

## OVERLAPPING SCHWARZ WAVEFORM RELAXATION FOR CONVECTION-DOMINATED NONLINEAR CONSERVATION LAWS\*

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**Abstract.** We analyze the convergence of the overlapping Schwarz waveform relaxation algorithm applied to convection-dominated *nonlinear* conservation laws in one spatial dimension. For two subdomains and bounded time intervals we prove superlinear asymptotic convergence of the algorithm in the parabolic case and convergence in a finite number of steps in the hyperbolic limit. The convergence results depend on the overlap, the viscosity, and the length of the time interval under consideration, but they are independent of the number of subdomains, as a generalization of the results to many subdomains shows. To investigate the behavior of the algorithm for a long time, we apply it to the Burgers equation and use a steady state argument to prove that the algorithm converges linearly over long time intervals. This result reveals an interesting paradox: while for the superlinear convergence rate on bounded time intervals decreasing the viscosity improves the performance, in the linear convergence regime decreasing the viscosity slows down the convergence rate and the algorithm can converge arbitrarily slowly, if there is a standing shock wave in the overlap. We illustrate our theoretical results with numerical experiments.

**Key words.** domain decomposition, waveform relaxation, Schwarz methods, nonlinear conservation laws

**AMS subject classifications.** 65M55, 35K55, 35L65

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**1. Introduction.** Overlapping Schwarz waveform relaxation is a class of domain decomposition algorithms to solve evolution problems in parallel. The classical way of applying domain decomposition methods to evolution problems is to discretize the time dimension first uniformly over the whole domain by an implicit scheme, and then to apply domain decomposition at each time step separately to solve the sequence of steady state problems obtained from the implicit time discretization. Numerical experiments for this approach with an overlapping Schwarz method for the two-dimensional heat equation can be found in [Meu91] and the application of the additive and multiplicative Schwarz preconditioners to the convection diffusion equation have been analyzed in [Cai91] and [Cai94], respectively. A nonoverlapping domain decomposition in this context using a Neumann–Dirichlet preconditioner at each time step was proposed in [Dry91], and an interesting variant, which uses an explicit method to advance the interface values in time and then an implicit method on each subdomain and thus avoids a subdomain iteration, is proposed in [DD91]; see also [RZ94] and [CL96]. For hyperbolic problems, the case of the wave equation with discontinuous bulk modulus and density fields per subdomain was analyzed in [BGT97] and the advantage of different grids in space due to the domain decomposition was emphasized (“the possibility of assigning to each subdomain its own space step makes numerical simulations much less expensive”), but due to the uniform time discretization, one cannot have an optimal space-time discretization close to the

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CFL condition in each subdomain in the classical approach. For a linear first-order hyperbolic equation in two dimensions, numerical results of the additive Schwarz preconditioner together with GMRES acceleration were reported in [WCK98] and a first analysis for the case of the Euler equations can be found in [DLN00]. As mentioned above, the main disadvantage of this classical approach is that one needs to use a uniform time discretization over the entire domain and thus loses one of the main features of domain decomposition methods, namely, to treat the problem on each subdomain numerically differently, with an appropriate discretization both in time and space adapted to the subdomain problems.

Overlapping Schwarz waveform relaxation does not have this disadvantage; the algorithm also uses an overlapping domain decomposition in space, like the classical Schwarz algorithm for steady state problems (see [Sch70]), but then the algorithm solves evolution problems on the subdomains and uses an iteration to converge to the solution of the original problem posed on the entire domain, like in waveform relaxation methods (see [LRSV82]), which explains the name of the algorithm. Since subdomain problems are solved both in space and time on subdomains, appropriate discretizations in space and time can be applied per subdomain. In addition, communication is not required at each time step; the computation can be performed over several time steps in a time window before information is exchanged with neighboring subdomains. This can be beneficial in a parallel environment where the startup cost of a connection with another processor is significant.

A very early use of this type of algorithm can be found in the research report [MS87], where it was applied to the one-dimensional wave equation, and it was shown that the algorithm converges in a finite number of steps in this case; see also [Gan97]. The algorithm has been analyzed in [BjØ95] for more general hyperbolic problems. The algorithm applied to parabolic problems was first analyzed for the heat equation in [GZ97, GZ02], for a semilinear model problem in [Gan98], and independently for convection diffusion problems in [GK02]. Applied to parabolic problems, the algorithm has the interesting property of having two different convergence regimes, depending on the length of the time evolution on the subdomains before information is exchanged with neighboring subdomains. The algorithm's performance can be further improved by using better transmission conditions; see, for example, [GHN99].

We analyze in this paper the convergence properties of overlapping Schwarz waveform relaxation applied to convection-dominated viscous conservation laws with nonlinear fluxes. Let  $T > 0$  be the end of the time interval of interest,  $t \in [0, T)$ , and  $\Omega \subseteq \mathbb{R}$  be the bounded or unbounded spatial domain. We analyze the overlapping Schwarz waveform relaxation method to compute solutions  $u = u(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}$  of the corresponding initial and initial boundary value problems

$$(1.1) \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon \frac{\partial^2 u}{\partial x^2} \quad \text{in } \Omega \times (0, T),$$

where  $\varepsilon \geq 0$  and  $f \in C^2(\mathbb{R})$  denotes a function which in general depends in a nonlinear way on  $u$ .

The initial boundary value problem for (1.1) is a scalar model problem for the physically important equations of viscous compressible flow: the compressible Navier–Stokes equations. Flow problems impose considerable difficulties if the Reynolds number becomes big. In our case we are therefore interested especially in the case  $0 < \varepsilon \ll 1$ . To understand this case it is essential to consider also the singular hyperbolic limit problem with  $\varepsilon = 0$ . We start by collecting in section 2 all the analytical background results we need in our analysis. In section 3 we analyze the behavior

of the overlapping Schwarz waveform relaxation algorithm for two subdomains over bounded time intervals,  $T < \infty$ . For  $\varepsilon > 0$  we prove convergence of the algorithm at a superlinear rate. The analysis allows us to track exactly how the rate depends on  $\varepsilon$  when  $\varepsilon$  tends to zero. In particular, we show the connection to the results of the Schwarz waveform relaxation algorithm in the case  $\varepsilon = 0$  which is analyzed separately. The convergence rate of the Schwarz waveform relaxation algorithm depends on the size of the overlap, the length of the time interval, and the viscosity term, and the algorithm becomes faster as the viscosity term becomes smaller, which leads in the limit to convergence in a finite number of steps. In section 4 we then generalize the superlinear convergence result to  $I > 2$  subdomains and show that the convergence rate is independent of the number of subdomains. To understand the long-time behavior of the algorithm ( $T \rightarrow \infty$ ), we study in section 5 the overlapping Schwarz algorithm for the steady Burgers equation, which can be considered as the long-time behavior of the evolution case. We show that the algorithm converges linearly for the two subdomain cases and that the convergence rate depends again on the size of the overlap and the viscosity term. In this case, however, when there is a standing shock in the overlap, we find that the smaller the viscosity is, the slower the convergence becomes, and the dependence is exponential. Our analytical results agree with the observations for long time computations for the Burgers equation that have been reported in [GK97, GK00]. There, it is shown that the solution depends in a *super-sensitive* way on the boundary conditions, if an internal layer is present. Moreover, an asymptotic analysis is presented and a two-dimensional version of the Burgers problem is considered. General analytical results using the Cole–Hopf transformation can be found in [LO95]. In section 6 we illustrate our results with numerical experiments.

**2. Initial and initial boundary value problem.** We consider for a given function  $f \in C^2(\mathbb{R})$  the scalar conservation law

$$(2.1) \quad \frac{\partial u^\varepsilon}{\partial t}(x, t) + \frac{\partial}{\partial x} f(u^\varepsilon(x, t)) = \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2}(x, t), \quad (x, t) \in \Omega \times (0, T),$$

where  $\Omega \subseteq \mathbb{R}$  is an open set,  $T > 0$ , and  $\varepsilon \geq 0$  is the viscosity parameter. We start with a review of basic analytical results for the parabolic case,  $\varepsilon > 0$ , and the (singular) hyperbolic case,  $\varepsilon = 0$ . In the first case we focus on the behavior of solutions for  $0 < \varepsilon \ll 1$ , which is important for the analysis of the Schwarz waveform relaxation algorithm later in section 3.

**2.1. The parabolic case,  $\varepsilon > 0$ .** In our analysis we need results both for the initial value problem (2.1) on the infinite domain  $\Omega = \mathbb{R}$  with the initial condition

$$(2.2) \quad u^\varepsilon(x, 0) = u_0(x), \quad x \in \Omega,$$

where we assume that  $u_0 \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , and for the initial boundary value problem (2.1) on the half line  $\Omega = (-\infty, 0)$  with the initial and boundary conditions

$$(2.3) \quad u^0(x, 0) = u_0(x), \quad x \in \Omega, \quad u^0(0, t) = g(t), \quad t \in [0, T],$$

where we suppose that  $u_0 \in C^2([-\infty, 0]) \cap L^\infty((-\infty, 0])$ ,  $g \in C^2([0, T])$ , and that the appropriate compatibility condition holds for  $u_0$  and  $g$  at  $x = t = 0$ . For both problems we consider classical solutions, i.e., functions that are smooth enough to satisfy the conservation law and the constraints pointwise.

THEOREM 2.1. For  $\varepsilon > 0$  we have the following two results.

- (i) For the initial value problem (2.1), (2.2), let the numbers  $\underline{u} = \inf_{x \in \mathbb{R}} \{u_0(x)\}$  and  $\bar{u} = \sup_{x \in \mathbb{R}} \{u_0(x)\}$  be given. Then there exists a unique classical solution  $u^\varepsilon \in C^1(0, T; C^2(\mathbb{R}))$  of (2.1), (2.2) that satisfies

$$\underline{u} \leq u^\varepsilon(x, t) \leq \bar{u}, \quad (x, t) \in \mathbb{R} \times [0, T], \quad \text{and} \quad \varepsilon \|u_x^\varepsilon\|_{L^\infty(\mathbb{R} \times [0, T])} \leq C_{IVP}.$$

Furthermore, there exists a function  $u^0 \in L^\infty(\mathbb{R} \times [0, T])$  such that for each compact set  $Q \subset \mathbb{R}$  we have

$$\lim_{\varepsilon \rightarrow 0} \|u^0 - u^\varepsilon\|_{L^1(Q \times [0, T])} = 0.$$

- (ii) For the initial boundary value problem (2.1), (2.3), let

$$\underline{u} = \min\{\inf_{x \leq 0} \{u_0(x)\}, \inf_{t \in [0, T]} \{g(t)\}\},$$

$$\bar{u} = \max\{\sup_{x \leq 0} \{u_0(x)\}, \sup_{t \in [0, T]} \{g(t)\}\}.$$

Then there exists a unique classical solution  $u^\varepsilon \in C^1(0, T; C^2((-\infty, 0]))$  of (2.1), (2.3) that satisfies

$$\underline{u} \leq u^\varepsilon(x, t) \leq \bar{u}, \quad (x, t) \in (-\infty, 0] \times [0, T]$$

and

$$\varepsilon \|u_x^\varepsilon\|_{L^\infty((-\infty, 0] \times [0, T])} \leq C_{IBVP}.$$

Furthermore, there exists a function  $u^0 \in L^\infty((-\infty, 0] \times [0, T])$  such that for each compact set  $Q \subset (-\infty, 0]$  we have

$$\lim_{\varepsilon \rightarrow 0} \|u^0 - u^\varepsilon\|_{L^1(Q \times [0, T])} = 0.$$

The constants  $C_{IVP}, C_{IBVP} > 0$  depend on  $f, u_0, g$ , and  $T$ , but not on  $\varepsilon$ .

