x_c} \\ &\leq c_\alpha x_c (B_{T+1}^{x_c})^2 + B_{T+1}^{x_c} - B_T^{x_c}, \end{aligned}$$

so

$$c_\alpha x_c (B_{T+1}^{x_c})^2 + B_{T+1}^{x_c} \geq B_T^{x_c}.$$

By induction, it is easy to check that

$$B_T^{x_c} \geq \frac{\min[B_1^{x_c}, 1/(c_\alpha x_c)]}{T}$$

for every  $T \geq 1$ , implying

$$Z(x_c) \geq \sum_{T>0} B_T^{x_c} = +\infty.$$

This completes the proof of the inequality  $\mu \geq x_c^{-1} = \sqrt{2 + \sqrt{2}}$ .

Let us turn to the other needed inequality  $\mu \leq x_c^{-1}$ . A *bridge* of width  $T$  is a self-avoiding walk in  $S_T$  from one side to the opposite side, defined up to vertical translation. The partition function of bridges of width  $T$  is  $B_T^x$ . Using (14.4), we can bound  $B_T^{x_c}$  by 1. Noting that a bridge of width  $T$  has length at least  $T$ , we obtain for  $x < x_c$

$$B_T^x \leq \left(\frac{x}{x_c}\right)^T B_T^{x_c} \leq \left(\frac{x}{x_c}\right)^T.$$

Thus, the series  $\sum_{T>0} B_T^x$  converges and so does the product  $\prod_{T>0} (1 + B_T^x)$ . Let us assume the following fact: any self-avoiding walk can be canonically decomposed into a sequence of bridges of widths  $T_{-i} < \dots < T_{-1}$  and  $T_0 > \dots > T_j$ . Furthermore, if one fixes the starting mid-edge and the first vertex visited, the decomposition uniquely determines the walk. This decomposition was first introduced by Hammersley and Welsh in [HW62] (for a modern treatment, see Section 3.1 of [MS93]). Applying the decomposition to walks starting at  $a$  (the first visited vertex is 0 or -1), we conclude

$$Z(x) \leq 2 \sum_{\substack{T_{-i} < \dots < T_{-1} \\ T_j < \dots < T_0}} \left( \prod_{k=-i}^j B_{T_k}^x \right) = \prod_{T>0} (1 + B_T^x)^2 < \infty.$$

The factor 2 is due to the fact that there are two possibilities for the first vertex once we fix the starting mid-edge. Therefore,  $Z(x) < +\infty$  whenever  $x < x_c$  and  $\mu \leq x_c^{-1} = \sqrt{2 + \sqrt{2}}$ . To complete the proof of the theorem it only remains to prove that such a decomposition into bridges does exist. Once again, this fact is well-known [MS93, HW62], we include the proof nevertheless.

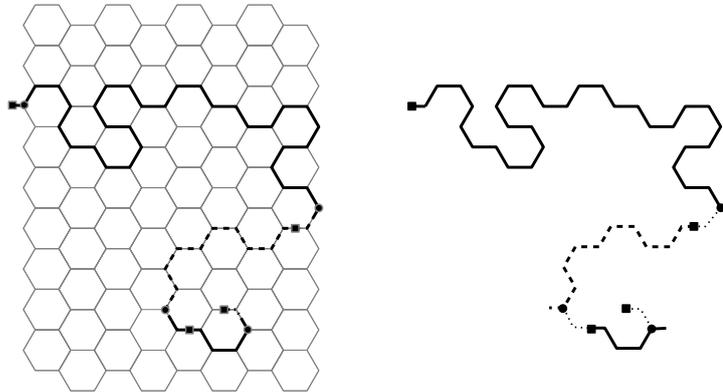


Figure 14.4: **Left:** Decomposition of a half-plane walk into four bridges with widths  $8 > 3 > 1 > 0$ . The first bridge corresponds to the maximal bridge containing the origin. Note that the decomposition contains one bridge of width 0. **Right:** The reverse procedure. If the starting mid-edge and the first vertex are fixed, the decomposition is unambiguous.

First assume that  $\tilde{\gamma}$  is a half-plane self-avoiding walk, meaning that the start of  $\tilde{\gamma}$  has extremal real part: we prove by induction on the width  $T_0$  that the walk admits a

canonical decomposition into bridges of widths  $T_0 > \dots > T_j$ . Without loss of generality, we assume that the start has minimal real part. Out of the vertices having the maximal real part, choose the one visited last, say after  $n$  steps. The  $n$  first vertices of the walk form a bridge  $\tilde{\gamma}_1$  of width  $T_0$ , which is the first bridge of our decomposition when prolonged to the mid-edge on the right of the last vertex. We forget about the  $(n + 1)$ -th vertex, since there is no ambiguity in its position. The consequent steps form a half-plane walk  $\tilde{\gamma}_2$  of width  $T_1 < T_0$ . Using the induction hypothesis,  $\tilde{\gamma}_2$  admits a decomposition into bridges of widths  $T_1 > \dots > T_j$ . The decomposition of  $\tilde{\gamma}$  is created by adding  $\tilde{\gamma}_1$  before the decomposition of  $\tilde{\gamma}_2$ .

If the walk is a reverse half-plane self-avoiding walk, meaning that the end has extremal real part, set the decomposition to be the decomposition of the reverse walk in the reverse order. If  $\gamma$  is a self-avoiding walk in the plane, one can cut the trajectory into two pieces  $\gamma_1$  and  $\gamma_2$ : the vertices of  $\gamma$  up to the first vertex of maximal real part, and the remaining vertices. The decomposition of  $\gamma$  is given by the decomposition of  $\gamma_1$  (with widths  $T_{-i} < \dots < T_{-1}$ ) plus the decomposition of  $\gamma_2$  (with widths  $T_0 > \dots > T_j$ ).

Once the starting mid-edge and the first vertex are given, it is easy to check that the decomposition uniquely determines the walk by exhibiting the reverse procedure, see Fig. 14.4 for the case of half-plane walks. □

**Remark 14.6.** *The proof provides bounds for the number of bridges from a to the right side of the strip of width  $T$ , namely,*

$$c/T \leq B_T^{x_c} \leq 1.$$

*In Sections 3.4.2 and 3.4.3 of [LSW04b], precise behaviors are conjectured for the number of self-avoiding walks between two points on the boundary of a domain. It easily implies the following estimate:*

$$\sum_{\gamma \subset S_T: 0 \rightarrow T+iyT} x_c^{\ell(\gamma)} \approx T^{-5/4} H(0, 1+iy)^{5/4}$$

*where  $H$  is the boundary derivative of the Poisson kernel. Integrating with respect to  $y$ ,  $B_T^{x_c}$  can be shown to decay as  $T^{-1/4}$  when  $T$  goes to infinity. Similar (conjectured) asymptotics are available for walks in  $S_T$  from 0 to  $iyT$ .*

### 3 Conjectures

In [Nie82], Nienhuis predicted that there exists  $A > 0$  such that

$$c_n \sim A n^{\gamma-1} \sqrt{2+\sqrt{2}}^n \tag{14.7}$$

where  $\gamma = 43/32$ . He also conjectured that the so-called mean-square displacement  $\langle |\gamma(n)|^2 \rangle$  would satisfy

$$\langle |\gamma(n)|^2 \rangle = \frac{1}{c_n} \sum_{\gamma \text{ } n\text{-step SAW}} |\gamma(n)|^2 \sim B n^{2\nu} \tag{14.8}$$

where  $\nu = 2/3$  and  $B$  is a constant. Despite the precision of the predictions (14.7) and (14.8), the best rigorously known bounds are very far from tight and almost 50 years old (see [MS93] for a complete account). For this reason, the derivation of these exponents is one of the most challenging problems in probability.

It was shown in [LSW04b] that  $\gamma$  and  $\nu$  could be computed if the scaling limit of self-avoiding walks was conformally invariant. More precisely, let  $\Omega \neq \mathbb{C}$  be a simply connected domain in the complex plane  $\mathbb{C}$  with two points  $a$  and  $b$  on the boundary. For  $\delta > 0$ , consider the discrete approximation given by the largest finite domain  $\Omega_\delta$  of  $\delta\mathbb{H}$  included in  $\Omega$ , and  $a_\delta$  and  $b_\delta$  to be the vertices of  $\Omega_\delta$  closest to  $a$  and  $b$  respectively. A probability measure  $\mathbb{P}_{x,\delta}$  is defined on the set of self-avoiding trajectories  $\gamma$  between  $a_\delta$  and  $b_\delta$  that remain in  $\Omega_\delta$  by assigning to  $\gamma$  a weight proportional to  $x^{\ell(\gamma)}$ . We obtain a random curve denoted  $\gamma_\delta$ . Conformal invariance of self-avoiding walks can be stated in the following form:

**Conjecture 14.7.** *Let  $\Omega$  be a simply connected domain (not equal to  $\mathbb{C}$ ) with two distinct points  $a, b$  on its boundary. For  $x = x_c$ , the law of  $\gamma_\delta$  in  $(\Omega_\delta, a_\delta, b_\delta)$  converges when  $\delta \rightarrow 0$  to the (chordal) Schramm-Loewner Evolution with parameter  $8/3$  in  $\Omega$  from  $a$  to  $b$ .*

A possible approach to proving Conjecture 14.7 might be the following. First, prove a tightness result for self-avoiding walks. Then, by taking a subsequence, the discrete curves  $\gamma_\delta$  can be assumed to converge to a continuous one (in fact, the limiting object would need to be a Loewner chain). The second step would consist in identifying the possible limits. The parafermionic observable should play a crucial role in this step. Indeed, define  $F_\delta(\cdot) = F(a_\delta, \cdot, x_c, 5/8)$  to be the parafermionic observable in the domain  $(\Omega_\delta, a_\delta)$ . If  $F_\delta$  converges when rescaled to an explicit function, one could use the martingale technique introduced in [Smi10a] to verify that the only possible limit is SLE(8/3).

Regarding the convergence of  $F_\delta$ , first recall that in the discrete setting contour integrals should be performed along dual edges. For  $\mathbb{H}$ , the dual edges form a triangular lattice, and Lemma 14.3 has the enlightening interpretation that the contour integral vanishes along any elementary dual triangle. Any simply connected area enclosed by a discrete closed dual contour is a union of elementary triangles, and hence the integral along any discrete closed contour also vanishes. This is a discrete analogue of Morera's theorem. It implies that *if* the limit of  $F_\delta$  (properly rescaled) exists and is continuous, then it is automatically holomorphic. By studying the boundary conditions, it is even possible to identify the limit. This leads to the following conjecture, which is based on ideas in [Smi10a].

**Conjecture 14.8.** *Let  $\Omega$  be a simply connected domain (not equal to  $\mathbb{C}$ ), let  $z \in \Omega$ , and let  $a, b$  be two distinct points on the boundary of  $\Omega$ . Assume that the boundary of  $\Omega$  is smooth near  $b$ . For  $\delta > 0$ , let  $F_\delta$  be the holomorphic observable in the domain  $(\Omega_\delta, a_\delta)$  approximating  $(\Omega, a)$ , and let  $z_\delta$  be the closest point in  $\Omega_\delta$  to  $z$ . Then*

$$\lim_{\delta \rightarrow 0} \frac{F_\delta(z_\delta)}{F_\delta(b_\delta)} = \left( \frac{\phi'(z)}{\phi'(b)} \right)^{5/8} \tag{14.9}$$

where  $\Phi$  is a conformal map from  $\Omega$  to the upper half-plane mapping  $a$  to  $\infty$  and  $b$  to  $0$ .

The right-hand side of (14.9) is well-defined, since the conformal map  $\phi$  is unique up to multiplication by a real factor. Answering this conjecture would be a major step toward Conjecture 14.7 and the derivation of critical exponents.

## 4 Connective constant of the $3.12^2$ lattice

It is easy to deduce the connective constant of another lattice in an elementary way, as was observed in [JG98]. Consider the lattice  $\mathcal{L}$  obtained from the hexagonal lattice by replacing every vertex by a triangle, see Fig. 14.5. This lattice is called the  $3.12^2$  lattice in physics literature.

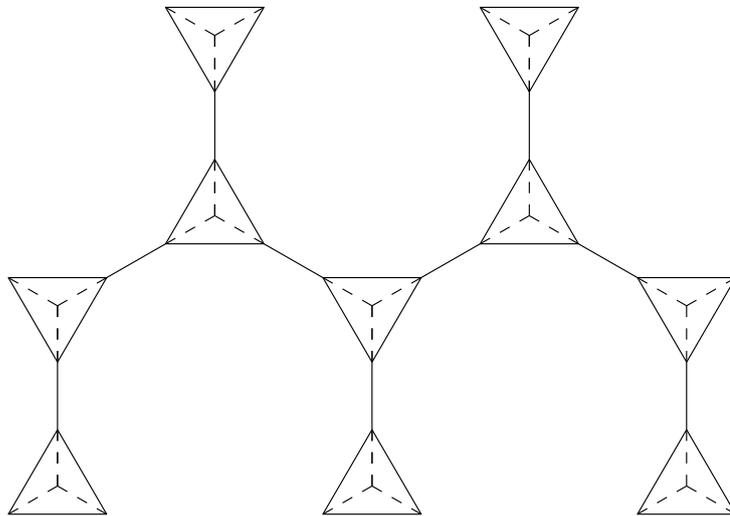


Figure 14.5: The  $3.12^2$  lattice  $\mathcal{L}$ .

**Theorem 14.9.** *Connective constants  $\mu(\mathcal{L})$  is the positive root of*

$$x^3 - \sqrt{2 + \sqrt{2}} x = \sqrt{2 + \sqrt{2}}.$$

In particular, it is algebraic of order 12, and can be computed explicitly (even though we will not impose such a pain to the reader).

**Proof** Set  $G_{\mathbb{H}}$  for the partition function of self-avoiding walks starting from mid-edges in the hexagonal lattice. Call a vertex  $v$  of a self-avoiding walk  $\omega$  in  $\mathcal{L}$  *pivot* if it is the center of an edge not in a triangle. The sequence of pivots forms a self-avoiding walk on the mid-edges of the hexagonal lattice. Moreover, the possibilities between two pivots are limited: the part of the walk is either composed of two edges forming the geodesic

between the two pivots, or it contains three edges, then the last two are using the other two edges of the triangle associated to the second pivot. Then, the partition function  $G_{\mathcal{L}}$  of self-avoiding walks in  $\mathcal{L}$  satisfies

$$G_{\mathcal{L}}(z) = G_{\mathbb{H}}(z^2 + z^3)$$

which implies that  $\mu(\mathcal{L})^{-3} + \mu(\mathcal{L})^{-2} = \mu(\mathbb{H})^{-1}$ . □



# Chapter 15

## Supercritical self-avoiding walks are space-filling

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**Abstract:** Supercritical self-avoiding walks are proved to be space-filling on any lattice with sufficient symmetries (in the following exposition, we restrict ourselves to  $\mathbb{Z}^d$ ). This chapter is inspired by the article *The supercritical self-avoiding walk is space-filling* written with Gady Kozma and Ariel Yadin [DCKY11].

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Let  $\Omega$  be a (nice) simply connected domain in  $\mathbb{R}^d$  with two points  $a, b$  on the boundary. For  $\delta > 0$ , recall that  $\Omega_\delta := \Omega \cap \delta\mathbb{L}$  and  $a_\delta, b_\delta$  are the two sites of  $\Omega_\delta$  closest to  $a$  and  $b$  respectively. We further assume that  $\Omega_\delta$  is connected (this is true for  $\delta$  small enough if  $\Omega$  is sufficiently nice). We think of  $(\Omega_\delta, a_\delta, b_\delta)$  as being an approximation of  $(\Omega, a, b)$ .

**Definition 15.1.** Let  $x > 0$ . On  $(\Omega_\delta, a_\delta, b_\delta)$ , define a probability measure on the finite set of self-avoiding walks in  $\Omega_\delta$  from  $a_\delta$  to  $b_\delta$  by the formula

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}(\gamma) = \frac{x^{|\gamma|}}{Z_{(\Omega_\delta, a_\delta, b_\delta)}(x)} \quad (15.1)$$

where  $|\gamma|$  is the length of  $\gamma$  (i.e. the number of edges), and  $Z_{(\Omega_\delta, a_\delta, b_\delta)}(x)$  is a normalizing factor. A random curve  $\gamma_\delta$  with law  $\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}$  is called the self-avoiding walk with parameter  $x$  in  $(\Omega_\delta, a_\delta, b_\delta)$ .

$Z_{(\Omega_\delta, a_\delta, b_\delta)}(x) = \sum_\gamma x^{|\gamma|}$  (the sum is over all self-avoiding walks in  $\Omega_\delta$  from  $a_\delta$  to  $b_\delta$ ) is sometimes called the *partition functions* (or *generating function*) of self-avoiding walks from  $a_\delta$  to  $b_\delta$  in the domain  $\Omega_\delta$ .

When the domain  $(\Omega, a, b)$  is fixed, we are interested in the scaling limit of the family  $(\gamma_\delta)$ , i.e. its geometric behavior when  $\delta$  goes to 0. The qualitative behavior is expected to differ drastically depending on the value of  $x$ . A phase transition occurs at the value  $x_c = 1/\mu$ , where  $\mu$  is the connective constant:

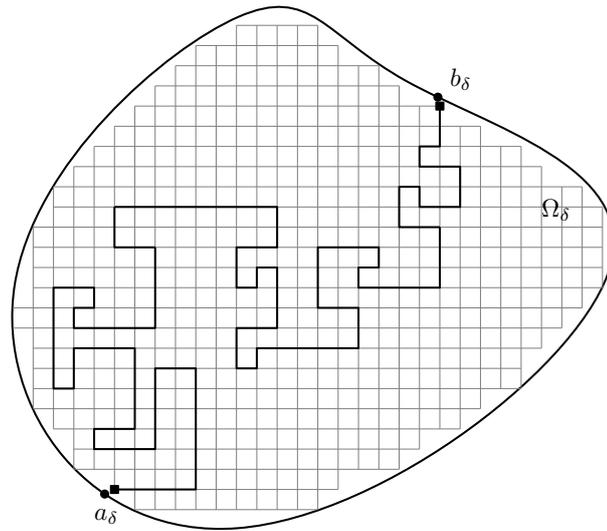


Figure 15.1: A domain  $\Delta$  with two points  $a$  and  $b$  on the boundary (circle dots). The set  $\Omega_\delta$  with points  $a_\delta$  and  $b_\delta$ . An example of possible walk from  $a_\delta$  to  $b_\delta$  is presented. Note that there are a finite number of them.

**When  $x < 1/\mu$ :**  $\gamma_\delta$  converges to a deterministic curve corresponding to the geodesic between  $a$  and  $b$  in  $\Delta$ . When rescaled,  $\gamma_\delta$  has Gaussian fluctuation of order  $\sqrt{\delta}$  around the geodesic. We refer to [Iof98] for a deeper study of this regime.

**When  $x = 1/\mu$ :**  $\gamma_\delta$  should converge to a random simple curve. In dimensions five and higher, it was proved in [HS91, HS92] (see also the book [MS93]) that the limit is a Brownian motion from  $a$  to  $b$  conditioned to stay in the domain  $\Omega$ . In fact, the literature studies the self-avoiding walk in the whole plane, but the reasoning can be applied to study self-avoiding walks in a domain. The behavior in dimension four should be the same, yet the investigation is much more difficult (it is the so-called critical dimension), see [BIS09] and references within. As mentioned before, in dimension two, the scaling limit is conjectured to be SLE(8/3). Finally, dimension three remains a mystery, and no clear candidate is known for the scaling limit of self-avoiding walks.

**When  $x > 1/\mu$ :**  $\gamma_\delta$  is expected to become space-filling in the following sense: for any open set  $U \subset \Omega$ ,

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}[\gamma_\delta \cap U = \emptyset] \rightarrow 0$$

when  $\delta$  goes to 0. On the one hand, let us mention that  $(\gamma_\delta)$  is not predicted to have a scaling limit when  $d \geq 3$ . On the other hand, the scaling limit is expected [Smi06] to exist in dimension two (it should be the Schramm-Loewner Evolution of parameter 8, which is conformally invariant).

At a microscopic level,  $\gamma_\delta$  cannot be space-filling. Nevertheless, one can quantify the size of the biggest hole not visited by the walk. The subject of this paper is the proof of

a result which quantifies how  $\gamma_\delta$  becomes space filling. Here is the precise formulation.

**Theorem 15.2.** *Let  $(\Omega, a, b)$  be a bounded domain with two points on the boundary. For every  $x > 1/\mu$ , there exist  $\xi = \xi(x) > 0$  and  $c = c(x) > 0$  such that*

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)} \left[ \text{the biggest component of } \Omega_\delta \setminus \Gamma_\delta^\xi \text{ is larger than } c \log(1/\delta) \right] \rightarrow 0$$

when  $\delta \rightarrow 0$ , where  $\Gamma_\delta^\xi$  is the set of sites in  $\Omega_\delta$  at graph distance less than  $\xi$  of  $\gamma_\delta$ .

The strategy of the proof is fairly natural. We first prove that in the supercritical phase, one can construct self-avoiding polygons in a prescribed box. Then, we show that the self-avoiding walk cannot leave holes that are too large, since adding polygons in the big holes would decrease the energy (of course, one needs to be careful about the classical energy/entropy competition). We present the proof **only in the case**  $d = 2$ , even though the reasoning carries over to all dimensions without difficulty (see Remark 15.6). One can also extend the result to other lattices with sufficient symmetry in a straightforward way (for instance to the hexagonal lattice).

## 1 Self-avoiding polygons in a square

In this section, we think of a walk as being indexed by (discrete) time  $t$  from 0 to  $n$ . For  $m > 0$ , let  $P_m$  be the set of self-avoiding polygons in  $[0, 2m + 1]^2$  that touch the middle of every face of the cube: more formally, such that the edges  $[(m, 0), (m + 1, 0)]$ ,  $[(2m + 1, m), (2m + 1, m + 1)]$ ,  $[(m, 2m + 1), (m + 1, 2m + 1)]$  and  $[(0, m), (0, m + 1)]$  belong to the polygon, see Fig 15.4. For  $x > 0$ , let  $Z_m(x)$  be the partition function (with parameter  $x$ ) of  $P_m$ , i.e.

$$Z_m(x) = \sum_{\gamma \in P_m} x^{|\gamma|}.$$

**Proposition 15.3.** *Let  $x > 1/\mu$ , we have  $\limsup_{m \rightarrow \infty} Z_m(x) = \infty$ .*

It is classical that the number of self-avoiding walks with certain constraints grows with the same exponential speed as the number of self-avoiding walks without constraints. For instance, the number  $b_n$  of *self-avoiding bridges* of length  $n$ , meaning self-avoiding walks  $\gamma$  of length  $n$  such that  $\text{Im}(\gamma_0) = \min_{t \in [0, n]} \text{Im}(\gamma_t)$  and  $\text{Im}(\gamma_n) = \max_{t \in [0, n]} \text{Im}(\gamma_t)$  satisfies

$$e^{-c\sqrt{n}} \mu^n \leq b_n \leq \mu^n \tag{15.2}$$

for every  $n$  [HW62] via Corollary 13.6. In the following, we need a class of walks with even more restrictive constraints. A *squared walk (of span  $k$ )* is a self-avoiding walk such that  $\gamma_0 = (0, 0)$ ,  $\gamma_n = (k, k)$  and  $\gamma \subset [0, k]^2$ .