*Proof.* For the existence part in (i) and (ii) we refer the reader to, e.g., [LSU68]. The  $L^\infty$ -estimates follow from an application of the maximum principle.

For the proof of the vanishing viscosity limits we refer the reader to [MNRR96, Chapter 2].  $\square$

**2.2. The hyperbolic limit  $\varepsilon = 0$ .** We now consider for the limiting case  $\varepsilon = 0$  the initial value problem (2.1), (2.2) on  $\Omega = \mathbb{R}$  with  $u_0 \in L^\infty(\mathbb{R})$  and the initial boundary value problem (2.1), (2.3) on  $\Omega = (-\infty, 0)$  with  $u_0 \in L^\infty((-\infty, 0])$  and  $g \in L^\infty([0, T])$ . It is well known that initial (boundary) value problems for nonlinear functions  $f$  do not have time-global classical solutions for arbitrary, even smooth functions  $u_0, g$ : within finite time singularities, i.e., shock waves, develop. The well-posedness theory for the singular problems (2.1), (2.2) and (2.1), (2.3) relies therefore on a weaker notion of a solution: *the entropy solution*. It can be proved that in both cases there exists a unique entropy solution which satisfies the boundary data in an appropriate sense. These are the solutions we consider throughout the paper, although we do not introduce the precise definitions here, since they are not needed in an essential way later. It is important that the vanishing viscosity limits  $u^0$  from Theorem 2.1 coincide with the entropy solutions for (2.1), (2.2) and (2.1), (2.3) in the case  $\varepsilon = 0$ . We refer the reader interested in the background on nonlinear

conservation laws and the precise definition of entropy solutions to [MNRR96]. The properties of the entropy solutions we need in our analysis are summarized in the following theorem.

**THEOREM 2.2.** *For  $\varepsilon = 0$  we have the following three results.*

- (i) *For the initial value problem (2.1), (2.2), denote  $\underline{u} = \text{essinf}_{x \in \mathbb{R}}\{u_0(x)\}$  and  $\bar{u} = \text{esssup}_{x \in \mathbb{R}}\{u_0(x)\}$ . Then the function  $u^0 \in L^\infty(\mathbb{R} \times [0, T])$  from (i) in Theorem 2.1 is the unique entropy solution of (2.1), (2.2). It satisfies for almost all  $(x, t) \in \mathbb{R} \times [0, T]$  the estimates*

$$(2.4) \quad \underline{u} \leq u^0(x, t) \leq \bar{u}.$$

- (ii) *For the initial boundary value problem (2.1), (2.3), let the numbers*

$$\begin{aligned} \underline{u} &= \min \left\{ \text{essinf}_{x \leq 0} \{u_0(x)\}, \text{essinf}_{t \in [0, T]} \{g(t)\} \right\}, \\ \bar{u} &= \max \left\{ \text{esssup}_{x \leq 0} \{u_0(x)\}, \text{esssup}_{t \in [0, T]} \{g(t)\} \right\} \end{aligned}$$

*be given. Then the function  $u^0 \in L^\infty(\mathbb{R} \times [0, T])$  from (ii) in Theorem 2.1 is the unique entropy solution of (2.1), (2.3). It satisfies for almost all  $(x, t) \in (-\infty, 0] \times [0, T]$  the estimates*

$$(2.5) \quad \underline{u} \leq u^0(x, t) \leq \bar{u}.$$

- (iii) *Let  $u_{01}, u_{02} \in L^\infty(-\infty, 0]$ ,  $g_1, g_2 \in L^\infty([0, T])$ , and let  $I = [a, b] \subset (-\infty, 0]$  be a bounded interval. Suppose that  $u_1^0, u_2^0 \in L^\infty((-\infty, 0] \times [0, T])$  are the entropy solutions of the initial boundary value problem (2.1), (2.3) with  $u_0 = u_{01}$ ,  $u_0 = u_{02}$ ,  $g = g_1$ , and  $g = g_2$ , respectively. We then have for almost all  $t \in [0, T]$  the inequality*

$$\begin{aligned} &\int_a^b |u_1(x, t) - u_2(x, t)| dt \\ &\leq \int_{a-\lambda t}^{\min\{0, b+\lambda t\}} |u_{01}(x) - u_{02}(x)| dx + \lambda \int_0^{b/\lambda+t} |g_1(s) - g_2(s)| ds. \end{aligned}$$

*The number  $\lambda > 0$  is given by*

$$\lambda = \max_{\underline{u} \leq v \leq \bar{u}} \{|f'(v)|\},$$

*where*

$$\underline{u} = \min \left\{ \text{essinf}_{x \leq 0} \{u_{01}(x)\}, \text{essinf}_{t \in [0, T]} \{g_1(t)\}, \text{essinf}_{x \leq 0} \{u_{02}(x)\}, \text{essinf}_{t \in [0, T]} \{g_2(t)\} \right\}$$

*and  $\bar{u}$  is defined analogously.*

*Proof.* All statements can be found in Chapter 2 of [MNRR96]. □

Theorem 2.1 gives the basic existence results for classical solutions and states that  $u^\varepsilon$  converges for  $\varepsilon \rightarrow 0$  pointwise almost everywhere to the entropy solutions  $u^0$  of the corresponding hyperbolic problems given in Theorem 2.2. Thus, in the particularly interesting singularly perturbed cases  $0 < \varepsilon \ll 1$  we have to take into account the viscous counterparts of shock waves, i.e., internal layers of width  $\mathcal{O}(\varepsilon)$  and amplitude  $\mathcal{O}(1)$ . Precisely the  $L^\infty$ -estimates on the derivative  $u_x^\varepsilon$  in Theorem 2.1 allow us to compute the dependency on  $\varepsilon$  explicitly in all our estimates.

We note that statement (ii) in Theorem 2.1 and statements (ii), (iii) in Theorem 2.2—appropriately adapted—hold true for more general domains, particularly for all kinds of quarter spaces.

**3. Overlapping Schwarz waveform relaxation for two subdomains.** We analyze now the overlapping Schwarz waveform relaxation algorithm for the initial value problem (2.1), (2.2) and two subdomains. We start with the parabolic case,  $\varepsilon > 0$ , in subsection 3.1. We pay close attention in this analysis to the dependence of the convergence results on  $\varepsilon$ , since we are also interested in the convergence behavior of the algorithm in the limit when  $\varepsilon \rightarrow 0$ . In section 3.2 we then analyze directly the behavior of the algorithm in the hyperbolic limit when  $\varepsilon = 0$ , and we show how *nonlinear* flux functions lead to convergence properties of the algorithm which are not present when it is applied to linear problems. To simplify the notation we will skip the index  $\varepsilon$  in  $u^\varepsilon$  in what follows.

**3.1. The parabolic case,  $\varepsilon > 0$ .** The overlapping Schwarz waveform relaxation algorithm applied to the initial value problem (2.1), (2.2) with the two subdomains  $\Omega_1 = (-\infty, L)$  and  $\Omega_2 = (0, \infty)$ ,  $L > 0$ , is given for iteration index  $n \in \mathbb{N}$  by

$$(3.1) \quad \begin{aligned} \frac{\partial u_1^n}{\partial t} + f'(u_1^n) \frac{\partial u_1^n}{\partial x} &= \varepsilon \frac{\partial^2 u_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0, T), \\ u_1^n(\cdot, 0) &= u_0 && \text{in } \Omega_1, \\ u_1^n(L, \cdot) &= u_2^{n-1}(L, \cdot) && \text{on } [0, T], \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} \frac{\partial u_2^n}{\partial t} + f'(u_2^n) \frac{\partial u_2^n}{\partial x} &= \varepsilon \frac{\partial^2 u_2^n}{\partial x^2} && \text{in } \Omega_2 \times (0, T), \\ u_2^n(\cdot, 0) &= u_0 && \text{in } \Omega_2, \\ u_2^n(0, \cdot) &= u_1^{n-1}(0, \cdot) && \text{on } [0, T]. \end{aligned}$$

There is also a more sequential variant of this algorithm, where the interface values on the second domain are taken from the newer iterate  $u_1^n$  on the first subdomain, like in a Gauss–Seidel iteration, but we analyze only the Jacobi version given in (3.1) and (3.2); the Gauss–Seidel case can be analyzed similarly. For the analysis we also require that the iteration starts with the initial guess

$$(3.3) \quad \begin{aligned} u_1^0(x, t) &= \inf_{x' \in (-\infty, L]} \{u_0(x')\}, && (x, t) \in \Omega_1 \times (0, T), \\ u_2^0(x, t) &= \inf_{x' \in [0, \infty)} \{u_0(x')\}, && (x, t) \in \Omega_2 \times (0, T). \end{aligned}$$

We define the errors in the Schwarz waveform relaxation iteration by  $e_1^n := u - u_1^n$  on the left subdomain and  $e_2^n := u - u_2^n$  on the right subdomain for  $n \in \mathbb{N}_0$ . For  $n \in \mathbb{N}$ , we find with (3.1), (3.2) that the errors satisfy the equations

$$(3.4) \quad \begin{aligned} \frac{\partial e_1^n}{\partial t} + f'(u) \frac{\partial e_1^n}{\partial x} + \theta_1^n e_1^n &= \varepsilon \frac{\partial^2 e_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0, T), \\ e_1^n(\cdot, 0) &= 0 && \text{in } \Omega_1, \\ e_1^n(L, \cdot) &= e_2^{n-1}(L, \cdot) && \text{on } [0, T], \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} \frac{\partial e_2^n}{\partial t} + f'(u) \frac{\partial e_2^n}{\partial x} + \theta_2^n e_2^n &= \varepsilon \frac{\partial^2 e_2^n}{\partial x^2} && \text{in } \Omega_2 \times (0, T), \\ e_2^n(\cdot, 0) &= 0 && \text{in } \Omega_2, \\ e_2^n(0, \cdot) &= e_1^{n-1}(0, \cdot) && \text{on } [0, T], \end{aligned}$$

where the functions  $\theta_i^n : \Omega_i \rightarrow \mathbb{R}$  are given by

$$(3.6) \quad \theta_i^n(x, t) = \frac{\partial}{\partial x} u_i^n(x, t) \int_0^1 f''\left(u_i^n(x, t) - s(u(x, t) - u_i^n(x, t))\right) ds, \quad i = 1, 2.$$

For later use we also define here the functions  $K_{1,x}, K_{2,x}$  by

$$(3.7) \quad K_{1,x}(x, t) = -\frac{1}{2\sqrt{\pi}} \frac{x-L}{\varepsilon^{1/2} t^{3/2}} \exp\left(-\frac{(x-L)^2}{4\varepsilon t}\right),$$

$$(3.8) \quad K_{2,x}(x, t) = \frac{1}{2\sqrt{\pi}} \frac{x}{\varepsilon^{1/2} t^{3/2}} \exp\left(-\frac{x^2}{4\varepsilon t}\right).$$

Because of our particular choice of the starting values (3.3) and the comparison principle for parabolic differential equations (see [Fri64]), the errors on both subdomains stay nonnegative for all iterations  $n \in \mathbb{N}_0$ ,

$$(3.9) \quad e_1^n(x, t) \geq 0, \quad (x, t) \in \Omega_1 \times (0, T), \quad e_2^n(x, t) \geq 0, \quad (x, t) \in \Omega_2 \times (0, T).$$

It suffices therefore to derive upper bounds for the errors to obtain a bound on the convergence rate of the overlapping Schwarz waveform relaxation algorithm.

LEMMA 3.1 (supersolutions). *Let the families  $\{e_1^n\}_{n \in \mathbb{N}}$ ,  $\{e_2^n\}_{n \in \mathbb{N}}$  be given by (3.4), (3.5). Then for all  $n \in \mathbb{N}$  we have*

$$\begin{aligned} 0 \leq e_1^n(x, t) \leq \bar{e}_1^n(x, t) & \quad \forall (x, t) \in \Omega_1 \times (0, T), \\ 0 \leq e_2^n(x, t) \leq \bar{e}_2^n(x, t) & \quad \forall (x, t) \in \Omega_2 \times (0, T), \end{aligned}$$

where the supersolution  $\bar{e}_1^n$  is the solution of the linear, constant coefficient problem

$$(3.10) \quad \begin{aligned} \frac{\partial \bar{e}_1^n}{\partial t} + a_1 \frac{\partial \bar{e}_1^n}{\partial x} + b_1 \bar{e}_1^n &= \varepsilon \frac{\partial^2 \bar{e}_1^n}{\partial x^2} && \text{in } \Omega_1 \times (0, T), \\ \bar{e}_1^n(\cdot, 0) &= 0 && \text{in } \Omega_1, \\ \bar{e}_1^n(L, t) &= \exp(\sigma_1 t) \sup_{0 \leq \tau \leq t} e_2^{n-1}(L, \tau), && t \in [0, T], \end{aligned}$$

with the constants  $a_1, b_1, \sigma_1 \in \mathbb{R}$  given by

$$\begin{aligned} a_1 &:= \inf_{(x,t) \in \Omega_1} f'(u(x, t)), \\ b_1 &:= \inf_{(x,t) \in \Omega_2} \left\{ \theta_1^n(x, t) + (f'(u(x, t)) - a_1) \frac{a_1}{2\varepsilon} \right\}, \\ \sigma_1 &:= \begin{cases} -\frac{a_1^2}{4\varepsilon} - b_1 & \text{if } -\frac{a_1^2}{4\varepsilon} - b_1 \geq 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and the supersolution  $\bar{e}_2^n$  is the solution of the linear, constant coefficient problem

$$(3.11) \quad \begin{aligned} \frac{\partial \bar{e}_2^n}{\partial t} + a_2 \frac{\partial \bar{e}_2^n}{\partial x} + b_2 \bar{e}_2^n &= \varepsilon \frac{\partial^2 \bar{e}_2^n}{\partial x^2} && \text{in } \Omega_2 \times (0, T), \\ \bar{e}_2^n(x, 0) &= 0 && \text{in } \Omega_2, \\ \bar{e}_2^n(0, t) &= \exp(\sigma_2 t) \sup_{0 \leq \tau \leq t} e_1^n(0, \tau), && t \in [0, T], \end{aligned}$$

with the constants  $a_2, b_2, \sigma_2 \in \mathbb{R}$  given by

$$\begin{aligned} a_2 &:= \sup_{(x,t) \in \Omega_1} f'(u(x,t)), \\ b_2 &:= \inf_{(x,t) \in \Omega_2} \left\{ \theta_2^n(x,t) + (f'(u(x,t)) - a_2) \frac{a_2}{2\varepsilon} \right\}, \\ \sigma_2 &:= \begin{cases} -\frac{a_2^2}{4\varepsilon} - b_2 & \text{if } f' - \frac{a_2^2}{4\varepsilon} - b_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the numbers  $\sigma_1$  and  $\sigma_2$  are finite but they can be of order  $\mathcal{O}(\varepsilon^{-1})$  due to result (ii) in Theorem 2.1.

*Proof.* The proof is constructive. We first define for  $i = 1, 2$  the constants

$$(3.12) \quad p_i = \frac{a_i}{2\varepsilon}, \quad q_i = -\frac{a_i^2}{4\varepsilon} - b_i,$$

and the function

$$g_1 = g_1(t) = \exp(-p_1 L + (\sigma_1 - q_1)t) \sup_{0 \leq \tau \leq t} e_2^{n-1}(L, \tau),$$

which is nonnegative due to (3.9), and monotonically increasing because of our choice of  $\sigma_1$ . For the linear, constant coefficient problem (3.10) satisfied by the supersolution we have the closed form solution formula

$$(3.13) \quad \bar{e}_1^n(x, t) = \exp(p_1 x + q_1 t) \int_0^t K_{1,x}(x, t - \tau) g_1(\tau) d\tau,$$

where the kernel  $K_{1,x}(x, t)$  is given in (3.7). To show that  $\bar{e}_1^n$  is indeed a supersolution, we have to show that

$$(3.14) \quad d_1^n := \bar{e}_1^n - e_1^n \geq 0.$$

Now the difference function  $d_1^n$  satisfies the linear convection diffusion equation

$$(3.15) \quad \frac{\partial d_1^n}{\partial t} + f'(u) \frac{\partial d_1^n}{\partial x} + \theta_1^n d_1^n - \varepsilon \frac{\partial^2 d_1^n}{\partial x^2} = Q_1(x, t),$$

where the source term  $Q_1(x, t)$  is given by

$$\begin{aligned} Q_1(x, t) &= (f'(u(x, t)) - a_1) \frac{\partial \bar{e}_1^n}{\partial x} + (\theta_1^n(x, t) - b_1) \bar{e}_1^n(x, t) \\ &= (f'(u(x, t)) - a_1) \frac{e^{(p_1 x + q_1 t)}}{2\sqrt{\pi}} \int_0^t \frac{e^{(-\frac{(x-L)^2}{4\varepsilon(t-\tau)})}}{\varepsilon^{1/2}(t-\tau)^{3/2}} \left[ \frac{(x-L)^2}{2\varepsilon(t-\tau)} - 1 \right] g_1(\tau) d\tau \\ &\quad - \left( (f'(u(x, t)) - a_1) p_1 + \theta_1^n(x, t) - b_1 \right) (x-L) \\ &\quad \times \frac{e^{(p_1 x + q_1 t)}}{2\sqrt{\pi}} \int_0^t \frac{e^{(-\frac{(x-L)^2}{4\varepsilon(t-\tau)})}}{\varepsilon^{1/2}(t-\tau)^{3/2}} g_1(\tau) d\tau \\ &=: (f'(u(x, t)) - a_1) e^{(p_1 x + q_1 t)} Q_{11}(x, t) \\ &\quad + \left( (f'(u(x, t)) - a_1) p_1 + \theta_1^n(x, t) - b_1 \right) e^{(p_1 x + q_1 t)} (L-x) Q_{12}(x, t). \end{aligned}$$



If we can show that  $Q_{11}(x, t)$  and  $Q_{12}(x, t)$  are nonnegative for all  $(x, t) \in \Omega_1 \times (0, T)$ , we obtain  $Q_1(x, t) \geq 0$  for all  $(x, t) \in \Omega_1 \times (0, T)$  by the definition of  $a_1, b_1$  which implies (3.14) by the maximum principle for (3.15) with zero initial and boundary data. But  $Q_{12}$  is nonnegative since  $g_1$  from (3.13) is nonnegative by (3.9), and for  $Q_{11}$  we observe that it is the  $x$ -derivative of the solution  $w$  of the heat equation  $w_t = \varepsilon w_{xx}$  in  $\Omega_1 \times (0, T)$  which satisfies  $w(L, \cdot) = g_1$  and  $w(\cdot, 0) \equiv 0$ . Since  $g_1$  is nonnegative and monotonically increasing,  $Q_{11}$  must also be nonnegative, which concludes the proof that  $\bar{e}_1^n$  is a supersolution of  $e_1^n$ . Similarly one can also show that  $\bar{e}_2^n$  is a supersolution of  $e_2^n$ .  $\square$

**THEOREM 3.2** (superlinear convergence). *The overlapping Schwarz waveform relaxation algorithm (3.1), (3.2) with two subdomains for the convection-dominated nonlinear conservation law (2.1), (2.2) converges superlinearly. For each  $t > 0$  we have*

$$(3.16) \quad \sup_{x \in \Omega_1, 0 \leq \tau \leq t} \{e_1^{2n}(x, \tau)\} \leq e^{\frac{C(t+L)}{\varepsilon}n} \operatorname{erfc}\left(\frac{nL}{\sqrt{\varepsilon t}}\right) \sup_{0 \leq \tau \leq t} \{e_1^0(0, \tau)\},$$

$$(3.17) \quad \sup_{x \in \Omega_2, 0 \leq \tau \leq t} \{e_2^{2n}(x, \tau)\} \leq e^{\frac{C(t+L)}{\varepsilon}n} \operatorname{erfc}\left(\frac{nL}{\sqrt{\varepsilon t}}\right) \sup_{0 \leq \tau \leq t} \{e_2^0(0, \tau)\},$$

where the constant  $C$  is independent of  $\varepsilon, L, t$ , and  $n$ .

*Proof.* Using Lemma 3.1 and the explicit formula for the supersolutions, we obtain the estimates

$$(3.18) \quad \begin{aligned} e_1^n(x, t) &\leq e^{p_1(x-L)+q_1t} \int_0^t K_{1,x}(x, t - \tau) \sup_{0 \leq s \leq \tau} \{e_2^{n-1}(L, s)\} e^{(\sigma_1 - q_1)\tau} d\tau, \\ e_2^n(x, t) &\leq e^{p_2x+q_2t} \int_0^t K_{2,x}(x, t - \tau) \sup_{0 \leq s \leq \tau} \{e_1^{n-1}(0, s)\} e^{(\sigma_2 - q_2)\tau} d\tau, \end{aligned}$$

where  $K_{1,x}, K_{2,x}$  are defined in (3.7), (3.8), respectively, and the constants  $p_1, q_1, p_2$ , and  $q_2$  are defined in (3.12). We evaluate the second inequality in (3.18) at  $x = L$  and insert it into the first one to obtain

$$\begin{aligned} e_1^n(x, t) &\leq e^{p_1(x-L)+q_1t} \int_0^t K_{1,x}(x, t - \tau) e^{(\sigma_1 - q_1)\tau} \\ &\quad \times \sup_{0 \leq s \leq \tau} \left\{ e^{p_2L+q_2s} \int_0^s K_{2,x}(L, s - \tilde{s}) \sup_{0 \leq \tilde{s} \leq \tilde{s}} \{e_1^{n-2}(0, \tilde{s})\} e^{(\sigma_2 - q_2)\tilde{s}} d\tilde{s} \right\} d\tau. \end{aligned}$$