**Lemma 15.4.** *There exists  $c > 0$  such that the number  $a_n$  of squared walks of length  $n$  satisfies*

$$a_n \geq \mu^n e^{-c\sqrt{n}}.$$

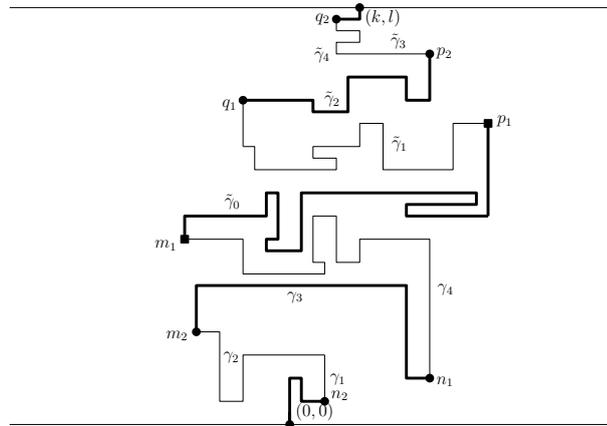


Figure 15.2: The decomposition of a bridge into walks. One can construct a squared walk in a rectangle by reflecting non-bold walks and then concatenate all the walks together.

**Proof**

**Step 1: Self-avoiding walks in rectangles** Let  $\Lambda_n$  be the set of self-avoiding bridges of length  $n$  starting at the origin. Let  $\Sigma_n$  be the set of  $n$ -step self-avoiding walks for which there exists  $(k, l)$  such that  $\gamma_0 = (0, 0)$ ,  $\gamma_n = (k, l)$  and  $\gamma \subset [0, k] \times [0, l]$ . We construct a map from  $\Lambda_n$  to  $\Sigma_n$ . Let  $x(v)$  and  $y(v)$  be the first and the second coordinates of the vertex  $v$ .

Fix  $\gamma \in \Lambda_n$  and denote by  $m_1$  the first time at which  $x(\gamma_{m_1}) = \min_{t \in [0, n]} x(\gamma_t)$ , see Fig. 15.2. Then, define  $n_1$  to be the first time at which  $x(\gamma_{n_1}) = \max_{t \in [0, m_1]} x(\gamma_t)$ . One can then define recursively  $m_k, n_k$ , by the formulæ

$$m_k = \min\{r \leq n_{k-1} : x(\gamma_r) = \min_{t \in [0, n_{k-1}]} x(\gamma_t)\}$$

$$n_k = \min\{r \leq m_k : x(\gamma_r) = \max_{t \in [0, m_k]} x(\gamma_t)\}$$

We stop the induction the first time  $m_k$  or  $n_k$  equals 0. For convenience, if the first time is  $n_k$ , we add a further step  $m_{k+1} = 0$ . We are then in the possession of a sequence of integers  $m_1 > n_1 > m_2 > ..m_r \geq n_r \geq 0$  and a sequence of walks  $\gamma_{2r-1} = \gamma[n_1, m_1]$ ,  $\gamma_{2r-2}[m_2, n_1], \dots, \gamma_1 = \gamma[0, m_r]$ . Note that the width of the walks  $\gamma_i$  is strictly increasing (see Fig. 15.2 again).

Similarly, let  $p_1$  be the last time at which  $x(\gamma_{p_1}) = \max_{t \in [m_1, n]} x(\gamma_t)$  and  $q_1$  the last time at which  $x(\gamma_{q_1}) = \min_{t \in [p_1, n]} x(\gamma_t)$ . Then define recursively  $p_k$  and  $q_k$  by the following formula

$$p_k = \max\{r \geq q_{k-1} : x(\gamma_r) = \max_{t \in [q_{k-1}, n]} x(\gamma_t)\}$$

$$q_k = \max\{r \leq p_k : x(\gamma_r) = \min_{t \in [p_k, n]} x(\gamma_t)\}$$

This procedure stops eventually and we obtain another sequence of walks  $\tilde{\gamma}_0 = \gamma[m_1, p_1]$ ,  $\tilde{\gamma}_1 = \gamma[p_1, q_1]$ , etc... This time, the width of the walks is strictly decreasing, see Fig. 15.2 one more time.

For a walk  $\omega$ , we set  $\sigma(\omega)$  to be its horizontal reflexion with respect to its starting point. Let  $f(\gamma)$  be the concatenation of  $\gamma_1, \sigma(\gamma_2), \gamma_3, \dots, \sigma(\gamma_r), \tilde{\gamma}_0, \sigma(\tilde{\gamma}_1), \tilde{\gamma}_2$  and so on. This walk is contained in the rectangle with corners being its endpoints so that  $f$  maps  $\Omega_n$  on  $\Sigma_n$ .

In order to estimate the cardinality of  $\Sigma_n$ , we remark that each element of  $\Sigma_n$  has a limited number of possible pre-images under  $f$ . More precisely, the map which gives  $f(\gamma)$  and the widths of the walks  $(\gamma_i)$  and  $(\tilde{\gamma}_i)$  is one-to-one (the reverse procedure is easy to identify). The number of possible widths for  $\gamma_i$  and  $\tilde{\gamma}_i$  is the number of pairs of *decreasing* sequences partitioning an integer  $l \leq n$ . This number is bounded by  $e^{c\sqrt{n}}$  (Theorem 13.4). Therefore, the number of possible pre-images under  $f$  is bounded by  $e^{c\sqrt{n}}$ . The cardinality of  $\Sigma_n$  is thus larger than  $e^{-c\sqrt{n}}b_n \geq e^{-2c\sqrt{n}}\mu^n$ .

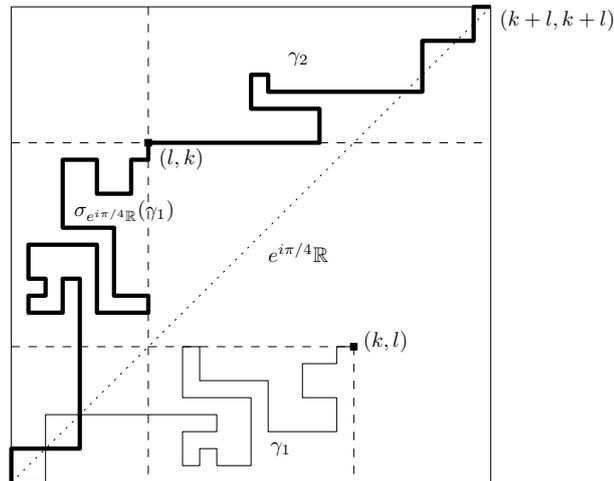


Figure 15.3: This figure depicts the passage of two walks in the rectangle  $[0, k] \times [0, l]$  to a walk in the square  $[0, k+l]^2$ .

**Step 2: Self-avoiding walks in squares** We have bounded from below the number of  $n$ -steps self-avoiding walks ‘contained in a rectangle’. We now extend this bound to the case of squares. There exist  $k, l \leq n$  such that the number of elements of  $\Sigma_n$  with  $(k, l)$  as an ending point is larger than  $e^{-2c\sqrt{n}}\mu^n/n^2$ . By taking two arbitrary walks of  $\Sigma_n$  ending at  $(k, l)$ , one can construct a  $2n$ -steps self-avoiding walk with  $\gamma_0 = (0, 0)$  and  $\gamma_{2n} = (k+l, k+l)$  contained in  $[0, k+l]^2$  by reflecting orthogonally to  $e^{i\pi/4}\mathbb{R}$  the first walk, and then concatenating both, see Fig. 15.3. We deduce that  $a_{2n} \geq \mu^{2n}e^{-4c\sqrt{n}}/n^4$ . The claim follows readily by choosing  $n$  large enough.  $\square$

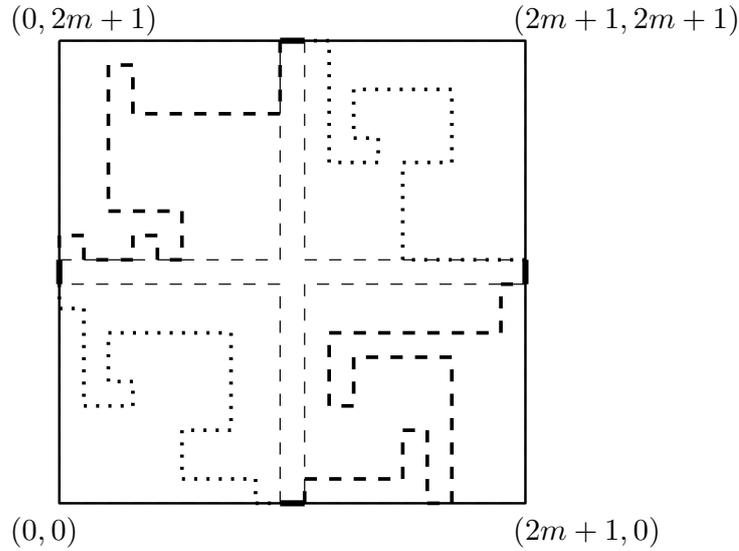


Figure 15.4: by concatenating four walks in squares of size  $m$  (plus four edges), one obtains a loop in the square of size  $2m + 1$  going through the middle of the sides.

**Proof of Proposition 15.3** Squared walks with length  $n$  were defined as walks between corners of some  $m \times m$  square, but  $m$  was not fixed. Fix now  $m$  to be such that the number of such walks is maximized (and then it is  $\geq a_n/n$  where  $a_n$  is the total number of squared walks). From any quadruplet  $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$  of such squared self-avoiding walks, one can construct a self-avoiding polygon of  $P_m$  as follows (See Figure 15.4):

- translate  $\gamma_1$  and  $\gamma_3$  by  $(m + 1, 0)$  and  $(0, m + 1)$  respectively,
- rotate  $\gamma_2$  and  $\gamma_4$  by an angle  $\pi/2$ , and then translate them by  $(m, 0)$  and  $(2m + 1, m + 1)$  respectively,
- add the four edges  $[(m, 0), (m + 1, 0)]$ ,  $[(2m + 1, m), (2m + 1, m + 1)]$ ,  $[(m, 2m + 1), (m + 1, 2m + 1)]$  and  $[(0, m), (0, m + 1)]$ .

Since each walk is contained in a square, one can easily check that we obtain a  $(4n + 4)$ -step polygon in  $P_m$ . Using Lemma 15.4, we obtain

$$Z_m(x) \geq x^{4n+4} \left(\frac{a_n}{n}\right)^4 \geq \left(\frac{x^{n+1} \mu^n e^{-c\sqrt{n}}}{n}\right)^4.$$

□

## 2 Proof of Theorem 15.2

The strategy is the following. We first show that for some hole (namely it will be a union of connected boxes of some size  $m$ ), the probability that the self-avoiding walk gets close

to it without intersecting it can be estimated in terms of  $Z_m(x)$ . This claim is the core of the argument, and is presented in Proposition 15.5. Next, we show that choosing  $m$  large enough (or equivalently  $Z_m(x)$  large enough), the probability to avoid some connected union of  $k$  boxes decays exponentially fast in  $k$ , thus implying the claim.

Let  $m > 0$ . A *cardinal edge* of a (square) box  $B$  of side length  $2m + 1$  is an edge of the lattice in the middle of one of the sides of  $B$ . For  $m \in \mathbb{N}$ , two boxes  $B$  and  $B'$  of side length  $2m + 1$  are said to be *adjacent* if they are disjoint and two of their cardinal edges  $[xy]$  and  $[zt]$  are such that  $x \sim z$ ,  $y \sim t$  (see Fig. 15.5). A family  $F$  of boxes is called *connected* if every two boxes can be connected by a path of adjacent boxes in  $F$ .

Let  $\delta > 0$ . Let  $\mathcal{F}(\Omega_\delta, m)$  be the set of connected families of boxes included in  $\Omega_\delta$ . For  $F \in \mathcal{F}(\Omega_\delta, m)$ , let  $\mathcal{V}_F$  be the set of vertices in boxes of  $F$ , and let  $\mathcal{E}_F$  be the set of edges with both end-points in  $\mathcal{V}_F$ . Let  $\text{dist}(\cdot, \cdot)$  be the graph distance on  $\mathbb{Z}^2$ .

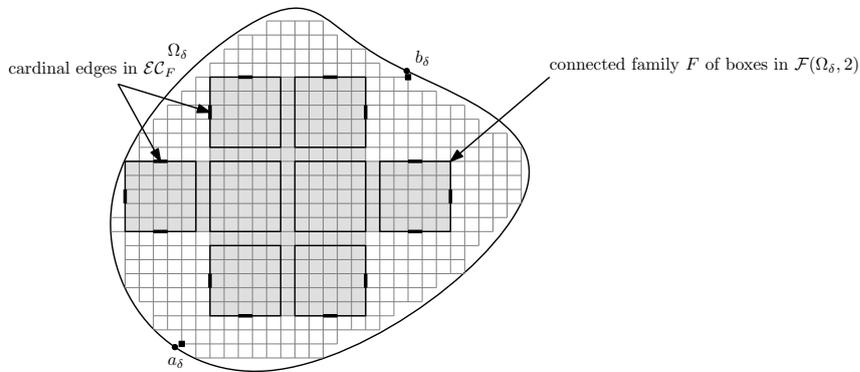


Figure 15.5: A discrete domain with a connected component of adjacent boxes of size 5 ( $m = 2$ ). Edges of  $\mathcal{E}_F$  lie in the gray area.

**Proposition 15.5.** *Let  $(\Omega, a, b)$  be a domain with two points on the boundary. Fix  $\delta > 0$  and  $m \in \mathbb{N}$ . There exists  $C = C(x, m) < \infty$  such that for every  $F \in \mathcal{F}(\Omega_\delta, m)$ ,*

$$\mathbb{P}_{\delta, x}(0 < \text{dist}(\gamma_\delta, \mathcal{V}_F) < 4m + 4) \leq CZ_m(x)^{-|F|}.$$

(recall that  $\mathbb{P}_{\delta, x}$  is our measure on self-avoiding walks with parameter  $x$ , on the discretized domain  $\Omega_\delta$ )

**Proof** For  $F \in \mathcal{F}(\Omega_\delta, m)$ , let  $\mathcal{E}_{C_F}$  be the set of *external cardinal edges* of  $F$  i.e. all cardinal edges in boxes of  $F$  which have neighbors outside of  $F$ . Let  $S_F$  be the set of self-avoiding polygons included in  $\mathcal{E}_{C_F}$  visiting all the edges in  $\mathcal{E}_{C_F}$ . Let  $Z_F(x)$  is the partition function of polygons in  $S_F$ . We have:

*Claim:* for  $F \in \mathcal{F}(\Omega_\delta, m)$ ,  $Z_F(x) \geq Z_m(x)^{|F|}$ .

*Proof Claim.* We prove the result by induction on the cardinality of  $F \in \mathcal{F}(\Omega_\delta, \xi)$ . If the cardinality of  $F$  is 1,  $Z_F(x) = Z_m(x)$  by definition. Consider  $F_0 \in \mathcal{F}(\Omega_\delta, \xi)$  and assume the statement true for every  $F \in \mathcal{F}(\Omega_\delta, \xi)$  with  $|F| < |F_0|$ . There exists a box  $B$  in  $F_0$

such that  $F_0 \setminus \{B\}$  is still connected. Therefore, for every couple  $(\gamma, \gamma') \in S_{\{B\}} \times S_{F_0 \setminus \{B\}}$ , one can associate a polygon in  $S_F$  in a one-to-one fashion. Indeed,  $B$  is adjacent to a box  $B' \in F_0 \setminus \{B\}$  so that one of the four cardinal edges (called  $[ab]$ ) of  $B$  is adjacent to a cardinal edge  $[cd]$  of  $B'$ . Note that  $[cd]$  belongs to  $\mathcal{EC}_{F_0 \setminus \{B\}}$ . Then, by changing the edges  $[cd]$  and  $[ab]$  of  $\gamma$  and  $\gamma'$  into the edges  $[ac]$  and  $[bd]$ , one obtains a polygon in  $S_F$ . Furthermore, the construction is one-to-one and we deduce

$$Z_{F_0}(x) \geq Z_{F_0 \setminus \{B\}}(x)Z_B(x) \geq Z_m^{|F_0 \setminus \{B\}|} Z_m(x) \geq Z_m(x)^{|F_0|}.$$

□

Consider the set  $\Theta_F$  of walks not intersecting  $F$  yet going to distance  $4m + 4$  of it. Since  $\gamma \in \Theta_F$  is at graph distance less than  $4m + 4$  of  $\mathcal{E}_F$ , it is at a distance less than  $5m + 4$  of some cardinal edge  $e \in \mathcal{EC}_F$  of a box in  $F$ . For each  $\gamma \in \Theta_F$ , consider a self-avoiding polygon  $\tilde{\gamma} = \tilde{\gamma}(\gamma)$  satisfying the three following properties:

- it contains  $e$  and is included in  $(\Omega_\delta \setminus \mathcal{E}_F) \cup \{e\}$ ,
- it intersects  $\gamma$  at one or two adjacent edges only,
- it has length smaller than  $100m$ .

One can easily check that such a polygon always exists, see Fig. 15.6.

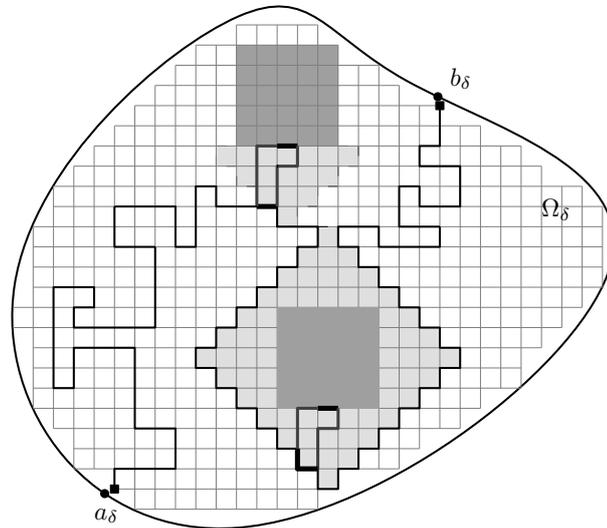


Figure 15.6: The polygon  $\tilde{\gamma}_1$  is in gray. It overlaps the curve in one edge exactly, except in the configuration depicted at the bottom, where we have no choice but overlapping the walk on two edges.

Now, consider the application that associates to  $(\gamma_1, \gamma_2) \in \Theta_F \times S_F$  the symmetric difference  $\gamma = f(\gamma_1, \gamma_2)$  of  $\gamma_1$ ,  $\tilde{\gamma}_1$  and  $\gamma_2$  (symmetric difference here meaning as sets of

edges). Note that the object that we obtain is a self-avoiding walk, which can be verified by noting that each vertex has degree 0 or 2 and that the set is connected. Further, its length is equal to  $|\gamma_1| + |\tilde{\gamma}_1| + |\gamma_2| - 1$  or  $|\gamma_1| + |\tilde{\gamma}_1| + |\gamma_2| - 2$  (depending on the fact that  $\gamma_1$  and  $\tilde{\gamma}_1$  intersect at one or two adjacent edges). Moreover, the application is one-to-one so that

$$\begin{aligned} Z_{\Theta_F}(x) \cdot Z_F(x) &= \left( \sum_{\gamma_1 \in \Theta_F} x^{|\gamma_1|} \right) \left( \sum_{\gamma_2 \in S_F} x^{|\gamma_2|} \right) \\ &\leq \max(1, x^{-100m}) \sum_{\gamma_1 \in \Theta_F, \gamma_2 \in S_F} x^{|\gamma_1| + |\tilde{\gamma}_1| + |\gamma_2|} \\ &\leq \max(1, x^{-100m}) \max(x, x^2) \sum_{\gamma_1 \in \Theta_F, \gamma_2 \in S_F} x^{|f(\gamma_1, \gamma_2)|} \\ &\leq \max(x^2, x^{-100m+1}) \sum_{\gamma \in f(\Theta_F \times S_F)} x^{|\gamma|} \\ &\leq \max(x^2, x^{-100m+1}) Z_{(\Omega_\delta, a_\delta, b_\delta)}(x), \end{aligned}$$

where in the second line we have used the fact that  $\tilde{\gamma}_1$  has length smaller than  $100m$ , in the third line the fact that  $|f(\gamma_1, \gamma_2)|$  equals  $|\gamma_1| + |\tilde{\gamma}_1| + |\gamma_2| - 1$  or  $|\gamma_1| + |\tilde{\gamma}_1| + |\gamma_2| - 2$ , and in the fourth the fact that  $f$  is one-to-one. Using the claim, the previous inequality implies

$$\mathbb{P}_{\delta, x}(0 < \text{dist}(\gamma_\delta, \mathcal{V}_F) < 4m + 4) = \frac{Z_{\Theta_F}(x)}{Z_{(\Omega_\delta, a_\delta, b_\delta)}(x)} \leq \frac{C}{Z_F(x)} \leq \frac{C}{Z_m(x)^{|F|}}.$$

□

**Proof of Theorem 15.2 in dimension 2** Let  $x > 1/\mu$  and  $(\Omega, a, b)$  a domain with two points on the boundary be from the statement of the theorem. Let  $A_n$  be the number of connected subsets of  $\mathbb{Z}^d$  containing 0. It is well known that  $\overline{\lim} \sqrt[n]{A_n}$  is finite (see e.g. Theorem 4.20 in [Gri99]). Let therefore  $\lambda = \lambda(d)$  satisfy  $A_n \leq \lambda^n$  for all  $n$ . We now apply Proposition 15.3 and get some  $m = m(x, d)$  such that  $Z_m(x) > 2\lambda$ . With this  $m$  set  $\xi = 4m + 4$ .