Evaluating this inequality at  $x = 0$  we get

$$\begin{aligned} e_1^n(0, t) &\leq e^{-p_1L+q_1t} \int_0^t K_{1,x}(0, t - \tau) e^{(\sigma_1 - q_1)\tau} \\ &\quad \times \sup_{0 \leq s \leq \tau} \left\{ e^{p_2L+q_2s} \int_0^s K_{2,x}(L, s - \tilde{s}) \sup_{0 \leq \tilde{s} \leq \tilde{s}} \{e_1^{n-2}(0, \tilde{s})\} e^{(\sigma_2 - q_2)\tilde{s}} d\tilde{s} \right\} d\tau \\ &\leq \tilde{C}(t) \int_0^t K_{1,x}(0, t - \tau) \int_0^\tau K_{2,x}(L, \tau - \tilde{s}) \sup_{0 \leq \tilde{s} \leq \tilde{s}} \{e_1^{n-2}(0, \tilde{s})\} d\tilde{s} d\tau \\ &= \tilde{C}(t) \int_0^t K_{2,x}(L, t - \tau) \int_0^\tau K_{2,x}(L, \tau - \tilde{s}) \sup_{0 \leq \tilde{s} \leq \tilde{s}} \{e_1^{n-2}(0, \tilde{s})\} d\tilde{s} d\tau, \end{aligned}$$

where we used that

$$K_{2,x}(L, t) = K_{1,x}(0, t) = \frac{L}{2\sqrt{\pi}\varepsilon^{1/2}t^{3/2}}e^{-\frac{L^2}{4\varepsilon t}}$$

and the function  $\tilde{C}$  given by

$$\tilde{C}(t) := \max\{1, e^{q_2 t}, e^{q_1 t}, e^{(q_1+q_2)t}\}e^{(\sigma_1+\sigma_2-q_1-q_2)t}e^{(p_2-p_1)L}.$$

Note that  $\sigma_1 + \sigma_2 - q_1 - q_2 \geq 0$ . The definition of  $p_1, p_2, q_1, q_2, \sigma_1, \sigma_2$  and Theorem 2.1 imply that there is a constant  $C > 0$  independent of  $\varepsilon, t, L$  such that

$$(3.19) \quad \tilde{C}(p_1, p_2, q_1, q_2, \sigma_1, \sigma_2, t) \leq e^{\frac{C(t+L)}{\varepsilon}}, \quad t > 0.$$

By induction we obtain

$$\sup_{0 \leq \tau \leq t} e_1^{2n}(0, \tau) \leq e^{\frac{C(t+L)}{\varepsilon}n} \sup_{0 \leq \tau \leq t} \{e_1^0(0, \tau)\}K^{2n}(t),$$

where  $K^{2n}$  is the  $2n$ -fold convolution of  $K_{2,x}(L, \cdot)$ , that is, the integral term

$$\begin{aligned} & \int_0^t K_{2,x}(L, t - s_1) \int_0^{s_1} K_{2,x}(L, s_1 - s_2) \\ & \cdots \int_0^{s_{2n-2}} K_{2,x}(L, s_{2n-2} - s_{2n-1}) \int_0^{s_{2n-1}} K_{2,x}(L, s_{2n-1} - s_{2n}) ds_{2n} ds_{2n-1} \cdots ds_2 ds_1. \end{aligned}$$

Since the Laplace transform  $\mathcal{L}[K_{2,x}(L, \cdot)]$  is  $e^{-L\sqrt{s/\varepsilon}}$ , we obtain for the  $2n$ -fold convolution

$$\mathcal{L}[K^{2n}](s) = \frac{1}{s} \left( \mathcal{L}[K_{2,x}(L, \cdot)](s) \right)^{2n} = \frac{1}{s} e^{-2nL\sqrt{s/\varepsilon}}.$$

The inverse Laplace transformation then yields

$$\sup_{0 \leq \tau \leq t} e_1^{2n}(0, \tau) \leq e^{\frac{C(t+L)}{\varepsilon}n} \sup_{0 \leq \tau \leq t} \{e_1^0(0, \tau)\} \int_0^t K_{2,x}(2nL, t - s) ds,$$

and the right-hand side of the last inequality can be simplified by a transform of variables to

$$\sup_{0 \leq \tau \leq t} e_1^{2n}(0, \tau) \leq e^{\frac{C(t+L)}{\varepsilon}n} \sup_{0 \leq \tau \leq t} \{e_1^0(0, \tau)\} \operatorname{erfc} \left( \frac{nL}{\sqrt{\varepsilon t}} \right),$$

where the error function complement  $\operatorname{erfc}$  is defined by

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_x^\infty e^{-y^2} dy.$$

We thus have proved convergence of the algorithm on the interface  $x = 0$  and with a similar argument convergence also follows on the interface  $x = L$ . Applying the maximum principle for problem (3.4) then shows that the error  $e_1^{2n}$  in the interior of the domain  $\Omega_1 \times (0, t)$  is amplified at most by a factor  $\exp(\bar{\theta}t)$  with

$\bar{\theta} = \sup_{(x,\tau) \in (-\infty, L) \times (0, t)} \{|\theta_1(x, \tau)|\}$ . Theorem 2.1, the definition of  $\theta_1$  in (3.6), and (3.19) show that there exists a constant  $C \geq 0$ , independent of  $\varepsilon$ ,  $n$ ,  $t$ , and  $L$ , such that (3.16) holds. The estimate (3.17) follows along the same lines when starting with the second inequality in (3.18).  $\square$

*Note 3.3.* If we apply the expansion  $\sqrt{\pi} \operatorname{erfc}(z) = e^{-z^2}(z^{-1} + \mathcal{O}(z^{-3}))$  for large values  $z > 0$  in the estimate (3.16), we obtain

$$(3.20) \quad \sup_{0 \leq \tau \leq t} \{e_1^{2n}(x, \tau)\} \approx \frac{1}{\sqrt{\pi}} e^{\frac{cn(t+L)}{\varepsilon} - \frac{n^2 L^2}{\varepsilon t}} \frac{\sqrt{\varepsilon t}}{nL} \sup_{0 \leq \tau \leq t} \{e_1^0(0, \tau)\}.$$

For fixed  $T, L, \varepsilon > 0$  we observe that the algorithm converges superlinearly for  $n \rightarrow \infty$  and  $t \leq T$ . In other words, we obtain the same asymptotic result as for the heat equation with linear convection and linear or nonlinear source terms; see [GZ97, Gan98].

*Note 3.4.* Let  $n$  in (3.20) be fixed. Then there exists a  $T = T(n)$  such that (a) the algorithm converges for  $\varepsilon \rightarrow 0$  and  $t < T(n)$ , and (b) the estimate for the error  $e_1^{2n}$  does not converge to 0 for  $\varepsilon \rightarrow 0$  and  $t > T(n)$ . This scenario does not happen for the heat equation with linear convection. In section 3.2 below we apply the Schwarz waveform relaxation algorithm to the hyperbolic case,  $\varepsilon = 0$ , and we will see that in this case it can happen that for all  $n > 0$  there exists a time  $T^* = T^*(n) > 0$  such that the error of the iteration vanishes for  $t < T^*$  and is nonzero for  $t > T^*$ . It turns out that this behavior can only occur for nonlinear convection. Thus the scenario described in (a) and (b) is not an artifact of the proof.

**3.2. The hyperbolic case  $\varepsilon = 0$ .** The Jacobi version of the overlapping Schwarz waveform relaxation algorithm with two subdomains  $\Omega_1 = (-\infty, L)$  and  $\Omega_2 = (0, \infty)$  for the initial value problem (2.1), (2.2) in the hyperbolic limit,  $\varepsilon = 0$ , is given by

$$(3.21) \quad \begin{aligned} \frac{\partial u_1^n}{\partial t} + \frac{\partial}{\partial x} f(u_1^n) &= 0 && \text{in } \Omega_1 \times (0, T), \\ u_1^n(\cdot, 0) &= u_0 && \text{in } \Omega_1, \\ u_1^n(L, \cdot) &= u_2^{n-1}(L, \cdot) && \text{on } [0, T], \end{aligned}$$

and

$$(3.22) \quad \begin{aligned} \frac{\partial u_2^n}{\partial t} + \frac{\partial}{\partial x} f(u_2^n) &= 0 && \text{in } \Omega_2 \times (0, T), \\ u_2^n(\cdot, 0) &= u_0 && \text{in } \Omega_2, \\ u_2^n(0, \cdot) &= u_1^{n-1}(0, \cdot) && \text{on } [0, T], \end{aligned}$$

for iteration index  $n \in \mathbb{N}$  and we are denoting by  $u_i^n$  the entropy solution in each iteration. The initial iterates  $u_1^0$  and  $u_2^0$  are chosen as in (3.3) taking the essential infimum.

The iterative algorithm (3.21), (3.22) is a priori not well defined. Note that problems (3.21), (3.22) are uniquely solvable by Theorem 2.2 for  $u_0 \in L^\infty(\mathbb{R})$  and  $u_2^{n-1}(L, \cdot), u_1^{n-1}(0, \cdot) \in L^\infty((0, T])$ . However it is not clear whether the traces  $u_2^{n-1}(L, \cdot)$  and  $u_1^{n-1}(0, \cdot)$  exist in an appropriate way as  $L^\infty$ -functions. This is assumed subsequently. Note that the local regularity of entropy solutions is a delicate issue which is out of the scope of this paper (cf. [Daf00, Chapter 11.3], for instance).

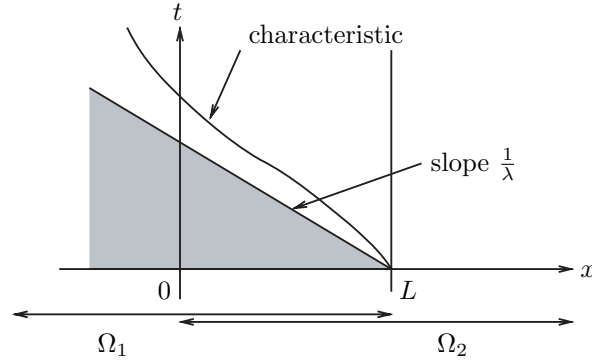


FIG. 1. Estimated region in space-time where the error vanishes in the first iteration on the subdomain  $\Omega_1$ .

Finally we note that the definition of the trace is no problem after discretization of the problem and that is how the algorithm is used in practice.

We define as in the viscous case the errors in the Schwarz iteration by  $e_1^n := u - u_1^n$  on the left subdomain and  $e_2^n := u - u_2^n$  on the right subdomain for  $n \in \mathbb{N}_0$ , where  $u$  is the entropy solution of the initial value problem (2.1), (2.2) in the hyperbolic limit  $\varepsilon = 0$ .

**THEOREM 3.5.** *Let  $\underline{u} = \text{essinf}_{x \in \mathbb{R}} \{u_0(x)\}$  and  $\bar{u} = \text{esssup}_{x \in \mathbb{R}} \{u_0(x)\}$ . For  $n \in \mathbb{N}$  let  $T^* = T^*(n, L) = nL/\lambda$  with*

$$\lambda = \sup_{\underline{u} \leq v \leq \bar{u}} \{|f'(v)|\}.$$

Then we have for  $t \in [0, T] \cap [0, T^*(n, L)]$

$$(3.23) \quad \|e_1^{n+1}\|_{L^1((-\infty, L] \times [0, t])} = \|e_2^{n+1}\|_{L^1([0, \infty) \times [0, t])} = 0.$$

Hence the algorithm converges in a finite number of steps.