Let  $\delta > 0$  and consider the event  $\mathcal{A}(s)$  that there exists a set connected  $S$  of cardinality  $s$  at distance larger than  $\xi$  of  $\gamma_\delta$ . There must exist a maximal connected family of  $s/(2m+1)^2$  boxes of size  $2m+1$  covering  $S$ . Since the family of boxes is maximal, the distance between the union of boxes and  $\gamma_\delta$  is smaller or equal to  $\xi = 4m + 4$ . Proposition 15.5 implies

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}[\mathcal{A}(s)] \leq \sum_{F \in \mathcal{F}(\Omega_\delta, \xi): |F| \geq s} C(x, m) [Z_m(x)]^{-|F|}.$$

By the definition of  $\lambda$ , the number of families of connected boxes of size  $K$  in  $\mathcal{F}(\Omega_\delta, \xi)$  is thus bounded by  $(C/\delta^2)\lambda^K$  (since up to translation they are connected subsets of a normalized square lattice), where  $C = C(\Omega)$  depends on the area of  $\Omega$ . Therefore, for  $c > 0$ ,

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)} \left[ \mathcal{A} \left( c \log \frac{1}{\delta} \right) \right] \leq C(x, m) \frac{C(\Omega)}{\delta^2} \sum_{i \geq c(\log \frac{1}{\delta})/\xi^2} \left( \frac{\lambda}{Z_m(x)} \right)^i$$

so that the claim follows as soon as  $c$  is chosen large enough.  $\square$

**Remark 15.6.** *Let us briefly describe what needs to be changed in higher dimensions. The notion of cardinal edge must be extended: in the box  $[0, 2m + 1]^d$ , cardinal edges for the face  $[0, 2m_1]^{d-1} \times \{0\}$  are all the edges joining vertices in  $\{m, m + 1\}^{d-1} \times \{0\}$  of the form  $[(\dots, m + 1, m, m, \dots, 0), (\dots, m + 1, m + 1, m, \dots, 0)]$ . We only consider part of the edges joining vertices in  $\{m, m + 1\}^{d-1} \times \{0\}$  because we want to contain all these edges, without forming a loop. Similarly cardinal edges can be defined for every face. It can be shown that the number of polygons included in some box  $[0, 2m + 1]^d$  and visiting all the cardinal edges grows exponentially at the same speed as the number of self-avoiding walks. The proofs then apply mutatis mutandis.*

### 3 Questions

The supercritical phase exhibits an interesting behavior. We know that the curve becomes space-filling, yet we have very little additional information. For instance, a natural question is to study the length of the curve. It is not difficult to show that the length is of order  $1/\delta^2$ , yet a sharper result would be interesting:

**Question 15.7.** *For  $x > 1/\mu$ , show that there exists  $\theta(x) > 0$  such that for every  $\varepsilon > 0$  and every  $(\Omega, a, b)$ ,*

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}(|\gamma_\delta| - \theta(x) \cdot |\Omega_\delta| | > \varepsilon |\Omega_\delta|) \longrightarrow 0 \quad \text{when } \delta \rightarrow 0.$$

The quantity  $\theta(x)$  would thus be an 'averaged density' of the walk. Note that the existence of  $\theta(x)$  seems natural since the space-filling curve should look fairly similar in different portions of the space.

Another challenge is to try to say something nontrivial on the critical phase. In other words:

**Question 15.8.** *When  $x = 1/\mu$ , show that the sequence  $(\gamma_\delta)$  does not converge to a geodesic, and that it does not become space-filling.*

Finally, we recall the conjecture made in [Smi06] concerning the two-dimensional limit in supercritical phase.

**Conjecture 15.9** (Smirnov). *Let  $(\Omega, a, b)$  be a simply connected domain of  $\mathbb{C}$  and consider approximations by the hexagonal lattice. The law of  $(\gamma_\delta)$  converges to the chordal Schramm-Loewner Evolution in  $(\Omega, a, b)$*

- with parameter  $8/3$  if  $x = 1/\mu$ ,
- with parameter  $8$  if  $x > 1/\mu$ .

# Chapter 16

## Bridge Decomposition of Restriction Measures

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**Abstract:** Motivated by Kesten’s bridge decomposition for two-dimensional self-avoiding walks in the upper half plane, we show that the conjectured scaling limit of the half-plane SAW, the SLE(8/3) process, also has an appropriately defined bridge decomposition. This continuum decomposition turns out to entirely be a consequence of the restriction property of SLE(8/3), and as a result can be generalized to the wider class of *restriction measures*. Specifically we show that the restriction hulls with index less than one can be decomposed into a Poisson Point Process of *irreducible bridges* in a way that is similar to Itô’s excursion decomposition of a Brownian motion according to its zeros. This chapter is inspired by the article *Bridge decomposition of restriction measures* written with Tom Alberts and published in Journal of Statistical Physics [ADC10].

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### 1 Introduction

One of the greatest successes of the Schramm-Loewner Evolution (SLE), and the broader study of two-dimensional conformally invariant stochastic processes that it enabled, has been the ability to describe the scaling limits of two-dimensional lattice models that arise in statistical mechanics. One of the most important open problems in the field is to prove that the scaling limit of the infinite *self-avoiding walk* in the upper half plane  $\mathbb{H}$  is given by SLE(8/3) [LSW04b]. It is known that *if* the scaling limit of half-plane SAWs exists *and* is conformally invariant, then the scaling limit must be SLE(8/3). Both the existence and conformal invariance are widely believed to be true, yet proofs remain elusive. Even without formally establishing the scaling limit result, it is often still possible to independently check that the various well-studied properties of half-plane SAWs carry over to the SLE(8/3) process. The main results of this chapter should be seen in this context. In [Kes63a] it is shown that half-plane SAWs admit what is called a *bridge decomposition*,

which raised the question of finding a similar decomposition for  $\text{SLE}(8/3)$ . In this chapter we will show that an appropriately defined continuum decomposition does exist, and we will describe some of its properties. A somewhat surprising aspect of the existence is that it depends only on the fact that  $\text{SLE}(8/3)$  satisfies the restriction property, and not on the fine details of the process itself. Specifically, the decomposition has no explicit reliance on the Loewner equation. Using this fact we are able to extend the continuum bridge decomposition beyond  $\text{SLE}(8/3)$  to a wider class of random sets whose laws are given by the so-called restriction measures. These probability measures were introduced and studied extensively in [LSW03], and they occupy an important position in the hierarchy of two-dimensional conformally invariant processes. We will give a more detailed description of restriction measures in Section 2, but we emphasize that the reader who is uninterested in general restriction measures will lose nothing by focusing on  $\text{SLE}(8/3)$  as the canonical one.

### 1.1 Motivation: Bridge Decomposition of SAWs

To motivate the continuum bridge decomposition, we first describe the corresponding decomposition for half-plane SAWs. This is thoroughly described in [MS93], along with many other interesting properties of the self-avoiding walk. In the discrete setting we will work exclusively on the lattice  $\mathbb{Z} + i\mathbb{Z}$ . Recall that an  $N$ -step self-avoiding walk  $\omega$  on  $\mathbb{Z} + i\mathbb{Z}$  is a sequence of lattice sites  $[\omega(0), \omega(1), \dots, \omega(N)]$  satisfying  $|\omega(j+1) - \omega(j)| = 1$  and  $\omega(i) \neq \omega(j)$  for  $i \neq j$ . We will write  $|\omega| = N$  to denote the length of  $\omega$ . Given walks  $\omega$  and  $\omega'$  of length  $N$  and  $M$  (respectively), the *concatenation* of  $\omega$  and  $\omega'$  is defined by

$$\omega \oplus \omega' = [\omega(0), \dots, \omega(N), \omega'(1) + \omega(N), \dots, \omega'(M) + \omega(N)].$$

Letting  $c_N$  denote the number of self-avoiding walks of length  $N$ , it is easy to see that

$$c_{N+M} \leq c_N c_M$$

since any SAW of length  $N + M$  can always be written as the concatenation of two SAWs of length  $N$  and  $M$ . A standard submultiplicativity argument then proves the existence of a constant  $\mu > 0$  such that

$$\lim_{N \rightarrow \infty} \frac{\log c_N}{N} = \log \mu, \tag{16.1}$$

or  $c_N \approx \mu^N$  in the common shorthand. The exact value of  $\mu$  is not known, nor is it expected to be any special value, but numerically it has been shown that  $\mu$  is close to 2.638 (see [MS93, Section 1.2]).

We will mostly deal with half-plane SAWs rooted at the origin, i.e. self-avoiding paths  $\omega$  such that  $\omega(0) = 0$  and  $\text{Im} \omega(j) > 0$  for all  $j > 0$ . Let  $\mathcal{H}$  denote the set of all such walks. The most commonly used probability measure on  $\mathcal{H}$ , and the one that we will consider throughout, is the weak limit of the uniform measure on  $\{\omega \in \mathcal{H} : |\omega| = N\}$ , as  $N \rightarrow \infty$ . This limit is proven to exist in [MS93], and again in the appendix of [LSW04b]. The key

element of both proofs is, in fact, the *bridge decomposition* of the walks in  $\mathcal{H}$ , the study of which was initiated by Kesten [Kes63a, Kes64a] and goes as follows. A *bridge of length*  $N$  is a self-avoiding walk  $\omega$  such that  $|\omega| = N$  and

$$\operatorname{Im} \omega(0) < \operatorname{Im} \omega(j) \leq \operatorname{Im} \omega(N), \quad 1 \leq j \leq N.$$

Note that the concatenation of any two bridges is still a bridge, but that not every bridge is the concatenation of two shorter ones. A bridge with the latter property is said to be *irreducible*, and such bridges are the basic building blocks of walks in  $\mathcal{H}$ . Indeed, given any  $\omega \in \mathcal{H}$ , one performs a bridge decomposition of  $\omega$  by searching for the smallest time  $j$  such that  $\operatorname{Im} \omega(k) \leq \operatorname{Im} \omega(j)$  for  $k \leq j$  and  $\operatorname{Im} \omega(k) > \operatorname{Im} \omega(j)$  for  $k > j$ . By the minimality of  $j$ , the subpath  $[w(0), w(1), \dots, w(j)]$  is an irreducible bridge, and the shifted subpath  $[0, w(j+1) - w(j), \dots, w(k) - w(j), \dots]$  for  $k \geq j$  is a new element of  $\mathcal{H}$  on which we may repeat this procedure. Iterating in this fashion produces the bridge decomposition of  $\omega$  into a sequence of irreducible bridges, and the decomposition is clearly unique<sup>1</sup>.

Much of the study of the infinite self-avoiding walk in the upper half plane therefore reduces to the study of irreducible bridges. Let  $\mathcal{B}$  be the set of all irreducible bridges rooted at the origin, and  $\lambda_N$  be the number of length  $N$  elements of  $\mathcal{B}$ . Using some clever tricks involving generating functions, Kesten was able to prove what is now called **Kesten's relation**:

$$\sum_{N \geq 1} \lambda_N \mu^{-N} = \sum_{\omega \in \mathcal{B}} \mu^{-|\omega|} = 1, \tag{16.2}$$

for the same  $\mu$  as in (16.1) (for proofs see [Kes63a] or [MS93, Section 4.3]). Kesten's relation shows that  $\mathbf{P}(\omega) := \mu^{-|\omega|}$  is a probability measure on  $\mathcal{B}$ , and by concatenating together an independent sequence of irreducible bridges each sampled from  $\mathbf{P}$ , a probability measure is induced on  $\mathcal{H}$ . In [MS93] and [LSW04b], the latter measure is shown to be the only possible candidate for the weak limit of the uniform measure on  $\{\omega \in \mathcal{H} : |\omega| = N\}$ , and therefore the question of existence of this weak limit is immediately settled.

The bridge decomposition shows that infinite half-plane SAWs have a renewal structure to them. At the end of each irreducible bridge the future path of the walk lies entirely in the half-plane above the horizontal line where the bridge ended. The future path is again a concatenation of a sequence of irreducible bridges, so that its law is the same as the law of the original path and the future path is independent of the past. In this sense the walk renews itself whenever it is at the end of an irreducible bridge, and it is appropriate to call such times renewal times. Note that the renewal times are functions of the *entire* half-plane SAW, since the algorithm for the bridge decomposition depends upon knowing the entire walk.

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<sup>1</sup>There is a minor technicality to point out here: if the walk oscillates infinitely often in the vertical direction without approaching some limit (including infinity) the decomposition algorithm will terminate after finitely many iterations and the remaining part of the walk will not be a bridge. However, we will see in the next paragraph that this is a probability zero event under the standard measure on  $\mathcal{H}$ , and that the vertical component of the SAW always goes to infinity with probability one.

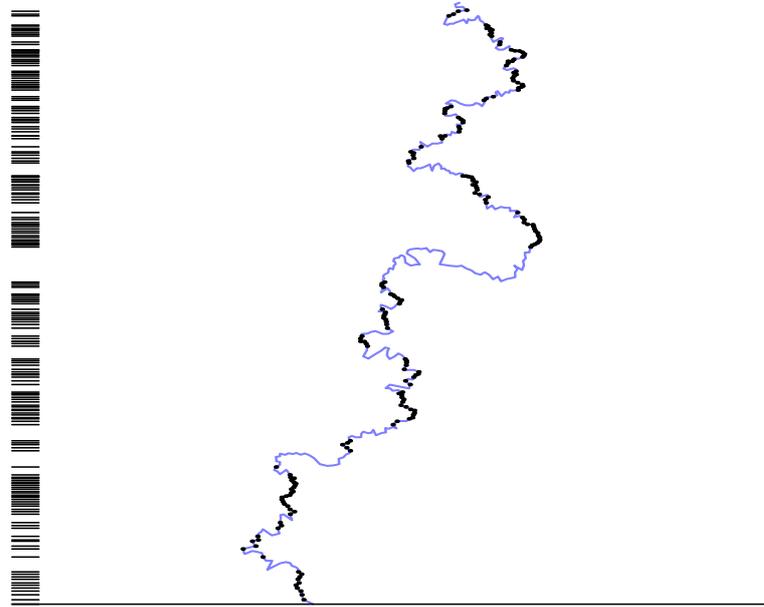


Figure 16.1: A sample SLE(8/3) curve in the lighter colour, with the bridge points superimposed in black. The bridge heights are plotted on the vertical axis. The SLE(8/3) curve is generated by Tom Kennedy’s algorithm and freely available graphics program; see [Ken07].

## 1.2 Statement of Results: The Continuum Bridge Decomposition

In the continuum we will show that an analogue of bridge times exists for the so-called *restriction hulls* in  $\mathbb{H}$ , and that these times are also renewal times. Using this renewal structure, we proceed to decompose the restriction hulls into countably many continuum irreducible bridges. This continuum decomposition most closely resembles the discrete one in the case of SLE(8/3), but we will see that it also holds for more general restriction hulls with parameter  $\alpha < 1$ . We will give a more in-depth description of the restriction hulls in Section 2, but provide a brief summary here.

Roughly speaking, a restriction hull is a stochastic process taking values in the space of unbounded hulls in  $\mathbb{H}$ . An *unbounded hull* is a closed, connected subset  $K \subset \overline{\mathbb{H}}$  such that  $\mathbb{H} \setminus K$  consists of exactly two connected components. The unbounded hulls that we will consider are closed, connected subsets of  $\overline{\mathbb{H}}$  that connect 0 and  $\infty$ , and intersect  $\mathbb{R}$  only at zero; moreover it will be possible to time parameterize them into a growing family  $(K_t, t \geq 0)$  of *hulls* (closed, connected subsets  $A$  of  $\overline{\mathbb{H}}$  such that  $\mathbb{H} \setminus A$  is simply connected with exactly one connected component) with  $K_\infty = K$ . This time parameterization is provided by the well-known construction of restriction hulls that was originally laid out in [LSW03] and [LW04]. Those chapters show that attaching the filled-in loops from a realization of the Brownian Loop Soup to an independent SLE curve induces a restriction law on unbounded hulls in  $\mathbb{H}$ . By changing the  $\kappa$  parameter for the SLE and the intensity

parameter for the loop soup (in a specific way) an entire family  $\mathbb{P}_\alpha$  of restriction measures on unbounded hulls is created. Here  $\alpha$  is a real parameter with  $\alpha \geq 5/8$ .

The definition of a continuum bridge is motivated by the algorithm for decomposing half-plane SAWs into irreducible bridges, which essentially searches for horizontal lines that separate the future path from the past.

**Definition 16.1.** *Let  $K$  be a hull (unbounded or not).*

- *Call  $L > 0$  a bridge height for  $K$  if the horizontal line  $y = L$  intersects  $K$  at exactly one point, i.e. if  $K \cap \{y = L\}$  is a singleton.*
- *If  $z \in \mathbb{H}$  is such a singleton then we call it a bridge point. Let  $C$  be the set of bridge points of  $K$ , and let  $D$  be the set of bridge heights (note that  $D = \{Im z : z \in C\}$ ).*
- *Let  $G$  be the set of bridge times at which the hull is at a bridge point, which can be written as  $G := \{t \geq 0 : K_t \setminus K_{t-} \cap C \neq \emptyset\}$ .*
- *A continuum bridge is a segment of the bridge between two bridge times, i.e. if  $s, t \in G$  with  $s < t$  then the hull  $K_t \setminus K_{s-}$  is a bridge. A continuum bridge is said to be irreducible if it contains no bridge points (other than the starting and ending points).*

Note that bridge heights, points and times are all functions of the *entire* hull  $K$ . A subset of  $K$  is, by itself, not enough to determine  $C, D$  or  $G$ . At any fixed time  $t \geq 0$  it is possible to determine what are the bridge points of the hull  $K_t$ , but not which of those are bridge points of the entire hull  $K_\infty = K$ , since some of the bridge points of  $K_t$  may ultimately be destroyed by the future hull as it grows.

There are two main steps behind the continuum bridge decomposition. The first is to show that bridge points actually exist for hulls with  $\alpha < 1$ , which is not a priori clear. We do this by calculating the almost sure Hausdorff dimensions of  $C$  and  $D$  and showing that they are strictly larger than zero (and in fact the same). Specifically we will show the following:

**Theorem 16.2.** *Suppose  $K$  has the law of  $\mathbb{P}_\alpha$ , then*

1. *the laws of  $C$  and  $D$  are scale invariant (i.e.  $rC \equiv C$  and  $rD \equiv D$  for all  $r > 0$ ),*
2.  *$C$  and  $D$  are almost surely perfect (i.e. closed and without isolated points),*
3. *the Hausdorff dimensions of both  $C$  and  $D$  are constant,  $\mathbb{P}_\alpha - a.s.$ ,*
4.  *$\dim_{\mathbb{H}} C = \dim_{\mathbb{H}} D = \max(2 - 2\alpha, 0)$ ,  $\mathbb{P}_\alpha - a.s.$ ,*
5.  *$C$  and  $D$  are empty,  $\mathbb{P}_\alpha - a.s.$  if and only if  $\alpha \geq 1$ .*

The proof of Theorem 16.2 is taken up in Section 3, but we will mention here that the key element is the **restriction formula**:

$$\mathbb{P}_\alpha(K \cap A = \emptyset) = \phi'_A(0)^\alpha, \tag{16.3}$$

where  $A$  is a hull that does *not* contain zero, and  $\phi_A$  is a conformal map from  $\mathbb{H} \setminus A$  to  $\mathbb{H}$  such that  $\phi_A(z) \sim z$  as  $z \rightarrow \infty$ . Most of the proof of Theorem 16.2 is based on an analysis of  $\phi'_A(0)$  for a specific choice of the hull  $A$ . The proof of part (5) builds upon the  $\alpha = 1$  case, which is related to Brownian excursions, and uses the fact that the vertical component of a Brownian excursion is a Bessel-3 process.

Given that bridge points exist for  $\alpha < 1$ , the next step is to prove an analogue of the renewal theory for half-plane SAWs. In Section 4 we show that the restriction hulls have an extended Markov property with respect to the information gained by observing the hull as it grows along with the *global* bridge points of  $K$  as they appear, and as a corollary we show that the bridge times are actually renewal times for the hull process. In Section 5 we will use this Markov property and Theorem 16.2 to show the existence of a “local time” for the time spent by a restriction hull at its bridge points, and the local time can then be used to prove:

**Theorem 16.3.** *There exists a local time  $\lambda$  supported on bridge heights such that  $\theta_\lambda(K_\lambda \setminus K_{\lambda-})$  is a Poisson Point Process, where  $\theta_t$  is an operator that shifts back to the origin the part of the hull that comes after time  $t$ . Moreover, the local time is the inverse of a stable subordinator of index  $2 - 2\alpha$ .*

The general theory of Poisson Point Processes then implies the existence of a sigma-finite measure  $\nu_\alpha$  on continuum irreducible bridges that is the analogue of the measure  $\mathbf{P}$  on irreducible bridges for half-plane SAWs. In Section 5 we mention some basic properties of this measure. We also show that the Poisson Point Process can be used to recover the restriction hull, so that as in the discrete case, the irreducible bridges are the building blocks of the restriction hull processes.

We should mention that most of these ideas are similar in spirit to the excursion decomposition of a one-dimensional Brownian motion according to its zeros, as was first described by Itô. In recent years, similar two-dimensional conformally invariant decompositions of this type have also been considered by Dubédat [Dub06] and Virág [Vir03]. They provide decompositions of unbounded hulls arising from certain variants of  $\text{SLE}(\kappa, \rho)$  and Brownian excursions, respectively, although their decompositions are at cutpoints rather than bridge points (i.e. points that, if removed from the set, would disconnect it into two pieces). Clearly bridge points are cutpoints but not vice versa, and there does not appear to be any direct relationship between our decomposition and theirs. In one sense their decompositions are more involved than ours, since their hulls refresh at cutpoints only after conformally mapping away the past, whereas our hulls refresh at bridge points after a simple shift of the future hull back to the origin. This difference is mostly cosmetic, however, and in spirit all these decompositions are quite similar.

The chapter is organized as follows: in Section 2 we give the necessary background on restriction measures and introduce some notation. Section 3 is devoted to proving the

existence of bridge points and Theorem 16.2, while Section 4 proves an extended Markov property and a refreshing property of the restriction hulls with respect to the filtration generated by bridge points as they appear. Section 5 then uses these results to prove the decomposition of Theorem 16.3. Finally, in Section 6 we present a series of open questions that were raised by our work.

## 2 Restriction Measures

In this section we review the basic construction and properties of restriction measures. We include no proofs but give references to the appropriate sources. For thorough overviews of the subject see [Law05] or [LSW03, LW04]. The reader interested only in the bridge decomposition for SLE(8/3), and not for general restriction measures, can entirely ignore the presence of the loops in this section.

To begin with, consider a simply connected domain  $D$  in the complex plane  $\mathbb{C}$  (other than the whole plane itself) and two boundary points  $z, w \in \partial D$ . A *chordal restriction measure* corresponding to the triple  $(D, z, w)$  is a probability measure  $\mathbb{P}^{(D, z, w)}$  on closed subsets of  $\overline{D}$ . The measures are supported on closed, connected subsets of  $K \subset \overline{D}$  such that  $K \cap \partial D = \{z, w\}$  and  $D \setminus K$  has exactly two components (for the triple  $(\mathbb{H}, 0, \infty)$  we call these sets unbounded hulls, for obvious reasons). The restriction measures satisfy the following properties, which essentially characterize them uniquely:

- **Restriction property:** for all simply connected subsets  $D'$  of  $D$  such that  $D \setminus D'$  is also simply connected and bounded away from  $z$  and  $w$ , the law of  $\mathbb{P}^{(D, z, w)}$ , conditioned on  $K \subset D'$ , is  $\mathbb{P}^{(D', z, w)}$ ,
- **Conformal invariance:** if  $f : D \rightarrow D'$  is conformal and  $K$  has  $\mathbb{P}^{(D, z, w)}$  as its law, then  $f(K)$  is distributed according to  $\mathbb{P}^{(f(D), f(z), f(w))}$ .

It turns out that for a given triple  $(D, z, w)$  there is only a one-parameter family of such laws, indexed by a real number  $\alpha$ . We denote the law by  $\mathbb{P}_\alpha^{(D, z, w)}$ , and due to the conformal invariance property it is enough to define the restriction measure for a single triple  $(D, z, w)$ . The canonical choice is  $(\mathbb{H}, 0, \infty)$ , and for shorthand we will write  $\mathbb{P}_\alpha$  for  $\mathbb{P}_\alpha^{(\mathbb{H}, 0, \infty)}$ . In [LSW03] it is shown that these restriction measures exist only if the parameter  $\alpha$  satisfies  $\alpha \geq 5/8$ , and that the measure is supported on simple curves only if  $\alpha = 5/8$ . In the latter case the restriction measure is simply the SLE(8/3) law from  $z$  to  $w$  in  $D$ . For  $\alpha = 1$  it turns out that the restriction measure coincides with the law of filled-in Brownian excursions in  $D$  from  $z$  to  $w$ .

For all  $\alpha \geq 5/8$ , one of the fundamental constructions of [LSW03] is that restriction measures can be realized by adding to an SLE( $\kappa$ ) curve the filled-in loops that it intersects from an independent realization of the Brownian loop soup, for an appropriate choice of  $\kappa$  for the curve and intensity parameter  $\lambda$  for the loop soup. Let

$$\kappa = \frac{6}{2\alpha + 1}, \quad \lambda = (8 - 3\kappa)\alpha,$$

and let  $\gamma$  be a chordal SLE( $\kappa$ ) and  $\mathcal{L}_\lambda$  be an independent realization of the Brownian loop soup (in  $\mathbb{H}$ ) with intensity parameter  $\lambda$ . The individual loops in  $\mathcal{L}_\lambda$  will be generically denoted by  $\eta$ , they can be thought of as continuous curves  $\eta : [0, t_\eta] \rightarrow \mathbb{H}$  such that  $\eta(0) = \eta(t_\eta)$ . Throughout we will use  $\gamma$  and  $\eta$  to denote the curves as well as their traces, i.e.  $\gamma[0, \infty)$  and  $\eta[0, t_\eta]$ , respectively. It will be clear from the context which we are referring to. Let  $K$  be the hull generated by the union of  $\gamma$  and all the (filled-in)  $\eta \in \mathcal{L}_\lambda$  such that  $\eta \cap \gamma \neq \emptyset$ . Then [LSW03] (along with [LW04]) proves that  $K$  is distributed according to  $\mathbb{P}_\alpha$ .

This construction allows us to identify restriction hulls with pairs  $(\gamma, \mathcal{L})$ , where  $\gamma : [0, t_\gamma] \rightarrow \mathbb{C}$  is a continuous, simple curve and  $\mathcal{L}$  is a set of loops. Furthermore, the curve plus loops structure gives a clean way of time parameterizing the hulls. Letting  $K$  be a restriction hull, which we identify with  $(\gamma, \mathcal{L})$ , we define  $K_t$  to be the hull generated by  $\gamma[0, t]$  plus the union of all filled-in loops  $\eta \in \mathcal{L}_\lambda$  such that  $\eta \cap \gamma[0, t] \neq \emptyset$ . Then  $(K_t)_{t \geq 0}$  is a growing family of hulls that increases to  $K_\infty = K$ . It is important for us to have such a time parameterization so that we may properly describe the renewal theory for the restriction hulls, but the particular time parameterization is not especially important since we are mostly interested in the restriction hull as a topological object. We remark that this growing family is not continuous with respect to the time parametrization, since loops are added “all at once”, but again it does not really matter for our purposes (nevertheless, notice that the parameterization is right continuous). The only issue to point out is that the bridge points of a restriction hull will always be a subset of the underlying (simple) curve  $\gamma$ , and therefore to each bridge point there is a corresponding unique bridge time. Hence the set of bridge times  $G$  is a well defined object.

The curve-plus-loops structure also makes it easy to define various operations on hulls. Given two pairs  $(\gamma, \mathcal{L})$  and  $(\gamma^*, \mathcal{L}^*)$  with  $\gamma(0) = \gamma^*(0) = 0$ , their *concatenation* is defined by

$$(\gamma, \mathcal{L}) \oplus (\gamma^*, \mathcal{L}^*) = (\gamma \oplus \gamma^*, \mathcal{L} \cup (\gamma(t_\gamma) + \mathcal{L}^*)),$$

where  $\gamma \oplus \gamma^*$  is the usual concatenation of curves given by

$$(\gamma \oplus \gamma^*)(t) = \begin{cases} \gamma(t), & 0 \leq t \leq t_\gamma \\ \gamma^*(t - t_\gamma) + \gamma(t_\gamma), & t_\gamma \leq t \leq t_\gamma + t_{\gamma^*} \end{cases}$$

We also define a time shift for the hulls. For  $t \leq s \leq t_\gamma$ , define the curve  $\gamma^{t,s}$  by  $\gamma^{t,s}(t') := \gamma(t + t')$  for  $0 \leq t' \leq s - t$ , and let

$$\mathcal{L}^{t,s} := \{\eta \in \mathcal{L} : \eta \cap \gamma^{t,s} \neq \emptyset, \eta \cap \gamma[0, t] = \emptyset\}.$$

Then we define  $\Lambda_{t,s}K := (\gamma^{t,s}, \mathcal{L}^{t,s})$ , which is the future hull between times  $t$  and  $s$ , and  $\theta_{t,s}K := \Lambda_{t,s}K - \gamma(t)$ , which shifts the future hull to start at the origin. If  $s = t_\gamma$ , which usually for us means  $s = \infty$ , we write  $\Lambda_t$  and  $\theta_t$  for these operators. In the case that  $K$  is an unbounded hull in  $\mathbb{H}$  and  $t$  is a bridge time for  $K$ , it is easy to see that  $\theta_t K$  is also an unbounded hull in  $\mathbb{H}$ . At non-bridge times  $\theta_t K$  does not remain in  $\mathbb{H}$ .

Imagine a walker moving along the hull that has discovered  $K_t$  at time  $t$ . The information that is progressively revealed to the walker is encapsulated by the filtration

$$\mathcal{F}_t := \sigma(K_s; 0 \leq s \leq t).$$

With respect to this filtration, the following Domain Markov property is true:

$$\text{The conditional law of } \Lambda_t K, \text{ given } \mathcal{F}_t, \text{ is } \mathbb{P}_\alpha^{(\mathbb{H} \setminus \gamma[0,t], \gamma(t), \infty)}. \quad (16.4)$$

This is similar to the Domain Markov property for regular SLE, where the future curve is an independent  $\text{SLE}(\kappa)$  curve from  $\gamma(t)$  to  $\infty$  in  $\mathbb{H} \setminus \gamma[0,t]$ , except that in the case of restriction measures one also attaches to the curve the filled-in loops of an independent realization of the Brownian loop soup in the domain  $\mathbb{H} \setminus \gamma[0,t]$ . Note, however, that both the future curve and loops are sampled from the laws corresponding to the domains  $\mathbb{H} \setminus \gamma[0,t]$ , *not* the laws corresponding to  $\mathbb{H} \setminus K_t$ . In short, the future curve and future loops are allowed to intersect the past loops but *not* the past curve  $\gamma[0,t]$ .

For the domain  $(\mathbb{H}, 0, \infty)$  recall that the restriction measures satisfy the restriction formula (16.3):

$$\mathbb{P}_\alpha(K \cap A = \emptyset) = \phi'_A(0)^\alpha,$$

where  $A$  is a hull in  $\mathbb{H}$  that is a positive distance from zero, and  $\phi_A$  is a conformal map from  $\mathbb{H} \setminus A$  onto  $\mathbb{H}$  satisfying  $\phi_A(z) \sim z$  as  $z \rightarrow \infty$ . In fact, specifying the above probabilities for a sufficiently large class of hulls  $A$  (so-called *smooth hulls*) uniquely determines  $\mathbb{P}_\alpha$ , see [LSW03] for a proof of this fact. For general triples  $(D, z, w)$ , the restriction formula is

$$\mathbb{P}_\alpha^{(D, z, w)}(K \cap A = \emptyset) = \phi'_{f(A)}(0)^\alpha, \quad (16.5)$$

where  $A$  is a hull in  $D$  not containing  $z$ , and  $f$  is a conformal map from  $D$  onto  $\mathbb{H}$  that sends  $z$  to 0 and  $w$  to  $\infty$ .

The restriction formula will be heavily used throughout this chapter. For a given hull  $A$  there are various techniques from both complex analysis and probability theory that can be used to compute  $\phi'_A(0)$ . We will exclusively use probabilistic techniques involving Brownian motion; these are described in the next section.

### 3 Bridge Lines and Bridge Points

The main focus of this section is proving Theorem 16.2. Specifically, we establish the existence of bridge points and lines for restriction hulls with  $\alpha < 1$ , and also prove the non-existence for  $\alpha \geq 1$ .

First observe that part (1) of Theorem 16.2 is trivial. The scale invariance of  $C$  and  $D$  follows immediately from the scale invariance of the restriction hulls (which itself follows from the scale invariance of SLE and of the loop soup). To prove part (2), first recall that

bridge points of a restriction hull are always on the SLE curve itself and never on a loop, and that there is always a unique bridge time corresponding to every bridge point. We refer to the end of the section for the proof.

The most involved proofs are for calculating the Hausdorff dimensions of  $C$  and  $D$ . The computation of the Hausdorff dimensions in Theorem 16.2 follows standard “one-point” and “two-point” arguments, as in, for example, [AS08, Bef08b, Law96, SZ07]. The idea behind this argument is to approximate  $C$  and  $D$  by “thickened” sets  $\varepsilon C$  and  $\varepsilon D$ , and then obtain estimates on the probability that a given set of points belongs to the thickened sets. A specific bound on the probability that one point belongs to the thickened set gives an upper bound on the Hausdorff dimension, and a similar bound on the probability that two points are in the thickened sets, together with the order of magnitude of the one-point estimate, gives a lower bound on the dimension. We recall the result that we will use in the remainder; throughout this chapter we use the notation  $f(\varepsilon) \asymp g(\varepsilon)$  to indicate that there exists constants  $C_1$  and  $C_2$  independent of  $\varepsilon$  such that  $C_1 g(\varepsilon) \leq f(\varepsilon) \leq C_2 g(\varepsilon)$ , for all  $\varepsilon$  sufficiently small.

**Proposition 16.4.** *Let  $H$  be a random subset of  $\mathbb{C}$  and  $\varepsilon H$  be the set of points at distance less than  $\varepsilon$  from  $H$ . Suppose that the two following conditions are fulfilled for some  $s \geq 0$  and constant  $c > 0$ :*

- for all  $z \in \mathbb{H}$ ,  $\mathbb{P}(z \in \varepsilon H) \asymp \varepsilon^s$ ,
- for all distinct  $w, z \in \mathbb{H}$ ,  $\mathbb{P}(w, z \in \varepsilon H) \leq c\varepsilon^s \wedge c(\varepsilon^{2s}/|w-z|^s)$ .

*Then  $\dim_{\mathbb{H}} H \leq 2 - s$  with probability one, and with some strictly positive probability we also have  $\dim_{\mathbb{H}} H \geq 2 - s$ . If  $H$  is a random subset of  $\mathbb{R}$  then the same conclusion holds with  $2 - s$  replaced by  $1 - s$ .*

Note that Proposition 16.4 by itself is not enough to conclude that the Hausdorff dimension of  $H$  is a constant, since the lower bound only holds on some event of positive probability. In our situation we are able to conclude that the Hausdorff dimension of  $C$  and  $D$  is constant by using a 0-1 law. The argument that follows uses the Blumenthal 0-1 Law and is modified from [Law96].

**Proof of Theorem 16.2, part (3)** We will prove the result for  $C$ , a similar argument holds for  $D$ . For  $0 \leq t \leq s$ , define  $C_t(s) := \{\text{bridge points of } K_s\} \cap K_t$ . For a fixed  $d > 0$ , let  $W_t(s) := \{\dim_{\mathbb{H}} C_t(s) \geq d\}$ . It is enough to show that  $\mathbb{P}_\alpha(W_\infty(\infty)) = 0$  or 1.

First note that for fixed  $s$ , both the sets  $C_t(s)$  and  $W_t(s)$  are increasing in  $t$ , while for fixed  $t$  they are decreasing in  $s$ . Defining

$$V_s := \bigcap_{n=1}^{\infty} W_{\frac{1}{n}}(s) = \{\dim_{\mathbb{H}} C_t(s) \geq d \ \forall 0 < t \leq s\},$$

it follows that  $V_s$  is also decreasing in  $s$ . For each element of the event  $V_s \setminus V_\infty$ , there exists a  $t_0$  such that  $0 < t_0 \leq s$  and for all  $0 < t \leq t_0$ ,

$$\dim_{\mathbb{H}} C_t(\infty) < d \leq \dim_{\mathbb{H}} C_t(s).$$

But this can only happen if for every  $0 < t \leq t_0$ , the future hull  $\Lambda_s K$  destroys bridge points of  $K_s$  that are in  $K_t$ , and since this happens for every  $0 < t \leq t_0$  and  $K_t \rightarrow \{0\}$  as  $t \rightarrow 0$ , this forces that the future hull comes arbitrarily close to the real axis. But this is clearly an event of measure zero. Hence for every  $s > 0$ ,  $\mathbb{P}_\alpha(V_s \setminus V_\infty) = 0$ , from which it immediately follows that

$$\mathbb{P}_\alpha\left(\bigcap_{n=1}^{\infty} V_{\frac{1}{n}}\right) = \mathbb{P}_\alpha(V_\infty).$$

However, the intersection of the  $V_{1/n}$  is  $\mathcal{F}_{0+}$ -measurable, and in the case of SLE(8/3) it follows that  $\mathbb{P}_{5/8}(V_\infty) = 0$  or 1 by the Blumenthal 0-1 Law, since the corresponding measure  $\mathbb{P}_{5/8}$  is a pushforward of Wiener measure through the Loewner equation. For general  $\alpha > 5/8$ , the same type of Blumenthal 0-1 Law holds via the usual argument. Indeed, the Domain Markov property implies that  $\phi_{K_t}(\Lambda_t K)$  is a restriction hull that is independent of  $\mathcal{F}_t$ , hence for  $A \in \mathcal{F}_{0+}$  and  $t > 0$  and any bounded, continuous function  $f$  on hulls we have

$$\mathbf{E}[f(\phi_{K_t}(\Lambda_t K)) \mathbf{1}_A] = \mathbf{E}[f(\phi_{K_t}(\Lambda_t K))] \mathbb{P}_\alpha(A)$$

Taking a limit of both sides as  $t \downarrow 0$  and using the fact that  $f$  is continuous and  $\phi_{K_t}$  goes continuously to the identity we get that

$$\mathbf{E}[f(K) \mathbf{1}_A] = \mathbf{E}[f(K)] \mathbb{P}_\alpha(A),$$

which shows that  $A$  is independent of all elements of  $\mathcal{F}_\infty$ , and therefore of itself.  $\square$

We now use Proposition 16.4 to prove part (4) of Theorem 16.2. We use the following events to define our thickened sets.

**Definition 16.5.** For  $z \in \mathbb{H}$  and  $\epsilon > 0$ , let  $I(z, \epsilon)$  be the horizontal line  $y = \text{Im} z$  with the gap of width  $2\epsilon$  centered around  $z$  removed. That is

$$I(z, \epsilon) := \{w \in \mathbb{H} : \text{Im} w = \text{Im} z, |\text{Re}(w - z)| \geq \epsilon\}.$$

Define the sets  $\varepsilon C$  and  $\varepsilon D$  by

$$\varepsilon C := \{z \in \mathbb{H} : I(z, \epsilon) \cap K = \emptyset\}, \quad \varepsilon D := \{L > 0 : I(n\epsilon + iL, \epsilon) \cap K = \emptyset \text{ for some } n \in \mathbb{Z}\}.$$

**Lemma 16.6.** With the definitions above, the following is true  $\mathbb{P}_\alpha$ -a.s.:

$$C = \bigcap_{\epsilon > 0} \varepsilon C, \quad D = \bigcap_{\epsilon > 0} \varepsilon D.$$

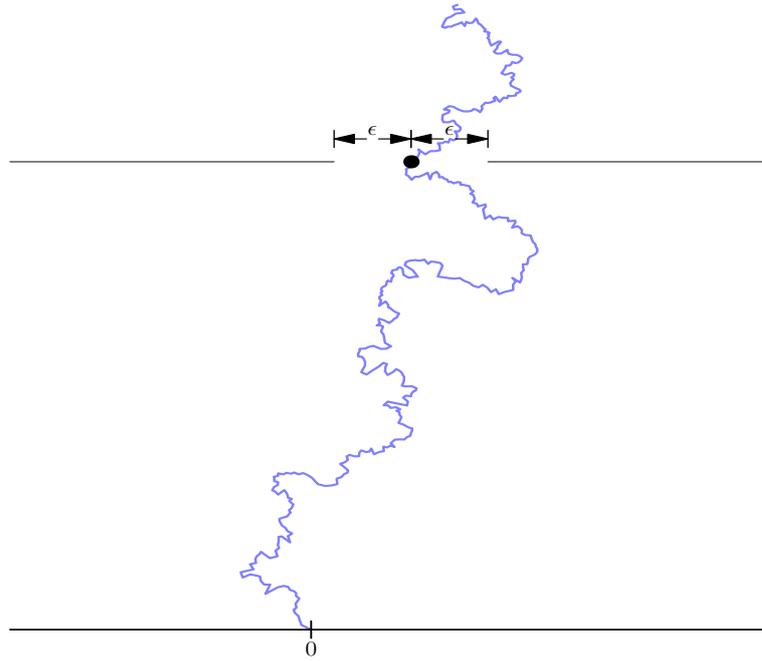


Figure 16.2: The dotted point is  $z$  and the two horizontal lines on either side form the set  $I(z, \epsilon)$ . This figure depicts the event that an SLE(8/3) avoids the hull  $I(z, \epsilon)$ .

**Proof** Recall that  $C$  consists of  $z \in \mathbb{H}$  for which  $K \cap \{y = \text{Im } z\} = \{z\}$ . Hence if  $z \in C$  then  $z \in \varepsilon C$  for all  $\epsilon > 0$ . To prove the converse, note that if  $z \in \varepsilon C$  for every  $\epsilon > 0$  then  $z$  is the only possible element in the set  $K \cap \{y = \text{Im } z\}$ . But the latter set is always non-empty, since restriction hulls are connected and their vertical component goes from zero to infinity ( $\mathbb{P}_\alpha$ -a.s.), and therefore with  $\mathbb{P}_\alpha$ -probability 1 the set  $K \cap \{y = L\}$  is non-empty for all  $L > 0$ . The proof for  $D$  is exactly the same.  $\square$

The restriction formula makes it easy to compute the probability that a point  $z \in \mathbb{H}$  is in  $\varepsilon C$ . Indeed, by formula (16.3) we have

$$\mathbb{P}_\alpha(z \in \varepsilon C) = \mathbb{P}_\alpha(I(z, \epsilon) \cap K = \emptyset) = \phi'_{I(z, \epsilon)}(0)^\alpha,$$

where  $\phi_{I(z, \epsilon)}$  is a conformal map from  $\mathbb{H} \setminus I(z, \epsilon)$  onto  $\mathbb{H}$  such that  $\phi_{I(z, \epsilon)}(w) \sim w$  as  $w \rightarrow \infty$ . Similarly,

$$\mathbb{P}_\alpha(w, z \in \varepsilon C) = \phi'_{I(w, \epsilon) \cup I(z, \epsilon)}(0)^\alpha.$$

By Proposition 16.4, the Hausdorff computation for  $C$  and  $D$  therefore comes down to an estimate of the derivative of these conformal maps at zero. We list three possible methods for these estimates. One deals only with conformal maps and is entirely analytic. The others use probabilistic techniques. We recall the analytic method but do not enter into details.

**Analytic Method:** While it is not possible to write down  $\phi_{I(z,\epsilon)}$  explicitly, one can write down the general form of its inverse. Let

$$f_{z,\epsilon}(w) := \lambda w + \frac{\operatorname{Im} z}{\pi} (\log(w - a) - \log(w - b) + \pi i),$$

where the imaginary part of the logarithm is zero along the positive real axis and  $\pi$  on the negative real axis. For appropriate choices of real constants  $\lambda, a,$  and  $b$  (with  $a < b,$   $\lambda > 0$ ),  $f_{z,\epsilon}$  maps  $\mathbb{H}$  onto  $\mathbb{H} \setminus I(z, \epsilon)$ . These constants implicitly depend on  $z$  and  $\epsilon$ , although it is difficult to give closed-form expressions for them. Close analysis of the asymptotic behavior of  $\lambda, a,$  and  $b$  could be used to get estimates on  $\phi'_{I(z,\epsilon)}(0)$  as  $\epsilon \downarrow 0$ , but we will mostly avoid this strategy. We will, however, mention that  $a$  and  $b$  are determined mostly by  $z$ , while  $\lambda$  is proportional to  $\epsilon^{-2}$ .

**Brownian Excursion Method:** The first probabilistic method uses a well-known formula, due to Bálint Virág [Vir03], for *Brownian excursions* in the upper half plane. Recall that a Brownian excursion in  $\mathbb{H}$  can be thought of as a Brownian motion that is started at zero and conditioned to have a positive imaginary part at all later times. Such excursions can be realized by a random path whose horizontal component is a one-dimensional Brownian motion and whose vertical component is an independent Bessel-3 process.

**Lemma 16.7.** ([Vir03]) *Let  $A$  be a compact hull in the upper half plane such that  $\mathbb{H} \setminus A$  is simply connected and  $\operatorname{dist}(0, A) > 0$ , and  $\phi_A$  be a conformal map from  $\mathbb{H} \setminus A$  into  $\mathbb{H}$  such that  $\phi_A(0) = 0$  and  $\phi_A(z) \sim z$  as  $z \rightarrow \infty$ . If  $BE$  denotes the path of a Brownian excursion in  $\mathbb{H}$  from 0 to  $\infty$ , then*

$$\phi'_A(0) = \mathbb{P}(BE \text{ does not intersect } A).$$

In particular, this lemma shows that the filling in of a Brownian excursion has the law of a restriction measure with index 1. It can also be used to get the estimates of Proposition 16.4, but we prefer the following method that produces asymptotic results (even if they are not necessary in our setting).