*Proof.* The proof relies on the finite speed of propagation property of entropy solutions which manifests itself in the statement (iii) of Theorem 2.2. To apply this statement for  $n = 1$  in  $\Omega_1 \times (0, T)$ , we compare the subdomain solution at the first iteration on  $\Omega_1 \times (0, T)$  with the exact solution restricted to  $\Omega_1 \times (0, T)$ . With the notation of Theorem 2.2, let

$$u_{01} = u_{02} = u_0|_{(-\infty, L]}, \quad g_1 = u_2^0(\cdot, L), \quad g_2 = u(\cdot, L).$$

The unique entropy solutions of the corresponding initial boundary value problems are  $u_1^1$  and  $u|_{\Omega_1 \times (0, T)}$ . Using statement (iii) of Theorem 2.2 and varying the interval  $I$ , we get in particular that  $e_1^1 = u_1^1 - u|_{\Omega_1 \times (0, T)} = 0$  almost everywhere in  $\mathcal{C}_1^1$ , where

$$\mathcal{C}_1^n = \{(x, s) \in \Omega_1 \times (0, T) \mid \lambda s \leq nL - x\}, \quad n \in \mathbb{N};$$

see Figure 1. The analogous argument for  $\Omega_2$  gives  $e_2^1 = 0$  almost everywhere in  $\mathcal{C}_2^1$  given by

$$\mathcal{C}_2^n = \{(x, s) \in \Omega_2 \times (0, T) \mid \lambda s \leq nL + x\}, \quad n \in \mathbb{N}.$$

Thus the errors  $e_1^1$  and  $e_2^1$  vanish on the interfaces in the time interval  $[0, \frac{L}{\lambda}]$  (almost everywhere). Doing the next iteration we find for  $t \leq T^*(1, L)$

$$\|e_1^2\|_{L^1((-\infty, L] \times [0, t])} = \|e_2^2\|_{L^1([0, \infty) \times [0, t])} = 0,$$

and the general result for arbitrary  $n \in \mathbb{N}$  follows by induction.  $\square$

*Example 3.6.* We consider the linear case,  $f(u) = au$ , for a convection speed  $a > 0$ . Taking any initial data and any initial guess for the iterates  $u_1^0$  and  $u_2^0$ , one can easily see that the algorithm (3.21), (3.22) converges in two steps: for  $n = 1$  the error  $e_1^1$  vanishes in  $\Omega_1 \times (0, T)$  since the solution is completely determined by the (correct) initial data, and no boundary data for  $x = L$  can be prescribed for  $a > 0$ . Thus the third equation in (3.21) is ignored. On the second subdomain the error  $e_2^1$  will in general not vanish in the whole of  $\Omega_2 \times (0, T)$ , since the boundary data  $u_2^0$  is not correct in general. For  $n = 2$ , again  $e_1^2$  vanishes in  $\Omega_1 \times (0, T)$ , but now also  $e_2^2$  vanishes in  $\Omega_2 \times (0, T)$ , since  $u_1^1(\cdot, L)$  provides the correct boundary data at the interface. The same result also holds for all functions  $f$  with  $f'(v) \neq 0$  for  $\underline{u} \leq v \leq \bar{u}$ .

*Example 3.7.* For  $L > 0$  we consider the case  $f(u) = u^2/2$  together with the initial condition

$$(3.24) \quad u_0(x) = \begin{cases} 1 & : x < L/2, \\ -1 & : x > L/2. \end{cases}$$

The entropy solution is given by the discontinuous time-independent function  $u(x, t) = u_0(x)$  for  $t \in [0, T]$  and we do not have only one propagation direction in this case. Starting the overlapping Schwarz waveform relaxation algorithm with the initial guess  $u_1^0 = a$  and  $u_2^0 = -a$  for  $a > 1$ , we obtain the restriction of the entropy solution  $u$  to  $(-\infty, L] \times [0, T]$  or  $[0, \infty) \times [0, T]$  when computing the first iterates  $u_1^1$  and  $u_2^1$  on regions whose shape and size depend on  $a$ ; a sketch of the iterates is displayed in Figure 2. For example, on the left subdomain  $\Omega_1$ , the speed of the shock separating 1 from  $-a$  and  $-1$  from  $-a$  increases when  $a$  increases. Thus the part of the boundary  $x = 0$  where the correct data is obtained for the next iteration decreases when  $a$  increases and tends to zero when  $a \rightarrow \infty$ . The same argument also holds for the right subdomain (see Figure 2), which shows that the overlapping Schwarz waveform relaxation algorithm becomes slower as  $a$  increases. Stationary shocks are a generically nonlinear phenomenon which is not observed for linear equations and this example shows that the performance of the overlapping Schwarz waveform relaxation algorithm is affected by shocks.

One might argue that the choice for the initial guess does not agree with the choice for the first iterates  $u_1^0, u_2^0$  as in (3.3). However, one can easily change the initial data  $u_0$  far outside such that the solution in, say,  $[-L, 2L] \times [0, T]$  is not altered due to finite propagation speeds and takes values less than  $-1$ . Then the flux function outside  $[-1, 1]$  is altered such that the wave speeds  $f'(u_1^0)$  and  $f'(u_2^0)$  take the values  $-a$  and  $a$ . This results in the same effect as described above. But the nonconvex flux function would introduce complicated shock-rarefaction patterns, so we omit a detailed construction here.

**4. Overlapping Schwarz waveform relaxation for more than two subdomains.** We now extend the results we obtained for two subdomains of  $\Omega = \mathbb{R}$  in section 3 to the case of  $I > 2$  subdomains,  $I \in \mathbb{N}$ , of a bounded domain  $\Omega = (0, 1)$ . Skipping the index  $\varepsilon$ , we search for the classical solution  $u$  of the initial boundary

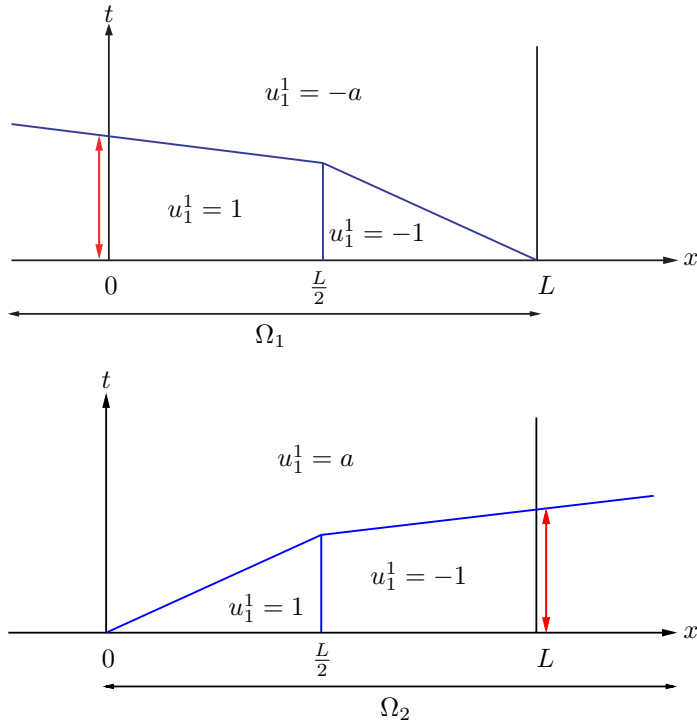


FIG. 2. The top figure displays the entropy solution  $u_1^1$  in  $\Omega_1$  taking the states 1,  $-1$ , and  $-a$  separated by shock waves. The double arrow indicates the part of the line  $x = 0$  where the correct boundary data for the next iteration is obtained. The bottom figure displays the corresponding situation for  $u_2^1$  in  $\Omega_2$  and the part of the line  $x = L$  where the correct boundary data for the next iteration is obtained.

value problem

$$\begin{aligned}
 (4.1) \quad & \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \varepsilon \frac{\partial^2 u}{\partial x^2} \quad \text{in } \Omega \times (0, T), \\
 & u(\cdot, 0) = u_0 \quad \text{in } \Omega, \\
 & u(0, \cdot) = g_0 \quad \text{on } [0, T], \\
 & u(1, \cdot) = g_1 \quad \text{on } [0, T].
 \end{aligned}$$

Here  $u_0 : \Omega \rightarrow \mathbb{R}$  and  $g_1, g_2 : [0, T] \rightarrow \mathbb{R}$  are given smooth functions. To represent the subdomains, we introduce the numbers  $L_1, \dots, L_I \in \mathbb{R}$  and  $R_1, \dots, R_I \in \mathbb{R}$  such that

$$0 = L_1 < L_2 < R_1 < L_3 < \dots < L_I < R_{I-1} < R_I = 1,$$

which leads to a decomposition of  $\Omega$  into overlapping subdomains  $\Omega_i = (L_i, R_i)$ ,  $i = 1, 2, \dots, I$ , as shown in Figure 3. The overlapping Schwarz waveform relaxation algorithm is then given by solving for  $i = 1, \dots, I$  and  $n = 1, 2, \dots$  the problems

$$\begin{aligned}
 (4.2) \quad & \frac{\partial u_i^n}{\partial t} + \frac{\partial}{\partial x} f(u_i^n) = \varepsilon \frac{\partial^2 u_i^n}{\partial x^2} \quad \text{in } \Omega_i \times (0, T), \\
 & u_i^n(\cdot, 0) = u_0 \quad \text{in } \Omega_i, \\
 & u_i^n(L_i, \cdot) = u_{i-1}^{n-1}(L_i, \cdot) \quad \text{on } [0, T], \\
 & u_i^n(R_i, \cdot) = u_{i+1}^{n-1}(R_i, \cdot) \quad \text{on } [0, T],
 \end{aligned}$$

where we identify  $u_0^{n-1}(L_1, \cdot)$  with  $g_0$  and  $u_{I+1}^{n-1}(R_I, \cdot)$  with  $g_1$ . In each  $\Omega_i$  the iteration is started with the constant initial guess

$$(4.3) \quad u_i^0(x, t) = \min \left\{ \inf_{x' \in \Omega} \{u_0(x')\}, \min_{t' \in [0, T]} \{g_0(t')\}, \min_{t' \in [0, T]} \{g_1(t')\} \right\}.$$

For  $i = 1, \dots, I$  we define the error  $e_i^n(x, t) := u(x, t) - u_i^n(x, t)$  in  $\Omega_i \times (0, T)$ . Note that  $e_0^n(0, t) = e_{I+1}^n(1, t) = 0$  and also  $e_i^n(x, 0) = 0$ . The choice (4.3) ensures the nonnegativity of the error for all iterations and all subdomains. We also introduce the maximum error  $e^n$  defined by

$$e^n = \max_{i=1, \dots, I} \left\{ \sup_{0 \leq t \leq T} \{e_{i+1}^n(R_i, t)\}, \sup_{0 \leq t \leq T} \{e_{i-1}^n(L_i, t)\} \right\}.$$

**THEOREM 4.1.** *Suppose that the minimum overlap between the subdomains is given by  $L > 0$ ,*

$$(4.4) \quad L := \min_{i=1, \dots, I-1} (R_i - L_{i+1}).$$

*Then the overlapping Schwarz waveform relaxation algorithm (4.2) with  $I > 2$  subdomains for the initial boundary value problem (4.1) converges superlinearly and independently of the number  $I$  of subdomains: for each  $T > 0$  we have*

$$(4.5) \quad e^{2n} \leq e^{\frac{Cn(T+L)}{\varepsilon}} \operatorname{erfc} \left( \frac{nL}{\sqrt{\varepsilon T}} \right) e^0.$$