**Brownian Motion Method:** Instead of using Brownian excursions to compute  $\phi'_A(0)$ , one can use Brownian motion directly. Oftentimes this is easier as it doesn't require dealing with the conditioning. In an appropriate sense,  $\phi'_A(0)$  is the exit density at zero (with respect to Lebesgue measure) of a Brownian motion in  $\mathbb{H} \setminus A$ , starting from  $\infty$ . This is also called the *excursion Poisson kernel* as seen from  $\infty$ . In what follows we let  $B$  be a complex Brownian motion.

**Definition 16.8.** *Given a simply connected domain  $D$  with  $z \in D, w \in \partial D$ , let  $H_D(z, w)$  denote the Poisson kernel. In the case  $D = \mathbb{H} \setminus A$ , we will often be interested in the “Poisson kernel as seen from infinity”, for which we introduce the notation*

$$H_{\mathbb{H} \setminus A}(\infty, w) := \lim_{L \uparrow \infty} LH_{\mathbb{H} \setminus A}(iL, w).$$

The following estimates will be useful when using Lemma 16.9 to estimate  $\phi'_A(0)$ . For  $x > 0$ ,  $H_{\mathbb{H}}(z, x) = \frac{1}{\pi} \text{Im}(z)/|z - x|^2$  and consequently  $H_{\mathbb{H}}(\infty, x) = \frac{1}{\pi}$ . Recall that under a conformal map  $f : D \rightarrow D'$ ,  $H_D(z, w)$  changes according to the scaling rule  $H_D(z, w) = |f'(w)|H_{f(D)}(f(z), f(w))$ . In particular, we have the scaling rule  $H_{\mathbb{H} \setminus A}(\infty, w) = H_{\mathbb{H} \setminus rA}(\infty, rw)$ .

The next lemma outlines how to use Brownian motion directly to estimate  $\phi'_A(0)$ . The method of proof is virtually identical to the one for Lemma 16.7, so we refer the reader to [Vir03] for details.

**Lemma 16.9.** *For a complex Brownian motion and a compact hull  $A$  in the upper half-plane such that  $\mathbb{H} \setminus A$  is simply connected and  $\text{dist}(0, A) > 0$ ,*

$$\phi'_A(0) = H_{\mathbb{H} \setminus A}(\infty, 0).$$

The computation of  $\phi'_{I(z, \epsilon)}(0)$  is thus reduced to some estimates on the exit density of a Brownian motion in the domain  $\mathbb{H} \setminus I(z, \epsilon)$ . In order to simplify the computations, we first estimate exit densities for an intermediate set  $\varepsilon S$ .

**Lemma 16.10.** *Let  $\varepsilon S = \mathbb{R} \times [0, 2i] \setminus I(i, \epsilon)$ . Then for  $x \in \mathbb{R}$  and  $\lambda \in [-1, 1]$ ,*

$$H_{\varepsilon S}(\lambda\epsilon + i, x) \sim \frac{\pi\sqrt{1-\lambda^2}}{8 \cosh^2(\pi x/2)} \epsilon \quad (16.6)$$

as  $\epsilon \downarrow 0$ , where “ $\sim$ ” means that the ratio of the two terms converges to 1 uniformly with respect to  $x$  and  $\lambda$ . In particular, the probability that the Brownian motion started at  $i$  exits  $S_\epsilon$  on  $\mathbb{R}$  is of order  $\epsilon$ .

**Proof** Let  $\varepsilon z = \lambda\epsilon + i$ . In this case, it is easy to find an explicit conformal map from  $\varepsilon S$  onto  $\mathbb{H}$ . A simple one is given by

$$\varepsilon f(z) = \left( \frac{e^{\pi z} + e^{\pi\epsilon}}{e^{\pi z} + e^{-\pi\epsilon}} \right)^{1/2}.$$

By the scaling rule for the Poisson kernel

$$H_{\varepsilon S}(\varepsilon z, x) = |\varepsilon f'(x)| H_{\mathbb{H}}(\varepsilon f(\varepsilon z), \varepsilon f(x)) = \frac{|\varepsilon f'(x)|}{\pi} \frac{\text{Im}(\varepsilon f(\varepsilon z))}{|\varepsilon f(\varepsilon z) - \varepsilon f(x)|^2}.$$

It is straightforward to verify that

$$\varepsilon f(x) \sim 1,$$

as  $\epsilon \downarrow 0$ , and

$$\begin{aligned} |\varepsilon f'(x)| &= \frac{1}{2f_\epsilon(x)} \frac{2\pi e^{\pi x} \sinh(\pi\epsilon)}{(e^{\pi x} + e^{-\pi\epsilon})^2} \\ &\sim \frac{\pi^2 \epsilon}{4 \cosh^2(\pi x/2)} \end{aligned}$$

Similarly

$$\begin{aligned}\varepsilon f(\varepsilon z) &= \left( \frac{e^{\pi\varepsilon} - e^{\pi\lambda\varepsilon}}{e^{-\pi\varepsilon} - e^{\pi\lambda\varepsilon}} \right)^{1/2} \\ &\sim \left( \frac{1 - \lambda}{-1 - \lambda} \right)^{1/2} \\ &= i \left( \frac{1 - \lambda}{1 + \lambda} \right)^{1/2}.\end{aligned}$$

Assembling the pieces proves (16.6), and then integrating (16.6) over  $x$  proves the last statement.  $\square$

**Lemma 16.11.** *Let  $x \in \mathbb{R}$  and  $\lambda \in [-1, 1]$ . Then*

$$H_{\mathbb{H} \setminus I(i, \varepsilon)}(\lambda\varepsilon + i, x) \sim H_{S_\varepsilon}(\lambda\varepsilon + i, x)$$

as  $\varepsilon \downarrow 0$ .

**Proof** If a Brownian motion started at  $\lambda\varepsilon + i$  exits  $\varepsilon S$  at  $x$ , then it also exits  $\mathbb{H} \setminus I(i, \varepsilon)$  at  $x$ . Consequently, the Poisson kernel on the left hand side is bigger than the one on the right. They are not the same because the Brownian motion in  $\mathbb{H} \setminus I(i, \varepsilon)$  can hit the line  $y = 2i$  before hitting zero, which the Brownian motion in  $\varepsilon S$  is not allowed to do. Asymptotically this event contributes nothing; indeed there is only an  $O(\varepsilon)$  chance that the Brownian motion even makes it up to  $y = 2i$ , and then another  $O(\varepsilon)$  chance that it passes back through the gap. Overall this makes the event of order  $\varepsilon^2$  (uniformly in  $x$  and  $\lambda$ ), which, by Lemma 16.10, is negligible compared to  $H_{S_\varepsilon}(\lambda\varepsilon + i, x)$ .  $\square$

**Proposition 16.12.** *For  $z = y(x + i) \in \mathbb{H}$ ,*

$$\phi'_{I(z, \varepsilon)}(0) \sim U(z)\varepsilon^2$$

as  $\varepsilon \downarrow 0$ , where

$$U(y(x + i)) = \frac{\pi}{16y^2 \cosh^2(\pi x/2)}.$$

**Proof** It suffices to prove the result in the case  $z = x + i$ , for the general form use the scaling rule. We use Brownian motion coming down from infinity as in Lemma 16.9. In order to reach 0, the Brownian motion coming down from infinity must first pass through the gap of width  $2\varepsilon$  centered at  $z$ , and then from the gap it must transition to zero while

avoiding  $I(z, \epsilon)$ . The two events are independent by the Strong Markov property, and each one is  $O(\epsilon)$ . More precisely, by Lemmas 16.10 and 16.11,

$$\begin{aligned}
 \phi'_{I(z, \epsilon)}(0) &= H_{\mathbb{H} \setminus I(x+i, \epsilon)}(\infty, 0) \\
 &= \int_{[-\epsilon, \epsilon]} H_{\mathbb{H}}(\infty, x+y) H_{\mathbb{H} \setminus I(x+i, \epsilon)}(x+y+i, 0) dy \\
 &= \int_{-\epsilon}^{\epsilon} \frac{1}{\pi} H_{\mathbb{H} \setminus I(i, \epsilon)}(y+i, -x) dy \\
 &= \frac{\epsilon}{\pi} \int_{-1}^1 H_{\mathbb{H} \setminus I(i, \epsilon)}(\lambda\epsilon + i, -x) d\lambda \\
 &\sim \frac{\epsilon^2}{8 \cosh^2(\pi x/2)} \int_{-1}^1 \sqrt{1-\lambda^2} d\lambda.
 \end{aligned}$$

□

From Proposition 16.12 and the restriction formula, it is easy to derive the probability that a bridge point is within distance  $\epsilon$  of a given point  $z$  decays like  $\epsilon^{2\alpha}$ . From this the first part of Proposition 16.4 follows easily, but we need a last proposition in order to derive the two point estimate.

**Proposition 16.13.** *Let  $z, w \in \mathbb{H}$ , with  $\text{Im}(z) > \text{Im}(w)$ , and  $\epsilon_z, \epsilon_w > 0$ . Let  $A = I(z, \epsilon_z) \cup I(w, \epsilon_w)$ . Then*

$$\phi'_A(0) \asymp U(z-w)U(w)\epsilon_z^2\epsilon_w^2,$$

as  $\epsilon_z, \epsilon_w \downarrow 0$ .

**Proof** The argument is virtually the same as for the one-point estimate in Proposition 16.12, the only difference being that the Brownian motion, after passing through the first gap at  $z$  then has to pass through a second gap at  $w$ . The probability of the latter event can be estimated using Proposition 16.12; indeed, after temporarily shifting  $w$  to zero, there is a  $U(z-w)\epsilon_z^2\epsilon_w$  chance that the Brownian motion hits in an  $\epsilon_w$  neighbourhood of  $w$  (and therefore also the second gap). With some positive probability it hits in the middle of the second gap, where the probability of moving to zero is, up to a constant, given by  $U(w)\epsilon_w$ . These two probabilities multiply since, by the Strong Markov property, the path before the second gap is independent of the path after the second gap. □

**Remark 16.14.** *By carefully decomposing the path according to the points it passes through in the gaps and then integrating, the statement of Proposition 16.13 could be strengthened to an asymptotic result rather than just up to constants. For our purposes, however, this is not required.*

**Proof of Theorem 16.2, part (4)** Propositions 16.12 and 16.13 combine with Proposition 16.4 to prove the result for  $C$ .

For  $D$ , the key observation is that if two gaps on a horizontal line do not overlap, then the curve can only avoid the line by going through one of them. Consequently, for  $n \neq m$ , the events  $I(n\epsilon + iL, \epsilon/2) \cap K = \emptyset$  and  $I(m\epsilon + iL, \epsilon/2) \cap K = \emptyset$  are disjoint, and therefore

$$\begin{aligned} \mathbb{P}_\alpha(L \in \varepsilon D) &= \mathbb{P}_\alpha\left(\bigcup_{n \in \mathbb{Z}} \{I(n\epsilon + iL, \epsilon/2) \cap K = \emptyset\}\right) \\ &= \sum_{n \in \mathbb{Z}} \mathbb{P}_\alpha(I(n\epsilon + iL, \epsilon/2) \cap K = \emptyset) \\ &\sim \frac{1}{L^{2\alpha}} \sum_{n \in \mathbb{Z}} U\left(\frac{n\epsilon}{L} + i\right)^\alpha \left(\frac{\epsilon^{2\alpha}}{4^\alpha}\right) \\ &\sim \frac{\epsilon^{2\alpha-1}}{4^\alpha L^{2\alpha-1}} \int_{\mathbb{R}} U(x + iL)^\alpha dx \\ &\sim \frac{\pi^\alpha \epsilon^{2\alpha-1}}{32^\alpha L^{2\alpha-1}} \int_{\mathbb{R}} \cosh^{-2\alpha}(\pi x/2) dx. \end{aligned}$$

The transition from sum to integral is a Riemann sum approximation. By  $2\alpha > 1$ , the integral is a finite constant depending only on  $\alpha$ . This gives the one-point estimate for  $D$ .

Similarly, for  $0 < L < L'$ ,

$$\begin{aligned} \mathbb{P}_\alpha(L, L' \in \varepsilon D) &= \mathbb{P}_\alpha\left(\bigcup_{m, n \in \mathbb{Z}} \{n\epsilon + iL, m\epsilon + iL' \in C_{\epsilon/2}\}\right) \\ &= \sum_{m, n \in \mathbb{Z}} \mathbb{P}_\alpha(n\epsilon + iL, m\epsilon + iL' \in C_{\epsilon/2}) \\ &\asymp \sum_{m, n \in \mathbb{Z}} \epsilon^{4\alpha} U[(m-n)\epsilon + i(L'-L)]^\alpha U(n\epsilon + iL)^\alpha \\ &\asymp \epsilon^{4\alpha-2} \int_{\mathbb{R}} U(x + i(L'-L))^\alpha dx \int_{\mathbb{R}} U(x + iL)^\alpha dx \\ &\asymp \frac{\epsilon^{4\alpha-2}}{L^{2\alpha-1}(L'-L)^{2\alpha-1}} \end{aligned}$$

We use the same transition from sum to integral as in the one-point bound. Proposition 16.4 now completes the proof.  $\square$

We show that  $C$  and  $D$  are almost surely empty for  $\alpha \geq 1$ . For  $\alpha < 1$ , the Hausdorff dimension is strictly positive and the set is non empty.

**Proof of Theorem 16.2, part (5)** For  $\alpha = 1$ , recall that the imaginary part of a Brownian excursion is a Bessel(3) process, and a bridge height for the hull necessarily corresponds to a point of increase for the Bessel(3) process. However, it is well known that Bessel(3) has no point of increase since, for example, a Bessel(3) process reversed from its last passage time of a level has the same law as a Brownian motion up to its first hitting time of zero, and Brownian motion is known to have no points of increase (see [RY99] for details of both facts).

For  $\alpha > 1$  consider the rectangle  $R = [-1, 1] \times [1/2, 1]$ . Cover it with  $2^{2n}$  squares each of side length  $2^{-n}$ , and let  $\{S_i\}_{1 \leq i \leq 2^{2n}}$  be the boxes and  $z_i$  be their centers. Then, by Proposition 16.12, the expected number of squares containing a bridge point decays exponentially fast since

$$\begin{aligned} \mathbb{E}_\alpha \left[ \sum_{i=1}^{2^{2n}} \mathbf{1} \{C \cap S_i \neq \emptyset\} \right] &\asymp \sum_{i=1}^{2^{2n}} \mathbb{P}_\alpha (I(z_i, 2^{-n}) \cap K \neq \emptyset) \\ &\asymp \sum_{i=1}^{2^{2n}} U(z_i) (2^{-n})^{2\alpha} \\ &= 2^{(2-2\alpha)n} 2^{-2n} \sum_{i=1}^{2^{2n}} U(z_i) \\ &\leq C 2^{(2-2\alpha)n}, \end{aligned}$$

for some constant  $C > 0$ . The last inequality is a simple consequence of the fact that  $U$  is Riemann integrable and hence

$$2^{-2n} \sum_{i=1}^{2^{2n}} U(z_i) \rightarrow \int_R U(z) dA(z) < \infty,$$

where  $dA(z)$  is two-dimensional Lebesgue measure. The Borel-Cantelli lemma then proves that  $R$  almost surely contains no bridge points. By scale invariance any scaled version of  $R$  also contains no bridge points. Translates of  $R$  in the horizontal direction also contain no bridge points, since clearly the expected number of bridge points in translates of  $R$  decreases as the rectangle is moved away from the imaginary axis. Finally, since the entire half-plane can be covered with countably many scaled and translated versions of  $R$ , the entire plane must almost surely be free of bridge points.  $\square$

We end this section with the proof of part (2) of Theorem 16.2. The lack of isolated points in  $C$  and  $D$  is also a consequence of the renewal property of restriction hulls at bridge points, so we defer the proof of this fact until the end of Section 4.

**Proof of Theorem 16.2, part (2)** We prove the result for  $D$ ; the proof for  $C$  is similar. To prove that  $D$  is closed, suppose that  $L$  is a limit point of  $D$ . Without loss of generality we may assume that the limiting sequence of bridge heights  $L_n$  that converges to  $L$  is strictly increasing. If  $t$  is the bridge time corresponding to  $L$ , then the restriction hull after time  $t$  must reside in the domain  $\text{Im } z \geq L$  (since each  $L_n$  is a bridge height). Then  $L$  is not in  $D$  if and only if the future hull touches the line  $\text{Im } z = L$  but does not cross it, which is clearly an event of probability zero. Indeed, for two points  $z$  and  $w$  on the same horizontal line let us define  $A(z, \epsilon_z, w, \epsilon_w)$  to be the event that the hull goes through the balls  $B(z, \epsilon_z)$  and  $B(w, \epsilon_w)$  while avoiding  $I(z, \epsilon_z) \cap I(w, \epsilon_w)$ . The estimates of Proposition 16.13 can be used to show that the probability of  $A(z, \epsilon_z, w, \epsilon_w)$  is of order  $\epsilon_z^{2\alpha} \epsilon_w^{2\alpha}$ , which easily implies the result since  $\alpha > 1/2$ .  $\square$

## 4 Renewal at Bridge Lines

In this section we show that the restriction hulls renew themselves at bridge heights. Most of the section is technical, so first we would like to give the intuition behind the renewal property. It is almost entirely a consequence of restriction. Suppose that  $K$  is a restriction hull with the law  $\mathbb{P}_\alpha$ . Given  $\mathcal{F}_t$ , the Domain Markov property (16.4) says that the future hull has the restriction law corresponding to the domain  $(\mathbb{H} \setminus \gamma[0, t], \gamma(t), \infty)$ . But if we also know that  $t$  is a bridge time, then the future hull is separated from the past by the bridge line that the hull is currently at. The future hull is therefore conditioned not to go below this bridge line, and this conditioning is, by the restriction property, “equivalent” to sampling the future hull from the restriction measure corresponding to the half plane above the bridge line. Shifting the bridge point back to the origin, this means that the shifted future hull  $\theta_t K$  also obeys the law  $\mathbb{P}_\alpha$  and is independent of  $\mathcal{F}_t$ .

There are two main technical obstacles to this intuition. The first is that the event that  $t$  is a bridge time for  $K$  is not measurable with respect to  $\mathcal{F}_t$ , since the set of bridge times is a function of the entire hull. To address this problem and still have a meaningful notion of renewal, we simply expand our filtration to a larger one  $\mathcal{G}_t$  that tells us which bridge heights of  $K_t$  are also bridge heights of  $K$ . The second and more problematic technicality is that  $t$  being a bridge time is an event of measure zero, and so conditioning on it requires some care. Theorem 16.19 deals with this latter problem by showing that the restriction hulls obey a certain Domain Markov property with respect to  $\mathcal{G}_t$ , and from this concludes that they refresh themselves at  $\mathcal{G}_t$ -stopping times  $\tau$  such that  $\mathbb{P}_\alpha(\tau \in G) = 1$  (recall that  $G$  is the set of bridge times).

We make the following definitions:

**Definition 16.15.** For  $t \geq 0$ , let  $D_t$  be the set of bridge heights of  $K_t$ . Note that  $D_t$  is  $\mathcal{F}_t$ -measurable and  $D_\infty = D$ . Observe that  $D_t \cap D$  is the set of bridge heights of  $K_t$  that are also bridge heights of  $K$ , and  $D_t \setminus D$  is the set of bridge heights of  $K_t$  that are not bridge heights of  $K$ . We also define

$$L_t := \sup D_t \cap D, \quad L'_t := \inf D_t \setminus D.$$

Note that neither of these quantities, nor  $D_t \cap D$  or  $D_t \setminus D$ , are  $\mathcal{F}_t$ -measurable. However, they are measurable with respect to the enlarged filtration

$$\mathcal{G}_t := \sigma(K_s, D_s \cap D; 0 \leq s \leq t).$$

Clearly  $\mathcal{F}_t \subset \mathcal{G}_t$ , and in this larger filtration the bridge lines (and points, and times) of  $K$  that belong to  $K_t$  are measurable objects.

Notice that  $D_t \cap D$  is almost surely closed, and therefore  $L_t$  is actually a maximum rather than a supremum (i.e.  $L_t \in D_t \cap D$ ). Hence  $L_t$  is the largest bridge height of  $K_t$  that is also a bridge height of  $K$ . Clearly  $L_t \leq L'_t$ . The next result follows easily from these definitions.

**Proposition 16.16.** The  $\sigma$ -algebra  $\mathcal{G}_t$  is generated by  $K_t$  and  $L_t$ , i.e.

$$\mathcal{G}_t = \sigma(\mathcal{F}_t, L_t).$$

**Proof** Clearly  $\sigma(\mathcal{F}_t, L_t) \subset \mathcal{G}_t$ , since  $L_t$  is determined by  $D_t \setminus D$ . For the other direction, it is clear that  $D_t \cap D = \{L \in D_t : L \leq L_t\}$ . Hence  $D_t \cap D$  is determined by both  $D_t$  (which is itself determined by  $K_t$ ) and  $L_t$ . This is sufficient because for  $s < t$  we have  $D_s \cap D \subset D_t \cap D$ , and hence  $D_s \cap D$  is the intersection of  $D_s$ , which is  $\mathcal{F}_s$ -measurable, and  $D_t \cap D$ , which we have just shown is  $\sigma(\mathcal{F}_t, L_t)$ -measurable.  $\square$

**Proposition 16.17.** *For a fixed  $t > 0$ ,  $L_t < L'_t$  with probability one.*

**Proof** First observe that  $t$  is almost surely not a bridge time. It is easy to see that the distance between  $\gamma[t, \infty)$  and the last bridge line  $\text{Im}(z) = L_t$  is strictly positive (for instance, there must exist another bridge height higher than  $L_t$ , and between, it is a continuous compact curve). But a bridge height for  $\gamma[0, t)$  that is not a bridge height for the whole curve must be greater than  $\inf \text{Im}(\gamma[t, \infty))$ . We deduce that  $L'_t$  is strictly greater than  $L_t$ .  $\square$

**Definition 16.18.** *Given a subset  $K$  of  $\mathbb{C}$ , define  $J(K) := \inf \{\text{Im} z : z \in K\}$ .*

With this definition in hand we state the chapter's main technical theorem.

**Theorem 16.19.** *Suppose  $K = (\gamma, \mathcal{L})$  obeys the law  $\mathbb{P}_\alpha$ , and let  $\tau$  be a  $\mathcal{G}_t$ -stopping time. On the event that  $\tau$  is a bridge time the  $\mathcal{G}_\tau$ -conditional law of  $\theta_\tau K$  is simply the law of a restriction hull in  $\mathbb{H}$ . If  $\tau$  is not a bridge time then the conditional law of  $\Lambda_\tau K$ , given  $\mathcal{G}_\tau$ , is the same as the law of a restriction hull  $K'$  in  $\mathbb{H} \setminus \gamma[0, \tau]$  whose distribution is the restriction measure corresponding to the triple  $(\mathbb{H} \setminus \gamma[0, \tau], \gamma(\tau), \infty)$ , but further conditioned on the event  $L_\tau < J(K') \leq L'_\tau$ .*

**Remark 16.20.** *Note that if  $\tau$  is a bridge time then  $L_\tau = \text{Im} \gamma(\tau)$  and  $L_{\tau'} = \infty$ . In this situation the notation  $L_\tau < J(K') < L_{\tau'}$  can be interpreted as meaning that the future hull lies strictly above the bridge line, which is an event of measure zero. To fully emphasize this very important point we have handled this case with a separate statement at the beginning of the theorem.*

Theorem 16.19 should be seen as the extension of the Domain Markov property (16.4) to the enlarged filtration  $\mathcal{G}_t$ . In words, it simply says that the extra information in  $\mathcal{G}_\tau$  forces the future restriction hull to go below the horizontal line  $y = L'_\tau$  but stay above the horizontal line  $y = L_\tau$ . This extra conditioning stops  $L'_\tau$  from being a bridge height for  $K$  but preserves  $L_\tau$  as a bridge height. A detailed proof of the theorem follows. It uses a standard procedure, which we modified from [Vir03], to bootstrap from the easy case of  $\tau$  being a deterministic time to the general case that  $\tau$  is a stopping time.

**Proof** To simplify notation, we will write

$$\mathbb{P}_\alpha^t := \mathbb{P}_\alpha^{(\mathbb{H} \setminus \gamma[0,t], \gamma(t), \infty)} (\cdot | L_t < J(K') \leq L'_t)$$

throughout this proof. The goal of the proof is to show that the  $\mathcal{G}_\tau$ -conditional law of  $\Lambda_\tau K$  is  $\mathbb{P}_\alpha^\tau$ .

Consider first the case that  $\tau$  is a deterministic time  $t$ . Recall that conditioning on  $\mathcal{G}_t$  is the same as conditioning on  $\mathcal{F}_t$  and  $L_t$ , by Proposition 16.16. Conditional on  $\mathcal{F}_t$ , the Domain Markov property (16.4) says that  $\Lambda_t K$  has the restriction law for the triple  $(\mathbb{H} \setminus \gamma[0,t], \gamma(t), \infty)$ . Conditioning again on  $L_t$  forces the future hull to stay above  $y = L_t$  but to go below  $y = L'_t$ , and since  $L_t < L'_t$  with positive probability this conditioning is well-defined. Hence the law conditioned on  $\mathcal{G}_t$  is exactly  $\mathbb{P}_\alpha^t$ .

Another way of stating the above is as follows: let  $X$  be a bounded, continuous<sup>2</sup> function on hulls. Then

$$\mathbf{E}_\alpha [X(\Lambda_t K) | \mathcal{G}_t] = \mathbf{E}_\alpha^t [X], \quad (16.7)$$

where  $\mathbb{E}_\alpha$  and  $\mathbb{E}_\alpha^t$  denote expectations with respect to  $\mathbb{P}_\alpha$  and  $\mathbb{P}_\alpha^t$ , respectively. To finish the proof we need to extend (16.7) to  $\mathcal{G}_t$ -stopping times instead of just fixed times. First suppose that  $\tau$  only takes values in some countable set  $\mathcal{T}$ . Then

$$\begin{aligned} \mathbf{E}_\alpha [X(\Lambda_\tau K) | \mathcal{G}_\tau] &= \sum_{t \in \mathcal{T}} \mathbf{E}_\alpha [X(\Lambda_\tau K) \mathbf{1}_{\{\tau = t\}} | \mathcal{G}_\tau] \\ &= \sum_{t \in \mathcal{T}} \mathbf{E}_\alpha [X(\Lambda_t K) \mathbf{1}_{\{\tau = t\}} | \mathcal{G}_t] \\ &= \sum_{t \in \mathcal{T}} \mathbf{1}_{\{\tau = t\}} \mathbf{E}_\alpha [X(\Lambda_t K) | \mathcal{G}_t] \\ &= \sum_{t \in \mathcal{T}} \mathbf{1}_{\{\tau = t\}} \mathbf{E}_\alpha^t [X] \\ &= \mathbf{E}_\alpha^\tau [X]. \end{aligned}$$

From this we can bootstrap up to the case of general  $\tau$ . Let  $\tau_n$  be the smallest element of  $2^{-n}\mathbb{N}$  that is greater than or equal to  $\tau$ . Then the last argument applies to  $\tau_n$ , so that

$$\mathbf{E}_\alpha [X(\Lambda_{\tau_n} K) | \mathcal{G}_{\tau_n}] = \mathbf{E}_\alpha^{\tau_n} [X]. \quad (16.8)$$

However, since  $\tau_n$  is determined at time  $\tau$  (i.e.  $\tau_n$  is  $\mathcal{G}_\tau$ -measurable),

$$\mathbf{E}_\alpha [X(\Lambda_{\tau_n} K) | \mathcal{G}_{\tau_n}] = \mathbf{E}_\alpha [X(\Lambda_{\tau_n} K) | \mathcal{G}_\tau].$$

Since  $\Lambda_{\tau_n} K \rightarrow \Lambda_\tau K$  as  $n \rightarrow \infty$ , and  $X$  is bounded and continuous, it follows that the left hand side of (16.8) converges to

$$\mathbf{E}_\alpha [X(\Lambda_\tau K) | \mathcal{G}_\tau].$$

---

<sup>2</sup>The topology we consider is close to the Caratheodory topology and has been defined in [LSW03, Lemma 3.5]

Hence, if we can show that  $\mathbf{E}_\alpha^{\tau_n}[X]$  converges to  $\mathbf{E}_\alpha^\tau[X]$  then we are done. Since  $X$  is bounded and continuous, this is equivalent to showing that almost surely the law  $\mathbb{P}_\alpha^{\tau_n}$  converges weakly to  $\mathbb{P}_\alpha^\tau$ , which we prove in the next lemma.  $\square$

**Lemma 16.21.** *Let  $\tau$  be a  $\mathcal{G}_t$ -stopping time and  $\tau_n$  be the smallest element of  $2^{-n}\mathbb{N}$  that is greater than or equal to  $\tau$ . Then  $\mathbb{P}_\alpha^{\tau_n}$  converges weakly to  $\mathbb{P}_\alpha^\tau$  with probability one, where we define  $\mathbb{P}_\alpha^\tau(\cdot) := \mathbb{P}_\alpha(\theta_\tau \cdot)$  in the case that  $\tau$  is a bridge time.*

**Proof** Throughout this proof we will let  $H_t := (\mathbb{H} + iL_t) \setminus \gamma[0, t]$ .

As shown in [LSW03, Lemma 3.2], a probability measure on unbounded hulls in the plane is uniquely determined by the collection of probabilities

$$\mathbb{P}(K \cap A = \emptyset)$$

that is indexed by a sufficiently large class of hulls  $A$ . Hence it is enough to show that

$$\mathbb{P}_\alpha^{\tau_n}(K' \cap A = \emptyset) \rightarrow \mathbb{P}_\alpha^\tau(K' \cap A = \emptyset) \quad (16.9)$$

for all hulls  $A$  in this class, with probability one. In our case, it is sufficient to prove that for each fixed restriction hull in  $\mathbb{H}$ , the convergence (16.9) holds for all hulls  $A$  in  $H_\tau$  that are a positive distance from  $\gamma(\tau)$ . Note that since  $\tau_n \downarrow \tau$  and  $\gamma$  is continuous, for sufficiently large  $n$  one must have that  $A$  is at positive distance from  $\gamma(\tau_n)$  also. Hence the probabilities on both sides are well defined. We prove (16.9) in the two distinct cases that  $\tau$  is and is not a bridge time.

#### CASE 1: $\tau$ IS NOT A BRIDGE TIME

First observe that in the definition of  $\mathbb{P}_\alpha^t$ , the conditioning  $J(K') > L_t$  forces the hull  $K'$  to avoid the region  $\{\text{Im } z \leq L_t\}$ , and by the restriction property this can equally be achieved by sampling  $K'$  from the restriction measure corresponding to the triple  $(H_t, \gamma(t), \infty)$ . Thus we have the relation

$$\mathbb{P}_\alpha^{(\mathbb{H} \setminus \gamma[0, t], \gamma(t), \infty)}(\cdot | L_t < J(K') \leq L'_t) = \mathbb{P}_\alpha^{(H_t, \gamma(t), \infty)}(\cdot | J(K') \leq L'_t).$$

Let  $g_t$  be the conformal map from  $H_t$  onto  $\mathbb{H}$  such that  $g_t(\gamma(t)) = 0$  and  $g_t(z) \sim z$  as  $z \rightarrow \infty$ . Let  $R_t := \{z \in H_t : \text{Im } z \leq L'_t\}$ . Then

$$\mathbb{P}_\alpha^t(\cdot) = \mathbb{P}_\alpha^{(H_t, \gamma(t), \infty)}(\cdot | K' \cap R_t \neq \emptyset).$$

The first key observation is that for all  $n$  sufficiently large we have that  $L_{\tau_n} = L_\tau$ . This equality is clear since  $G$  is closed, and hence  $\tau_n$  must belong to the same connected component of  $G^c$  that  $\tau$  belongs to, for  $n$  sufficiently large. For these  $n$  we have  $L_{\tau_n} = L_\tau$ . For  $L'_\tau$  there are two distinct possibilities, which we now treat separately.

First note that necessarily  $L'_\tau < \infty$ . Indeed, the maximum of the imaginary part of  $\text{Im } K_\tau$  is always an element of  $D_\tau$ , and since  $\tau$  is not a bridge time this maximum cannot be in  $D$ . So first consider the case that  $L'_\tau < \text{Im } \gamma(\tau)$ . By formula (16.5), we have that

$$\begin{aligned} \mathbb{P}_\alpha^t (K' \cap A = \emptyset) &= \frac{\mathbb{P}_\alpha^{(H_t, \gamma(t), \infty)} (K' \cap A = \emptyset, K' \cap R_t \neq \emptyset)}{\mathbb{P}_\alpha^{(H_t, \gamma(t), \infty)} (K' \cap R_t \neq \emptyset)} \\ &= \frac{\phi'_{A_t}(0)^\alpha - \phi'_{A_t \cup S_t}(0)^\alpha}{1 - \phi'_{S_t}(0)^\alpha}. \end{aligned} \quad (16.10)$$

where  $A_t = g_t(A)$  and  $S_t = g_t(R_t)$  (this is justified since neither  $A$  nor  $R_\tau$  contains  $\gamma(\tau)$ ). Equation (16.10) shows that it is sufficient to prove

$$\phi'_{A_{\tau_n}}(0) \rightarrow \phi'_{A_\tau}(0), \quad \phi'_{A_{\tau_n} \cup S_{\tau_n}}(0) \rightarrow \phi'_{A_\tau \cup S_\tau}(0), \quad \phi'_{S_{\tau_n}}(0) \rightarrow \phi'_{S_\tau}(0). \quad (16.11)$$

For  $n$  large enough,  $L'_{\tau_n} = L'_\tau$  since for any neighborhood of  $\text{Im } \gamma(\tau)$  there is an  $n$  sufficiently large such that  $D_{\tau_n} \setminus D_\tau$  is contained within this neighborhood. Since  $L'_\tau < \text{Im } \gamma(\tau)$ , by making the neighborhood sufficiently small we get that  $D_{\tau_n} \setminus D$  and  $D_\tau \setminus D$  must have the same infimum; that is  $L'_{\tau_n} = L'_\tau$ . Hence,  $A_{\tau_n}$  and  $S_{\tau_n}$  are only decreasing as  $\gamma[0, \tau_n]$  decreases, and again since  $\gamma[0, \tau_n]$  is a simple curve that shrinks to  $\gamma[0, \tau]$  it follows that  $g_{\tau_n}$  converges uniformly to  $g_\tau$  on all subcompacts of  $H_\tau$ , from which the convergences of (16.11) follow (by Cauchy's derivative formula and the Schwarz reflection principle, see [LSW03]).

The second possibility is to have  $L'_\tau = \text{Im } \gamma(\tau)$ . On the one hand, the conditioning on  $K'$  going below  $\text{Im}(\gamma(\tau))$  is trivial so that  $\mathbb{P}_\alpha^\tau = \mathbb{P}_\alpha^{(H_\tau, \gamma(\tau), \infty)}$ . On the other hand,  $L'_{\tau_n}$  is greater than  $L'_\tau$  so that one can strengthen the conditioning of  $\mathbb{P}_\alpha^{\tau_n}$  by requiring that the future hull goes below  $L'_\tau$ . Since  $\gamma$  is a simple curve shrinking to 0, one again has that  $g_{\tau_n}$  converges uniformly to  $g_\tau$  on all subcompacts of  $H_\tau$ , which proves that the conditioning becomes trivial.

#### CASE 2: $\tau$ IS A BRIDGE TIME

In this case note that  $A$  is a hull in the domain  $\mathbb{H} + i\text{Im } \gamma(\tau) = \mathbb{H} + iL_\tau$ ; hence it is simply a translate of a hull in  $\mathbb{H}$ . Moreover  $g_\tau$  is simply the shift map  $z \rightarrow z - \gamma(\tau)$ , from which it follows that  $A_\tau = A - \gamma(\tau)$  and  $S_\tau = \mathbb{H}$ . Since  $\mathbb{P}_\alpha^\tau(\cdot) = \mathbb{P}_\alpha(\theta_\tau \cdot)$ , proving (16.9) amounts to showing that

$$\mathbb{P}_\alpha^{\tau_n} (K' \cap A = \emptyset) \rightarrow \phi'_{A_\tau}(0).$$

We use (16.10) to rewrite the left hand side. Define  $U_t = \phi_{A_t}(S_t \cap A_t^c)$  so that

$$\phi_{A_t \cup S_t} = \phi_{U_t} \circ \phi_{A_t},$$

from which it follows that

$$\phi'_{A_t \cup S_t}(0) = \phi'_{U_t}(0) \phi'_{A_t}(0).$$

Therefore

$$\mathbb{P}_\alpha^{\tau_n} (K' \cap A = \emptyset) = \phi'_{A_{\tau_n}}(0)^\alpha \frac{1 - \phi'_{U_{\tau_n}}(0)^\alpha}{1 - \phi'_{S_{\tau_n}}(0)^\alpha}.$$

The convergence of  $\phi'_{A_{\tau_n}}(0)$  to  $\phi'_{A_\tau}(0)$  is simple since it only involves the map  $g_{\tau_n}$ . Note that  $L_\tau \leq L_{\tau_n} \leq \text{Im } \gamma(\tau_n)$ , so that the domains  $H_{\tau_n}$  converge to  $H_\tau$ , and since  $\gamma$  is a simple curve it once again follows that  $g_{\tau_n}$  converges uniformly to  $g_\tau$  on all subcompacts of  $A_\tau$ . As before, this implies the convergence of  $\phi'_{A_{\tau_n}}(0)$  to  $\phi'_{A_\tau}(0)$ .

It remains to be shown that, as  $n \rightarrow \infty$ ,

$$\frac{1 - \phi'_{U_{\tau_n}}(0)^\alpha}{1 - \phi'_{S_{\tau_n}}(0)^\alpha} = \frac{\mathbb{P}_\alpha (K'' \cap U_{\tau_n} \neq \emptyset)}{\mathbb{P}_\alpha (K'' \cap S_{\tau_n} \neq \emptyset)} \rightarrow 1.$$

Observe that

$$\begin{aligned} \mathbb{P}_\alpha (K \cap U_{\tau_n} \neq \emptyset) &= \mathbb{P}_\alpha (K \cap \phi_{A_{\tau_n}}(S_{\tau_n} \cap A_{\tau_n}^c) \neq \emptyset) \\ &= \mathbb{P}_\alpha^{(\mathbb{H} \setminus A_{\tau_n}, 0, \infty)} (K \cap S_{\tau_n} \neq \emptyset) \\ &\sim \mathbb{P}_\alpha^{(\mathbb{H} \setminus A_\tau, 0, \infty)} (K \cap S_{\tau_n} \neq \emptyset). \end{aligned}$$

The last relation follows since  $g_{\tau_n}$  converges uniformly to  $g_\tau$  on all subcompacts of  $H_{\tau_n}$ , to which  $A$  eventually belongs, so that  $A_{\tau_n}$  converges to  $A_\tau$ . Next recall that  $S_{\tau_n} = g_{\tau_n}(R_{\tau_n})$ , and

$$0 < \sup \text{Im } R_{\tau_n} \leq L'_{\tau_n} - \text{Im } \gamma(\tau),$$

with the right hand side going to zero as  $n \rightarrow \infty$ . Since the distance of  $A_\tau$  from zero is positive, for  $n$  sufficiently large the probability that a restriction hull intersects  $S_{\tau_n}$  is of the order of  $\sup \text{Im } R_{\tau_n}$  and dominated by hulls that intersect  $S_{\tau_n}$  near zero. Since the set  $S_{\tau_n}$  is the same near zero in both  $\mathbb{H}$  and  $\mathbb{H} \setminus A_\tau$ , the ratio

$$\frac{\mathbb{P}_\alpha^{(\mathbb{H} \setminus A_\tau, 0, \infty)} (K \cap S_{\tau_n} \neq \emptyset)}{\mathbb{P}_\alpha (K \cap S_{\tau_n} \neq \emptyset)}$$

tends to 1. □

**Remark 16.22.** *Theorem 16.19 is most useful when  $\tau$  is a bridge time, meaning it almost surely takes values in  $G$ . In that case  $\gamma(\tau)$  is a bridge point for  $K$ , and the corresponding bridge line separates the future hull from the past. Shifting the future hull back to the origin by subtracting off  $\gamma(\tau)$ , we have the following:*

**Corollary 16.23.** *At  $\mathcal{G}_t$ -stopping times  $\tau$  that almost surely take values in  $G$ , the shifted future hull  $\theta_\tau K$  obeys the law  $\mathbb{P}_\alpha$ .*

Corollary 16.23 will be the key element in proving that the restriction hulls can be decomposed into a Poisson Point Process, which is the subject of the next section. Before doing that, we immediately apply the corollary to Theorem 16.2, part (2) by showing that  $C$  and  $D$  almost surely have no isolated points.

**Proof of Theorem 16.2, part (2)** We have already shown that  $C$  and  $D$  are closed, we prove that  $C$  has no isolated points. Almost surely, zero is not isolated in  $C$  because of the scale invariance and the fact that bridge points exist. For a rational number  $r$ , let  $\tau_r$  be the first bridge time after time  $r$ . Then by the previous corollary, we deduce that the law of  $\theta_{\tau_r}K$  obeys the law  $\mathbb{P}_\alpha$ . Since  $\gamma(\tau_r)$  shifts to zero under  $\theta_{\tau_r}$ , the previous remark shows that  $\gamma(\tau_r)$  is almost surely not isolated. From these facts we deduce that the event  $\{\gamma(\tau_r) \text{ is not isolated in } C \text{ for all rational } r\}$  has probability one. If a point  $\gamma(t) \in C$  were isolated then there would have to be an interval of time around  $t$  which contains no other bridge times, but since this interval contains a rational time we arrive at a contradiction.  $\square$

## 5 Local Time of the Decomposition

In this section we will show that there exists a natural local time on the bridge heights that we use to decompose the restriction hulls into a Poisson Point Process of irreducible bridges. All the results of this section derive from the theory of subordinators and regenerative sets, which is well described in [Ber99]. We briefly recall the definition of regenerative sets, which is taken from [Ber99, Chapter 2].