*The constant  $C$  does not depend on  $\varepsilon, T, I,$  and  $n$ .*

*Proof.* We first derive an explicit bound for the error by successive construction of supersolutions. The error  $e_i^n$  is a solution of the problem

$$(4.6) \quad \begin{aligned} \frac{\partial e_i^n}{\partial t} + f'(u) \frac{\partial e_i^n}{\partial x} + \theta_i^n e_i^n &= \varepsilon \frac{\partial^2 e_i^n}{\partial x^2} && \text{in } \Omega_i \times (0, T), \\ e_i^n(\cdot, 0) &= 0 && \text{in } \Omega_i, \\ e_i^n(L_i, \cdot) &= e_{i-1}^{n-1}(L_i, \cdot) && \text{on } [0, T], \\ e_i^n(R_i, \cdot) &= e_{i+1}^{n-1}(R_i, \cdot) && \text{on } [0, T], \end{aligned}$$

where the function  $\theta_i^n$  is defined analogously to (3.6). One can check that  $e_i^n = e_{iL}^n + e_{iR}^n$ , where  $e_{iL}^n$  satisfies

$$\begin{aligned} \frac{\partial e_{iL}^n}{\partial t} + f'(u) \frac{\partial e_{iL}^n}{\partial x} + \theta_i^n e_{iL}^n &= \varepsilon \frac{\partial^2 e_{iL}^n}{\partial x^2} && \text{in } \Omega_i \times (0, T), \\ e_{iL}^n(\cdot, 0) &= 0 && \text{in } \Omega_i, \\ e_{iL}^n(L_i, \cdot) &= 0 && \text{on } [0, T], \\ e_{iL}^n(R_i, \cdot) &= e_{i+1}^{n-1}(R_i, \cdot) && \text{on } [0, T], \end{aligned}$$

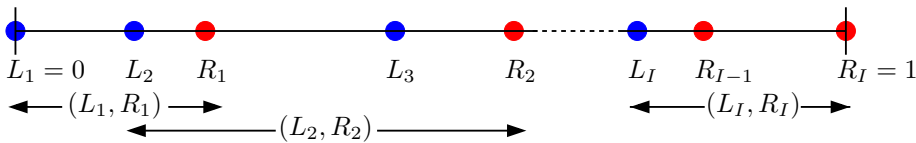


FIG. 3. Sketch of the multidomain decomposition for the interval  $(0, 1)$ .

and  $e_{iR}^n$  satisfies

$$\begin{aligned} \frac{\partial e_{iR}^n}{\partial t} + f'(u) \frac{\partial e_{iR}^n}{\partial x} + \theta_i^n e_{iR}^n &= \varepsilon \frac{\partial^2 e_{iR}^n}{\partial x^2} && \text{in } \Omega_i \times (0, T), \\ e_{iR}^n(\cdot, 0) &= 0 && \text{in } \Omega_i, \\ e_{iR}^n(L_i, \cdot) &= e_{i-1}^{n-1}(L_i, \cdot) && \text{in } [0, T], \\ e_{iR}^n(R_i, \cdot) &= 0 && \text{in } [0, T]. \end{aligned}$$

Furthermore, the classical solution  $\tilde{e}_{iL}^n : (-\infty, R_i] \times \mathbb{R} \rightarrow \mathbb{R}$  of the quarter-space problem

$$(4.7) \quad \begin{aligned} \frac{\partial \tilde{e}_{iL}^n}{\partial t} + f'(u) \frac{\partial \tilde{e}_{iL}^n}{\partial x} + \theta_i^n \tilde{e}_{iL}^n &= \varepsilon \frac{\partial^2 \tilde{e}_{iL}^n}{\partial x^2} && \text{in } (-\infty, R_i] \times (0, T), \\ \tilde{e}_{iL}^n(\cdot, 0) &= 0 && \text{in } (-\infty, R_i), \\ \tilde{e}_{iL}^n(R_i, \cdot) &= e_{i+1}^{n-1}(R_i, t) && \text{on } [0, T], \end{aligned}$$

with solutions decaying at infinity,  $\lim_{x \rightarrow -\infty} \tilde{e}_{iL}^n(x, t) = 0$  for  $t \in [0, T]$ , satisfies

$$\tilde{e}_{iL}^n(x, t) \geq e_{iL}^n(x, t), \quad x \in \Omega_i, \quad t \in (0, T).$$

The functions  $f(u)$  and  $\theta_i^n$  are so far defined in  $\Omega \times (0, T)$ ; outside this set  $f(u)$  and  $\theta_i^n$  in the quarter-plane problem (4.7) have to be understood as smooth extensions taking only values from the range of  $f(u)$  and  $\theta_i^n$  in  $[0, 1] \times [0, T]$ . Similarly we obtain a function  $\tilde{e}_{iR}^n$  with  $\tilde{e}_{iR}^n \geq e_{iR}^n$  in  $\Omega_i$ . In Lemma 3.1 we have constructed supersolutions for the classical solutions of quarter-space problems. Using the same technique here, we obtain the function

$$\bar{e}_{iL}^n(x, t) = \exp(p_{iL}x + q_{iL}t) \int_0^t K_{iL}(x, t - \tau) g_{iL}(\tau) d\tau,$$

which satisfies

$$(4.8) \quad \bar{e}_{iL}^n(x, t) \geq \tilde{e}_{iL}^n(x, t), \quad x \in \Omega_i, \quad t \in (0, T),$$

and solves the constant coefficient problem

$$\begin{aligned} \frac{\partial \bar{e}_{iL}^n}{\partial t} + a_{iL} \frac{\partial \bar{e}_{iL}^n}{\partial x} + b_{iL} \bar{e}_{iL}^n &= \varepsilon \frac{\partial^2 \bar{e}_{iL}^n}{\partial x^2} && \text{in } (-\infty, R_i) \times (0, T), \\ \bar{e}_{iL}^n(\cdot, 0) &= 0 && \text{in } (-\infty, R_i), \\ \bar{e}_{iR}^n(R_i, t) &= \exp(\sigma_{iL}t) \sup_{0 \leq \tau \leq t} e_{i+1}^{n-1}(R_i, \tau), && t \in [0, T], \end{aligned}$$

where the constants are defined as in Lemma 3.1 by

$$\begin{aligned} p_{iL} &:= \frac{a_{iL}}{2\varepsilon}, & q_i &:= -\frac{a_{iL}^2}{4\varepsilon} - b_{iL}, \\ a_{iL} &:= \inf_{(x,t) \in \Omega_i} \{f'(u(x, t))\}, & b_{iL} &:= \inf_{(x,t) \in \Omega_i} \left\{ \theta_i^n(x, t) + (f'(u(x, t)) - a_{iL}) \frac{a_{iL}}{2\varepsilon} \right\}, \\ \sigma_{iL} &:= \begin{cases} q_{iL} & : \quad q_{iL} \geq 0, \\ 0 & : \quad \text{otherwise,} \end{cases} \end{aligned}$$

and the kernel function and  $g_{iL}$  in the integral are given by

$$\begin{aligned} K_{iL,x}(x, t) &:= -\frac{1}{2\sqrt{\pi}} \frac{x - R_i}{\varepsilon^{1/2} t^{3/2}} \exp\left(-\frac{(x - R_i)^2}{4\varepsilon t}\right), \\ g_{iL}(t) &:= \exp(-p_{iL}R_i + (\sigma_{iL} - q_{iL})t) \sup_{0 \leq \tau \leq t} \{e_{i+1}^{n-1}(R_i, \tau)\}. \end{aligned}$$



With the analogous definitions for  $p_{iR}$ ,  $q_{iR}$ ,  $a_{iR}$ ,  $b_{iR}$ ,  $\sigma_{iR}$ , and

$$K_{iR,x}(x, t) := \frac{1}{2\sqrt{\pi}} \frac{x - L_i}{\varepsilon^{1/2} t^{3/2}} \exp\left(-\frac{(x - L_i)^2}{4\varepsilon t}\right),$$

$$g_{iR}(t) := \exp(-p_{iR}L_i + (\sigma_{iR} - q_{iR})t) \sup_{0 \leq \tau \leq t} \{e_{i-1}^{n-1}(L_i, \tau)\},$$

we define the function

$$\bar{e}_{iR}^n(x, t) = \exp(p_{iR}x + q_{iR}t) \int_0^t K_{iR}(x, t - \tau) g_{iR}(\tau) d\tau,$$

which is an upper bound on the quarter-plane problem solution,

$$(4.9) \quad \bar{e}_{iR}^n(x, t) \geq \tilde{e}_{iR}^n(x, t), \quad x \in \Omega_i, \quad t \in (0, T).$$

From (4.8), (4.9), and the explicit formulas for the supersolutions we derive

$$\begin{aligned} e_i^n(x, t) &\leq \bar{e}_{iL}^n(x, t) + \bar{e}_{iR}^n(x, t) \\ &= \exp(p_{iL}x + q_{iL}t) \int_0^t K_{iL,x}(x, t - \tau) g_{iL}(\tau) d\tau \\ &\quad + \exp(p_{iR}x + q_{iR}t) \int_0^t K_{iR,x}(x, t - \tau) g_{iR}(\tau) d\tau. \end{aligned}$$

Then there exists a constant  $\tilde{C} \geq 0$  that depends on  $p_{iL/R}$ ,  $q_{iL/R}$ ,  $a_{iL/R}$ ,  $b_{iL/R}$ ,  $\sigma_{iL/R}$ , the time  $T$ , and the decomposition  $L_1, R_1, \dots, L_I, R_I$  such that

$$\begin{aligned} &\max\{e_i^n(R_{i-1}, t), e_i^n(L_{i+1}, t)\} \\ &\leq \tilde{C} \left( \int_0^t K_{iL,x}(R_{i-1}, t - \tau) d\tau + \int_0^t K_{iR,x}(R_{i-1}, t - \tau) d\tau \right. \\ &\quad \left. + \int_0^t K_{iL,x}(L_{i+1}, t - \tau) d\tau + \int_0^t K_{iR,x}(L_{i+1}, t - \tau) d\tau \right) e^{n-1} \\ &\leq 2\tilde{C} \left( \int_0^t K_{iR,x}(R_{i-1}, t - \tau) d\tau + \int_0^t K_{iL,x}(L_{i+1}, t - \tau) d\tau \right) e^{n-1}. \end{aligned}$$

The last inequality is a consequence of the fact that the integrals are solutions of quarter-space problems for the heat equations where the boundary function is monotonically increasing. Therefore the solutions decay monotonically in space if one moves in the direction where the domain is unbounded. Thus the long-range error contribution involving  $K_{iL,x}(R_{i-1}, t - \tau)$  and  $K_{iR,x}(L_{i+1}, t - \tau)$  can be estimated by the short-range error contributions. Similarly one can now also estimate the contributions from the various overlap sizes using the minimum of the overlaps: letting  $K_{L,x}(t) := \frac{1}{2\sqrt{\pi}} \frac{L}{\varepsilon^{1/2} t^{3/2}} \exp(-\frac{L^2}{4\varepsilon t})$ , we obtain

$$\max\{e_i^n(R_{i-1}, t), e_i^n(L_{i+1}, t)\} \leq 4\tilde{C} \int_0^t K_{L,x}(t - \tau) d\tau e^{n-1}.$$

As in the proof of Lemma 3.1 we denote now by  $K_{L,x}^{2n}(t)$  the  $2n$ -fold convolution of the kernel  $K_{L,x}(t)$  and obtain

$$e^{2n} \leq (4\tilde{C})^{2n} K_{L,x}^{2n}(t) e^0 \leq (4\tilde{C})^{2n} \operatorname{erfc}\left(\frac{nL}{\sqrt{\varepsilon t}}\right) e^0,$$

which is formula (4.5).  $\square$

**5. Behavior of the algorithm over long time intervals.** The superlinear convergence estimates for the overlapping Schwarz waveform relaxation algorithm derived in the previous sections depend on the time interval under consideration. To get more insight into the behavior of the algorithm over long time intervals, we study in this section the limiting case of a steady state solution and we apply an overlapping Schwarz method to this steady state case. We study here only the Burgers equation to illustrate that the convergence rate can be arbitrarily slow in this case, and convergence can be lost if the viscosity goes to zero. Similar behavior has been in fact observed for long time calculations for the Burgers equation in [GK00].