**Definition 16.24.** *A random subset  $S$  of  $[0, \infty)$  is a regenerative set with respect to a filtration  $\mathcal{F}_t$  if for every  $s \geq 0$ , conditionally on  $M_s = \inf\{t > s : t \in S\} < \infty$ , the shifted set  $(S - M_s) \cap [0, \infty)$  has the same law as  $S$  and is independent of  $\mathcal{F}_{M_s}$ .*

Using the results of Sections 3 and 4, we can immediately prove:

**Proposition 16.25.** *The set  $D$  of bridge heights is regenerative with respect to  $\mathcal{D}_L := \sigma(D \cap [0, L])$ .*

**Proof** Consider  $L \geq 0$ . Since  $D$  is closed,  $M_L \in D$  almost surely. Then  $M_L$  is a bridge height, and the time  $\tau_L$  at which the curve reaches this bridge height is a  $\mathcal{G}_t$ -stopping time taking values in  $G$ . By Corollary 16.23, the  $\mathcal{G}_{\tau_L}$ -law of  $\theta_{\tau_L}K$  is the same as the original law of  $K$ . Consequently, the  $\mathcal{G}_{\tau_L}$ -law of  $D(\theta_{\tau_L}K) = D - M_L$  is the same as the law of  $D$ . Since  $\mathcal{D}_L \subset \mathcal{G}_{\tau_L}$  this completes the proof.  $\square$

Proposition 16.25 proved that the set  $D$  is regenerative, and consequently by [Ber99, Theorem 2.1] it is the closure of the image of some subordinator (and the subordinator is unique up to a linear change of its time scale). On the other hand, Theorem 16.2 showed that  $D$  is scale invariant, and it is an easy step to deduce from this that the subordinator must be stable. Recall that there is a one-parameter family of stable subordinators, indexed by the real numbers between 0 and 1, and, as shown in [Ber99, Chapter 5], the index of a stable subordinator is the same as the Hausdorff dimension of its image. Hence we have the following:

**Corollary 16.26.** *Under the law  $\mathbb{P}_\alpha$ , the set  $D$  is the closure of the image of a stable subordinator  $(\sigma_\lambda, \lambda \geq 0)$  of index  $2 - 2\alpha$ .*

The parameter  $\lambda$  can be thought of as the local time corresponding to the subordinator. Recall that the local time for  $\sigma$  is the function  $\lambda : [0, \infty) \rightarrow [0, \infty)$  defined by  $\lambda(s) := \inf\{t \geq 0 : \sigma_t > s\}$ , and it is well known in the subordinator literature that  $\lambda$  is an increasing, continuous function which increases only on  $D$ . This means that if we run the restriction hulls on the  $\lambda$  time scale, then the hull grows only when it is crossing bridge lines. For  $\lambda \geq 0$  we define

$$\tau_h := \inf\{t \geq 0 : \sup \text{Im}(K_t) = h\},$$

and

$$t(\lambda) := \tau_{\sigma_\lambda}.$$

Note that  $\sigma_\lambda$  is the bridge height at which  $\lambda$  units of local time are first accumulated, and then  $t(\lambda)$  is the time, in the original parameterization of the restriction hull, at which the local time first reaches  $\lambda$ . It follows that  $t(\lambda)$  is an increasing, right-continuous process for which the closure of its image is precisely the set of bridge times  $G$ . Intervals of  $\lambda$  on which the process  $t(\lambda)$  is flat correspond to times at which the restriction hull is between bridge heights. Using the  $t(\lambda)$  time-scale, we are able to define a Poisson Point Process taking values in the space of irreducible bridges rooted at the origin. Let  $\delta$  be the curve which starts and ends at zero in zero time (i.e.  $\delta : \{0\} \rightarrow \{0\}$ ). For  $\lambda \geq 0$ , define  $e_\lambda$  by

$$e_\lambda = \begin{cases} \theta_{t(\lambda-), t(\lambda)} K, & t(\lambda) > t(\lambda-) \\ \delta, & t(\lambda) = t(\lambda-) \end{cases} \quad (16.12)$$

From this we have the following:

**Proposition 16.27.**  *$e_\lambda$  is an  $(\mathcal{F}_{t(\lambda)})_{\lambda \geq 0}$  Poisson Point Process on the space of irreducible bridges.*

**Proof** Take a subset  $U$  of the set of irreducible bridges that doesn't contain  $\delta$ , and an interval  $I := [\lambda_1, \lambda_2]$ . As in [RY99, Chapter XII], one needs to show that the number of times that  $e_\lambda$  belongs to  $U$  for  $\lambda \in I$  is independent of  $\mathcal{F}_{t(\lambda_1)}$  and has the same law as the number of times that  $e_\lambda$  belongs to  $U$  for  $\lambda \in [0, \lambda_2 - \lambda_1]$ . But this is essentially a property of Corollary (16.23).  $\square$

We denote by  $\nu_\alpha$  the intensity measure of the Poisson Point Process  $e_\lambda$ , and we call it the **continuum irreducible bridge measure**. It conveniently encodes all the behavior of continuum irreducible bridges. For a set of irreducible bridges  $E$ ,  $\nu_\alpha(E)$  is simply the expected number of elements of  $E$  that occur in  $e[0, 1]$ , which may or may not be finite. For instance, if  $E_L$  is the set of irreducible bridges with height greater than  $L$ , then a

simple consequence of Corollary 16.26 is that  $\nu_\alpha(E_L) = c_\alpha L^{2\alpha-2}$  for some fixed constant  $c_\alpha$ , and furthermore,

$$\mathbf{P}_\alpha^L(\cdot) := \frac{\nu_\alpha(\cdot \cap E_L)}{\nu_\alpha(E_L)} \tag{16.13}$$

is exactly the law of the first irreducible bridge with height greater than  $L$ . To make the analogy with other well-known decompositions of stochastic processes,  $\nu_\alpha$  is the equivalent of Itô's measure on 1-dimensional Brownian excursions, or Balint Virág's measure on 2-dimensional Brownian Beads. Compared to half-plane SAWs,  $\nu_\alpha$  is the analogue of the measure  $\mathbf{P}(\omega) = \beta^{-|\omega|}$  on SAW irreducible bridges, although we point out that  $\mathbf{P}$  is a probability measure (by Kesten's relation), whereas  $\nu_\alpha$  is infinite but  $\sigma$ -finite.

In the case of half-plane SAWs, the measure on paths is realized by concatenating together an i.i.d. sequence of irreducible bridges, each distributed according to  $\mathbf{P}$ , and in the continuum a similar statement holds. If  $(e_\lambda)_{\lambda \geq 0}$  is a Poisson Point Process of irreducible bridges with intensity measure  $\nu_\alpha$ , then the concatenation

$$K = \bigoplus_{\lambda \geq 0} e_\lambda$$

has the law of an index  $\alpha$  restriction hull. Note, however, that we are not attempting to show that the irreducible bridges can be concatenated together in such a way as to reconstruct the sequence of growing hulls  $(K_t)_{t \geq 0}$ , even though this should be possible with enough care. Recall though that the time parameterization we are using for the restriction hulls is completely artificial to begin with, and therefore attempting to reconstruct it would mostly be an uninteresting and unuseful exercise.

## 6 Open Questions

In this final section we present some open questions that were raised by our work.

**Question 16.28.** *What other properties of the irreducible bridge measure  $\nu_\alpha$  can be derived?*

Our work has essentially determined only one main property of bridges: that the distribution of their vertical height is the same as the jump distribution for a stable subordinator of index  $2 - 2\alpha$  (up to a multiplicative constant). Ultimately we hope that much more can be said about irreducible bridges than this. It may be naturally difficult to say anything more, since even in the case of half-plane SAWs there is not much known about irreducible bridges (although in the “off-critical” case there are some results, see [MS93, Chapter 4]). For other two-dimensional decompositions, notably Virág's Brownian Beads, it appears similarly difficult to say anything about the bead measure.

**Question 16.29.** *Is there a constructive way of building irreducible bridges?*

In the case of SLE(8/3), for example, is there a driving term for the Loewner equation that outputs irreducible bridges (perhaps with at least some specified vertical height)? And for general restriction measures with  $\alpha < 1$ , can some driving term for the Loewner equation be combined with the Brownian loop soup to produce irreducible bridges for restriction hulls?

**Question 16.30.** *Is there a natural “length” that can be put on irreducible bridges?*

For half-plane SAWs the length of the walk is simply the number of steps in it, and many results on SAWs are expressed in terms of this length. We expect that there is some way of defining a similar natural length on irreducible bridges, and that this length is somehow the scaling limit of the length for SAWs. However, because the irreducible bridges are fractal objects it is not an easy matter to define a non-trivial length on them. In the case of SLE(8/3) specifically, this question is closely related to the problem of the “natural time parameterization” for SLE, which has recently been considered by Lawler and Sheffield [LS09]. The key idea of their time parameterization is to build a length measure on the curve (that also has some other desirable properties), and then reparameterize in such a way that the length of the curve at time  $t$  is  $t$ , as with the SAWs. Their length measure should also be a natural length measure for irreducible bridges.

**Question 16.31.** *Is there some sort of continuous analogue of Kesten’s relation?*

This is closely related to the problem of the natural length on irreducible bridges described above. Supposing that  $L(K)$  is the “natural length” of an irreducible bridge, and making an analogy with (16.2), we might expect that

$$\int_0^\infty \beta^{-l} \nu_\alpha(L(K) \in dl)$$

is finite for  $\beta < \mu$  but infinite for  $\beta > \mu$ , for some universal  $\mu$ , and then one can ask for the behavior at this critical  $\mu$ .

**Question 16.32.** *Can the restriction hulls be time parameterized in such a way that the time parameterization also refreshes itself at bridge points?*

Presently we are only showing that the hulls refresh themselves as sets and *not* as time parameterized objects. But it is entirely plausible that there is some time parameterization which refreshes itself at bridge points along with the geometrical objects, especially considering that the counting parameterization for half-plane SAWs has this property (at each bridge point, one simply starts counting off the number of steps anew). It is possible that the natural time parameterization of Lawler and Sheffield will have this property for SLE(8/3) but it is not immediately clear that this will be the case, since their time parameterization has no way of seeing that it is currently at a bridge point and therefore is unlikely to refresh at such bridge times.

**Question 16.33.** *Can some element of the bridge decomposition be used to prove the existence of, or at least heuristically deduce, critical exponents for half-plane SAWs or SAW bridges?*

For example, it is conjectured that the number of  $N$ -step SAW bridges grows asymptotically like  $N^{-\beta}u^N$  as  $N \rightarrow \infty$ , for the same  $\mu$  as in (16.1) and some unknown constant  $\beta$ . Recently, Neal Madras has privately communicated to us his conjecture that  $\beta = 7/16$ , although this quantity was likely known beforehand in the physics literature. He uses two different methods to derive this value, the first being based purely on some heuristics for half-plane SAWs, and the other making use of the relation (16.13) and the conjecture that the scaling limit of half-plane SAWs is SLE(8/3). Being able to answer further questions of this type would be extremely helpful for studying half-plane SAWs.

**Question 16.34.** *Do bridge heights and lines exist for SLE( $\kappa$ ) for values of  $\kappa$  different from 8/3. If so, what is the Hausdorff dimension of  $C$  and  $D$  and how does it depend on  $\kappa$ ?*

Currently we only know that at  $\kappa = 0$  and  $\kappa = 8/3$ , the Hausdorff dimensions of  $C$  and  $D$  are 1 and  $3/4$ , respectively (the  $\kappa = 0$  result is clear from the fact that the corresponding SLE curve is a vertical line). We conjecture that the Hausdorff dimensions of  $C$  and  $D$  are always the same, and they are a strictly decreasing, continuous function of  $\kappa$ . When  $\kappa = 4$  the Hausdorff dimension must certainly be zero since the SLE(4) curve comes arbitrarily close to the real line, but we do not know if this is the smallest  $\kappa$  for which the dimension is zero. We have no conjecture as to what that  $\kappa$  might be, other than it is somewhere between  $8/3$  and  $4$ .

We should briefly mention that, as a corollary of Theorem 16.2, we do have lower bounds on the Hausdorff dimension of  $C$  and  $D$  for  $2 \leq \kappa \leq 8/3$ . Since attaching loops to an SLE curve can only reduce the number of bridge points that the SLE curve has, we know

**Proposition 16.35.** *Let  $C$  and  $D$  be the set of bridge points and heights for an SLE( $\kappa$ ) curve, with  $2 \leq \kappa \leq 8/3$ . Then the Hausdorff dimensions of  $C$  and  $D$  are both almost surely constant, with  $\dim_{\text{H}} C \geq 3 - \frac{6}{\kappa}$ .*

This lower bound is probably far from sharp, since it is increasing with  $\kappa$  rather than decreasing. To prove that the Hausdorff dimensions of  $C$  and  $D$  are almost surely constant, Theorem 16.2 part (3) can be used without modification.



# Chapter 17

## Many questions and a few answers

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**Abstract:** In this chapter, we gather several open questions and recall some of the main conjectures in the domain.

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### 1 Ising model

#### 1.1 Universality of the Ising model

Until now, we considered only the square lattice Ising model. Nevertheless, normalization group theory predicts that the scaling limit should be universal. In other words, the limit of critical Ising models on planar graphs should always be the same. In particular, the scaling limit of interfaces in spin Dobrushin domains should converge to SLE(3).

Of course, one should be careful about the way the graph is drawn in the plane. For instance, the isotropic spin Ising model of Chapter 6, when considered on a stretched square lattice (every square is replaced by a rectangle), is not conformally invariant (it is not invariant under rotations). Isoradial graphs mentioned in Chapter 2 form a large family of graphs possessing a natural embedding on which a critical Ising model is expected to be conformally invariant. More details are now provided about this fact.

**Definition 17.1.** *A rhombic embedding of a graph  $G$  is a planar quadrangulation satisfying the following properties:*

- *the vertices of the quadrangulation are the vertices of  $G$  and  $G^*$ ,*
- *the edges connect vertices of  $G$  to vertices of  $G^*$  corresponding to adjacent faces of  $G$ ,*
- *all the edges of the quadrangulation have equal length, see Fig. 2.5.*

*A graph which admits a rhombic embedding is called isoradial.*

Isoradial graphs are fundamental for two reasons. First, discrete complex analysis on isoradial graphs was extensively studied (see *e.g.* [Mer01, Ken02, CS08]) as explained in Chapter 2. Second, the Ising model on isoradial graphs satisfies very specific integrability properties and a natural critical point can be defined as follows. Let  $J_{xy} = \operatorname{arctanh} [\tan(\theta/2)]$  where  $\theta$  is the half-angle at the corner  $x$  (or equivalently  $y$ ) made by the rhombus associated to the edge  $[xy]$ . One can define the *critical Ising model* with Hamiltonian

$$H(\sigma) = - \sum_{x \sim y} J_{xy} \sigma_x \sigma_y.$$

This Ising model on isoradial graphs (with rhombic embedding) is critical and conformally invariant in the following sense:

**Theorem 17.2** (Chelkak, Smirnov [CS09]). *The interfaces of the critical Ising model on isoradial graphs converge, as the mesh size goes to 0, to the chordal Schramm-Loewner Evolution with  $\kappa = 3$ .*

Note that the previous theorem is uniform on any rhombic graph discretizing a given domain  $(\Omega, a, b)$ , as soon as the edge-length of rhombi is small enough. This provides a first step towards universality for the Ising model.

**Question 17.3.** *Since not every topological quadrangulation admits a rhombic embedding [KS05], can another embedding with a sufficiently nice version of discrete complex analysis always be found?*

**Question 17.4.** *Is there a more general discrete setup where one can get similar estimates, in particular convergence of preholomorphic functions to the holomorphic ones in the scaling limit?*

In another direction, consider a biperiodic lattice  $\mathcal{L}$  (one can think of the universal cover of a finite graph on the torus), and define a Hamiltonian with periodic correlations  $(J_{xy}) \in (0, \infty)^{E(\mathcal{L})}$  by setting  $H(\sigma) = - \sum_{x \sim y} J_{xy} \sigma_x \sigma_y$ . The Ising model with this Hamiltonian makes perfect sense and there exists a critical inverse temperature separating the disordered phase from the ordered phase.

**Question 17.5.** *Prove that there always exists an embedding of  $\mathcal{L}$  such that the Ising model on  $\mathcal{L}$  is conformally invariant.*

Note that the question of universality is not restricted to the Ising case. It is classical in the case of random-walks, has been studied in the case of loop-erased random-walks [YY08]. Recently, progresses have been made in the case of bond percolation [GM11a, GM11b].

## 1.2 Full scaling limit of critical Ising model

It has been proved in [Smi10b] that the scaling limit of Ising interfaces in Dobrushin domains is SLE(3). Chelkak-Smirnov's proof of this fact was provided in Chapter 11. The

next question is to understand the full scaling limit of the interfaces. This question raises interesting technical problems. Consider the Ising model with free boundary conditions. Interfaces now form a family of loops. By consistency, each loop should look like a SLE(3). We believe that the strong form of crossing estimates proved in Chapter 10 would be useful to study these interfaces. In [HK11], Hongler and Kytolä made one step towards the complete picture by studying interfaces with  $+/-$  free boundary conditions.

Sheffield and Werner [SW10a, SW10b] introduced a one-parameter family of processes of non-intersecting loops which are conformally invariant – called the Conformal Loop Ensembles  $CLE(\kappa)$  for  $\kappa > 8/3$ . Not-surprisingly, loops of  $CLE(\kappa)$  are locally similar to  $SLE(\kappa)$  and these processes are natural candidates for the scaling limits of planar models of statistical physics. In the case of the Ising model, the limits of interfaces all together should be a  $CLE(3)$ .

### 1.3 Scaling relations for the Ising model

As mentioned in Chapter 12, the near-critical phase of the FK-Ising model exhibits striking phenomena. It would be interesting to understand the behavior of the increasing coupling between random-cluster measures, and to make sense of the scaling relations when the probability to be pivotal is replaced by the influence. Let us mention that most of the exponents for Ising model have been already computed. Still, the understanding of the mechanisms behind the scaling relations is of some value. A natural question to start with would be the following:

**Question 17.6.** *Does the derivative of the one-arm probability  $\pi_1^p(n) = \phi_{p,2}(0 \leftrightarrow \partial B_n)$  in FK-Ising behaves like  $\pi_1^p(n) \cdot \frac{d}{dp} \phi_{p,2}(B_n \text{ is crossed})$ ?*

### 1.4 Conformal invariance of Ising model when $\beta < \beta_c$

The high-temperature Ising model on the triangular lattice should also be conformally invariant. Each spin is either  $+$  or  $-$ , with probability  $1/2$  (in this case there is only one infinite-volume measure, hence the symmetry  $+/-$ ). Moreover, the correlations between spins decay exponentially fast (think that the random-cluster representation is subcritical). The scaling-limit of the interfaces  $+/-$  should be conformally invariant, and should satisfy the locality property, hence it should be SLE(6).

**Question 17.7.** *Prove conformal invariance (universality in the temperature parameter) of the high-temperature of the Ising model on the triangular lattice?*

This model is at the interface between critical Ising and percolation on the triangular lattice, two models for which conformal invariance is known.

## 2 Random-cluster model with cluster-weight $q \geq 0$

### 2.1 Identification of the critical point and crossing probabilities

On the square lattice, the random-cluster model with cluster-weight  $q \in (0, \infty)$  is conjectured to be critical for  $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$ . This result was proved in Chapter 4 for  $q \geq 1$ . Yet the proof strongly relies on positive association. In fact, the existence of a critical point is not even proved in the case  $q \in (0, 1)$ . These models are expected to be negatively correlated and the FKG inequality is not valid anymore<sup>1</sup>. In opposition with the theory of positively correlated models, negative correlation is very poorly understood.

**Question 17.8.** *Prove that there is a phase transition for  $q < 1$  for the square lattice? Prove that  $p_c(q) = \sqrt{q}/(1 + \sqrt{q})$ ?*

Note that critical points are not expected to have 'nice' values for general models. In particular, one should not expect to find close formulæ. Nevertheless, an important corollary of the proof in Chapter 4 is the exponential decay of correlations in subcritical phase and proving this fact makes perfect sense:

**Question 17.9.** *Prove that the phase transition of random-cluster models with  $q \geq 1$  is sharp on any periodic lattice.*

The argument developed in Chapter 4 is quite general in nature, yet quite fragile concretely. Indeed, the two ingredients of the proof (crossing probabilities estimates, also called RSW theorems, and sharp threshold theorems) are expected to be universal. This seems to suggest that the proof itself should be robust. Unfortunately, the proof of RSW relies heavily on strong forms of self-duality to counter the lack of independence which is not available for general random-cluster models. Although, it should be possible to obtain RSW at criticality abstractly, as it is the case for percolation. In fact, even in the case of percolation the proof of RSW is restricted to lattices with rotational symmetry (except a recent proof due to Grimmett and Manolescu [GM11a] for anisotropic models). This leads to the following question:

**Question 17.10.** *Understand RSW for general lattices (for percolation or for random-cluster models with bulk boundary conditions).*

A question which is strongly related to the previous one is the existence of so-called strong Russo-Seymour-Welsh theorems. More precisely, does there exist lower and upper bounds on crossing probabilities at criticality which are uniform in the size *and* the boundary conditions? In other words, the following is expected

**Conjecture 17.11.** *Consider the random-cluster model of parameter  $(p_{sd}(q), q)$  with  $1 \leq q < 4$  and let  $0 < \beta_1 < \beta_2$ . There exist two constants  $0 < c_-(q) \leq c_+(q) < 1$  such that for any rectangle  $R$  with side lengths  $n$  and  $m \in \llbracket \beta_1 n, \beta_2 n \rrbracket$ , one has*

$$c_-(q) \leq \mathbb{P}_{p_{sd}(q), q, R}^{\xi}(\mathcal{C}_v(R)) \leq c_+(q)$$

<sup>1</sup>Consider a graph with two vertices joined by two edges: the probability that both edges are open is smaller than the product of the probabilities of each one being open.

for any boundary conditions  $\xi$ .

Theorem 4.4 does not answer this question since it is restricted to wired boundary conditions at infinity or periodic boundary conditions (except when  $q = 1$ , since boundary conditions do not matter in this case). Theorem 9.1 however solves the question for  $q = 2$ . We have seen that even when  $q = 2$ , passing from free boundary conditions on a smooth boundary to free boundary conditions on a rough boundary is not easy, and that it requires additional arguments.

Uniform lower bounds holding true with free boundary conditions on the boundary of a rectangle is a discrete statement and is therefore hard to predict using the usual SLE-machinery. Nevertheless, it seems coherent to believe that Theorem 9.1 holds for any  $q < 4$ . When  $q > 4$ , we have seen that the phase transition should be of first order, and crossing probabilities with free boundary conditions should decay exponentially fast. The case of  $q = 4$  is interesting, since Theorem 9.1 is expected to fail, even though there are still circuits in annuli with arbitrary boundary conditions with positive probability. Note that the existence of a circuit in an annulus is an event which is measurable in terms of the CLE in the scaling limit. Since  $q = 4$  corresponds to  $\text{CLE}(4)$ , there should be loops surrounding the inner boundary with positive probability.

## 2.2 Order of the phase transition

Critical random-cluster is expected to exhibit a very rich phase transition, whose properties depend strongly on the value of  $q$  (see Fig. 17.1).

**Case  $q \leq 4$ .** In Chapter 5, the transition was shown to be second order when  $q < 4$  in the sense that the correlation length diverges when approaching the critical point. As was discussed previously, a very rich behavior can be expected at criticality in the case of such phase transitions. In terms of probabilities, the divergence of the critical point is not the best indicator of a second order phase transition. It would be more relevant to classify infinite-volume measures at criticality and to prove the following:

**Question 17.12.** *For  $q \leq 4$ , prove that there is a unique infinite-volume measure with parameter  $(p_{sd}(q), q)$ .*

Remember that it is sufficient to prove that there is no infinite cluster for  $\phi_{p_{sd},q}^1$  almost surely. In the case of percolation, an argument of Russo [Rus78] shows that the divergence of the susceptibility is equivalent to the absence of infinite cluster in the dual. For  $1 < q \leq 3$ , the mean-size of the cluster at the origin under  $\phi_{p_{sd},q}^0$  was shown to be infinite (corresponding to divergence of the susceptibility), which should be an indicator of the absence of dual cluster. Since the dual model is a random-cluster model at criticality with wired boundary conditions, the result would follow if Russo's argument could be extended to general random-cluster models. Even though the argument seems fairly rigid, we were unable to generalize it.

**Question 17.13.** *Show that an infinite susceptibility implies the absence of infinite-cluster in the dual.*

**Case  $q > 4$ .** The picture is very different: the phase transition is conjectured to be of first order : there are multiple infinite-volume measures at criticality. In particular, the critical random-cluster model with wired boundary conditions should possess an infinite cluster almost surely while the critical random-cluster model with free boundary conditions does not (in this case, the connectivity probabilities should even decay exponentially fast). This result is known only for  $q \geq 25.72$  (see [Gri06, LMMS<sup>+</sup>91, LMR86] and references therein).

**Question 17.14.** *Prove that there exists an infinite cluster for  $\phi_{p_{sd},q}^1$  whenever  $q > 4$ .*

This result was shown in a different geometry, since there is an infinite cluster on the infinite stairs (see Chapter 5). Bootstrapping information on measures in the plane would be very interesting, and would allow us to prove first order phase transition.

### 2.3 Conformal invariance for $q \in [0, 4]$

The parafermionic observable is now used to predict the critical behavior for general  $q \in [0, 4]$ . For  $q > 0$ , recall that the *parafermionic observable* is defined by

$$F(e) = \mathbb{E}_{\Omega_\delta^{a_\delta, b_\delta}} [e^{\sigma i W_\gamma(e, b_\delta)} \mathbf{1}_{e \in \gamma}], \tag{17.1}$$

where  $\sigma = \sigma(q)$  is called the *spin* ( $\sigma$  takes a special value described below). When  $q \leq 4$ , the spin is real and is expected to be related to the central charge of the conformal field theory describing the critical behavior. Less prosaically, the value of  $\sigma$  can be tuned in such a way that  $F$  satisfies integrability relations at criticality: consider the observable  $F$  at criticality with spin  $\sigma = 1 - \frac{2}{\pi} \arccos(\sqrt{q}/2)$ . For any medial vertex inside the domain,

$$F(N) - F(S) = i[F(E) - F(W)] \tag{17.2}$$

where  $N, E, S$  and  $W$  are the four medial edges adjacent to the vertex (See Proposition 5.8). These relations can be understood as Cauchy-Riemann equations around some vertices and  $F$  is weakly-preholomorphic (see Section 3.3 of Chapter 2). Importantly,  $F$  is not determined by these relations for general  $q$  (the number of variables exceeds the number of equations). For  $q = 2$ , which corresponds to  $\sigma = 1/2$ , the complex argument modulo  $\pi$  of the observable offers additional relations (Lemma 7.4) and it is then possible to obtain the preholomorphicity (Proposition 7.6).

Parafermionic observables can be defined on medial vertices by the formula

$$F(v) = \frac{1}{2} \sum_{e \sim v} F(e)$$

where the summation is over medial edges with  $v$  as an end-point. Even so they are only weakly-holomorphic, one still expects them to converge to a holomorphic function. The natural candidate for the limit is not hard to find:

**Conjecture 17.15.** *Let  $q \leq 4$  and  $(\Omega, a, b)$  be a simply connected domain with two points on its boundary. For every  $z \in \Omega$ ,*

$$\frac{1}{(2\delta)^\sigma} F_\delta(z) \rightarrow \phi'(z)^\sigma \quad \text{when } \delta \rightarrow 0 \quad (17.3)$$

where  $\sigma = 1 - \frac{2}{\pi} \arccos(\sqrt{q}/2)$ ,  $F_\delta$  is the observable at  $p_c(q)$  in discrete domains with spin  $\sigma$ , and  $\phi$  is any conformal map from  $\Omega$  to  $\mathbb{R} \times (0, 1)$  sending  $a$  to  $-\infty$  and  $b$  to  $\infty$ .

Being mainly interested in the convergence of interfaces, one could try to follow the same program as in Chapter 11:

- Prove compactness of the interfaces (done in Proposition 11.15).
- Show that sub-sequential limits are Loewner chains (with unknown random driving process  $W_t$ ).
- Prove the convergence of discrete observables (more precisely martingales) of the model.
- Extract from the limit of these observables enough information to evaluate the conditional expectation and quadratic variation of increments of  $W_t$  (in order to harness Lévy theorem). This would imply that  $W_t$  is the Brownian motion with a particular speed  $\kappa$  and so curves converge to SLE( $\kappa$ ).

The third step, corresponding to Conjecture 17.15, should be the most difficult. Note that the second step is also open for  $q \neq 0, 1, 2$  (the first step is Theorem 11.15). Even though the convergence of observables is still unproved, one can perform a computation similar to the proof of Proposition 11.13 in order to identify the possible limiting curves (this is the fourth step). The following conjecture is thus obtained:

**Conjecture 17.16.** *For  $q \leq 4$ , the law of critical random-cluster interfaces converges to the Schramm-Loewner Evolution with parameter  $\kappa = 4\pi / \arccos(-\sqrt{q}/2)$ .*

The conjecture was proved by Lawler, Schramm and Werner [LSW04a] for  $q = 0$ , when they showed that the perimeter curve of the uniform spanning tree converges to SLE(8). Note that the loop representation with Dobrushin boundary conditions still makes sense for  $q = 0$  (more precisely for the model obtained by letting  $q \rightarrow 0$  and  $p/q \rightarrow 0$ ). In fact, configurations have no loops, just a curve running from  $a$  to  $b$  (which then necessarily passes through all the edges), with all configurations being equally probable. The  $q = 2$  case corresponds to Theorem 11.2. All other cases are wide open. The  $q = 1$  case is particularly interesting, since it is actually bond percolation on the square lattice.

**Remark 17.17.** *The observable makes sense in the  $q > 4$  case. Interestingly, the spin  $\sigma$  is not real anymore and does not have any physical interpretation. A natural question would be to relate this change of behavior for  $\sigma$  to the transition between conformally invariant critical behavior and first order critical behavior.*

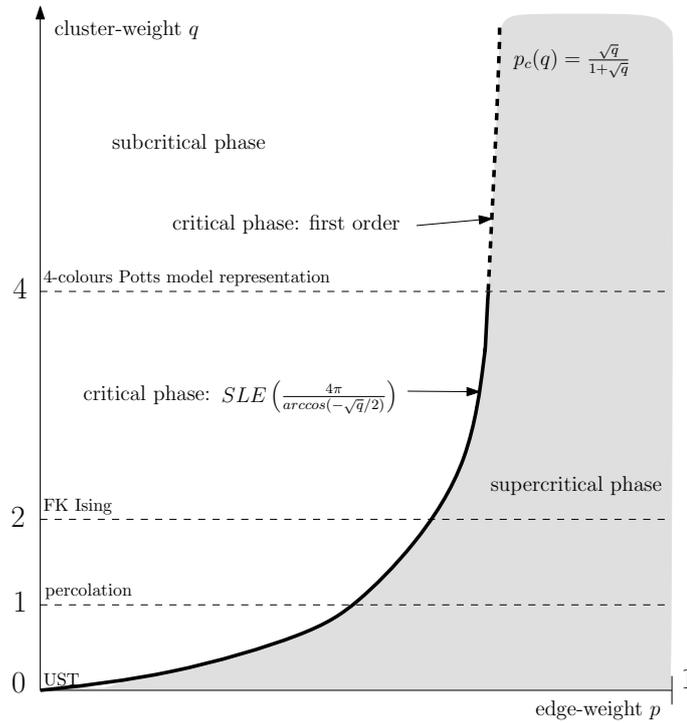


Figure 17.1: The phase diagram of the random-cluster model on the square lattice.

### 3 Self-avoiding walks and $O(n)$ models on the hexagonal lattice

#### 3.1 $O(n)$ models

The Ising fermionic observable was introduced in [Smi06] in the setting of general  $O(n)$  models on the hexagonal lattice. Exactly as in the case of the random-cluster model, one can extend the definition of the spin fermionic observable. For a discrete domain  $\Omega$  with two points on the boundary  $a$  and  $b$ , the *parafermionic observable* is defined on middle of edges by

$$F(z) = \frac{\sum_{\omega \in \mathcal{E}(a,z)} e^{-\sigma i W_\gamma(a,z)} x^{\#\text{ edges in } \omega} n^{\#\text{ loops in } \omega}}{\sum_{\omega \in \mathcal{E}(a,b)} e^{-\sigma i W_\gamma(a,b)} x^{\#\text{ edges in } \omega} n^{\#\text{ loops in } \omega}} \tag{17.4}$$

where  $\mathcal{E}(a, z)$  is the set of configurations of loops with one interface from  $a$  to  $z$ . One can easily prove that the observable satisfies local relations at the (conjectured) critical value if  $\sigma$  is chosen carefully.

**Proposition 17.18.** *If  $x = x_c(n) = 1/\sqrt{2 + \sqrt{2 - n}}$ , let  $F$  be the parafermionic observable with spin  $\sigma = \sigma(n) = 1 - \frac{3}{4\pi} \arccos(-n/2)$ , then*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0 \tag{17.5}$$

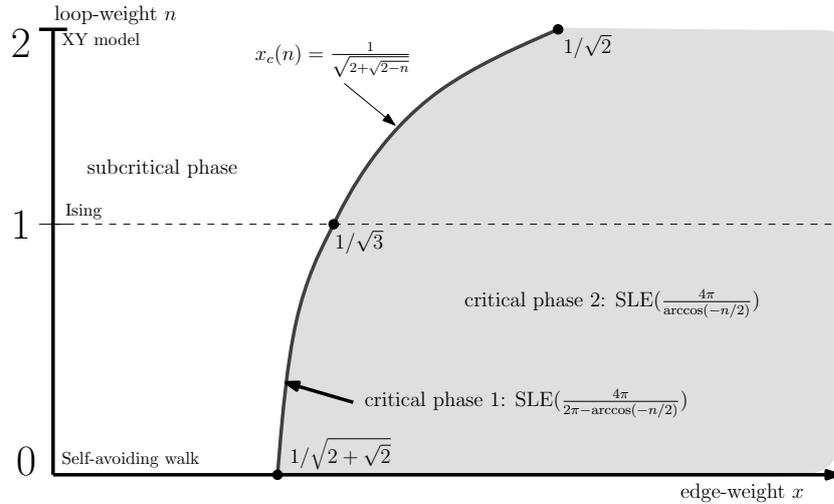


Figure 17.2: The phase diagram of the  $O(n)$  model on the hexagonal lattice.

where  $p, q$  and  $r$  are the three mid-edges adjacent to a vertex  $v$ .

This relation can be seen as a discrete version of the Cauchy-Riemann equation on the triangular lattice and the observable is weak-preholomorphic yet again. Once again, the relations do not determine the observable for a general  $n$ . Nonetheless, if the family of observables is precompact, then the limit should be holomorphic and it is natural to conjecture the following:

**Conjecture 17.19.** Let  $n \in [0, 2]$  and  $(\Omega, a, b)$  be a simply connected domain with two points on the boundary. For  $x = x_c(n)$ ,

$$F_\delta(z) \rightarrow \left( \frac{\psi'(z)}{\psi'(b)} \right)^\sigma \tag{17.6}$$

where  $\sigma = 1 - \frac{3}{4\pi} \arccos(-n/2)$ ,  $F_\delta$  is the observable in the discrete domain with spin  $\sigma$  and  $\psi$  is any conformal map from  $\Omega$  to the upper half plane sending  $a$  to  $\infty$  and  $b$  to  $0$ .

A conjecture on the scaling limit for the interface from  $a$  to  $b$  in the  $O(n)$  model can be also deduced from these considerations:

**Conjecture 17.20.** For  $n \in [0, 2)$  and  $x_c(n) = 1/\sqrt{2 + \sqrt{2 - n}}$ , as the lattice step goes to zero, the law of  $O(n)$  interfaces converges to the chordal Schramm-Loewner Evolution with parameter  $\kappa = 4\pi/(2\pi - \arccos(-n/2))$ .

This conjecture is only proved in the case  $n = 1$  (Theorem 11.3). The other cases are open. The case  $n = 0$  is especially interesting since it corresponds to self-avoiding walks. Proving the conjecture in this case would pave the way to the computation of many quantities, including the mean-square displacement exponent, see [LSW04b] for further details on this problem.

The phase  $x < x_c(n)$  is subcritical and not conformally invariant (the interface converges to the shortest curve between  $a$  and  $b$  for the Euclidean distance). The critical phase  $x \in (x_c(n), \infty)$  should be conformally invariant, and universality is predicted: the interfaces are expected to converge to the same SLE. The edge-weight  $\tilde{x}_c(n) = 1/\sqrt{2 - \sqrt{2 - n}}$ , which appears in Nienhuis's works [Nie82, Nie84], seems to play a specific role in this phase. Interestingly, it is possible to define a parafermionic observable at  $\tilde{x}_c(n)$  with a spin  $\tilde{\sigma}(n)$  other than  $\sigma(n)$ :

**Proposition 17.21.** *If  $x = \tilde{x}_c(n)$ , let  $F$  be the parafermionic observable with spin  $\tilde{\sigma} = \tilde{\sigma}(n) = -\frac{1}{2} - \frac{3}{4\pi} \arccos(-n/2)$ , then*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0 \quad (17.7)$$

where  $p$ ,  $q$  and  $r$  are the three mid-edges adjacent to a vertex  $v$ .

A convergence statement corresponding to Conjecture 17.19 for the observable with spin  $\tilde{\sigma}$  enables us to predict the value of  $\kappa$  for  $\tilde{x}_c(n)$ , and thus for every  $x > x_c(n)$  thanks to universality.

**Conjecture 17.22.** *For  $n \in [0, 2)$  and  $x \in (x_c(n), \infty)$ , as the lattice step goes to zero, the law of  $O(n)$  interfaces converges to the chordal Schramm-Loewner Evolution with parameter  $\kappa = 4\pi / \arccos(-n/2)$ .*

The case  $n = 1$  corresponds to the subcritical high-temperature expansion of the Ising model on the hexagonal lattice, which also corresponds to the supercritical Ising model on the triangular lattice via Kramers-Wannier duality. The interfaces should converge to SLE(6). In the case  $n = 0$ , the scaling limit should be SLE(8) which is space-filling. For both cases, a (slightly different) model is known to converge to the corresponding SLE (site percolation on the triangular lattice for SLE(6), and the perimeter curve of the uniform spanning tree for SLE(8)). Yet, known proofs do not extend to this context. Proving that the whole critical phase  $(x_c(n), \infty)$  has the same scaling limit would be an important example of universality (not on the graph, but on the parameter this time).

The two previous sections presented a program to prove convergence of discrete curves towards the Schramm-Loewner Evolution. It was based on discrete martingales converging to continuous SLE martingales. One can study directly SLE martingales (*i.e.* with respect to  $\sigma(\gamma[0, t])$ ). In particular,  $g'_t(z)^\alpha [g_t(z) - W_t]^\beta$  is a martingale for SLE( $\kappa$ ) where  $\kappa = 4(\alpha - \beta) / [\beta(\beta - 1)]$ . All the limits in these notes are of the previous forms, see *e.g.* Proposition 11.13. Therefore, the parafermionic observables are discretizations of very simple SLE martingales.

**Question 17.23.** *Can new preholomorphic observables be found by looking at discretizations of more complicated SLE martingales?*

Conversely, in [SS05], the harmonic explorer is constructed in such a way that a natural discretization of a SLE(4) martingale is a martingale of the discrete curve. This fact implied the convergence of the harmonic explorer to SLE(4).

**Question 17.24.** *Can this reverse engineering be done for other values of  $\kappa$  in order to find discrete models converging to SLE?*

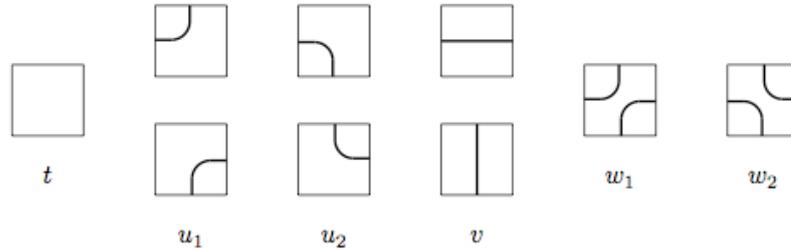


Figure 17.3: Different possible plaquettes with their associated weights.

### 3.2 Discrete observables in other models

The study can be generalized to a variety of lattice models, see the work of Cardy, Ikhlef, Riva, Rajabpour [IC09, RC07, RC06]. Unfortunately, the observable is only partially preholomorphic (satisfying only some of the Cauchy-Riemann equations) except for the Ising case. Interestingly, weights for which there exists a 'half-holomorphic' observable which is not degenerate in the scaling limit always correspond to weights for which the famous Yang-Baxter equality holds.

**Question 17.25.** *The approach to two-dimensional integrable models described here is in several aspects similar to the older approaches based on the Yang-Baxter relations [Bar89]. Can one find a direct link between the two approaches?*

Let us give the example of the  $O(n)$  model on the square lattice. We refer to [IC09] for a complete study of the following.

It is tempting to extend the definition of  $O(n)$  models to the square lattice in order to obtain a family of models containing self-avoiding walks on  $\mathbb{Z}^2$  and the high-temperature expansion of the Ising model. Nevertheless, difficulties arise when dealing with  $O(n)$  models on non-trivalent graphs. Indeed, the indeterminacy when counting intersecting loops prevents us from defining the model as in the previous paragraph.

One can still define a model of loops on  $G \subset \mathbb{L}$  by distinguishing between local configurations: faces of  $G^* \subset \mathbb{L}^*$  are filled with one of the nine plaquettes in Fig. 17.3. A weight  $p_v$  is associated to every face  $v \in G^*$  depending on the type of the face (meaning its plaquette). The probability of a configuration is then proportional to  $n^{\#\text{ loops}} \prod_{v \in \mathbb{L}^*} p_v$ .

**Remark 17.26.** *The case  $u_1 = u_2 = v = x$ ,  $t = 1$  and  $w_1 = w_2 = n = 0$  corresponds to vertex self-avoiding walks on the square lattice. The case  $u_1 = u_2 = v = \sqrt{w_1} = \sqrt{w_2} = x$  and  $n = t = 1$  corresponds to the high-temperature expansion of the Ising model. The case  $t = u_1 = u_2 = v = 0$ ,  $w_1 = w_2 = 1$  and  $n > 0$  corresponds to the random-cluster model at criticality with  $q = n$ .*

A parafermionic observable can also be defined on the medial lattice:

$$F(z) = \frac{\sum_{\omega \in \mathcal{E}(a,z)} e^{-i\sigma W_\gamma(a,z)} n^{\#\text{ loops}} \prod_{v \in \mathbb{L}^*} p_v}{\sum_{\omega \in \mathcal{E}} n^{\#\text{ loops}} \prod_{v \in \mathbb{L}^*} p_v} \tag{17.8}$$

where  $\mathcal{E}$  corresponds to all the configurations of loops on the graph, and  $\mathcal{E}(a, z)$  corresponds to configurations with loops and one interface from  $a$  to  $z$ .

One can then look for a local relation for  $F$  around a vertex  $v$ , which would be a discrete analogue of the Cauchy-Riemann equation:

$$F(N) - F(S) = i[F(E) - F(W)], \quad (17.9)$$

An additional geometric degree of freedom can be added: the lattice can be stretched, meaning that each rhombus is not a square anymore, but a rhombus with inside angle  $\alpha$ .

As in the case of random-cluster models and spin Ising, one can associate configurations by pairs, and try to check (17.9) for each of these pairs, thus leading to a certain number of complex equations. We possess degrees of freedom in the choice of the weights of the model, of the spin  $\sigma$  and of the geometric parameter  $\alpha$ . Very generally, one can thus try to solve the linear system and look for solutions. This leads to the following discussion:

**Case  $v = 0$  and  $n = 1$ :** There exists a non-trivial solution for every spin  $\sigma$ , which is in bijection with a so-called six-vertex model in the disordered phase. The height function associated with this model should converge to the Gaussian free field. This is an example of a model for which interfaces cannot converge to SLE (in [IC09], it is conjectured that the limit is described by  $\text{SLE}(4, \rho)$ ).

**Case  $v = 0$  and  $n \neq 1$ :** There exist unique weights associated to an observable with spin  $-1$ . This solution is in bijection with the random-cluster model at criticality with  $\sqrt{q} = n + 1$ . Nevertheless, physical arguments tend to show that the observable with this spin should have a trivial scaling limit. It would not provide any information on the scaling limit of the model itself, see [IC09] for additional details.

**Case  $v \neq 0$ :** Fix  $n$ , there exists a solution for  $\sigma = \frac{3\eta}{2\pi} - \frac{1}{2}$  where  $\eta \in [-\pi, \pi]$  satisfies  $-\frac{n}{2} = \cos 2\eta$ . Note that there are a priori four possible choices for  $\sigma$ . In general the following weights can be found:

$$\begin{cases} t &= -\sin(2\phi - 3\eta/2) + \sin(5\eta/2) - \sin(3\eta/2) + \sin(\eta/2) \\ u_1 &= -2 \sin(\eta) \cos(3\eta/2 - \phi) \\ u_2 &= -2 \sin(\eta) \sin(\phi) \\ v &= -2 \sin(\phi) \cos(3\eta/2 - \phi) \\ w_1 &= -2 \sin(\phi - \eta) \cos(3\eta/2 - \phi) \\ w_2 &= 2 \cos(\eta/2 - \phi) \sin(\phi) \end{cases}$$

where  $\phi = (1 + \sigma)\alpha$ . We now interpret these results:

When  $\eta \in [0, \pi]$ , the scaling limit has been argued to be described by a Coulomb gas with a coupling constant  $2\eta/\pi$ . In other words, the scaling limit should be the same as the corresponding  $O(n)$  model on the hexagonal lattice. In particular, interfaces should converge to the corresponding Schramm-Loewner Evolution.

When  $\eta \in [-\pi, 0]$ , the scaling limit curve cannot be described by SLE, and it provides yet another example of a two-dimensional model for which the scaling limit is not described via SLE.

### 3.3 Self-avoiding walks

We finish by mentioning open questions on self-avoiding walks which do not require scaling limits. As was mentioned before, Nienhuis predicted in [Nie82, Nie84] that there exists  $A > 0$  such that

$$c_n \sim A n^{\gamma-1} \sqrt{2+\sqrt{2}}^n \quad (17.10)$$

where  $\gamma = 43/32$ . He also conjectured that the so-called mean-square displacement  $\langle |\gamma(n)|^2 \rangle$  should satisfy

$$\langle |\gamma(n)|^2 \rangle = \frac{1}{c_n} \sum_{\gamma \text{ } n\text{-step SAW}} |\gamma(n)|^2 \sim B n^{2\nu} \quad (17.11)$$

where  $\nu = 2/3$  and  $B$  is a constant. It was shown in [LSW04b] that  $\gamma$  and  $\nu$  could be computed if the scaling limit of self-avoiding walks was conformally invariant, which brings us back to Conjecture 17.20. Without going that far, an interesting question is to obtain a polynomial bound on the correction term to  $c_n$ . Indeed, the best known result, due to Hammersley and Welsh [HW62] (see Chapter 13), is the following: there exists  $c > 0$  such that

$$\sqrt{2+\sqrt{2}}^n \leq c_n \leq e^{c\sqrt{n}} \sqrt{2+\sqrt{2}}^n.$$

**Question 17.27.** *Prove that there exists  $\Gamma < \infty$  such that*

$$c_n \leq n^\Gamma \sqrt{2+\sqrt{2}}^n.$$

The geometry of the critical self-avoiding walk seems to be completely out of reach as for today. Any non-trivial information on it would be of great value, in particular any information distinguishing between the subcritical, the supercritical and the critical phases. For instance, the following question corresponds to the fact that the self-avoiding walk is not space-filling nor ballistic:

**Question 17.28.** *There exists  $\varepsilon > 0$  such that*

$$n^{1+\varepsilon} \leq \langle |\gamma(n)|^2 \rangle \leq n^{2-\varepsilon}.$$

We conclude this manuscript by an aesthetic and diverting question.

**Question 17.29.** *What is the smallest connective constant for Cayley graphs not equal to  $\mathbb{Z}$ ?*

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