**5.1. Boundary value problem.** We consider the boundary value problem obtained for the steady state case of the Burgers equation on the domain  $\Omega = (-1, 1)$ ,

$$(5.1) \quad \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2} - \frac{1}{2} \frac{\partial (u^\varepsilon)^2}{\partial x} = 0, \quad u^\varepsilon(-1) = 1, \quad u^\varepsilon(1) = -1.$$

The general solution for the differential equation (5.1) can be computed in closed form,

$$(5.2) \quad u^\varepsilon(x) = -\sqrt{2}C \tanh\left(C \frac{x+D}{\sqrt{2}\varepsilon}\right), \quad C, D \in \mathbb{R}.$$

To satisfy the boundary conditions, the two constants  $C, D \in \mathbb{R}$  have to satisfy the nonlinear system of equations

$$\frac{-1}{\sqrt{2}C} = \tanh\left(C \frac{D-1}{\sqrt{2}\varepsilon}\right), \quad \frac{1}{\sqrt{2}C} = \tanh\left(C \frac{D+1}{\sqrt{2}\varepsilon}\right).$$

This implies in particular  $D = 0$ . In the limit as  $\varepsilon \rightarrow 0$  the constant  $C = C(\varepsilon)$  converges to  $\frac{1}{\sqrt{2}}$ ; thus in this regime it is bounded from above and below independently of  $\varepsilon$ . Therefore, the solution  $u^\varepsilon$  contains an *internal* layer for  $\varepsilon$  small. This type of solution is typical for *nonlinear* flux functions  $f$  with extrema. It does not exist for linear advection-diffusion problems which allow for (ordinary) boundary layers only. Note that we have for almost all  $x \in [-1, 1]$

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x) = u^0(x) \equiv \begin{cases} 1 & : x < 0, \\ -1 & : x > 0. \end{cases}$$

We consider now the corresponding singular boundary value problem

$$(5.3) \quad \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = 0, \quad u(-1) = 1, \quad u(1) = -1,$$

where the function  $u^0$  satisfies the boundary conditions and is a distributional solution of the differential equation (5.3). Such solutions are not unique; in fact all piecewise constant functions taking only the values  $\pm 1$  and satisfying the boundary conditions at  $x = \pm 1$  are solutions in that sense. This failure of uniqueness is crucial for the behavior of the overlapping Schwarz algorithm for (5.1) that we will analyze now.

**5.2. Overlapping Schwarz algorithm.** We first investigate the influence of the internal layer within the overlap for  $\varepsilon \ll 1$ . The overlapping Schwarz algorithm with the two subdomains  $\Omega_1 = (-1, L)$  and  $\Omega_2 = (-L, 1)$  for problem (5.1) is given

by

$$\begin{aligned}
 (5.4) \quad & \varepsilon \frac{\partial u_1^n}{\partial x^2} + u_1^n \frac{\partial u_1^n}{\partial x} = 0 && \text{in } \Omega_1, \\
 & u_1^n(-1) = 1, \\
 & u_1^n(L) = u_2^{n-1}(L), \\
 & \varepsilon \frac{\partial u_2^n}{\partial x^2} + u_2^n \frac{\partial u_2^n}{\partial x} = 0 && \text{in } \Omega_2, \\
 & u_2^n(1) = -1, \\
 & u_2^n(-L) = u_1^{n-1}(-L),
 \end{aligned}$$

where  $L > 0$  is a constant that defines the overlap  $2L$ . The iteration is started with  $u_1^0 = 1, u_2^0 = -1$ , and we have skipped the index  $\varepsilon$  to simplify the notation.

**THEOREM 5.1** (linear convergence). *The overlapping Schwarz algorithm defined in (5.4) converges for  $0 < L < 1$  linearly to the solution of (5.1). The asymptotic convergence rate as  $\varepsilon \rightarrow 0$  is*

$$(5.5) \quad 1 - 2e^{-\frac{1-L}{\varepsilon}} + o(e^{-\frac{1}{\varepsilon}}).$$

*Proof.* From (5.2) we obtain

$$u_1^n(x) = -\sqrt{2}C_1 \tanh\left(C_1 \frac{x+D_1}{\sqrt{2\varepsilon}}\right),$$

where the two constants of integration  $C_1$  and  $D_1$  are determined by the boundary conditions. This leads to the system of nonlinear equations for  $C_1$  and  $D_1$ ,

$$\begin{aligned}
 D_1 &= 1 - \frac{\sqrt{2\varepsilon}}{C_1} \operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_1}\right), \\
 u_2^{n-1}(L) &= -\sqrt{2}C_1 \tanh\left(C_1 \frac{L+D_1}{\sqrt{2\varepsilon}}\right).
 \end{aligned}$$

Inserting the first equation into the second one, we find the transcendental equation

$$(5.6) \quad u_2^{n-1}(L) = \sqrt{2}C_1 \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_1}\right) - \frac{C_1(1+L)}{\sqrt{2\varepsilon}}\right),$$

which defines the constant  $C_1$  implicitly. Denoting its solution by  $C_1(u_2^{n-1}(L))$ , we find over one iteration

$$(5.7) \quad u_1^n(x) = \sqrt{2}C_1(u_2^{n-1}(L)) \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_1(u_2^{n-1}(L))}\right) - \frac{C_1(u_2^{n-1}(L))(1+x)}{\sqrt{2\varepsilon}}\right).$$

Similarly we find for the solution on the second subdomain

$$u_2^n(x) = \sqrt{2}C_2(u_1^{n-1}(-L)) \tanh\left(\operatorname{arctanh}\left(\frac{C_2(u_1^{n-1}(-L))(1-x)}{\sqrt{2\varepsilon}} - \frac{1}{\sqrt{2}C_2(u_1^{n-1}(-L))}\right)\right),$$

where the function  $C_2(u_1^{n-1}(-L))$  is implicitly defined by the transcendental equation

$$(5.8) \quad u_1^{n-1}(-L) = -\sqrt{2}C_2 \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_2}\right) - \frac{C_2(1+L)}{\sqrt{2\varepsilon}}\right).$$

We are interested in the asymptotic convergence of this fixed point iteration on the interfaces  $x = -L$  and  $x = L$ . We could evaluate  $u_2^{n-1}(x)$  at  $x = L$  and insert the result into the relation for  $u_1^n(x)$  to find directly a nonlinear relation between  $u_1^n(-L)$  and  $u_1^{n-2}(-L)$  and analyze the fixed point of this iteration. But this would lead to

formulas of twice the complexity of the formulas of the single iteration. To avoid this, we define the two auxiliary functions  $g(C_1) := u_1^n(-L, C_1)$  and  $h(C_2) := u_2^{n-1}(L, C_2)$ . With those, the above double step can be written as

$$(5.9) \quad u_1^n(-L) = g(C_1(u_2^{n-1}(L))) = g(C_1(h(C_2(u_1^{n-2}(-L))))).$$

Now note that  $g(y) = -h(y)$  and  $C_1(y) = C_2(-y)$ , which leads to

$$\begin{aligned} u_1^n(-L) &= g(C_1(h(C_2(u_1^{n-2}(-L)))))) \\ &= g(C_1(-g(C_2(u_1^{n-2}(-L)))))) \\ &= g(C_2(g(C_2(u_1^{n-2}(-L))))), \end{aligned}$$

and hence it suffices to analyze the fixed point iteration

$$(5.10) \quad y^n = G(y^n) := g(C_2(y^n)),$$

where by (5.7) the function  $g(C_2)$  is defined by

$$(5.11) \quad g(C_2) = \sqrt{2}C_2 \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_2}\right) - \frac{C_2(1-L)}{\sqrt{2}\varepsilon}\right),$$

and the function  $C_2(y)$  is defined implicitly by

$$(5.12) \quad y = -\sqrt{2}C_2 \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_2}\right) - \frac{C_2(1+L)}{\sqrt{2}\varepsilon}\right);$$

see (5.8). To analyze the convergence of this fixed point iteration, we need to study the derivative

$$(5.13) \quad G'(y) = \frac{d}{dy}g(C_2(y)) = \frac{dg}{dC_2} \frac{dC_2(y)}{dy} = \frac{dg}{dC_2} \left(\frac{dy}{dC_2}\right)^{-1} =: \tilde{G}'(C_2)$$

at the fixed point  $y^*$  on the left or for the expression on the right at the corresponding  $C_2^*$ . Note that the fixed point iteration depends on  $\varepsilon$ ; we have  $\tilde{G}'(C_2, \varepsilon)$ , and we want to study this function at the fixed point  $C_2^*(\varepsilon)$  which also depends on  $\varepsilon$ . We thus need to expand  $\tilde{G}'(C_2^*(\varepsilon), \varepsilon)$  for small  $\varepsilon$ . Unfortunately the expression  $C_2^*(\varepsilon)$  is not available in closed form, but we have the inverse,  $\varepsilon(C_2^*)$  from the equation of the solution  $u(-1, C_2^*(\varepsilon)) = 1$ , which gives

$$1 = -\sqrt{2}C_2^* \tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_2^*}\right) - \frac{2C_2^*}{\sqrt{2}\varepsilon}\right)$$

or

$$(5.14) \quad \varepsilon = \varepsilon(C_2^*) = \frac{C_2^*}{\sqrt{2}\operatorname{arctanh}\left(\frac{1}{\sqrt{2}C_2^*}\right)}.$$

Hence we expand  $\tilde{G}'(C_2^*, \varepsilon(C_2^*))$  about  $C_2^* = \frac{1}{\sqrt{2}}$  which corresponds to  $\varepsilon = 0$ . To simplify the expansion, we perform a further change of variables,

$$(5.15) \quad d = C_2^* - 1/\sqrt{2},$$

with which we find, after simplifying, the expression

$$(5.16) \quad \tilde{G}'(d) = \frac{2d(\sqrt{2} + d)(T^2 - 1)(L - 1)A - (\sqrt{2}Td + T - 1)(T + \sqrt{2}d + 1)}{2d(\sqrt{2} + d)(T^2 - 1)(L + 1)A + (\sqrt{2}Td + T + 1)(T - \sqrt{2}d - 1)},$$

where the functions  $A = A(d)$  and  $T = T(d, L)$  are given by

$$(5.17) \quad A(d) = \operatorname{arctanh}\left(\frac{1}{\sqrt{2d+1}}\right), \quad T(d, L) = \tanh(LA(d)).$$

The difficulty in the expansion process lies in the fact that  $L$  is an arbitrary real number in the hyperbolic tangent in the function  $T(d, L)$ . We first expand  $A(d)$  for  $d$  small, taking into account the singularity,

$$(5.18) \quad A(d) = -\ln\left(\frac{\sqrt{d}}{2^{\frac{1}{4}}}\right) + \frac{1}{2\sqrt{2}}d - \frac{1}{8}d^2 + \mathcal{O}(d^3).$$

Hence  $A(d)$  tends to infinity as  $d$  goes to zero. We now use an asymptotic expansion of

$$\tanh(\ln(z)) = 1 - \frac{2}{z^2} + \frac{2}{z^4} - \frac{2}{z^6} + \mathcal{O}\left(\frac{1}{z^8}\right)$$

and insert  $z := \exp(LA)$  into this expansion. This leads to

$$(5.19) \quad T(d, L) = 1 - 2^{1-\frac{L}{2}}d^L + 2^{1-L}d^{2L} - 2^{1-\frac{3L}{2}}d^{3L} + \dots + L2^{\frac{1-L}{2}}d^{1+L} + o(d^{1+L}).$$

Using the structure of (5.16), we see that the numerator and the denominator contain the same terms, but some with the sign changed. Denoting by  $p = 2d(\sqrt{2} + d)(T^2 - 1)$  the factor in front of  $A$  and rearranging, we find

$$\tilde{G}'(d) = -\frac{(1 + \sqrt{2}d)(1 - T^2) - 2Td(\sqrt{2} + d) + p(L - 1)A}{(1 + \sqrt{2}d)(1 - T^2) + 2Td(\sqrt{2} + d) - p(L + 1)A},$$

where in both the numerator and the denominator the first term is  $O(d^L)$ , the second is  $O(d)$ , and the third is  $O(d^{1+L} \log(d))$ . Hence for  $d$  small, the last term can be neglected. Inserting the expressions (5.19) of  $T$  and (5.18) of  $A$  and expanding again for  $d$  small, we get for the leading order term

$$\tilde{G}'(d) \approx -1 + \sqrt{2}2^{L/2}d^{1-L}.$$

To find the asymptotic convergence rate as a function of  $\varepsilon$ , we expand the implicit relation between  $d$  and  $\varepsilon$  given in (5.14) and (5.15), which leads to

$$\varepsilon = \frac{1}{\ln(2) - \ln(\sqrt{2}d)} + \mathcal{O}(d),$$

and thus  $d \approx \sqrt{2}e^{-\frac{1}{\varepsilon}}$ , which inserted into the asymptotic convergence rate gives the result (5.5).  $\square$

Theorem 5.1 shows that the convergence rate of the Schwarz algorithm applied to this model problem tends exponentially fast to 1 in modulus as  $\varepsilon$  goes to zero. This indicates that the overlapping Schwarz waveform relaxation algorithm applied to nonlinear convection-dominated conservation laws over long time intervals can be slow if there are steady shock waves in the overlap.

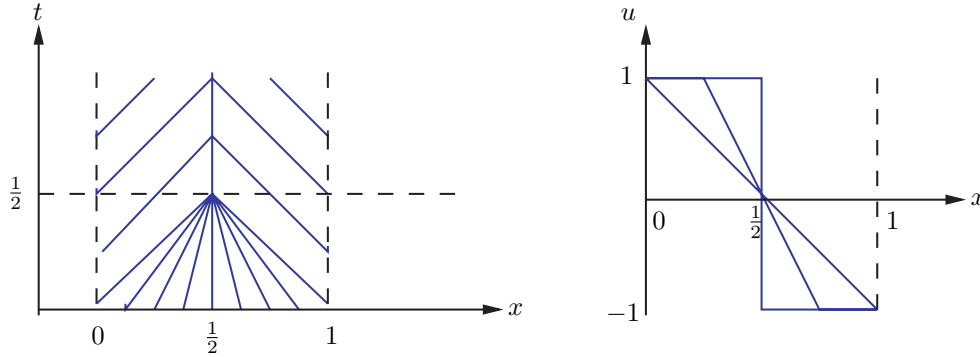


FIG. 4. The left picture displays the characteristics for the entropy solution  $u^0$  of (6.1) colliding at  $(1/2, t)$  for  $t \geq 1/2$ . The right graph shows  $u^0$  for  $t = 0, 1/4$  and  $t \geq 1/2$ .

**6. Numerical experiments.** We first illustrate the superlinear convergence rate of the overlapping Schwarz waveform relaxation algorithm given in Theorem 3.2. We solve the following initial boundary value problem for the Burgers equation with  $\varepsilon \geq 0$  on the bounded domain  $\Omega = (0, 1)$ :

$$\begin{aligned}
 (6.1) \quad & u_t^\varepsilon + \left(\frac{(u^\varepsilon)^2}{2}\right)_x = \varepsilon u_{xx}^\varepsilon && \text{in } \Omega \times (0, T), \\
 & u^\varepsilon(\cdot, 0) = 1 - 2x && \text{in } \Omega, \\
 & u^\varepsilon(0, \cdot) = 1 && \text{on } [0, T], \\
 & u^\varepsilon(1, \cdot) = -1 && \text{on } [0, T].
 \end{aligned}$$

For the case  $\varepsilon = 0$  we can compute the entropy solution of (6.1) explicitly: for  $(x, t) \in [0, 1] \times [0, 1/2)$  we find

$$u^0(x, t) = \begin{cases} 1 & : x < t, \\ \frac{2}{-1 + 2t} \left(x - \frac{1}{2}\right) & : t \leq x \leq 1 - t, \\ -1 & : x > 1 - t, \end{cases}$$

and for  $(x, t) \in [0, 1] \times [1/2, \infty)$  the entropy solution is

$$u^0(x, t) = \begin{cases} 1 & : x < 1/2, \\ -1 & : x > 1/2. \end{cases}$$

The solution is continuous (but not classical) up to  $t = 1/2$  and then produces a stationary shock for all times; see Figure 4. We use the overlapping Schwarz waveform relaxation algorithm with the two subdomains  $\Omega_1 = (0, \frac{1}{2} + L)$  and  $\Omega_2 = (\frac{1}{2} - L, 1)$  with the overlap parameter  $L = 0.1$ . We choose  $T = 0.6$  and a centered finite difference scheme in space, explicit for the nonlinear term and implicit for the Laplacian. The discretization parameters are  $\Delta x = 0.01$  and  $\Delta t = 0.003$ . Figure 5 shows the superlinear convergence behavior of the overlapping Schwarz waveform relaxation algorithm for various values of the viscosity parameter  $\varepsilon$ . This numerical experiment shows that the smaller the viscosity is, the faster the algorithm converges, as predicted by the analysis, when the algorithm is in the superlinear convergence regime.

In the next numerical experiment, we solve the steady viscous Burgers equation

on the domain  $\Omega = (-1, 1)$ ,

$$\begin{aligned} \left(\frac{(u^\varepsilon)^2}{2}\right)_x &= \varepsilon u_{xx} && \text{in } \Omega, \\ u(-1) &= 1, \\ u(1) &= -1, \end{aligned}$$

using an overlapping Schwarz method with the two subdomains  $\Omega_1 = (-1, L)$  and  $\Omega_2 = (-L, 1)$  and the overlap parameter  $L = 0.2$ . The solution of this problem consists of a shock centered at  $x = 0$ . We discretize the problem with the same centered finite difference scheme as in the evolution case and solve the nonlinear system of equations with iteration in time. Figure 6 shows the convergence behavior of the overlapping Schwarz method for various values of the viscosity parameter  $\varepsilon$ . This experiment shows that the overlapping Schwarz algorithm converges linearly, and one can see that the smaller the viscosity parameter  $\varepsilon$  is, the slower the algorithm becomes, as predicted by Theorem 5.1. Since the convergence rate is exponential in the viscosity, already for a moderate value of the viscosity parameter  $\varepsilon$ , the algorithm seems to stand still.

**7. Conclusions.** We have analyzed the performance of the overlapping Schwarz waveform relaxation algorithm applied to convection-dominated nonlinear conservation laws. We have proved that the algorithm’s asymptotic convergence rate is super-linear and independent of the number of subdomains. The convergence rate depends strongly on the viscosity parameter: the smaller the viscosity is, the faster the algorithm becomes. To learn more about the algorithm’s performance over long time intervals, we have also analyzed the convergence rate of the overlapping Schwarz method applied to a special steady case of the Burgers equation. This analysis revealed that the asymptotic convergence rate is linear and depends exponentially on the viscosity

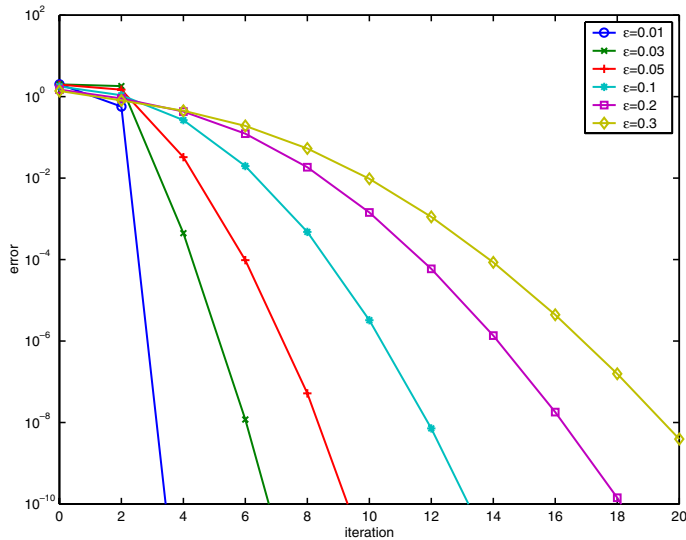


FIG. 5. Superlinear convergence behavior of the overlapping Schwarz waveform relaxation algorithm for the viscous Burgers equation with various values for the viscosity parameter  $\varepsilon$ .

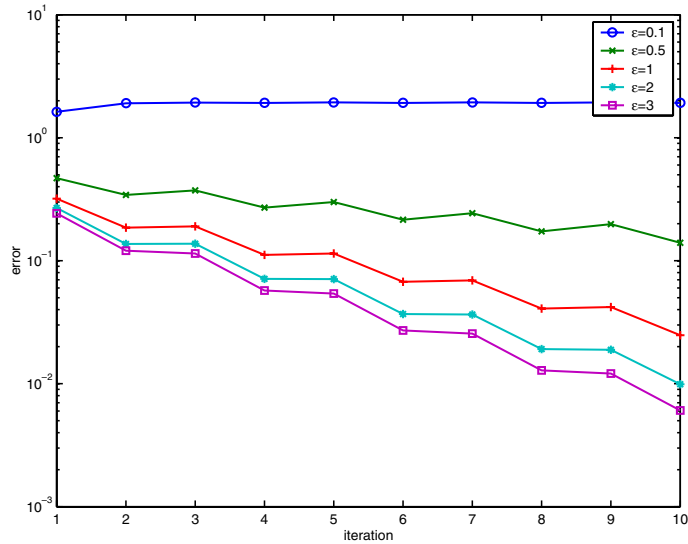


FIG. 6. Linear convergence behavior of the overlapping Schwarz method for the steady viscous Burgers equation for various values of the viscosity parameter  $\varepsilon$ .

parameter. In contrast to the superlinear case, however, we showed that the smaller the viscosity is, the slower the algorithm becomes.

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